(Most) Big mapping class groups fail the Tits Alternative.

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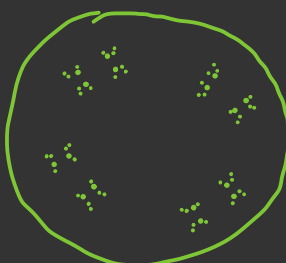
A big mapping class group means the MCG of a surface Σ whose π_1 is ∞ly generated.

∞ genus
(2 ends, both "accumulated by genus")

punctures



disk − (Cantor set)

Theorem If Σ has (i) ∞ genus, (ii) ∞ly many punctures, or (iii) Contains disk−(Cantor set) then its Mapping class gp $G = G_\Sigma$ fails the Tits alternative.

Means: contains a subgp (f.g. in our case) that is NOT virtually solvable & contains no free gp of rk ≥ 2.

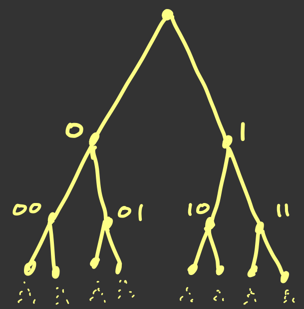
Refers to Tits' thm that GL_n(field) contains no such subgp.

Examples of such Σ found by Lanier-Loving & Patel-Aougab-Vlamis.

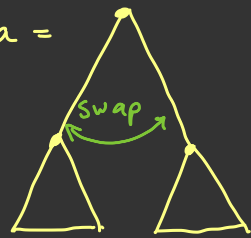
Finite-type mapping class gps do satisfy a strong form of T.A. (Ivanov; McCarthy)

Grigorchuk's gp

$\Gamma \le \mathrm{Aut}$(rooted binary tree T)
$= \langle a, b, c, d \rangle$

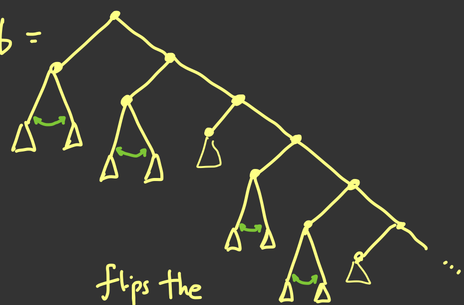


$a =$



swaps subtrees rigidly

$b =$



flips the bit after the first 0, if this bit's position is $\equiv 2, 0 \bmod 3$.

c, d similar with $2,0$ replaced by $0,1$ resp. $1,2$

Famous for being of intermediate growth.

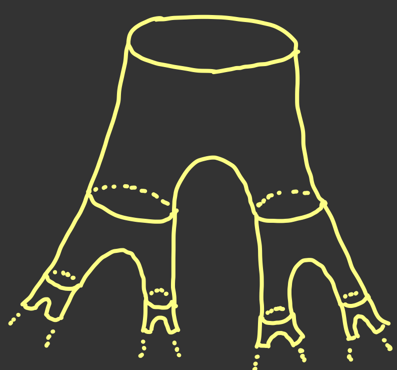
All we need: (i) f.g. but ∞
(ii) torsion
(iii) no free subgps.

(Γ actually acts on suitable Σ, embedding in G_Σ. But we don't use this) For Σ as in our thm, we find $\hat{\Gamma} \le G_\Sigma$ st.

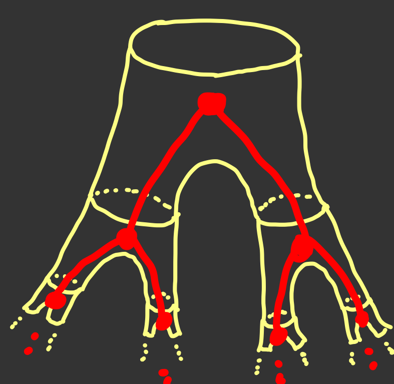
$$1 \longrightarrow \text{abelian} \longrightarrow \hat{\Gamma} \longrightarrow \Gamma \longrightarrow 1.$$

This structure $\Longrightarrow$ $\hat{\Gamma}$ not virtually solvable & has no F_2.
(Thm follows)

The Basic construction:



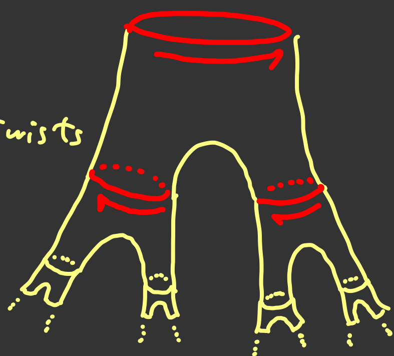
"looks like"



Convert Γ's generators to surface auts:

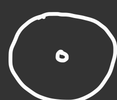
to swap two subtrees rigidly:

squares to composition of 3 Dehn twists



$\Longrightarrow$ MCG(Disk − (Cantor set)) fails Tits Alternative.

How to do Σ having ∞ly many punctures?

⌜Puncture = subsurface $\cong$  disk−(pt)⌟

Approximate T by a sequence of finite subtrees:

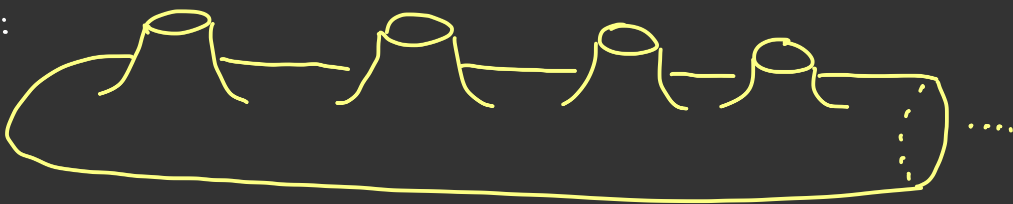
For each of $a, b, c, d \in \Gamma$, construct $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ $MCG(\Sigma)$ as we did for disk − (Cantor set), acting on each 2^n-punctured subsurface according to action on top n layers of tree T.

$\Longrightarrow$ If Σ has ∞ly many punctures then G_Σ fails T.A.
[& same works for ∞ genus.]

$\boxed{\text{Thm}}$ If Σ has ∞ type, but only finitely many ∂ components, then G_Σ fails T.A.

What's left? ∞ly many ∂ components:

Eg.:



Same arg works except that conventionally MCG is $\{$diffeos acting by 1 on $\partial\}$/isotopy. So can't permute. This is an artifact of defn of MCG, intended to "keep only the Dehn twists around ∂ components" [rather than a $\mathrm{Homeo}^+(S^1)$ for each ∂ comp.]

More natural to assume each ∂ circle of Σ comes with an identification with S^1, & these circles can be permuted subject to respecting these identifications.

$$1 \to \text{old MCG} \to \text{new MCG} \to \begin{Bmatrix} \text{permutations} \\ \text{of } \partial \text{ comps} \end{Bmatrix}$$

With this change, same pf works for ∞ly many ∂ circles.

Weirder examples: $\mathbb{R}^2 - (\infty\text{ly many open disks}) - (\text{pts on } \partial)$

Here the pt. is that MCG preserves each ∂ "circle."
Don't know if it satisfies Tits Alternative.

Thank you for your attention!