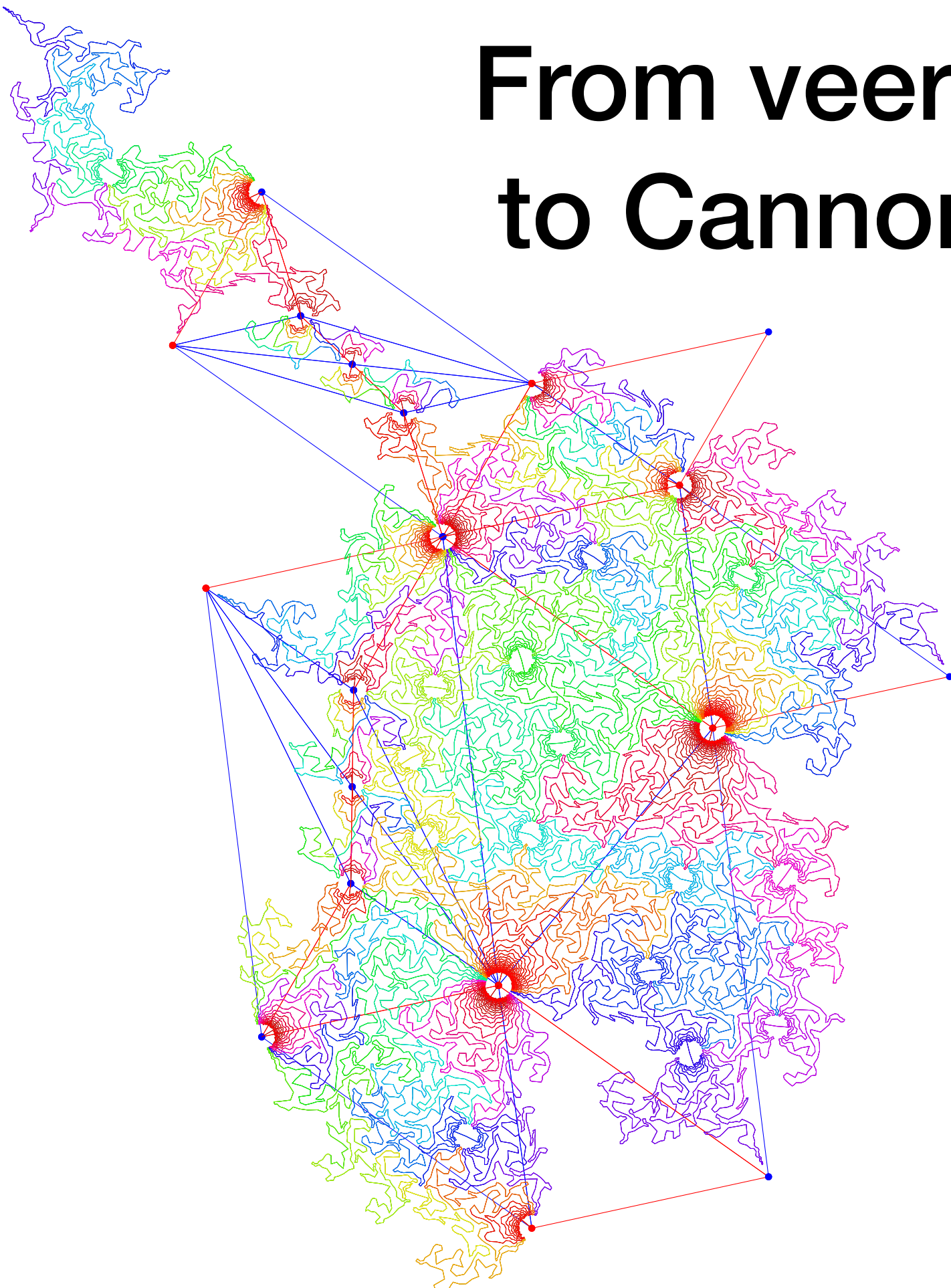


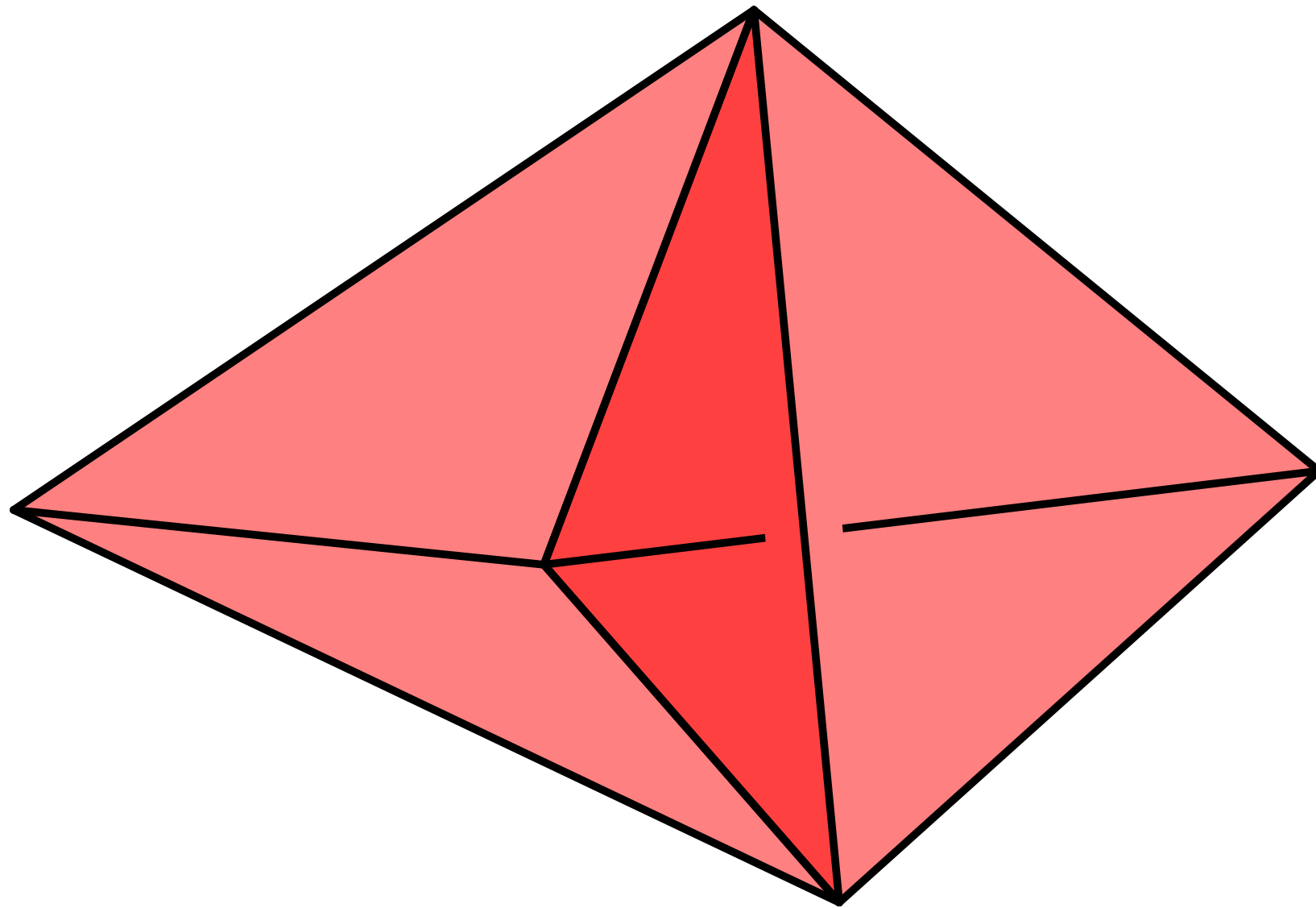
From veering triangulations to Cannon-Thurston maps



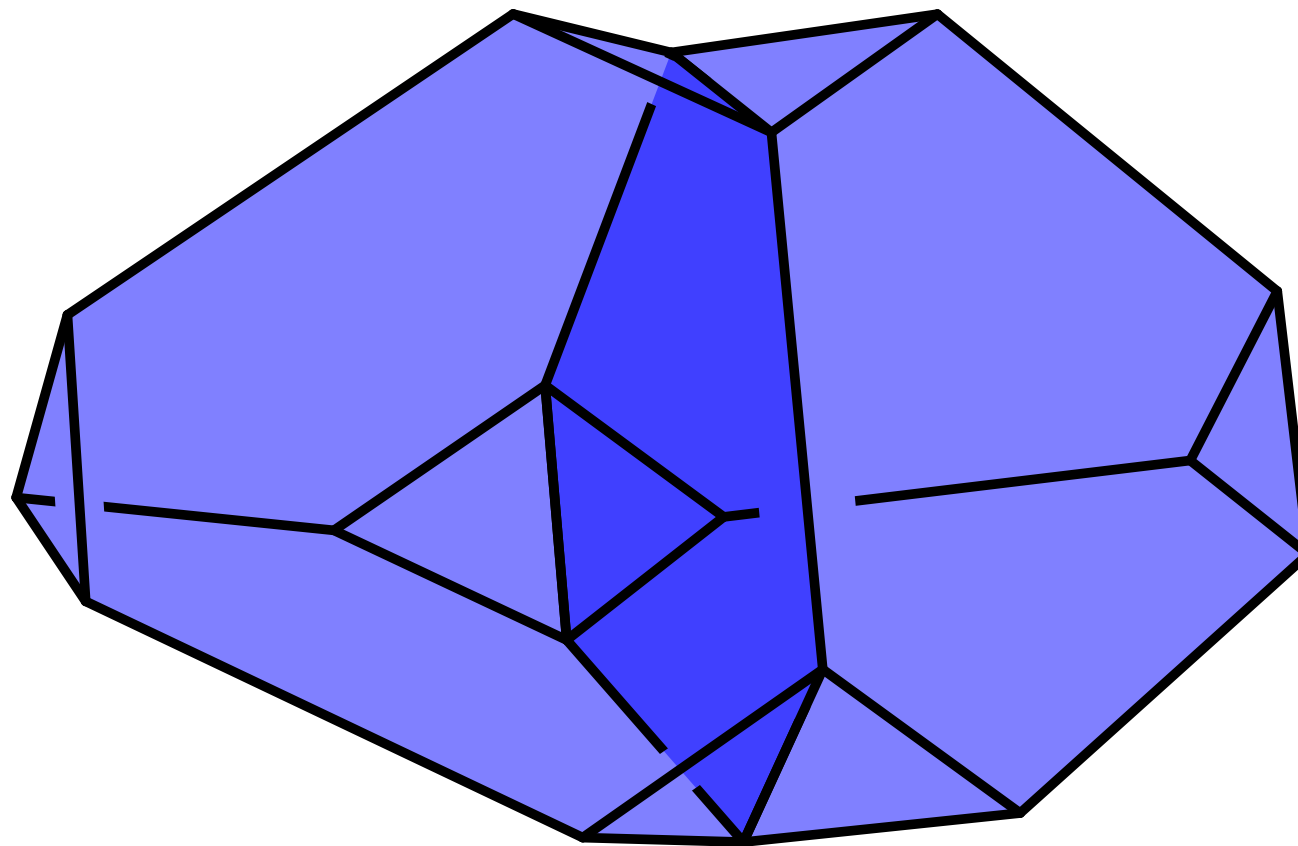
Henry Segerman
Oklahoma State University

joint work with
Jason Manning and
Saul Schleimer

Triangulations

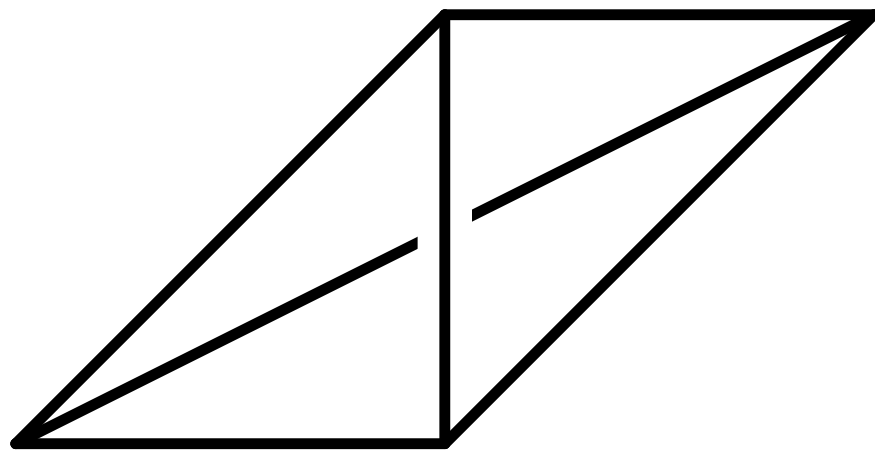
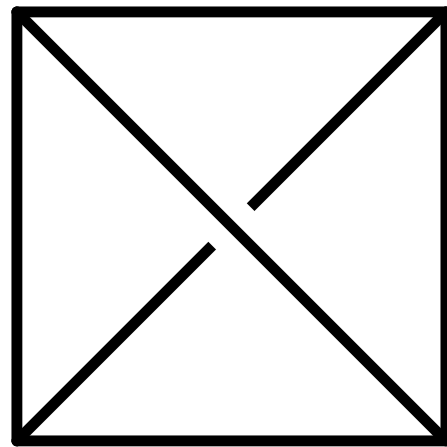


Ideal Triangulations (Thurston)



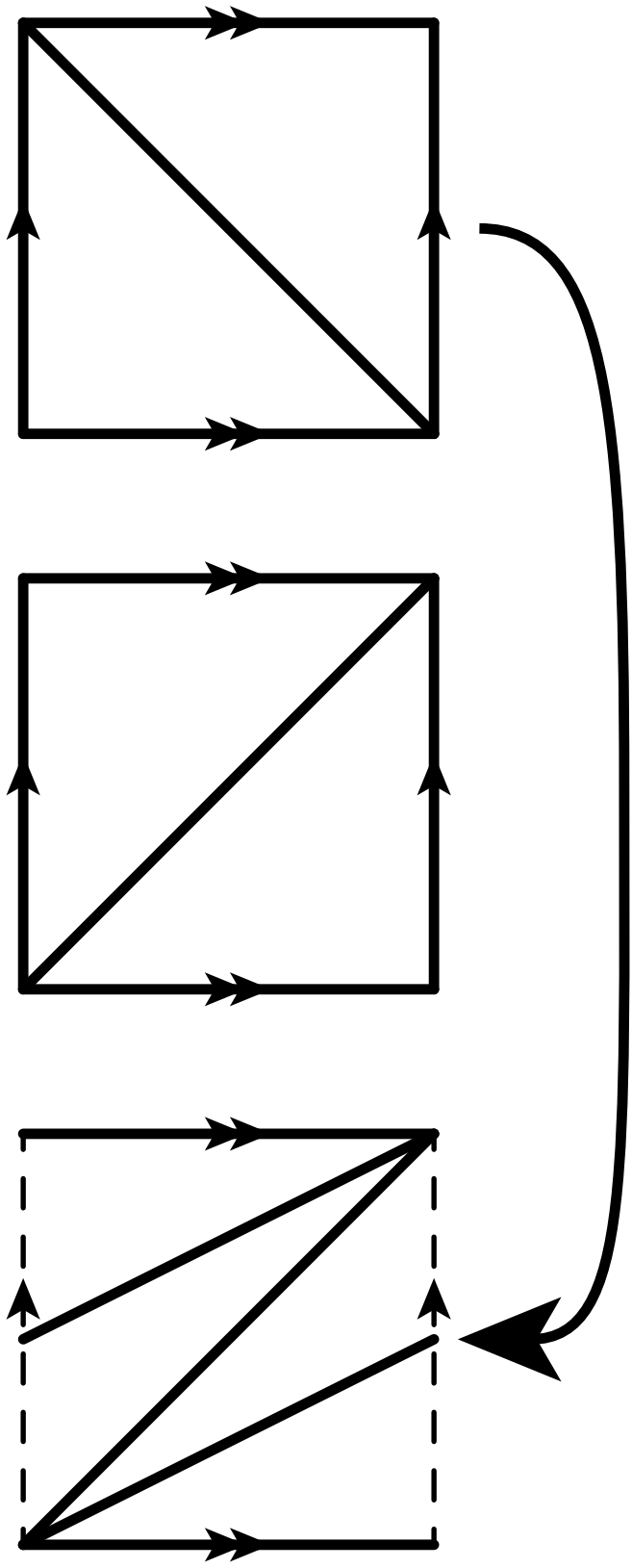
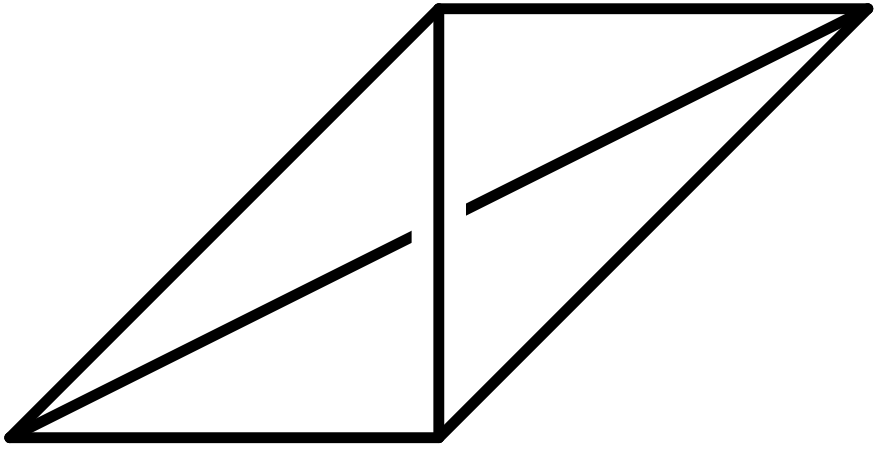
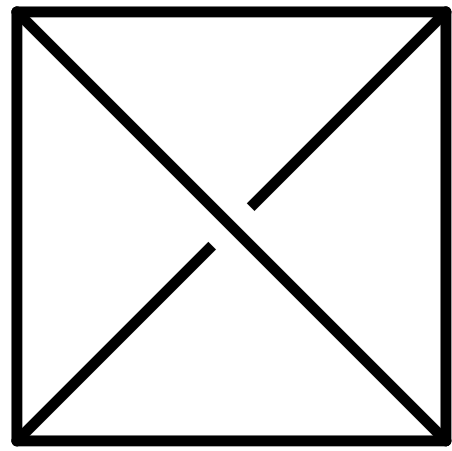
Example: the figure 8 knot complement

$$M = S^3 - \text{Figure 8}$$



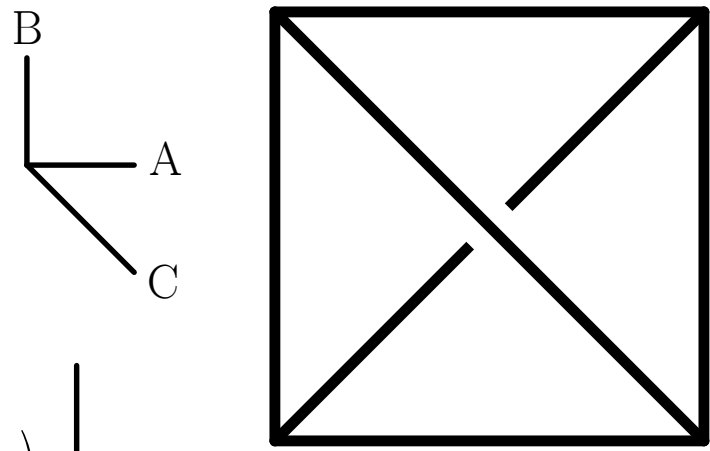
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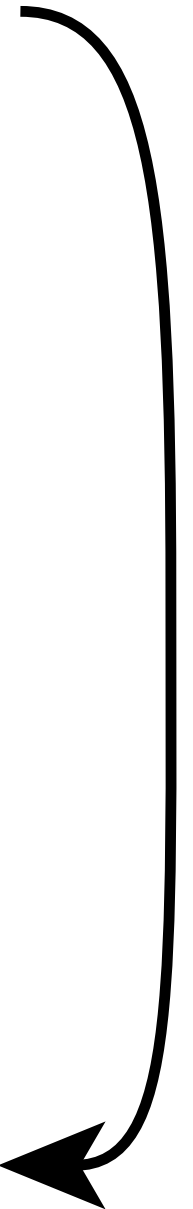
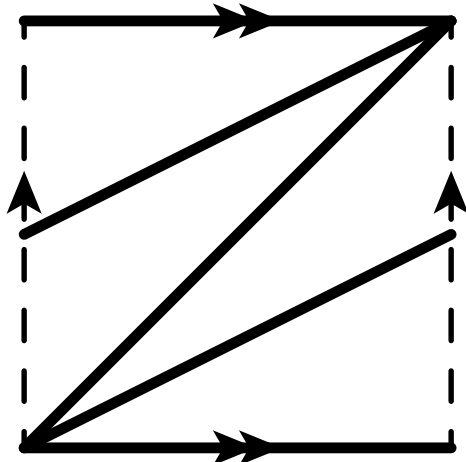
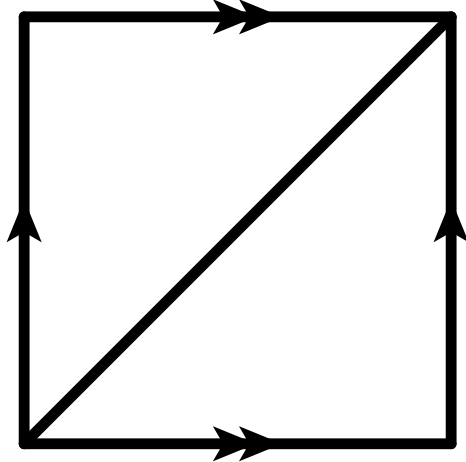
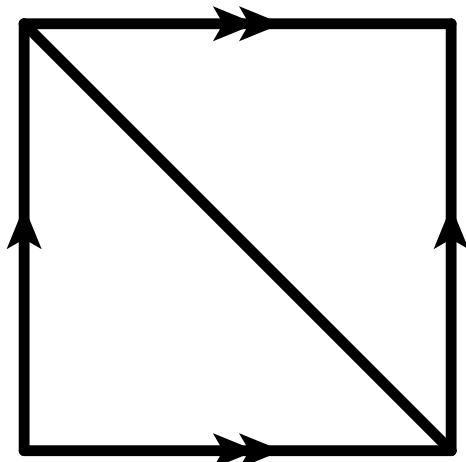
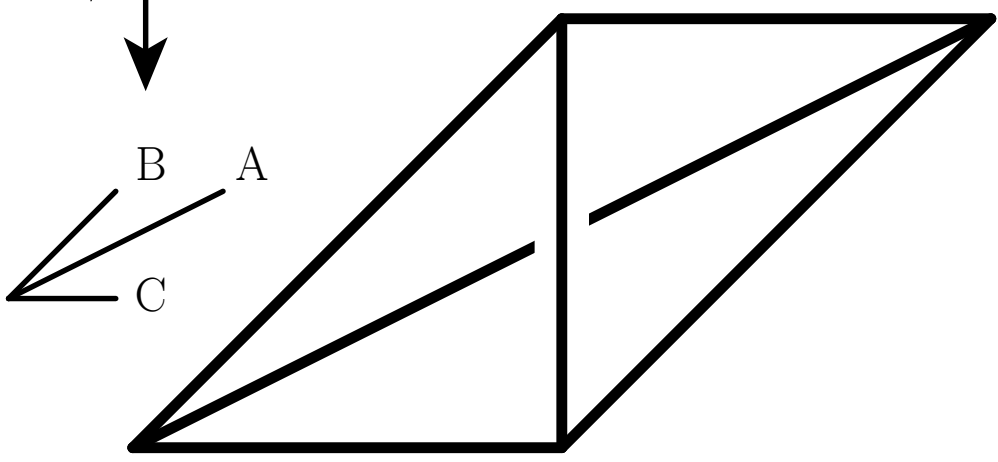


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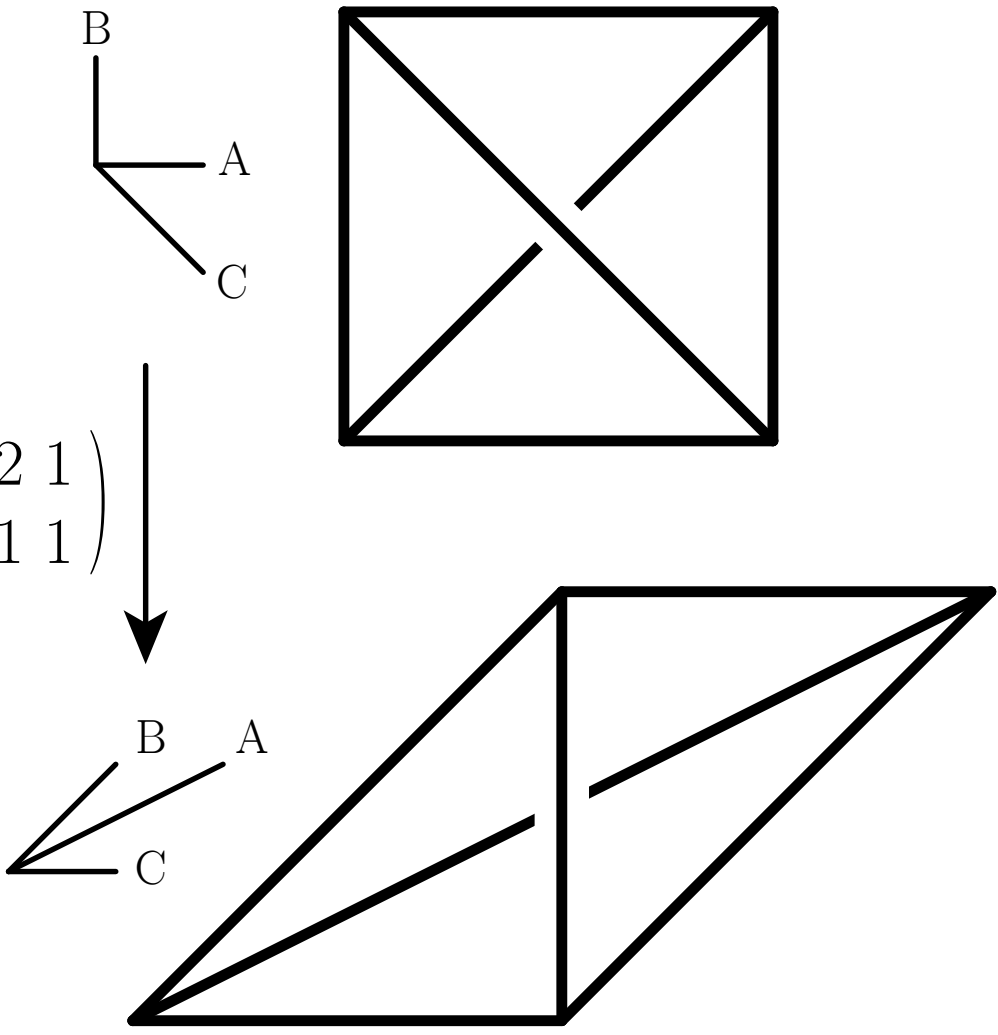
$$\varphi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$



φ

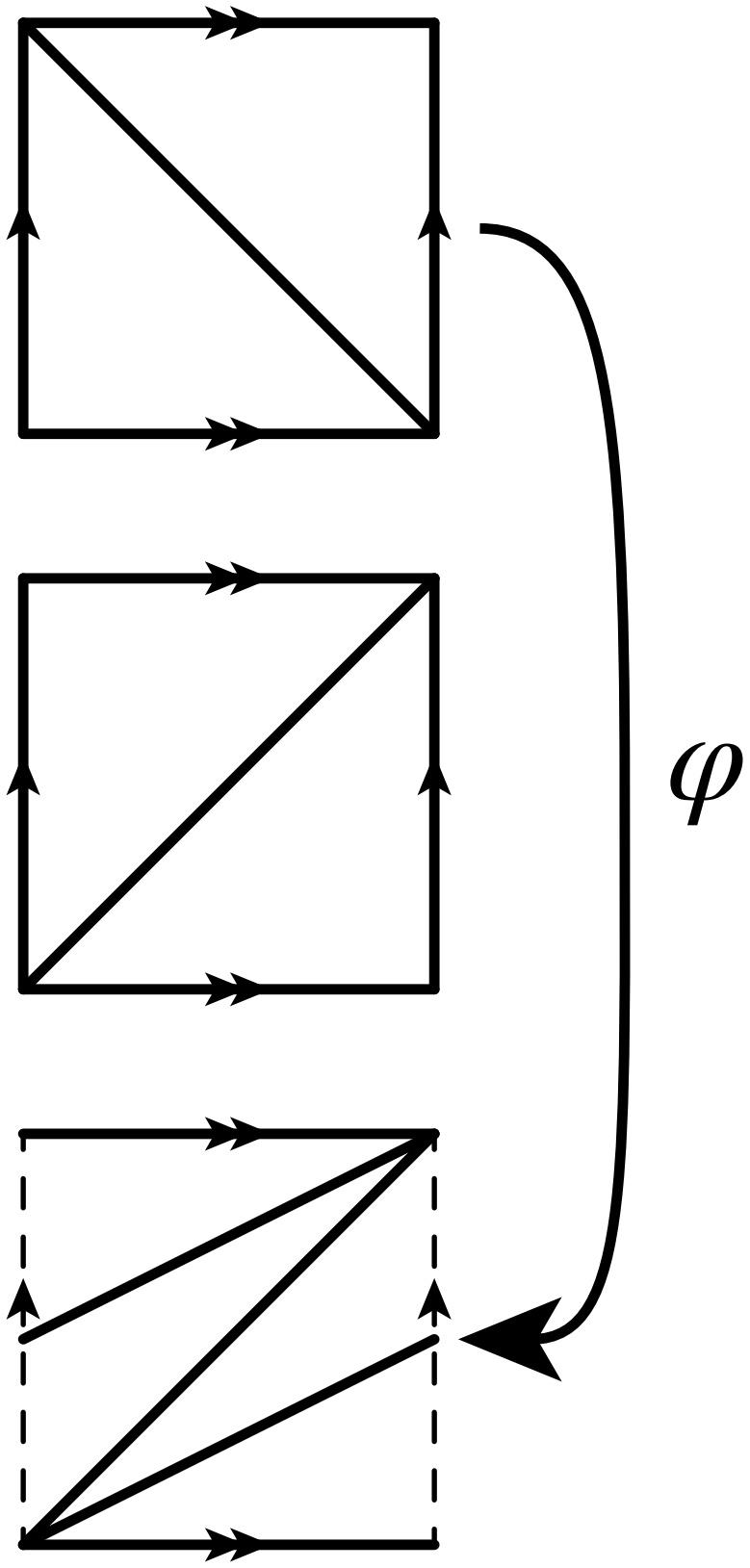
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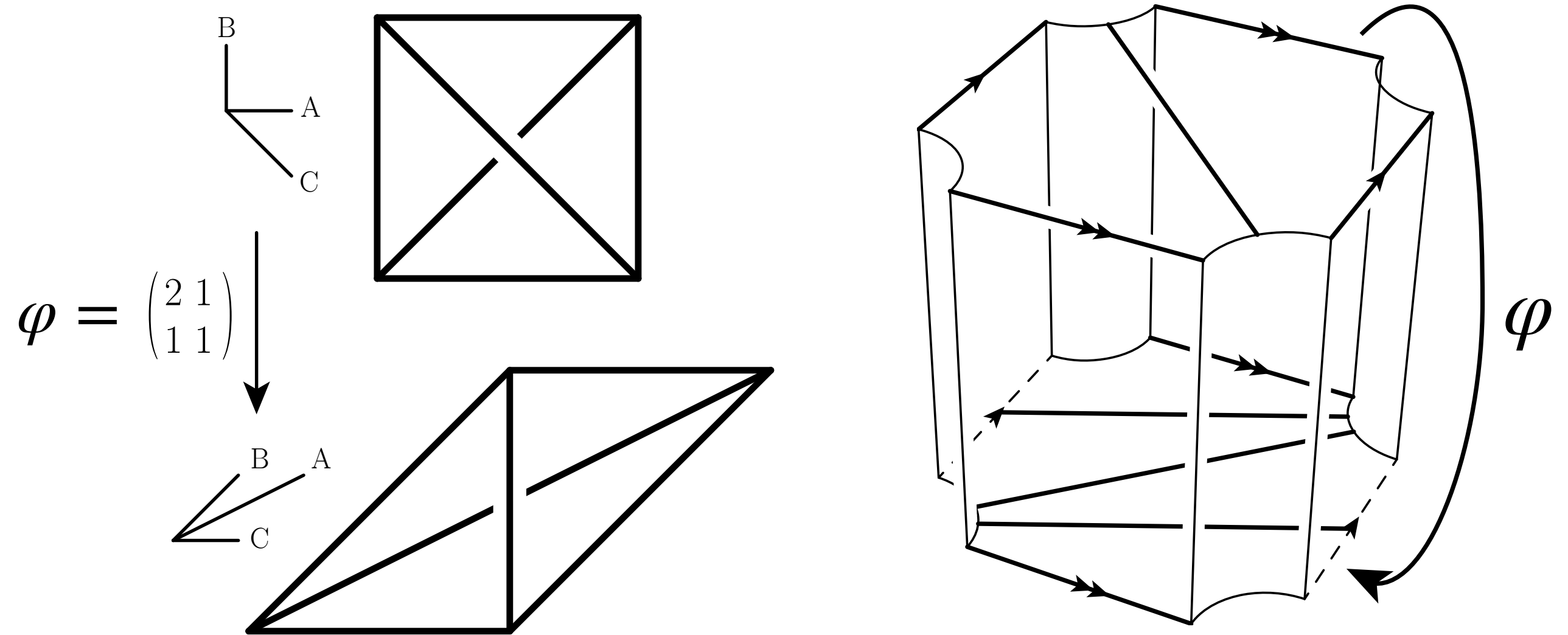
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This is a *layered* triangulation



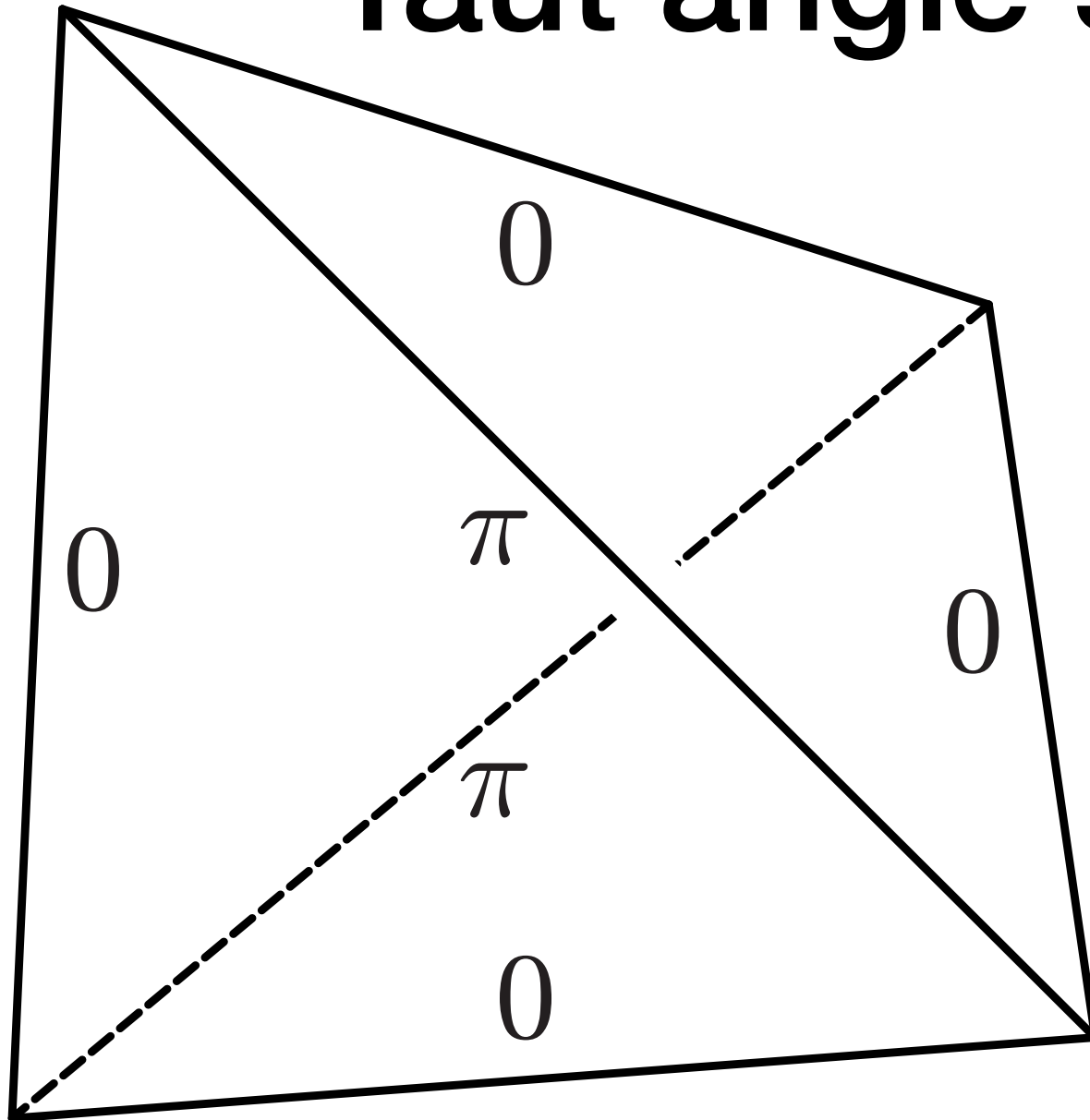
Example: the figure 8 knot complement

$$M = S^3 - \text{figure 8 knot}$$



This is a *layered* triangulation of the *surface bundle* $M \cong (T_*^2 \times I)/(x,1) \sim (\varphi(x),0)$

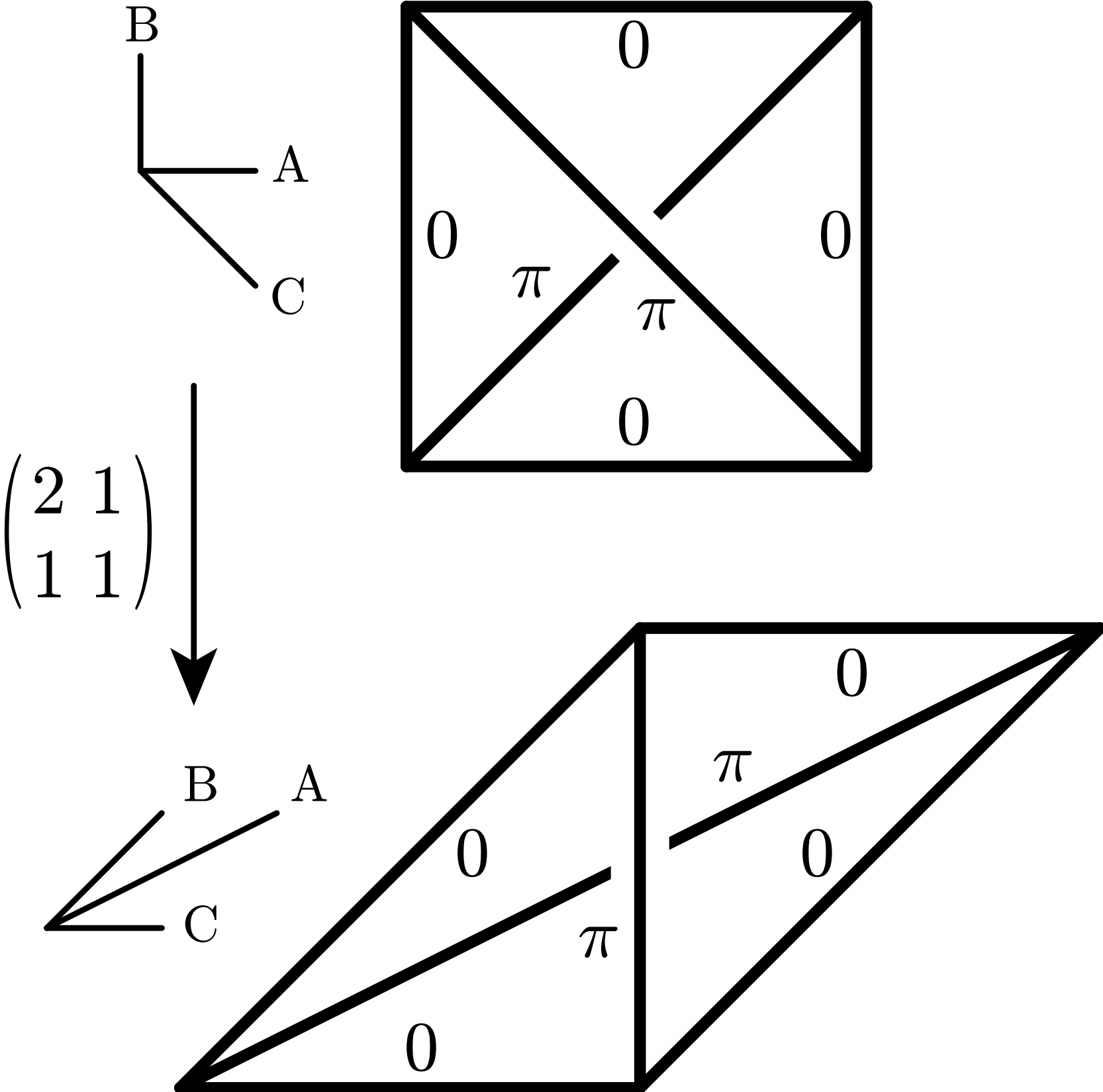
Taut angle structure (Lackenby)



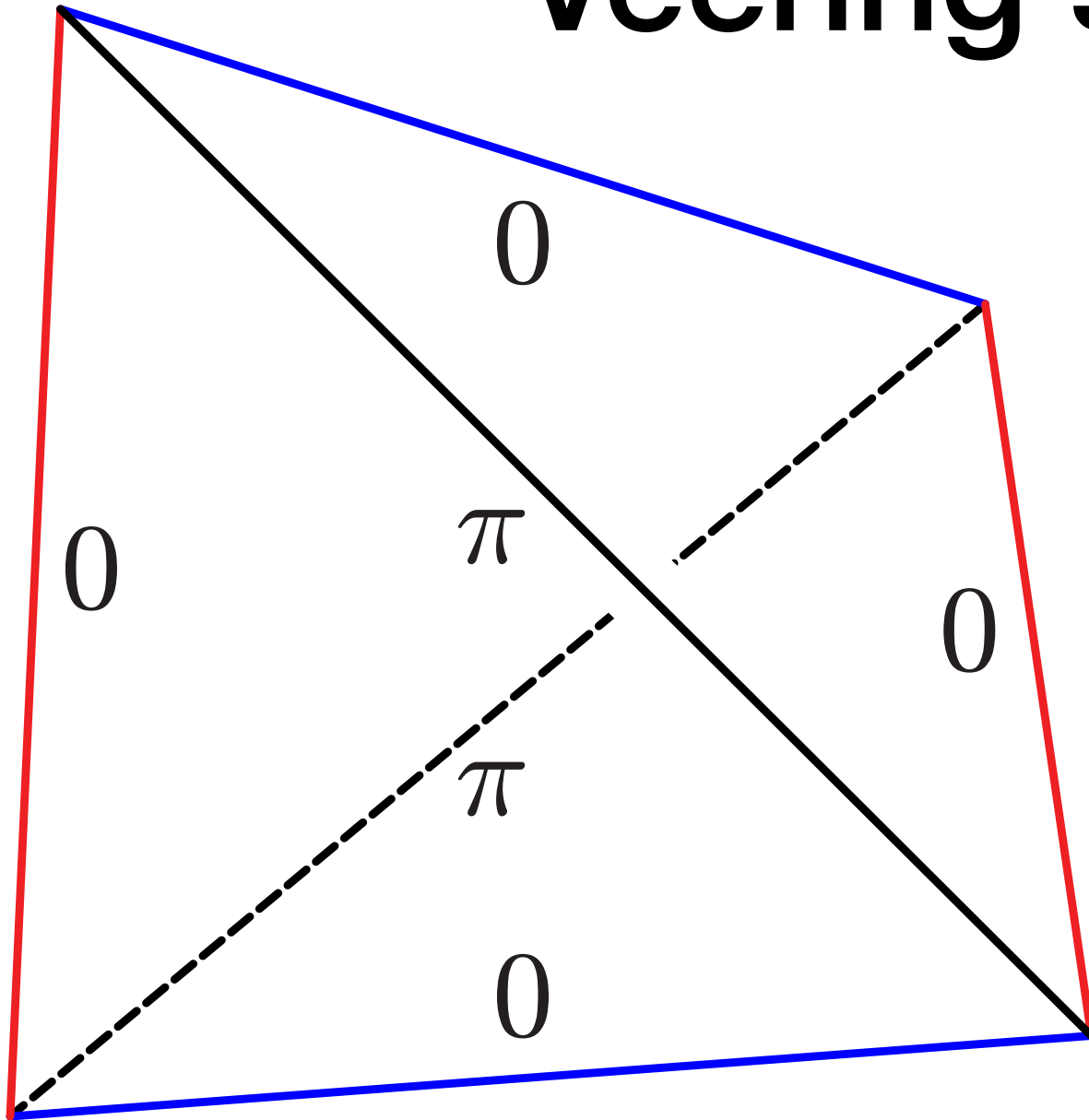
Assign angles 0 and π to the edges of each tetrahedron.

We require that the sum of angles around each edge of the triangulation is 2π .

Example: the figure 8 knot complement



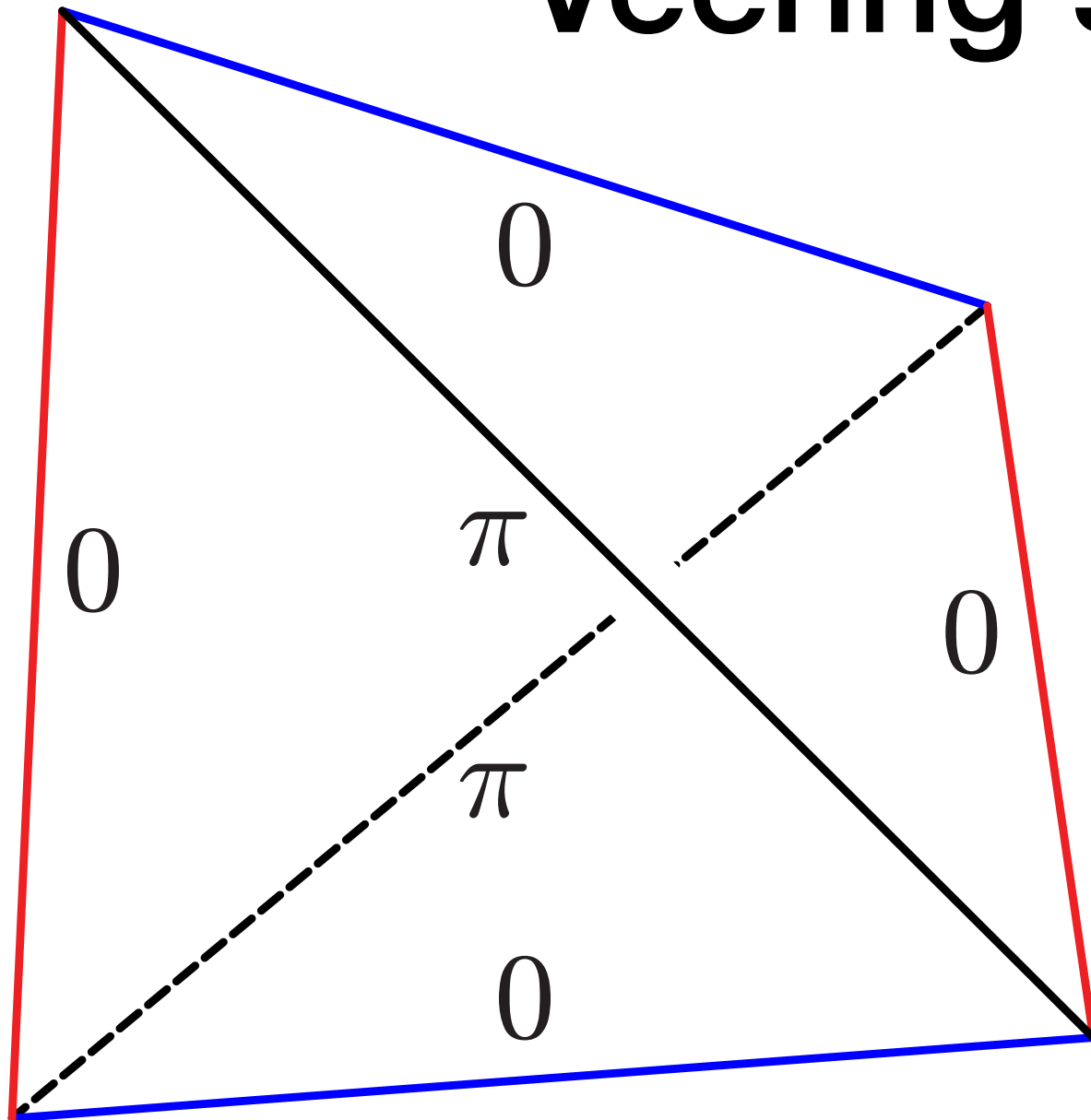
Veering structure (Agol)



taut angle structure, and:

Each tetrahedron colours its 0 angle edges **red** or **blue**.

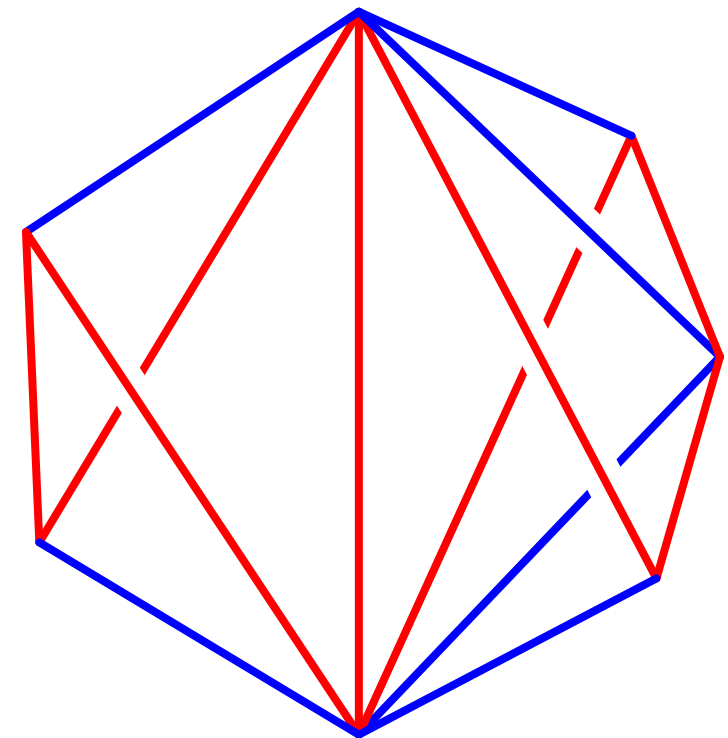
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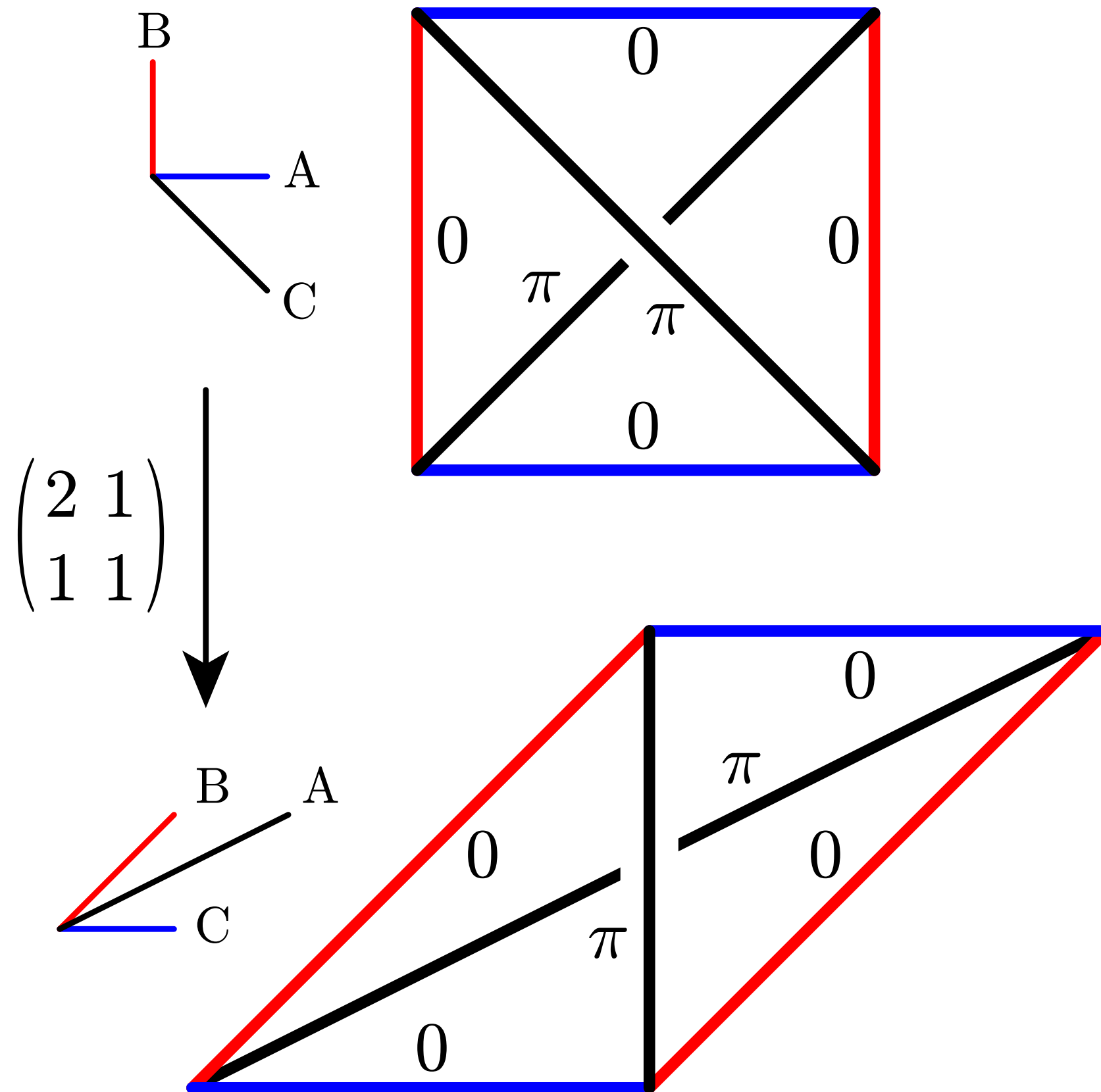
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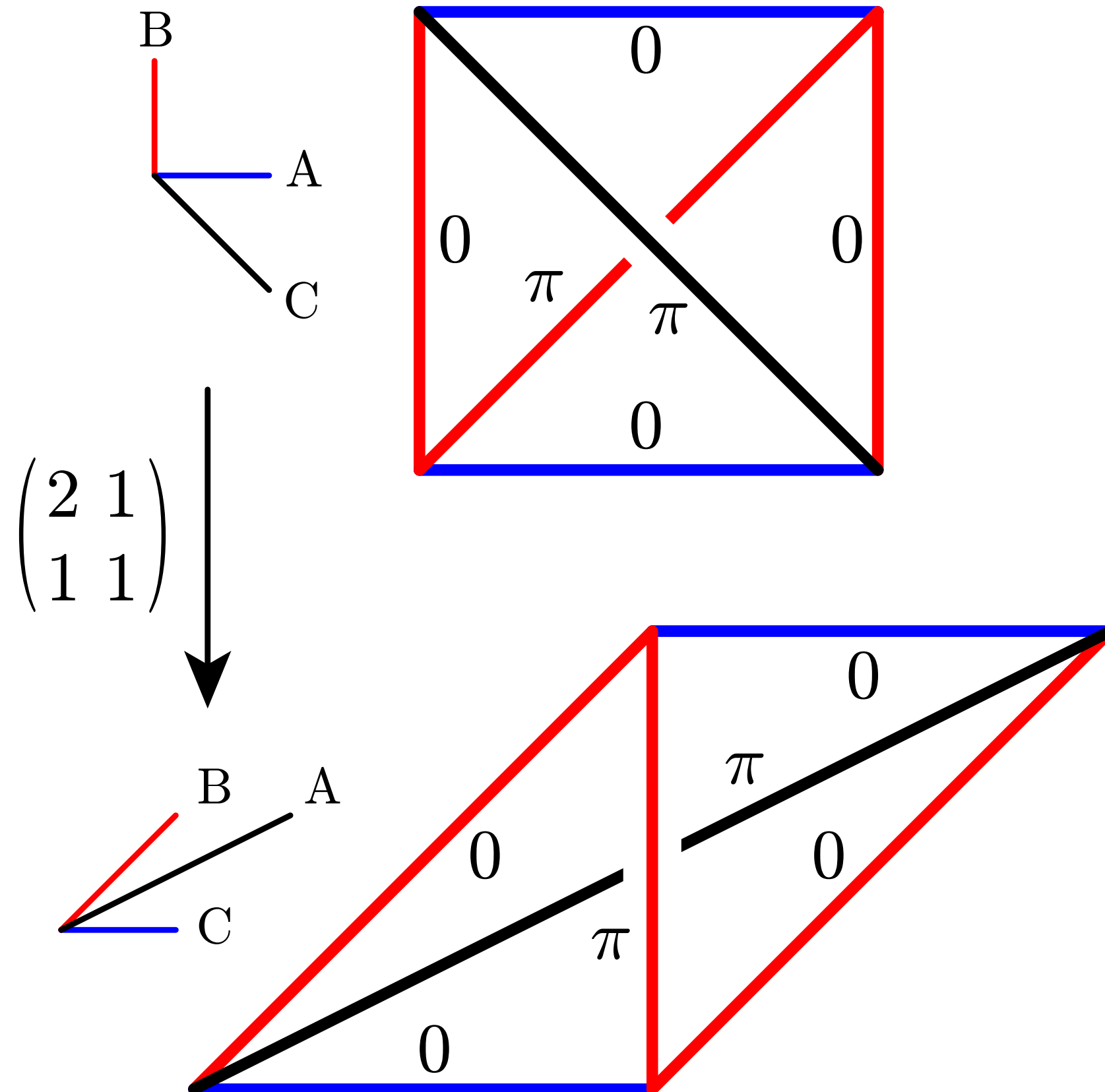
These colours must be consistent for all tetrahedra incident to the edge.



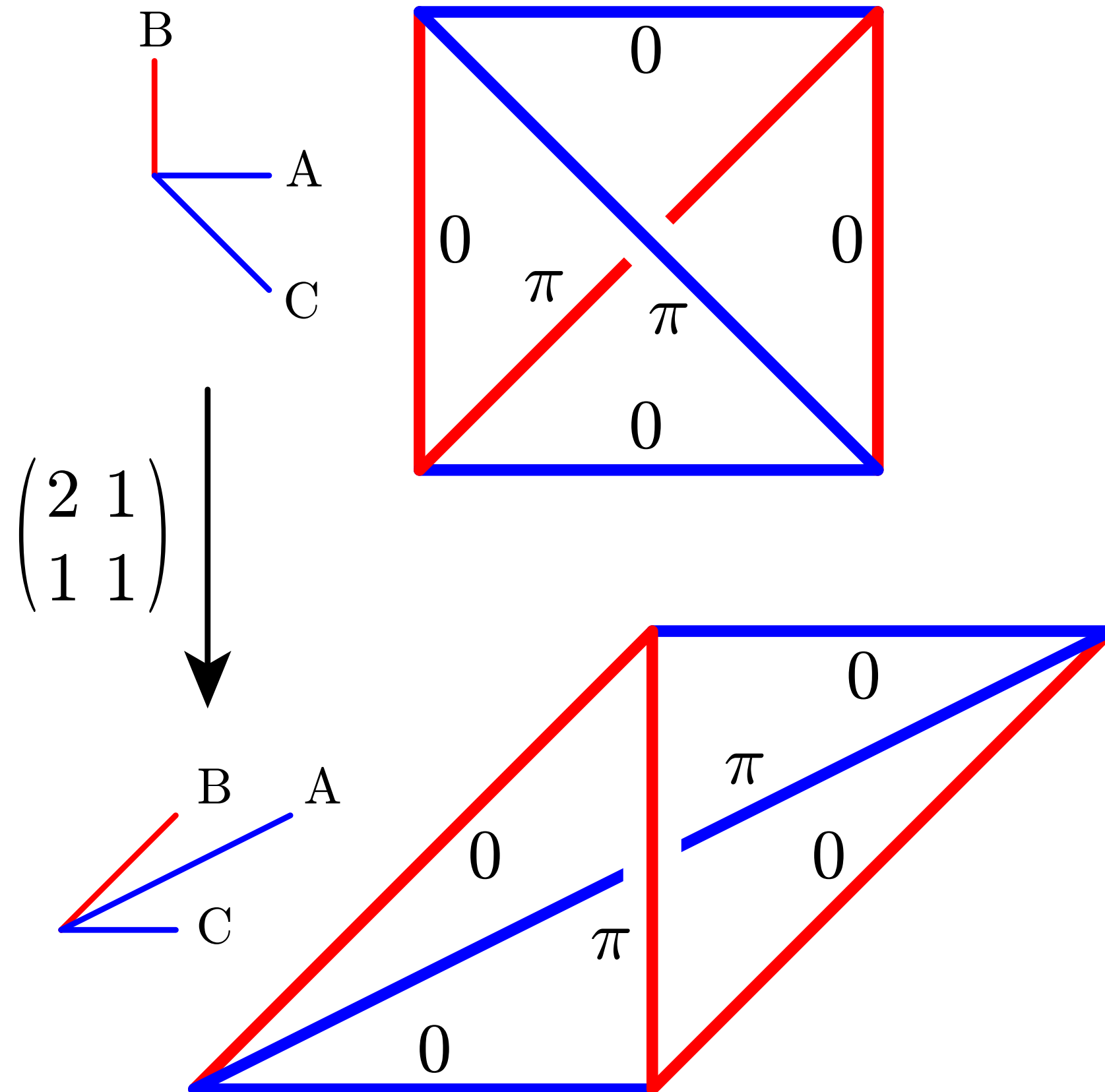
Ex: the figure 8 knot complement



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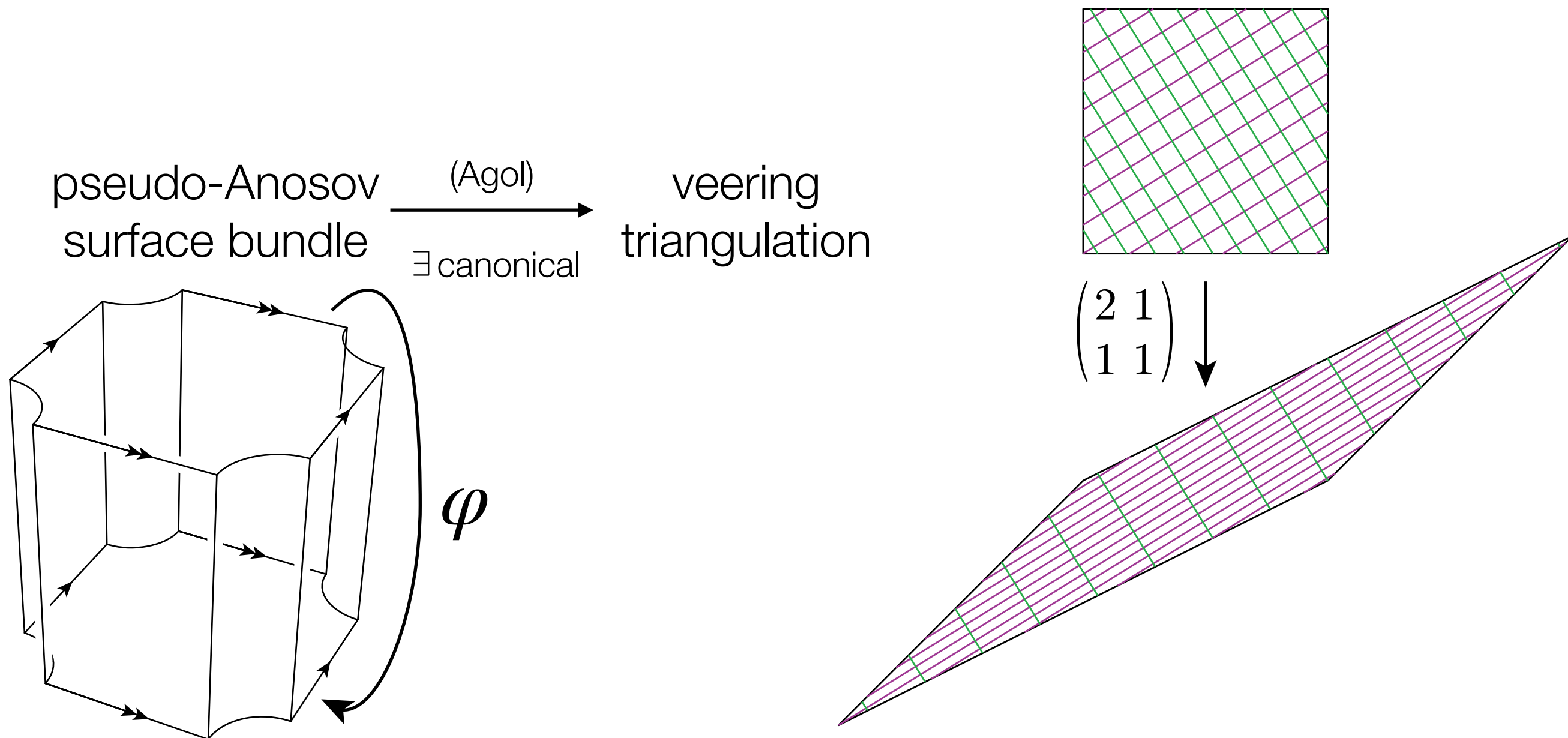
Ex: the figure 8 knot complement



(Agol 2010) Let M be a surface bundle with pseudo-Anosov monodromy φ . The result of drilling out singular orbits admits a veering triangulation canonically associated to φ .

pseudo-Anosov
surface bundle $\xrightarrow[\exists \text{ canonical}]{(\text{Agol})}$ veering
triangulation

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(Hodgson-Rubinstein-S-Tillmann 2011)

Found non-layered veering triangulations.

pseudo-Anosov surface bundle $\xrightarrow{\text{(Agol)}}$ veering triangulation
 \exists canonical

(Hodgson-Rubinstein-S-Tillmann 2011)

Found non-layered veering triangulations.

pseudo-Anosov surface bundle $\xrightarrow{\text{(Agol)}}$ veering triangulation
 \exists canonical

??



The diagram consists of two main parts. The top part shows a horizontal arrow pointing from the text 'pseudo-Anosov surface bundle' to 'veering triangulation'. Above the arrow is the text '(Agol)' and below it is '∃ canonical'. The bottom part shows the text '??' with an arrow pointing upwards and to the right towards the 'veering triangulation' text.

Cannon-Thurston maps

Let M be a hyperbolic surface bundle and $F \subset M$ be a fiber. Any elevation $\tilde{F} \subset \tilde{M}$ of F is a properly embedded disk.

Cannon and Thurston prove that this induces a continuous map $S^1 \cong \partial\tilde{F} \hookrightarrow \partial\tilde{M} \cong \partial\mathbb{H}^3 \cong S^2$. They deduce that the map is sphere-filling.*

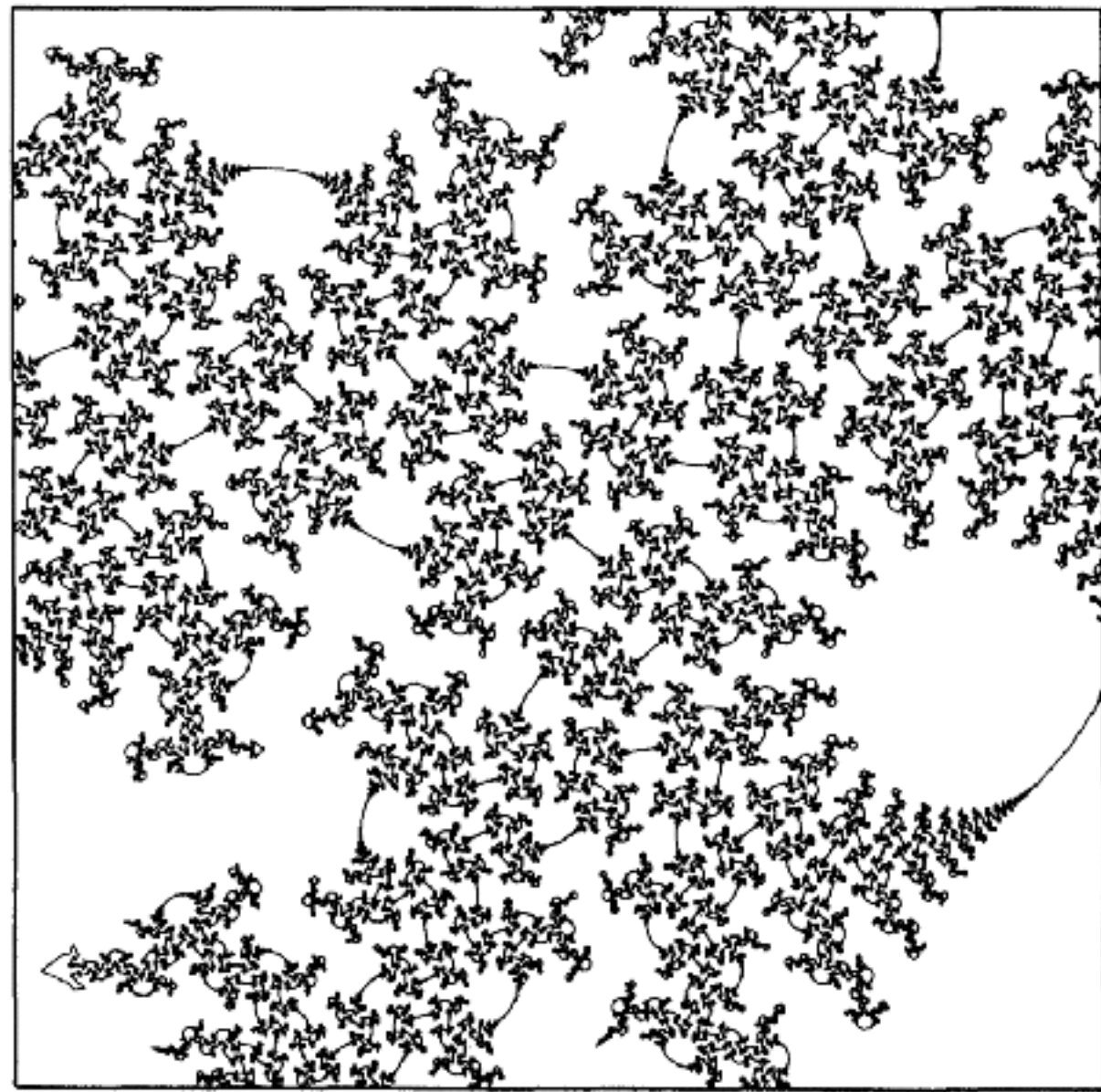


Figure by Bill Thurston, from *Three dimensional manifolds, kleinian groups and hyperbolic geometry*

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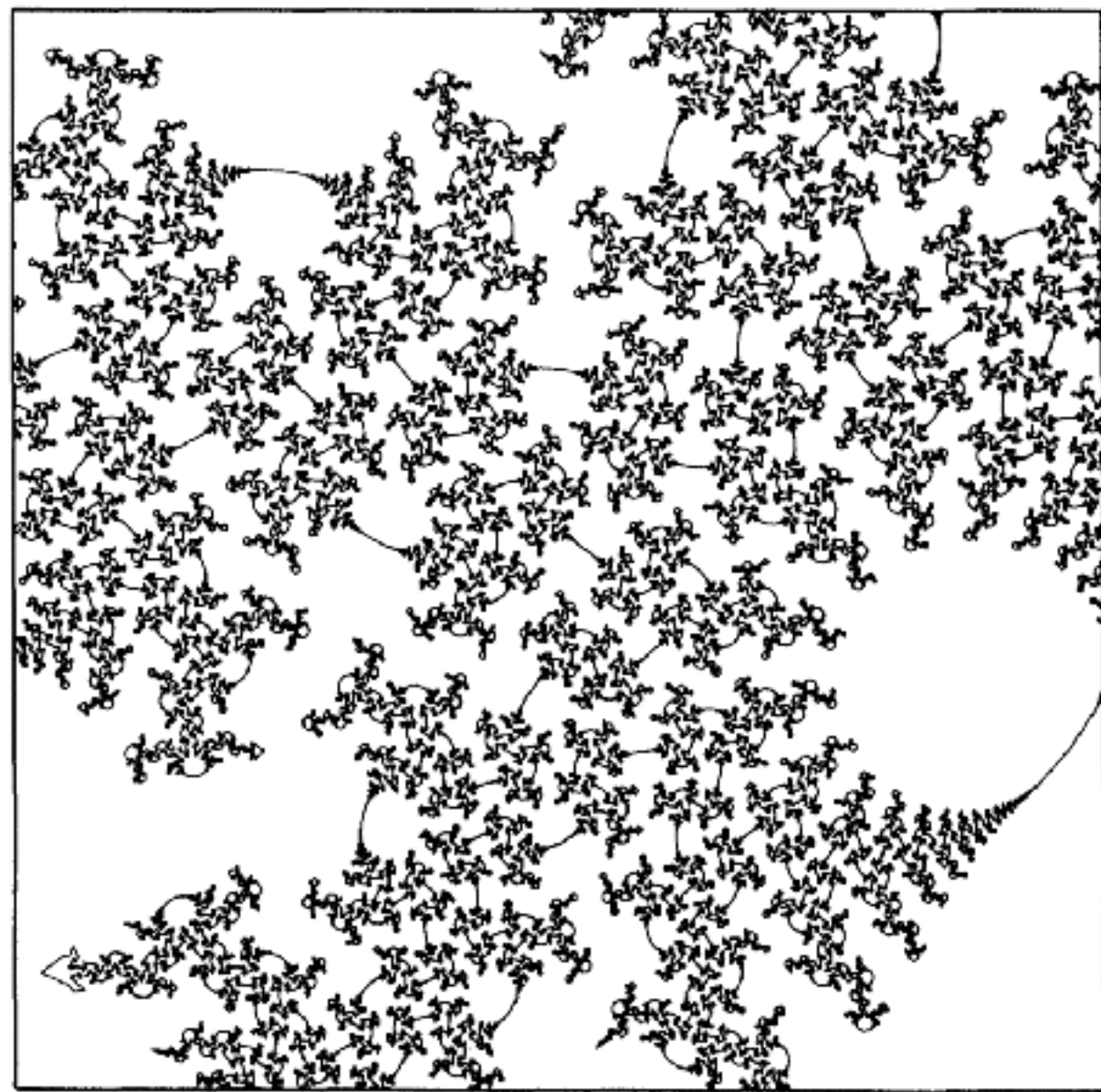


Figure by Bill Thurston, from *Three dimensional manifolds, kleinian groups and hyperbolic geometry*

*Many cases dealt with by many authors: Cannon-Thurston, Minsky, Alperin-Dicks-Porti, McMullen, Bowditch, Mitra.

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Theorem (Manning-Schleimer-Segerman): A veering structure induces a *Cannon-Thurston map* (even when the manifold is non-fibered).

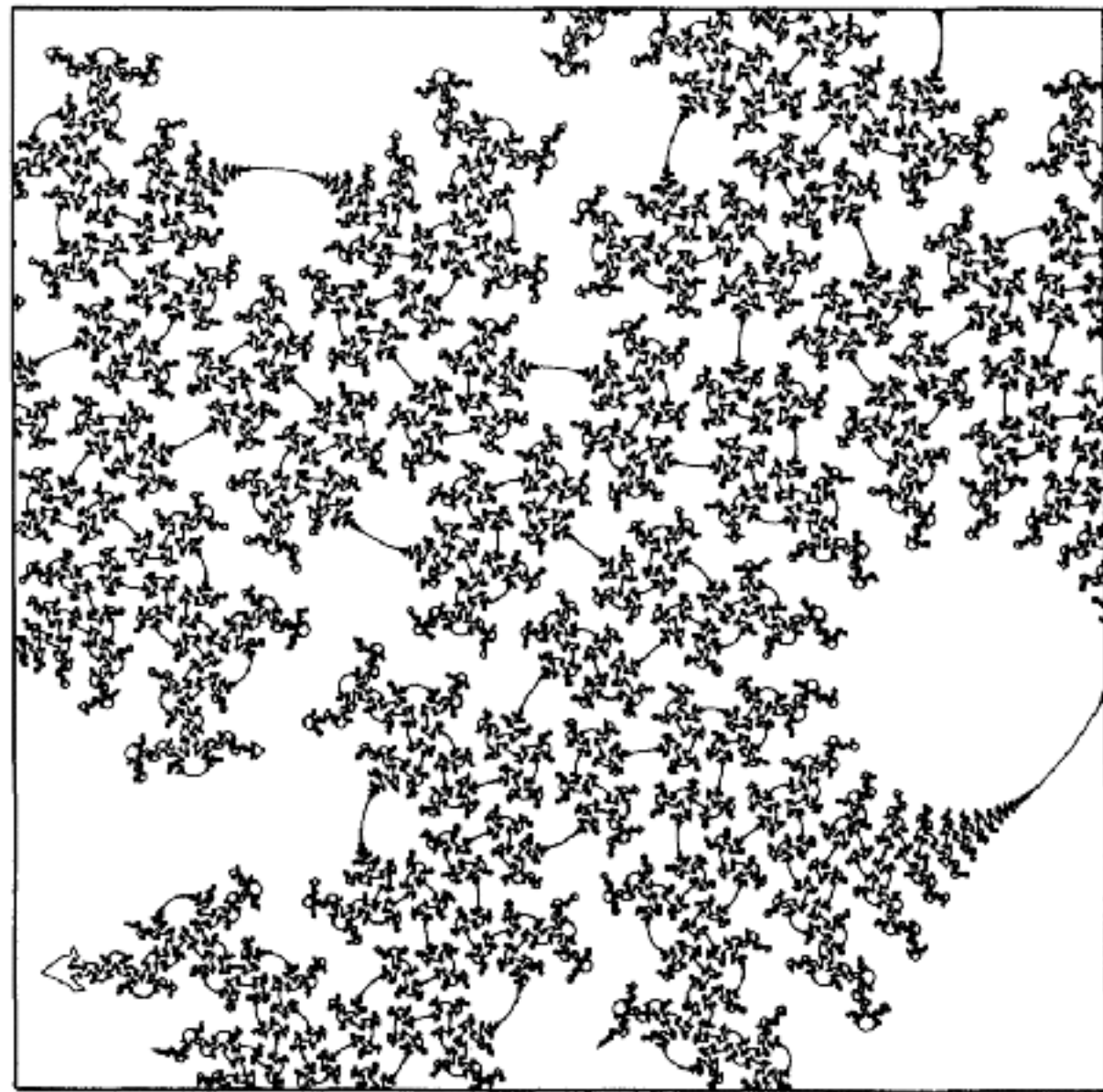
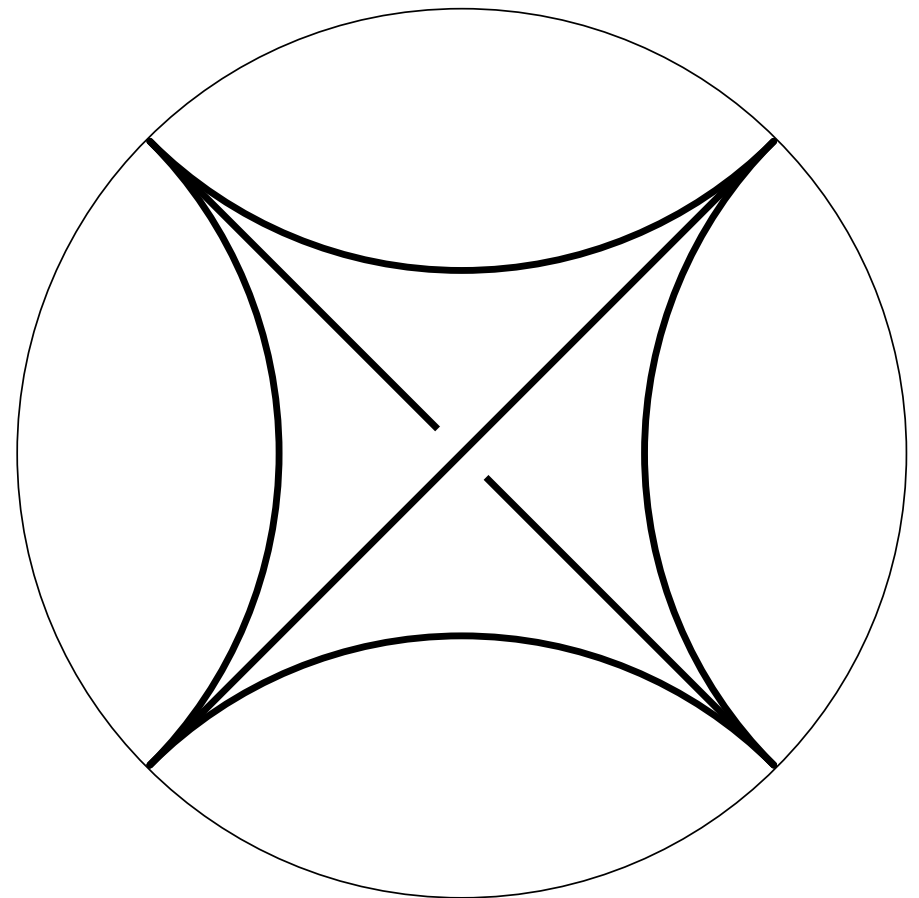


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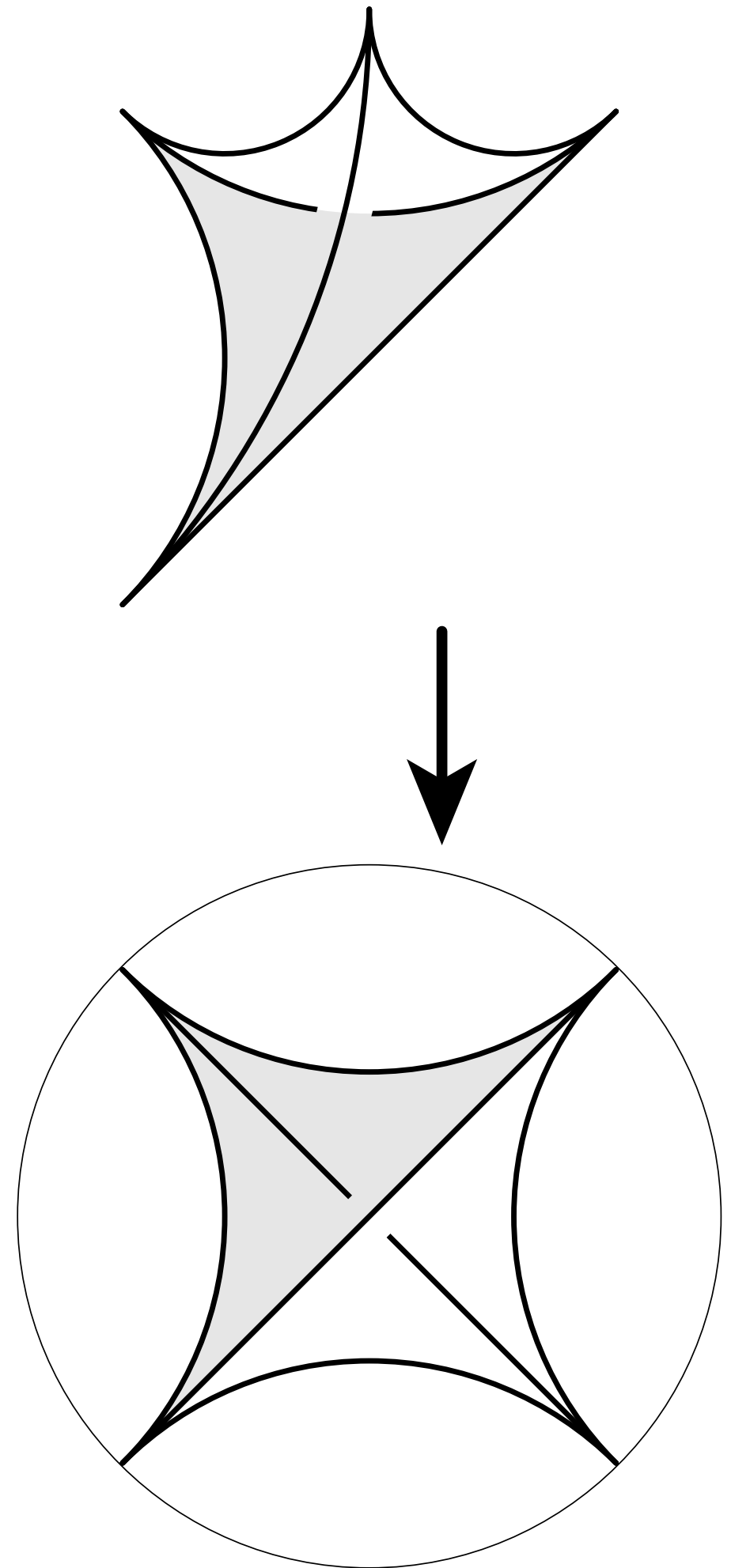
Layers and continents in the universal cover

Taut ideal tetrahedra layer to make
larger taut ideal polyhedra:
continents.



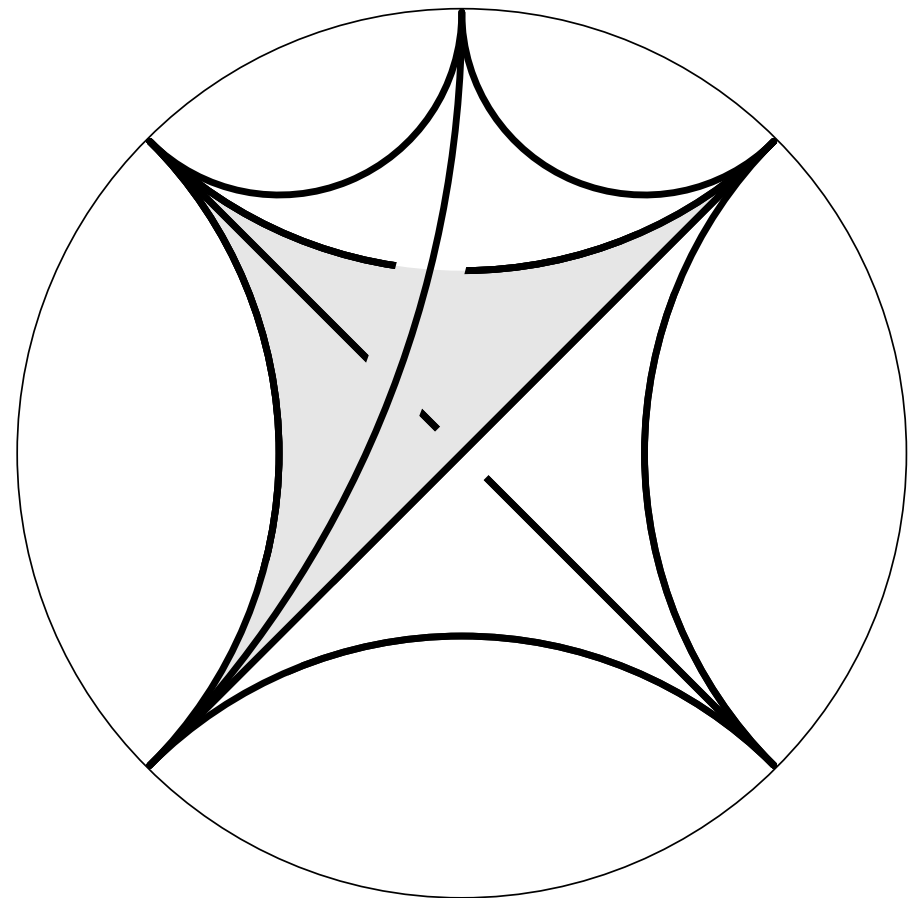
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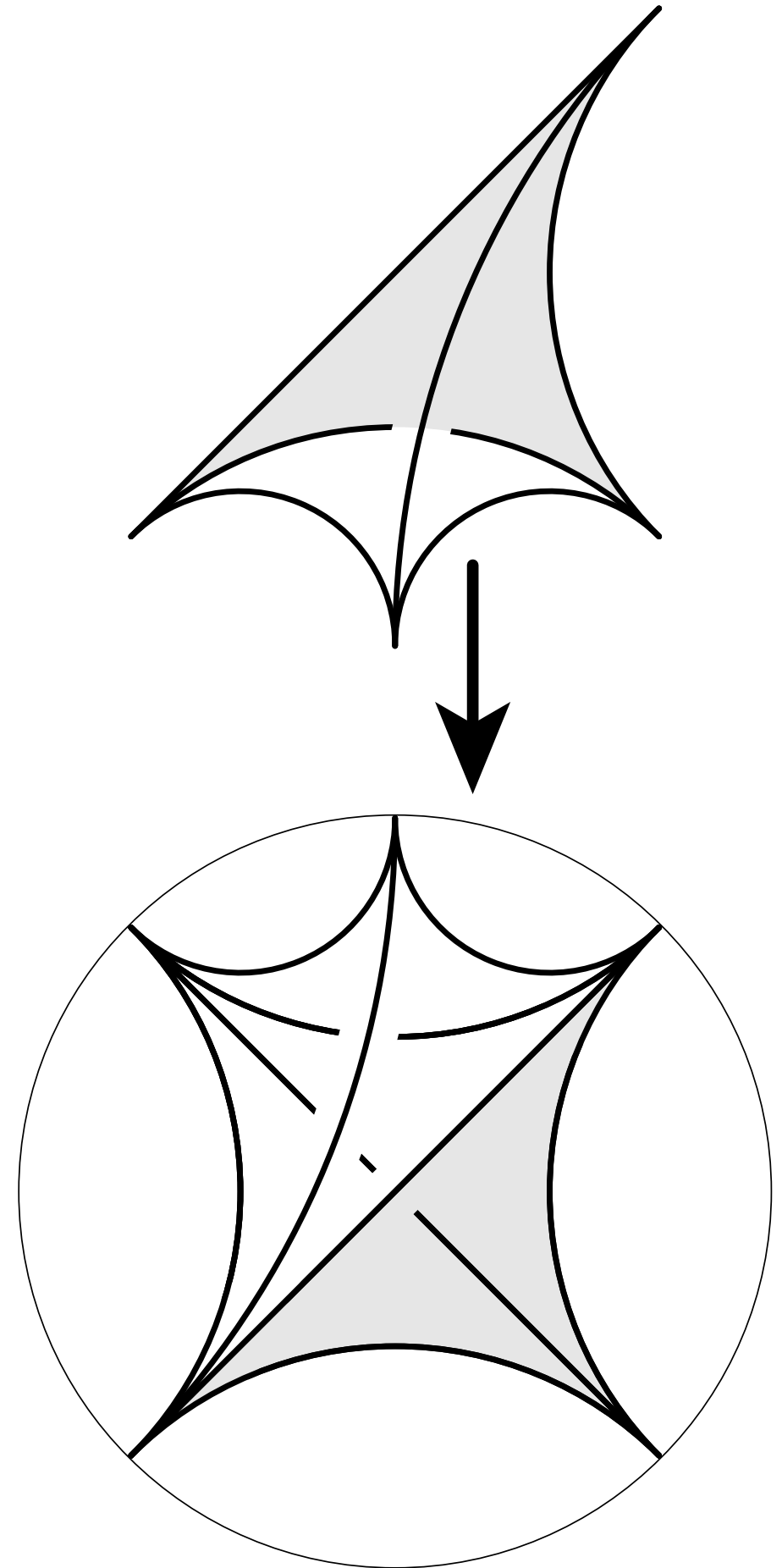
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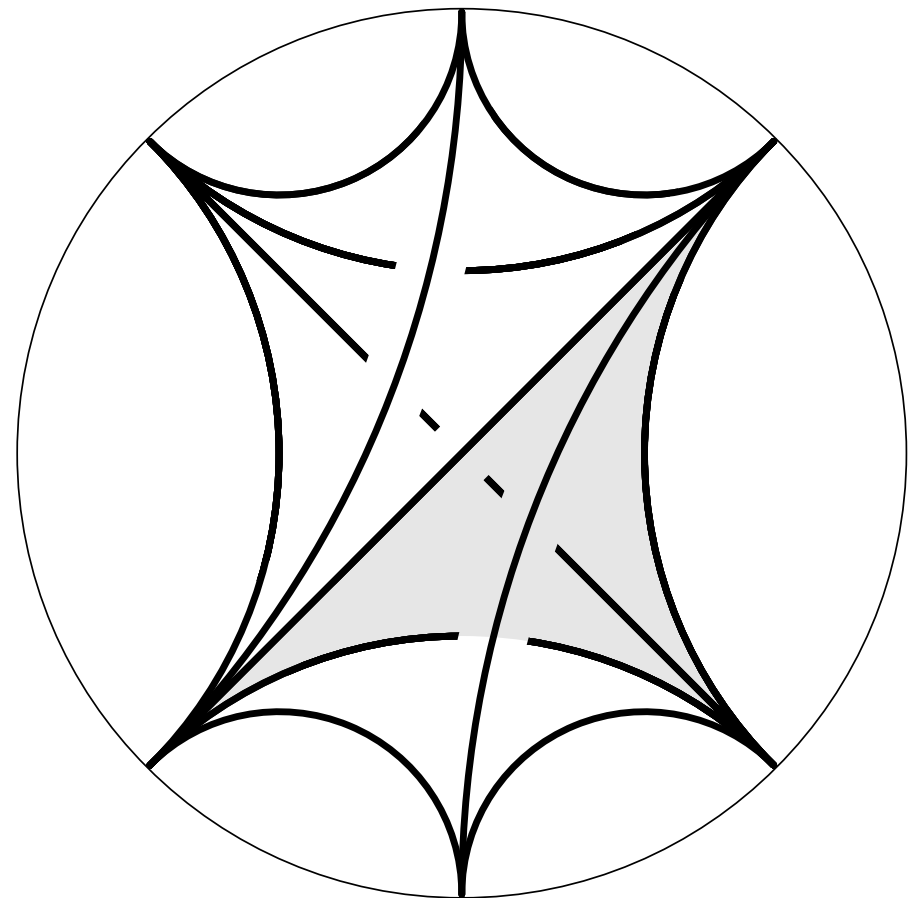
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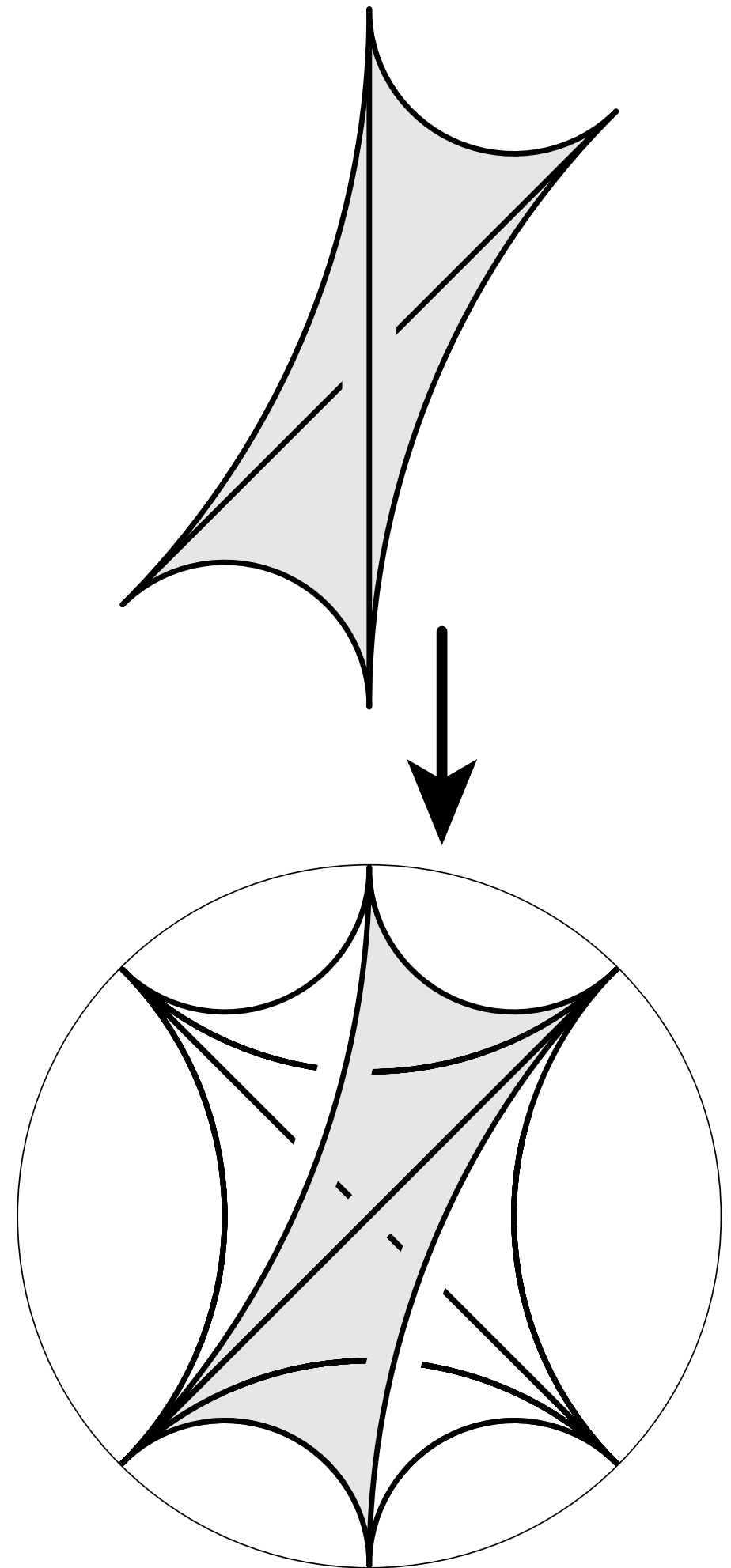
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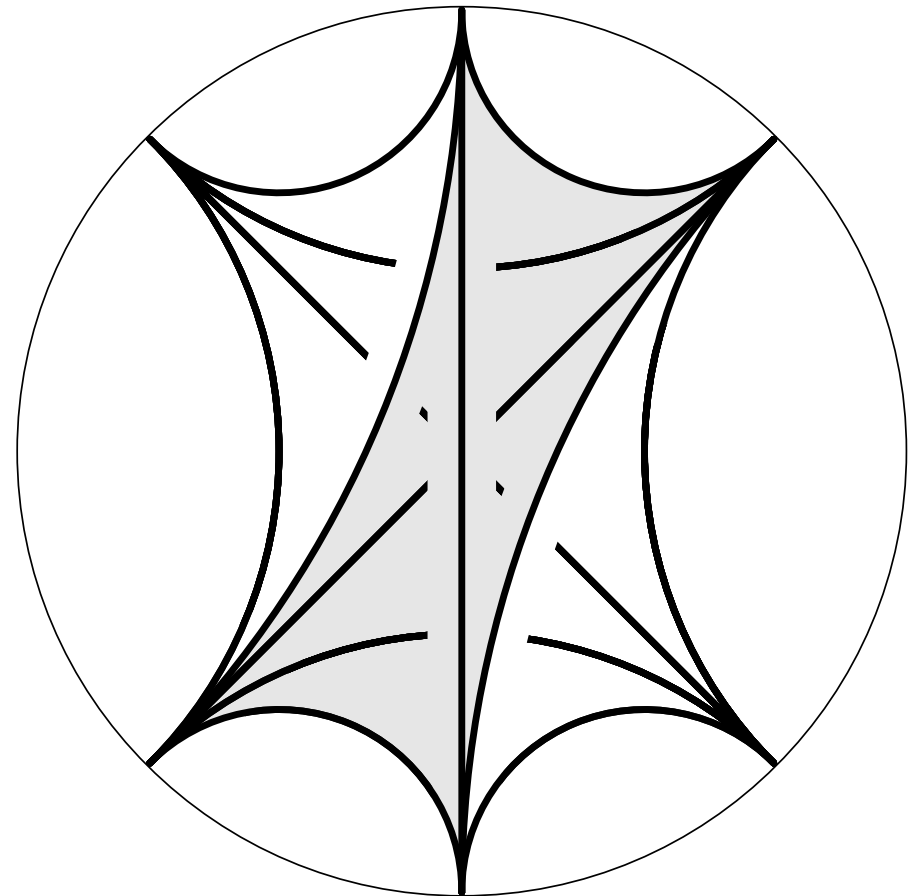
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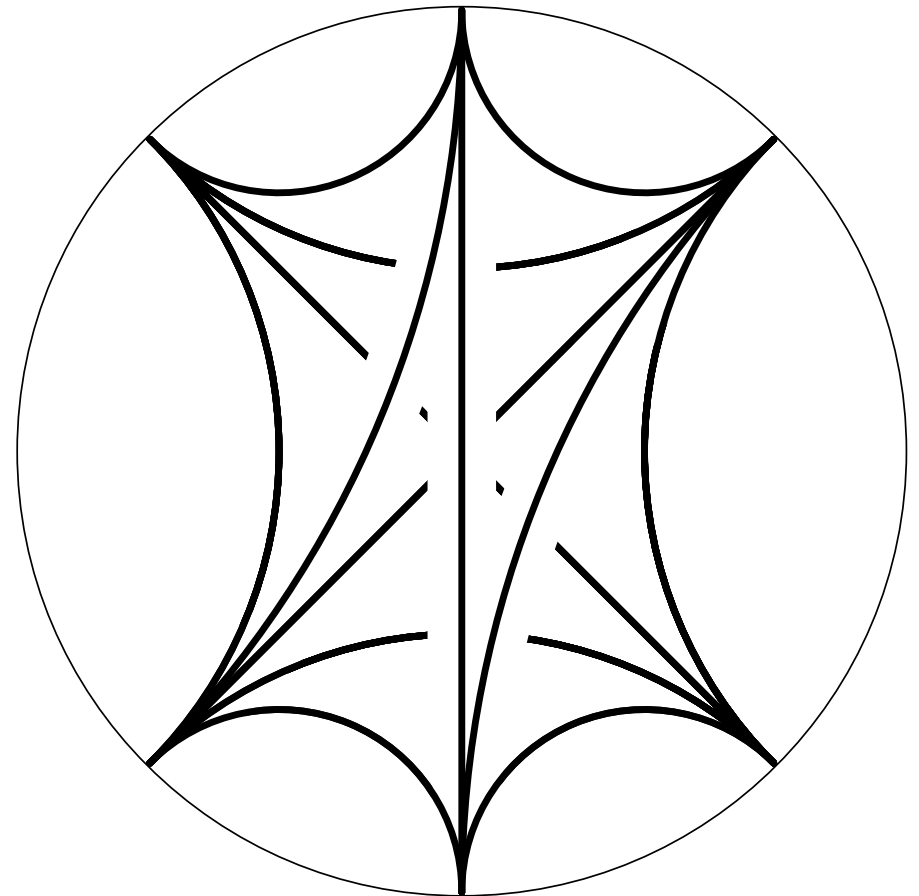
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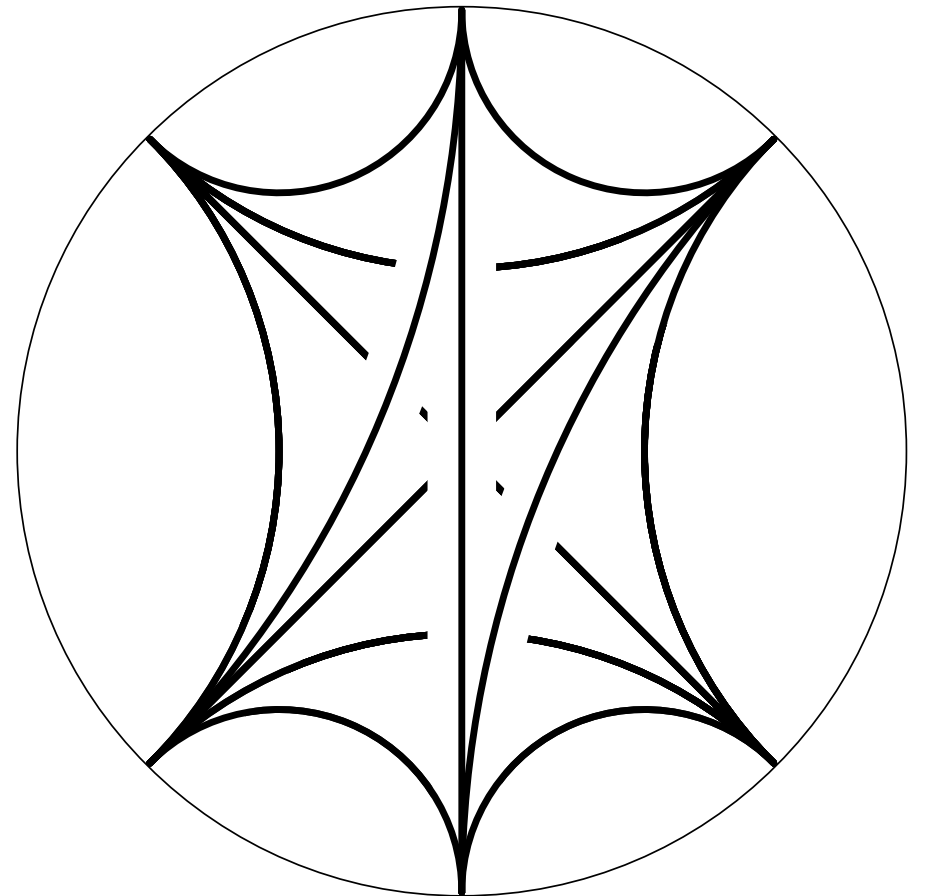
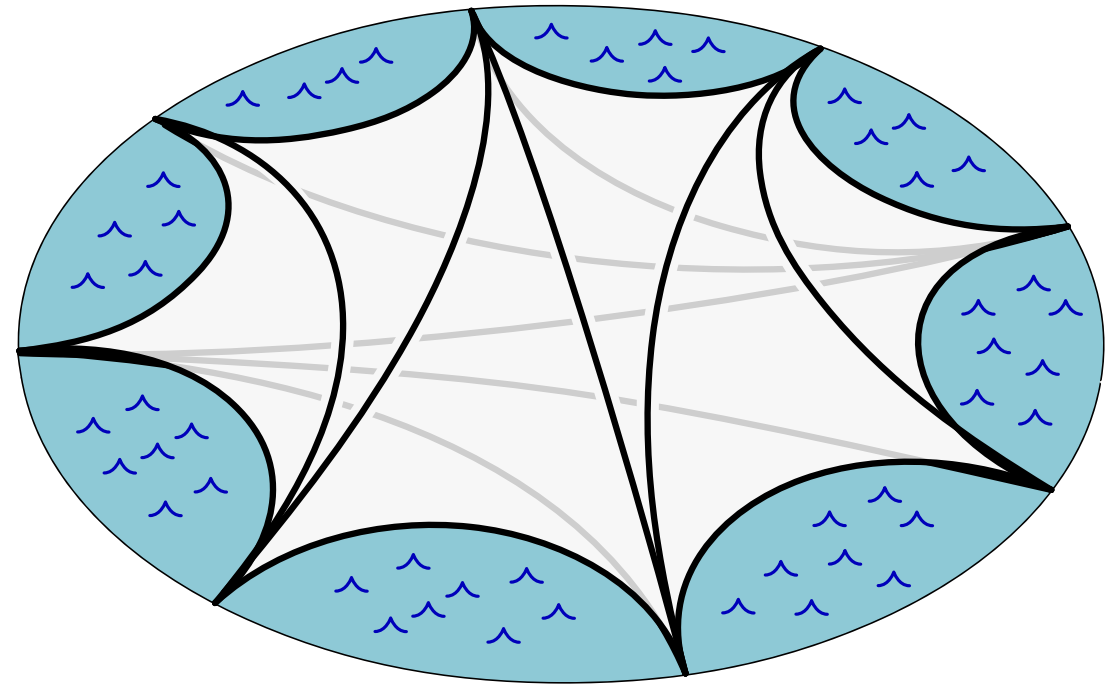
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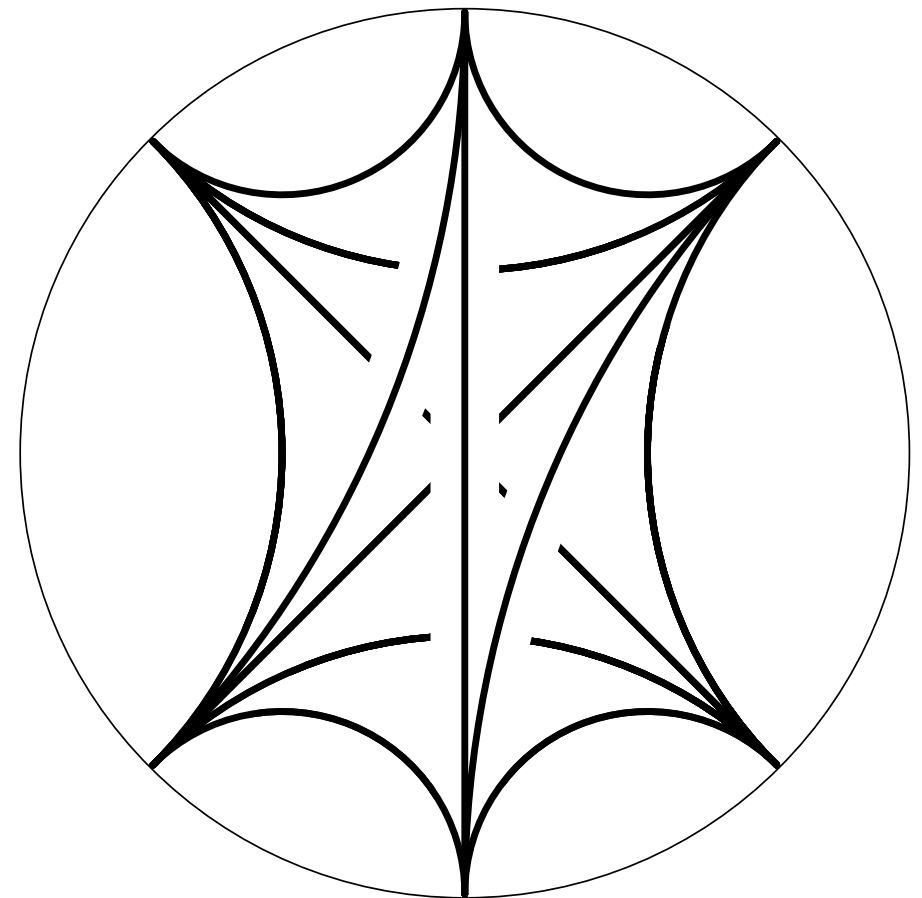
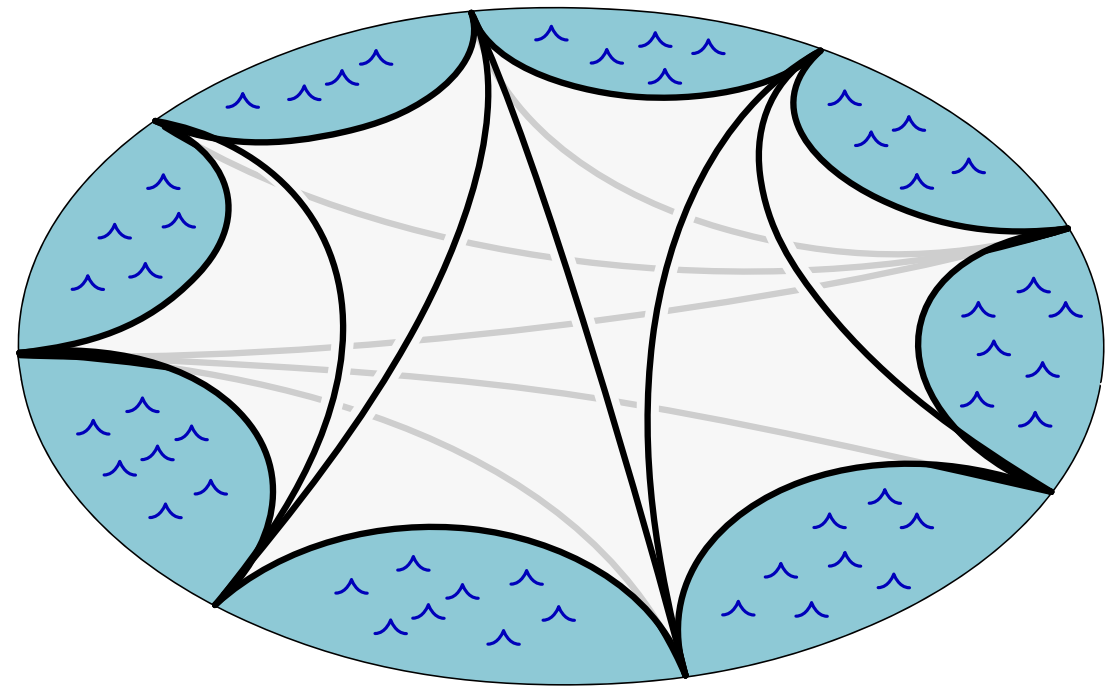
Taut ideal tetrahedra layer to make
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Layers and continents in the universal cover

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This gives a circular order to the
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Layers and continents in the universal cover

Taut ideal tetrahedra layer to make larger taut ideal polyhedra: *continents*.

This gives a circular order to the vertices of the tetrahedra, *compatible* with the taut angle structure.

Theorem A (Schleimer-S): A veering triangulation admits a *unique* compatible circular order on the vertices of the universal cover.

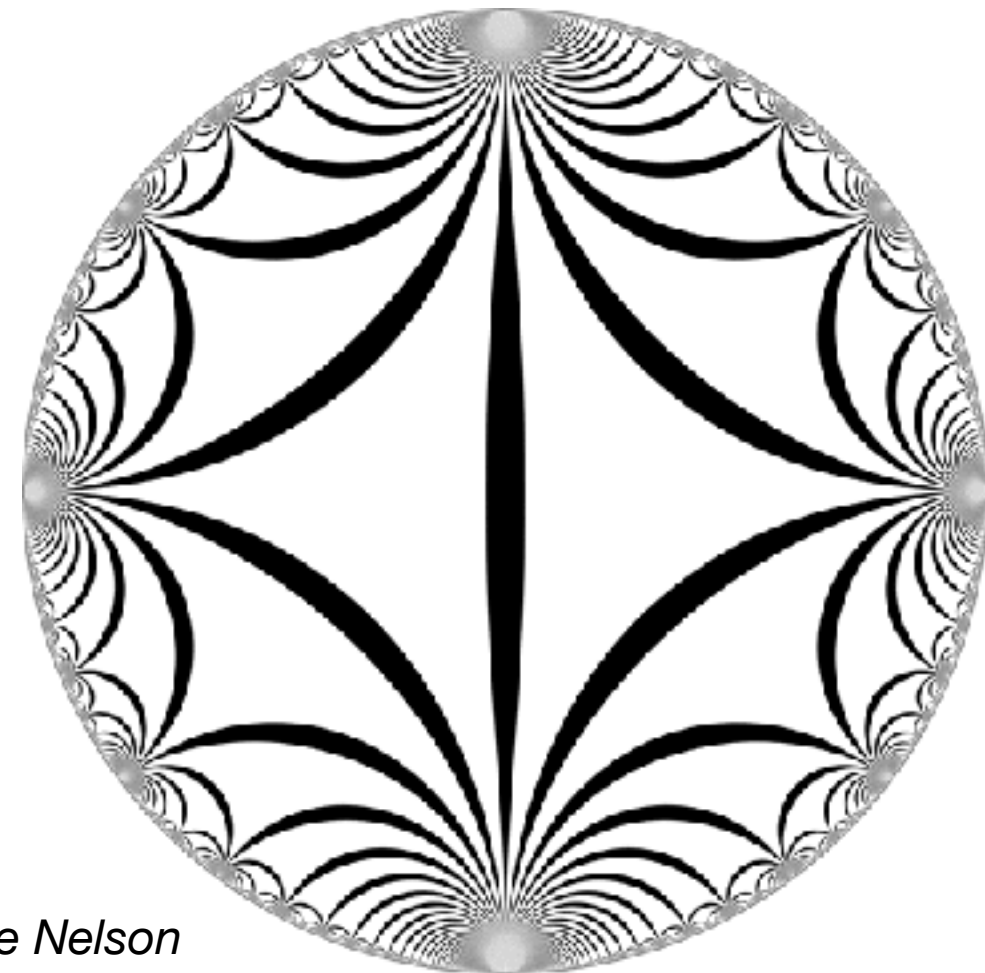
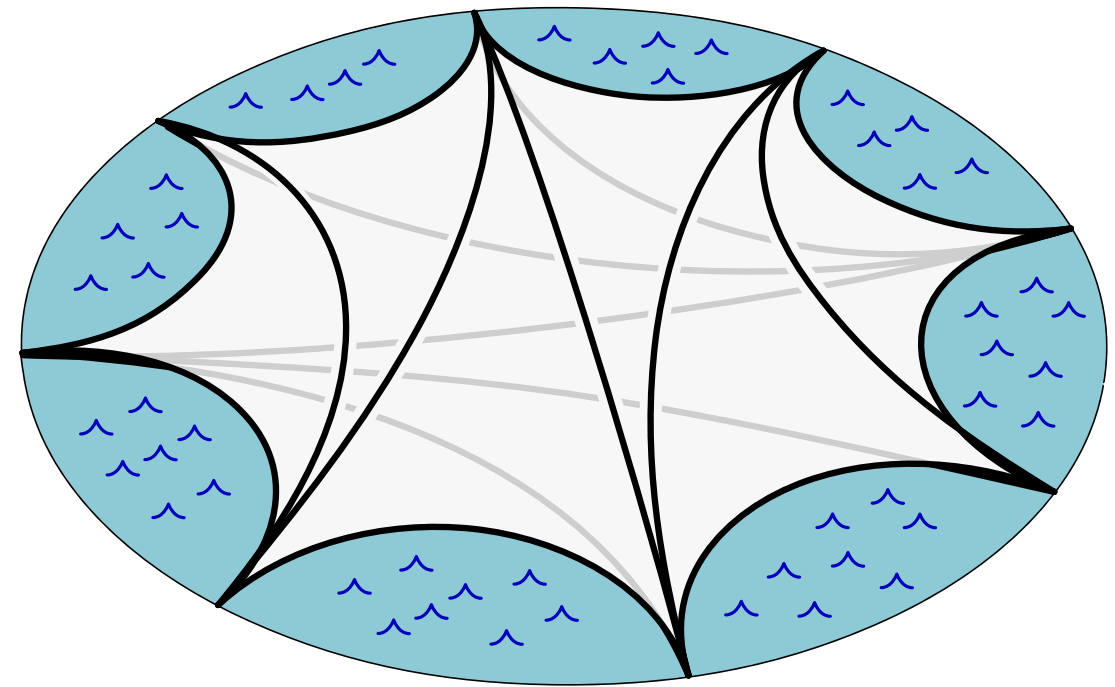
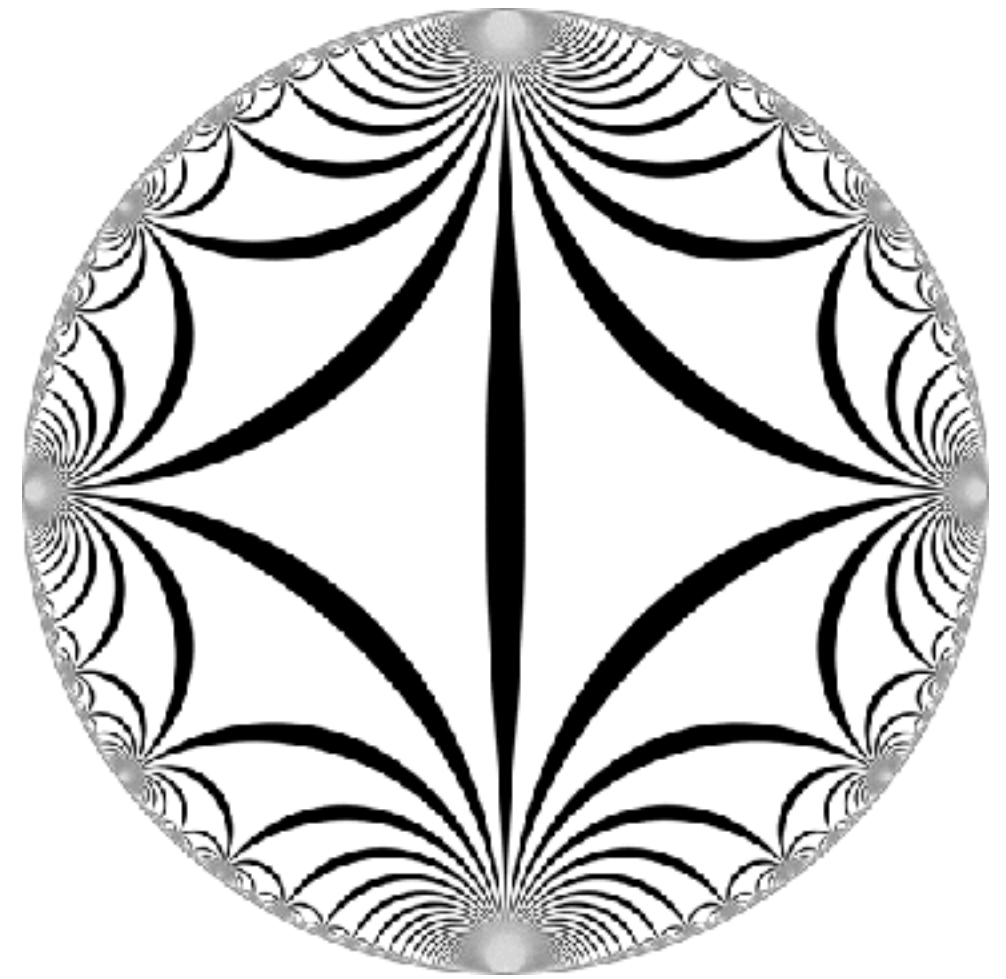


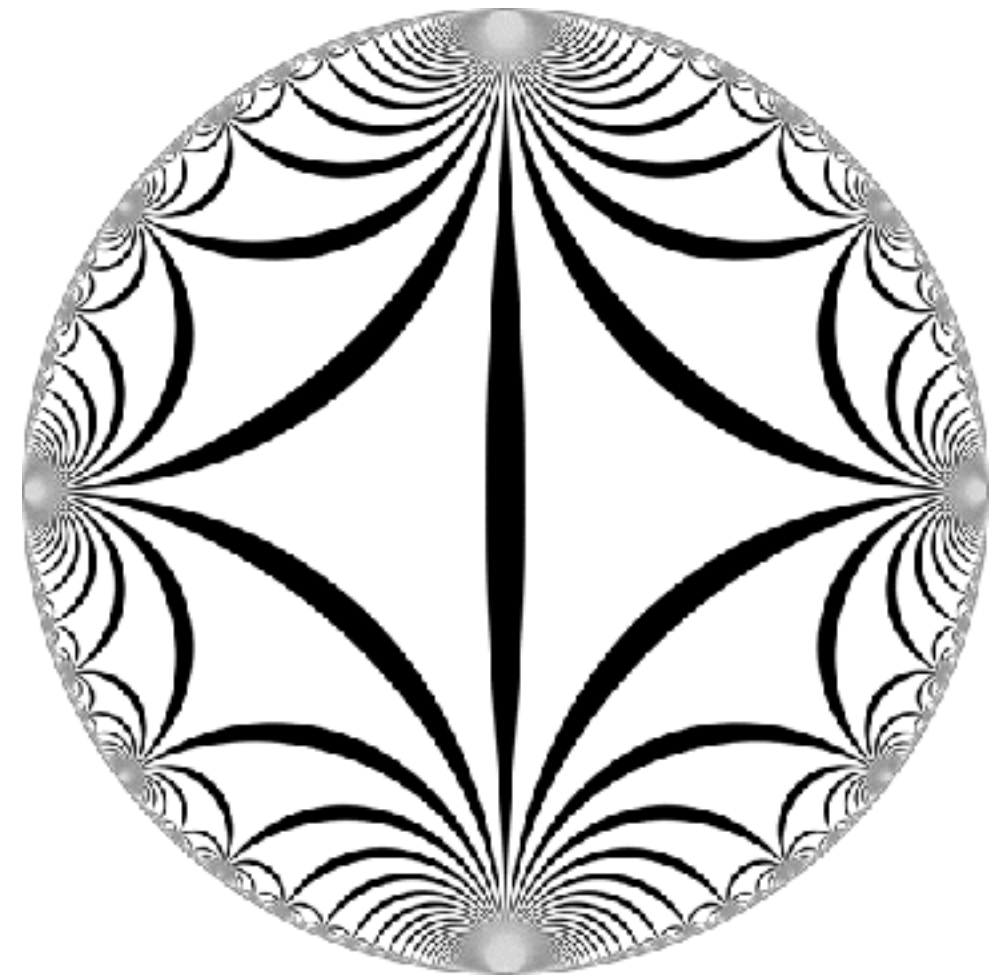
Figure by Roice Nelson

Theorem A (Schleimer, S): A veering triangulation admits a *unique* compatible circular order on the vertices of the universal cover.



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We complete this circular order to the *veering circle*.

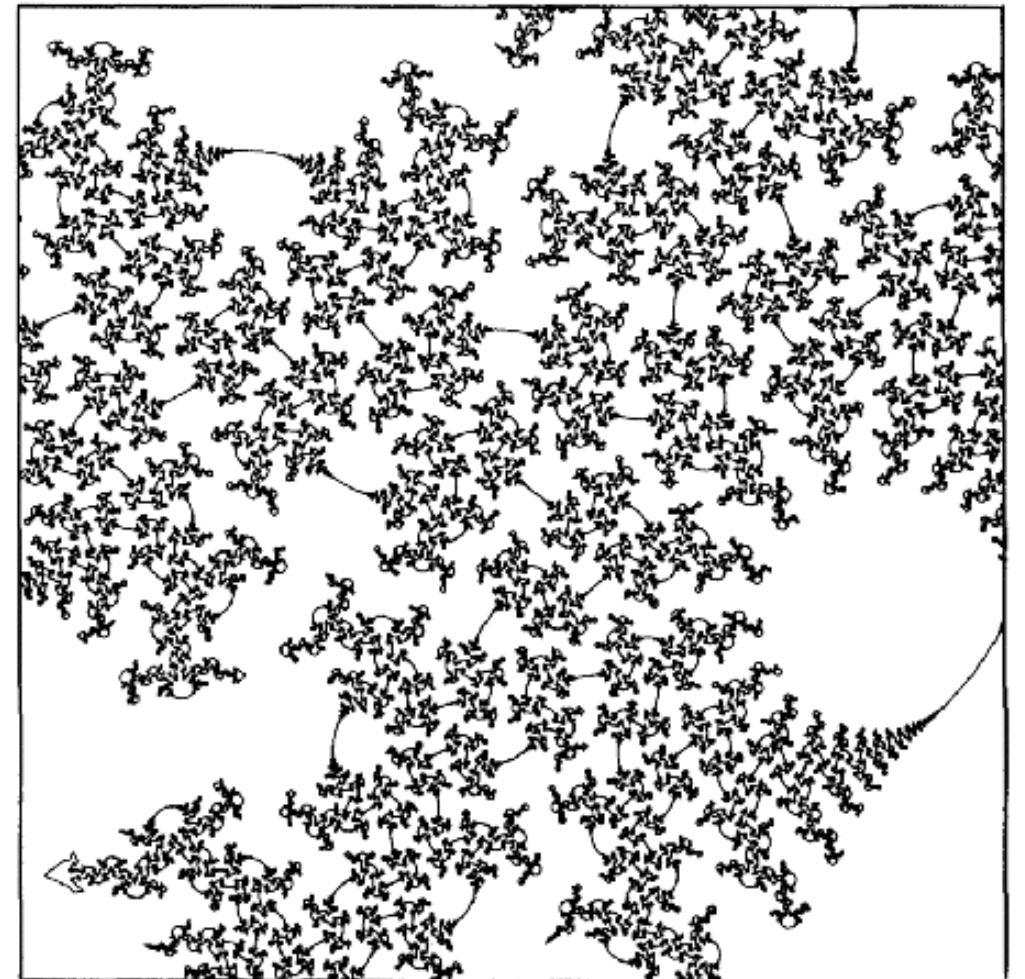
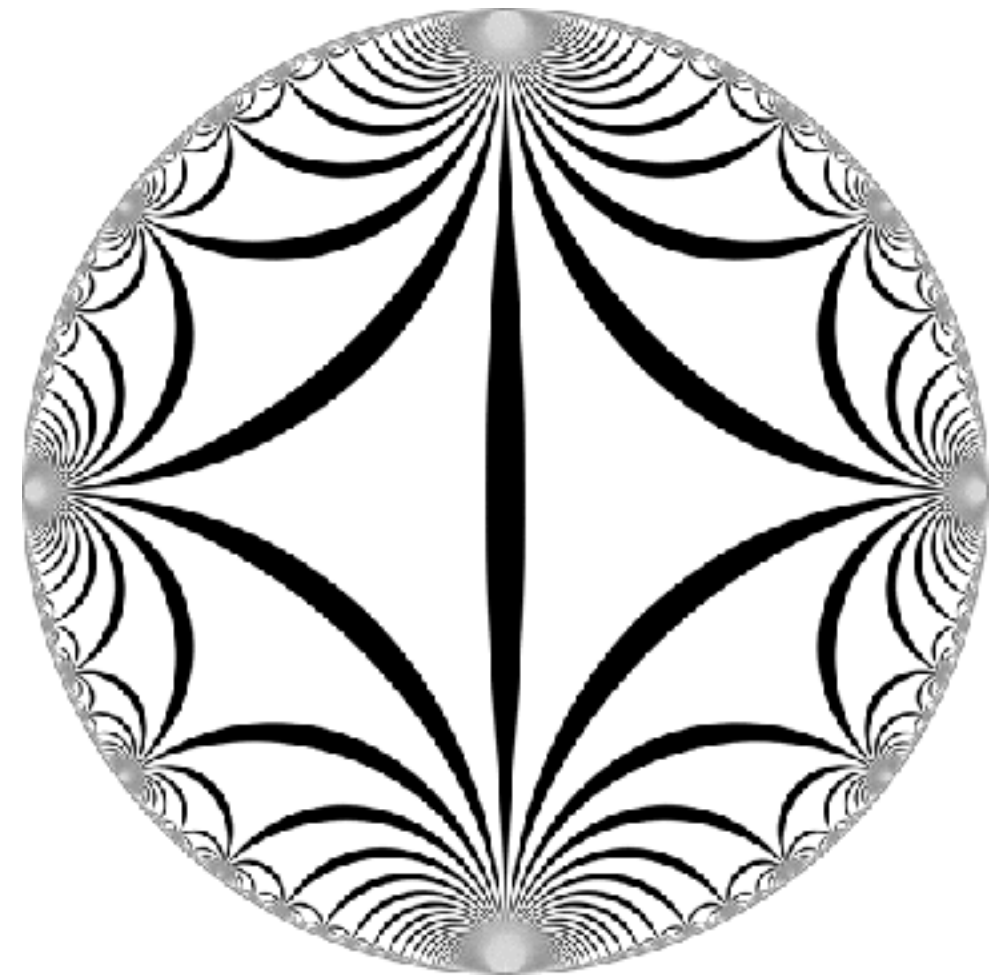


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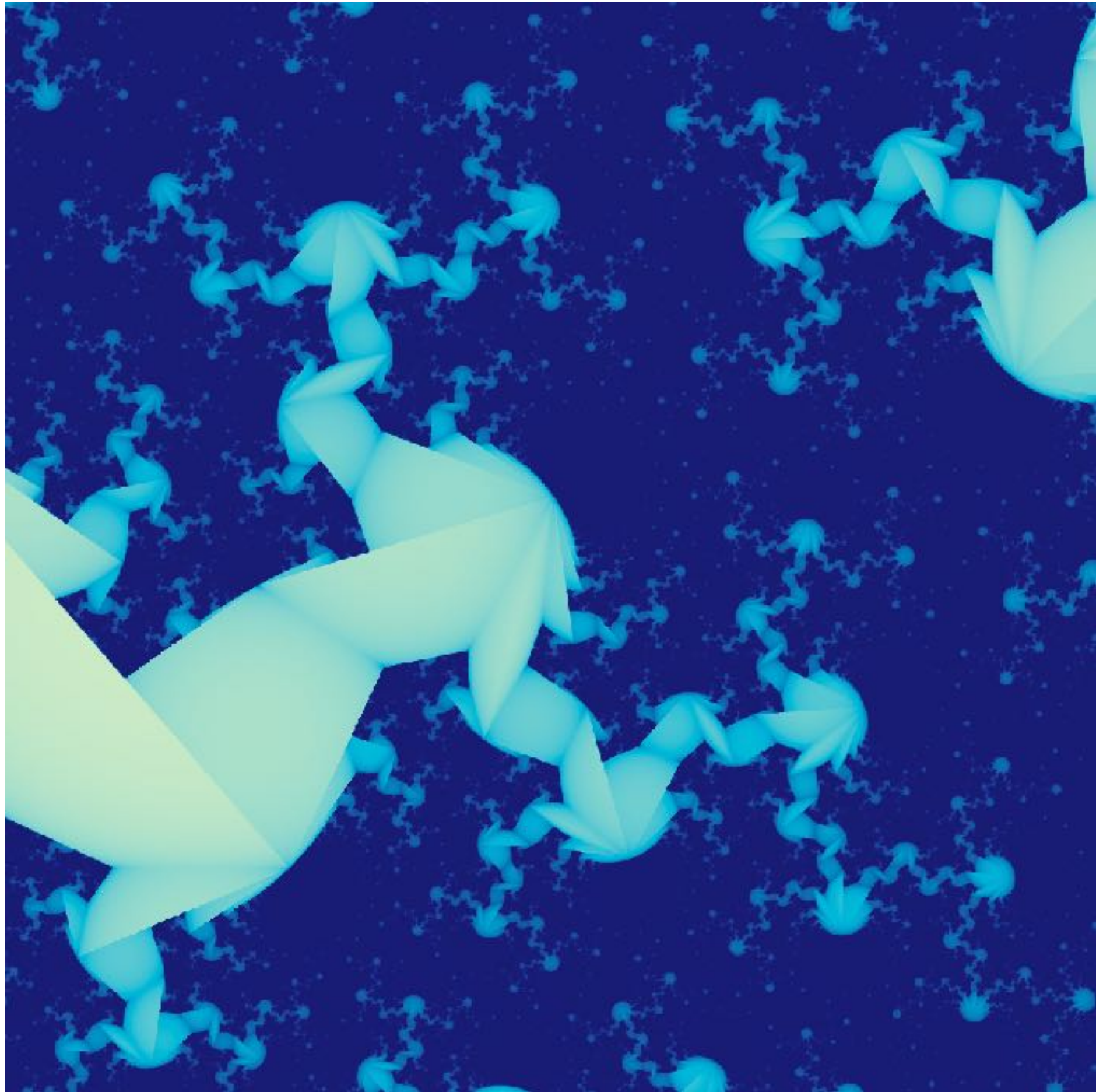
The veering circle is our equivalent to the boundary of the elevation $\partial\tilde{F} \hookrightarrow \partial\tilde{M} \cong \partial\mathbb{H}^3 \cong S^2$ in the usual Cannon-Thurston map.

Theorem B (Manning-Schleimer-S): There is a continuous equivariant sphere-filling map from the veering circle to $\partial\tilde{M} \cong \partial\mathbb{H}^3 \cong S^2$.

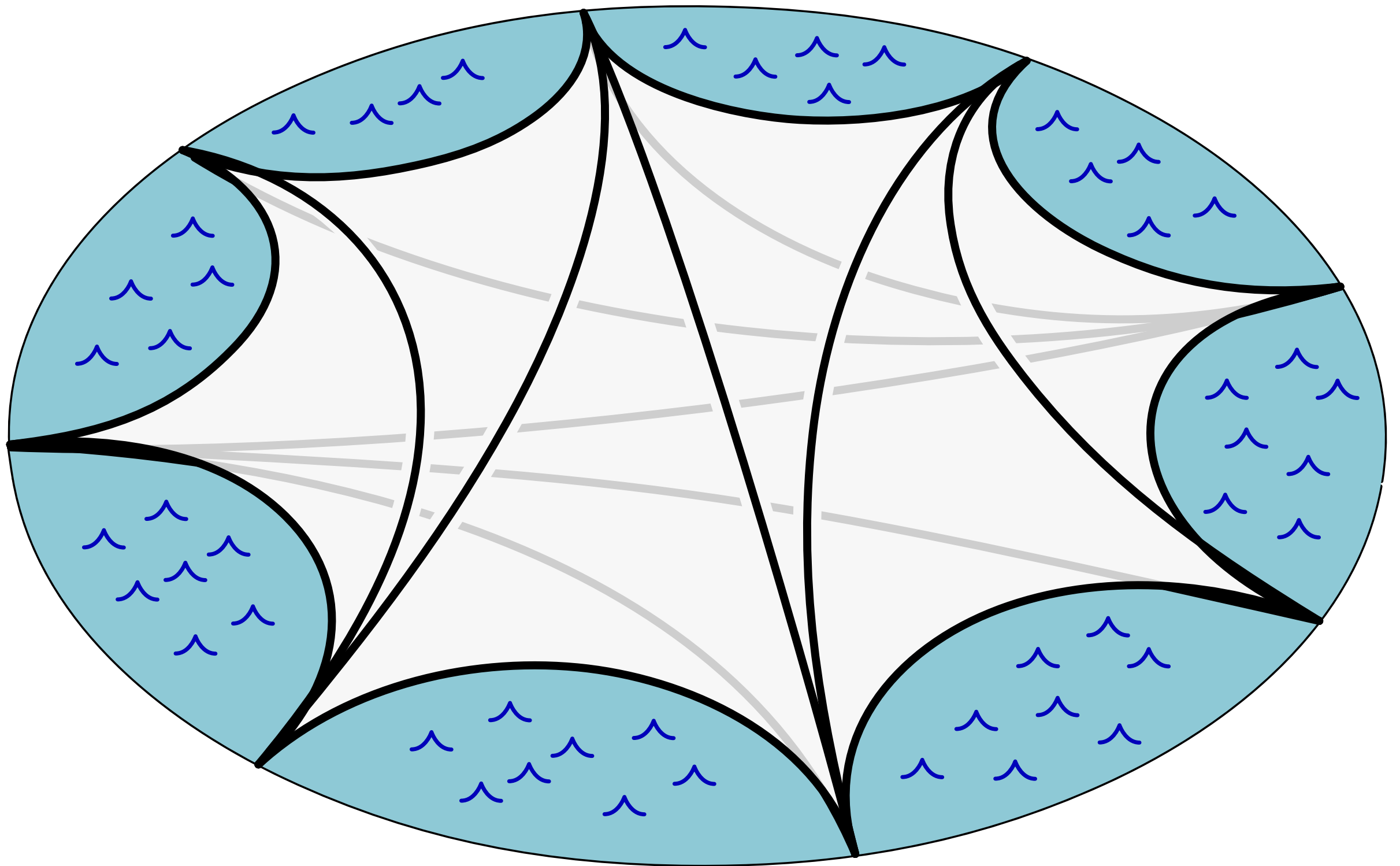


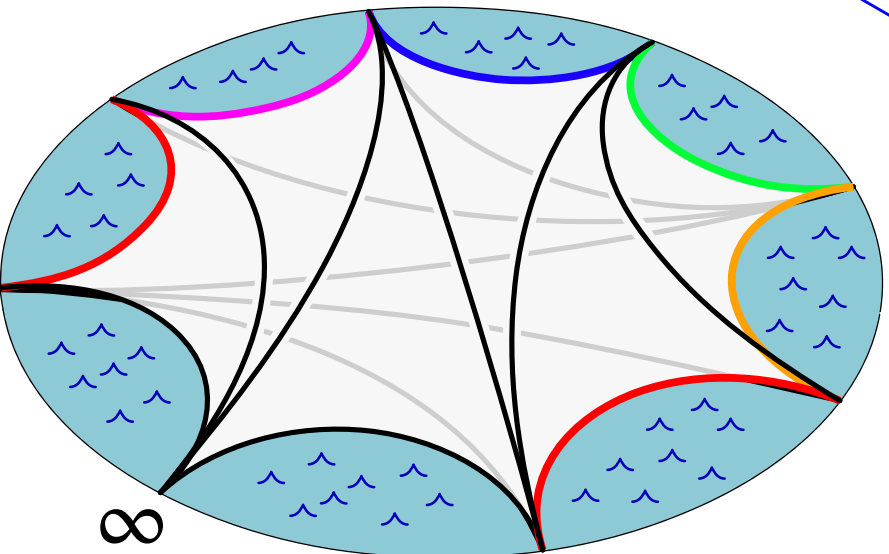
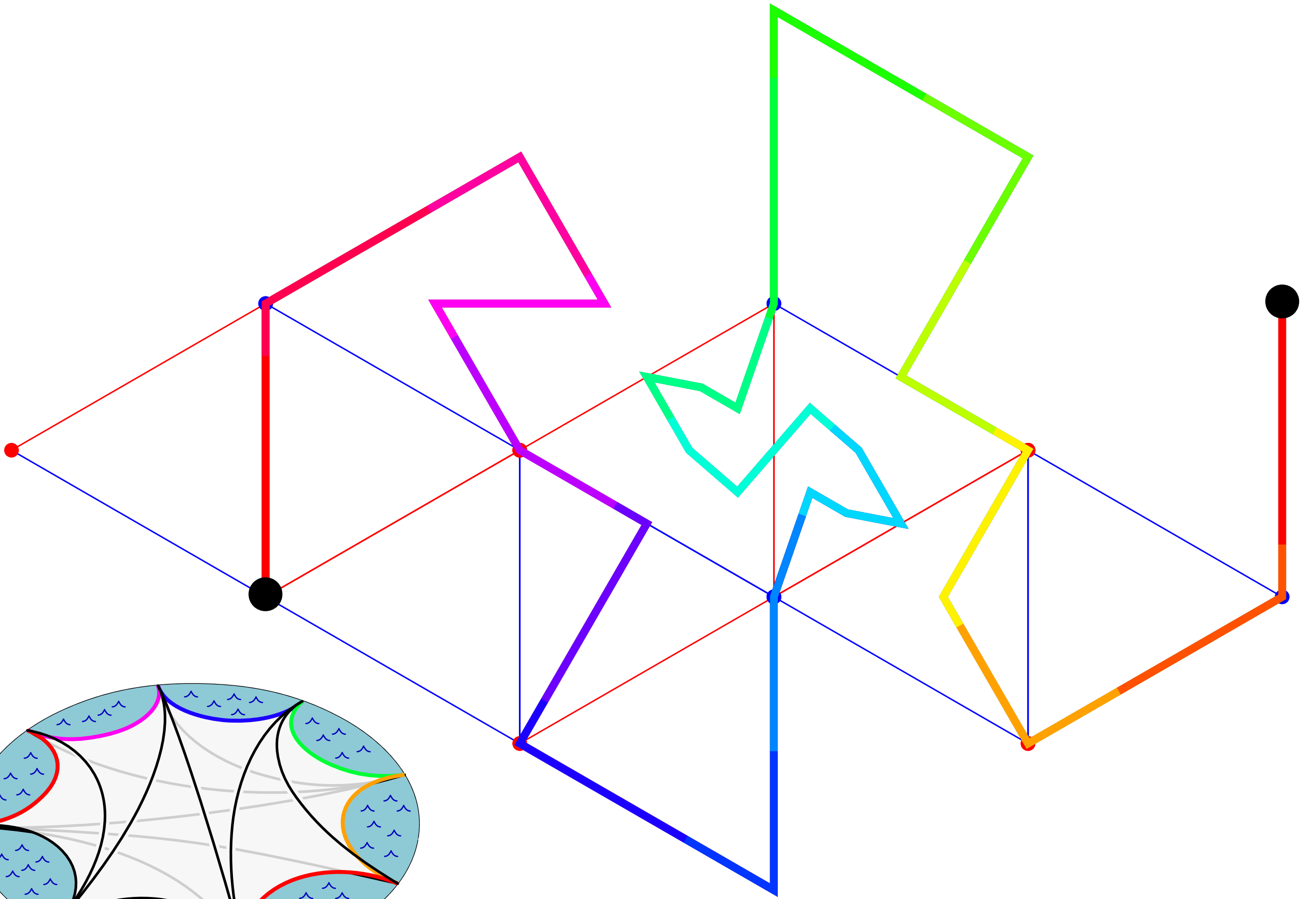
Before saying something about the proof, we can draw some approximations to our Cannon-Thurston maps.

Thurston's original approximation is made from a large disk $D \subset \tilde{F}$: draw ∂D in place of $\partial \tilde{F}$.

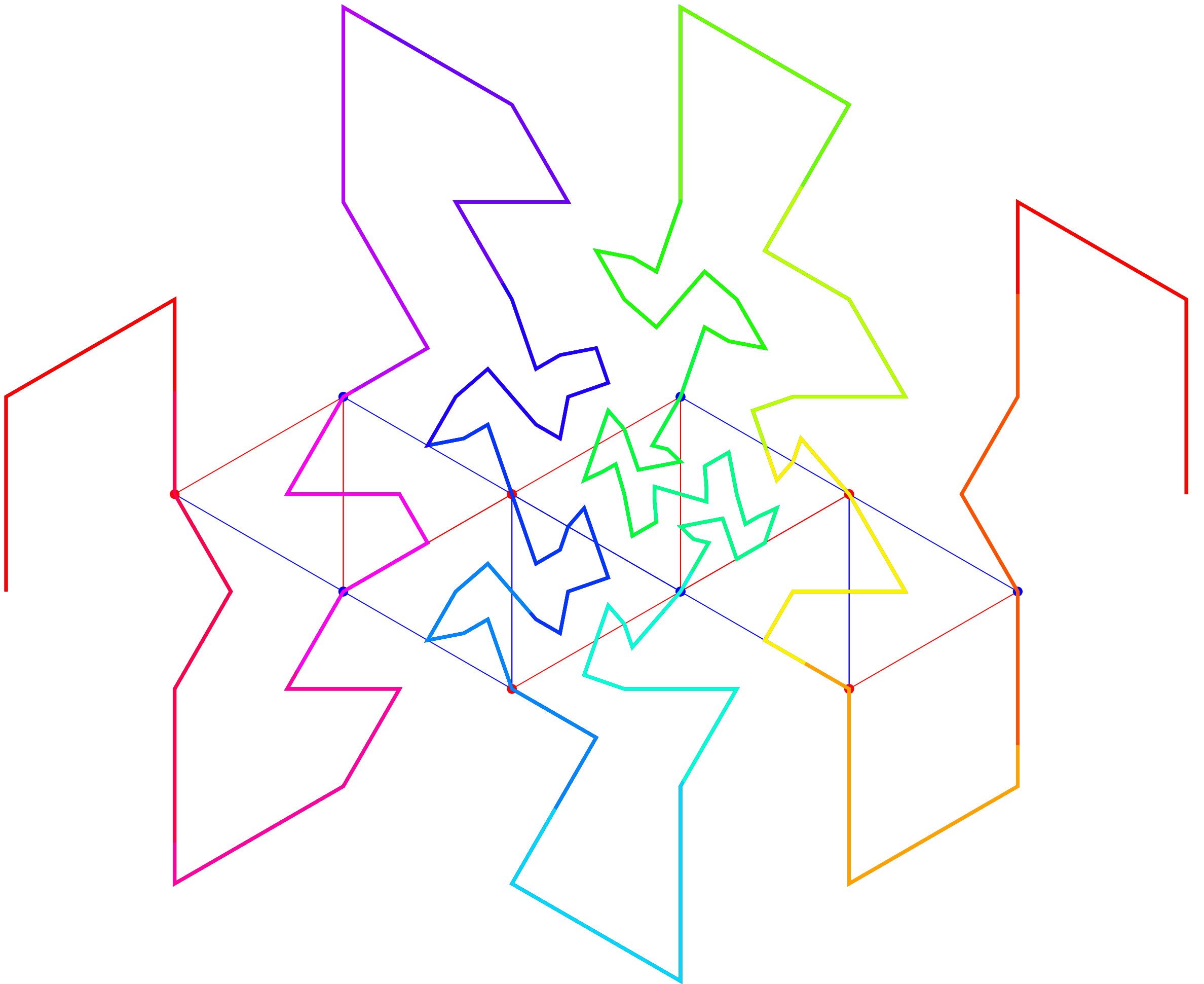


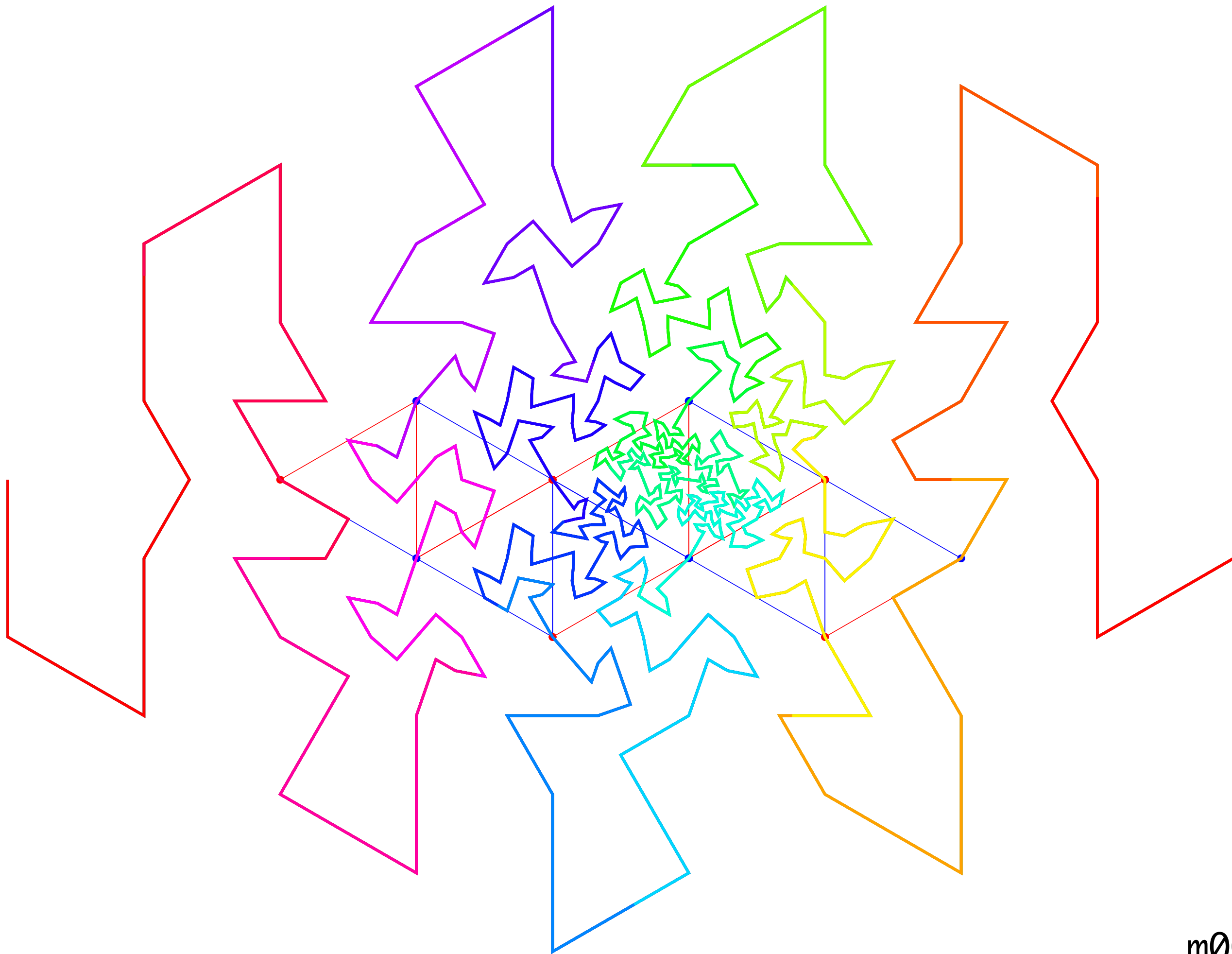
In place of a large disk D , we use a large continent.



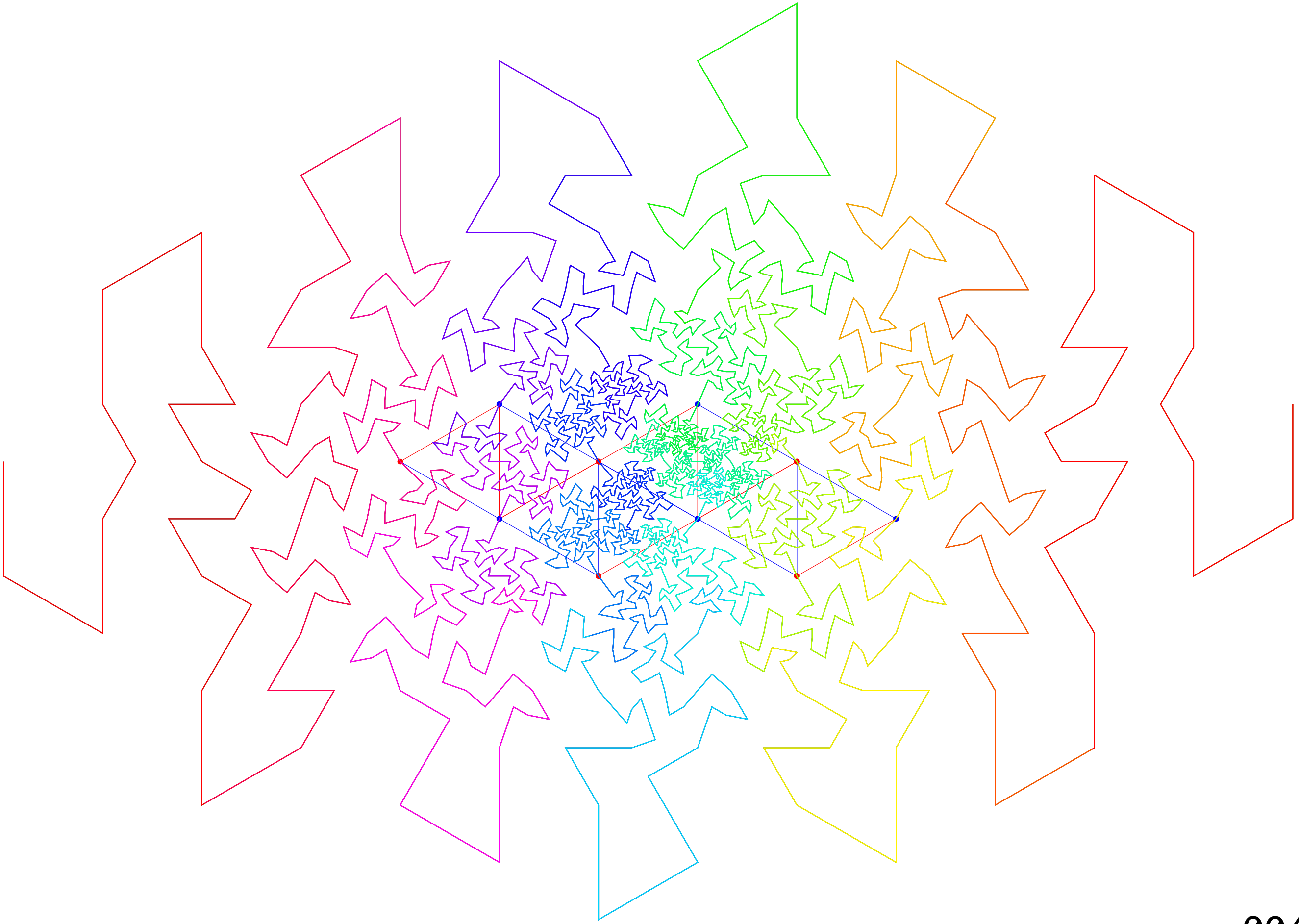


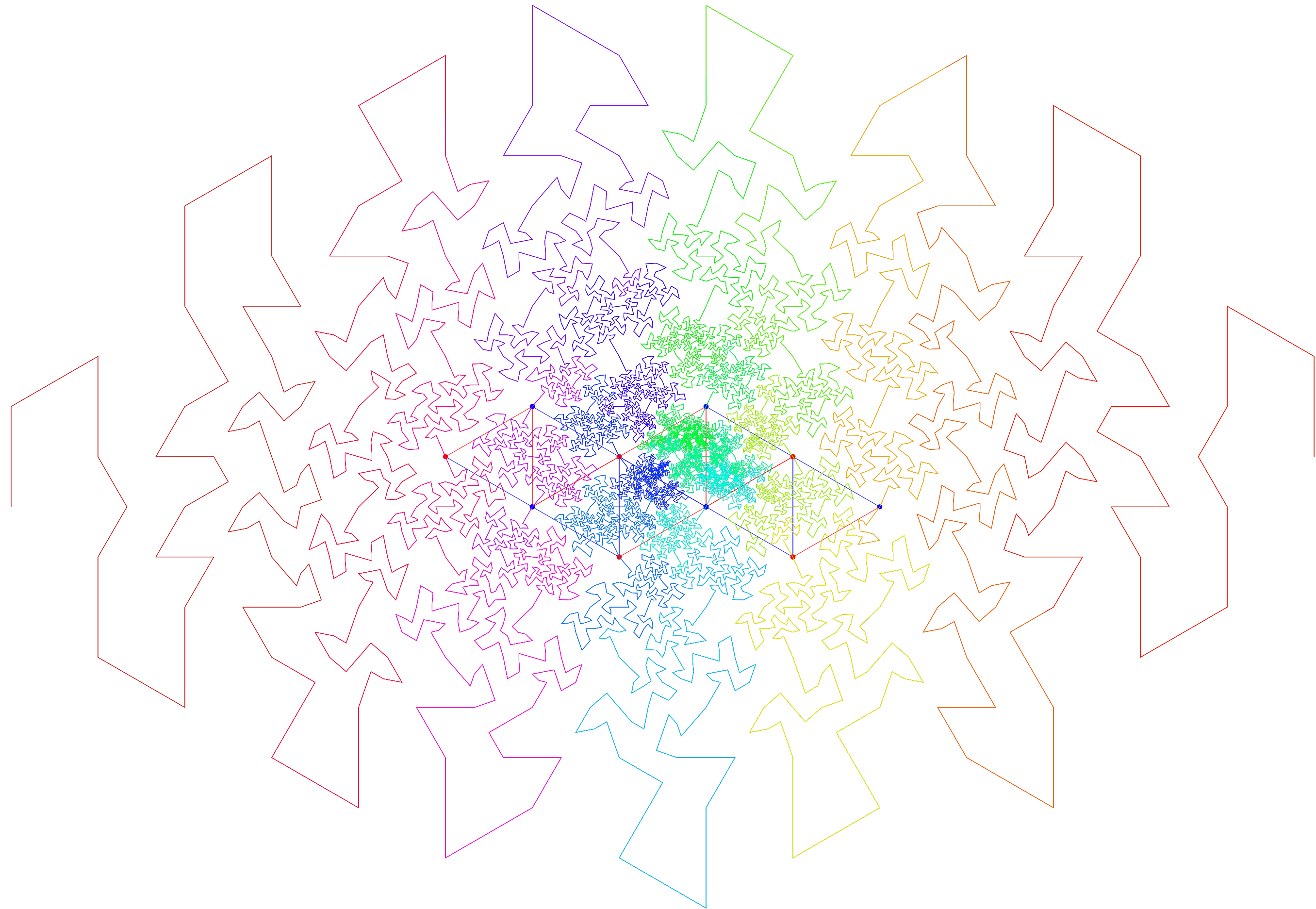
m004



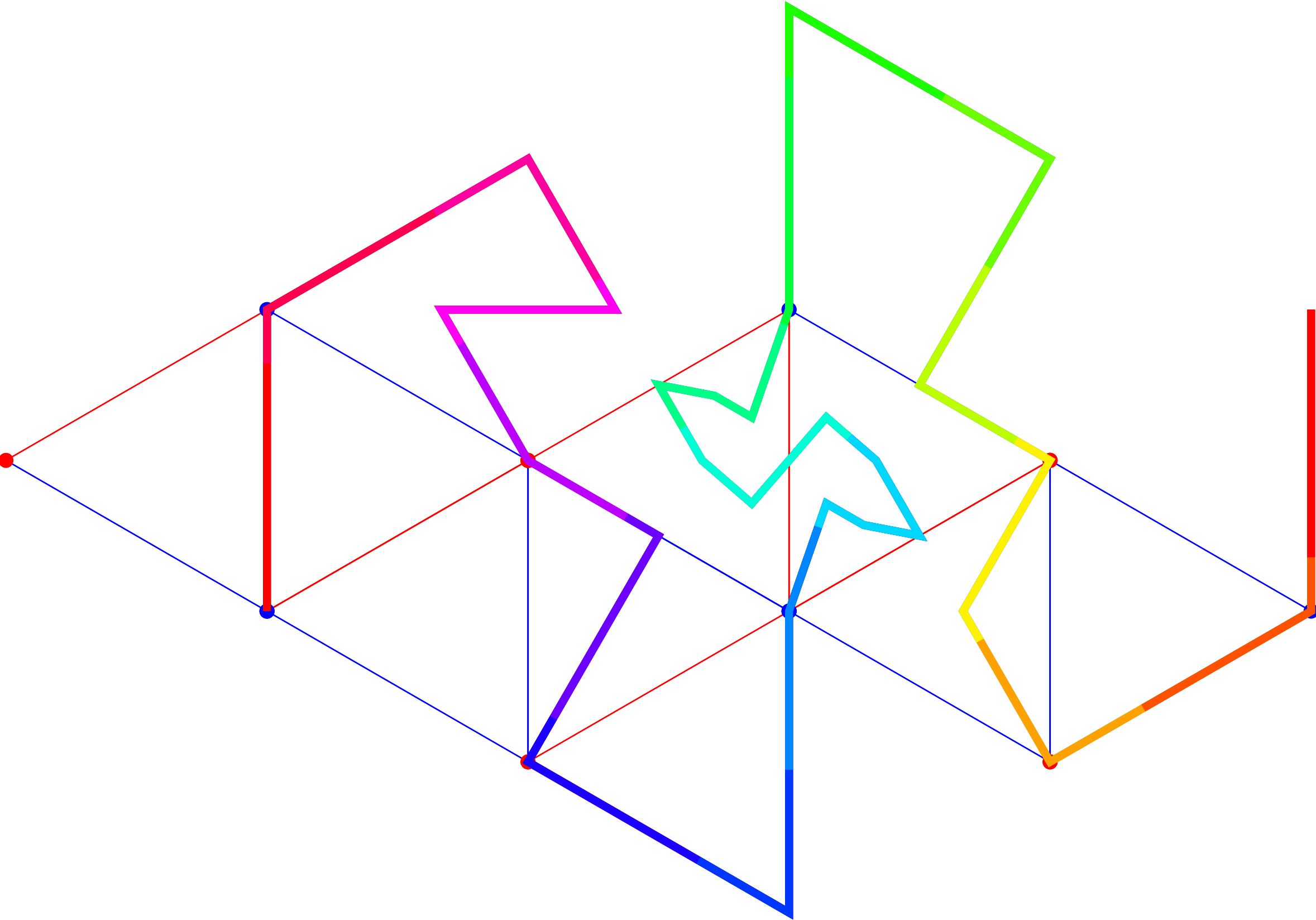


m004

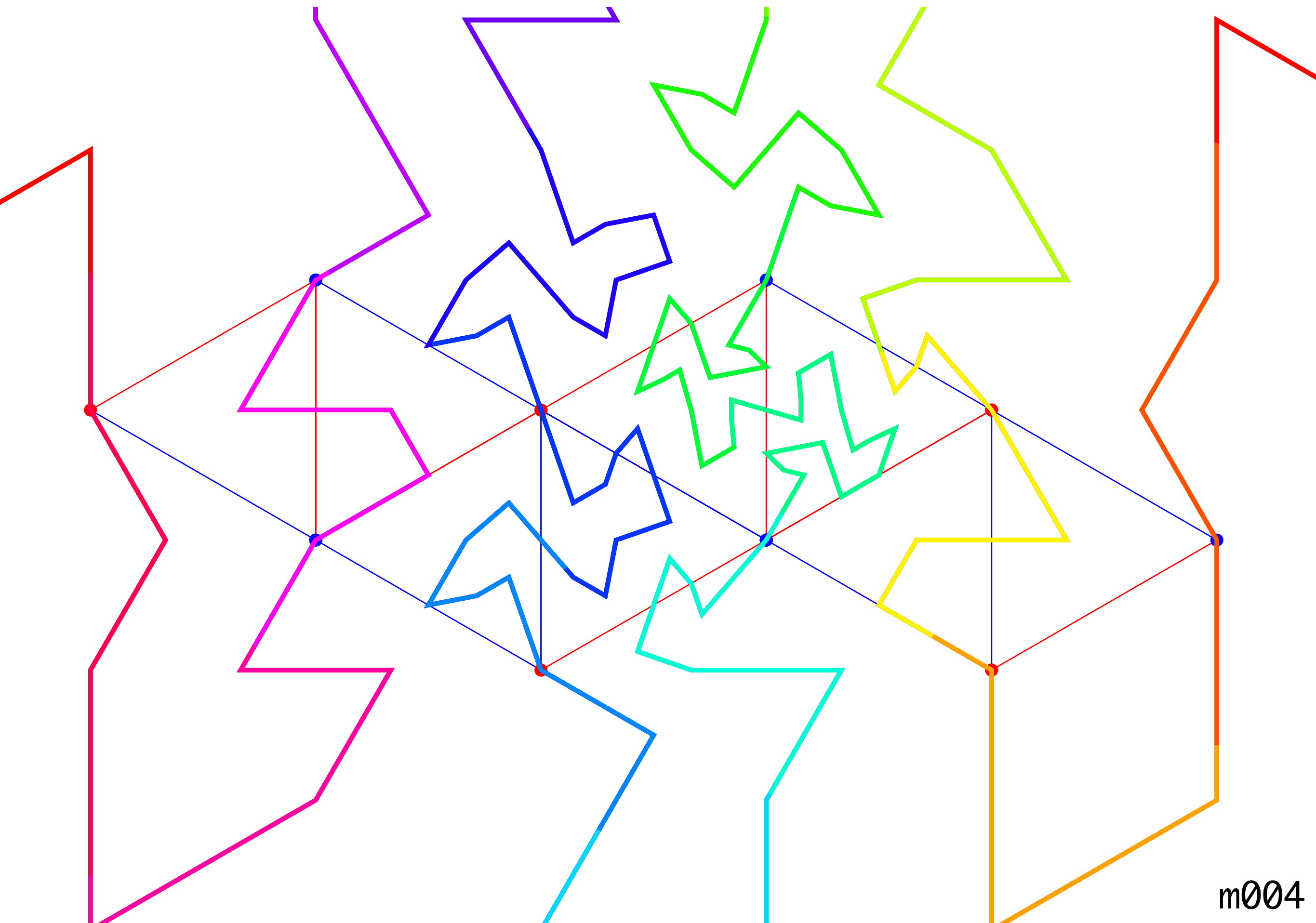




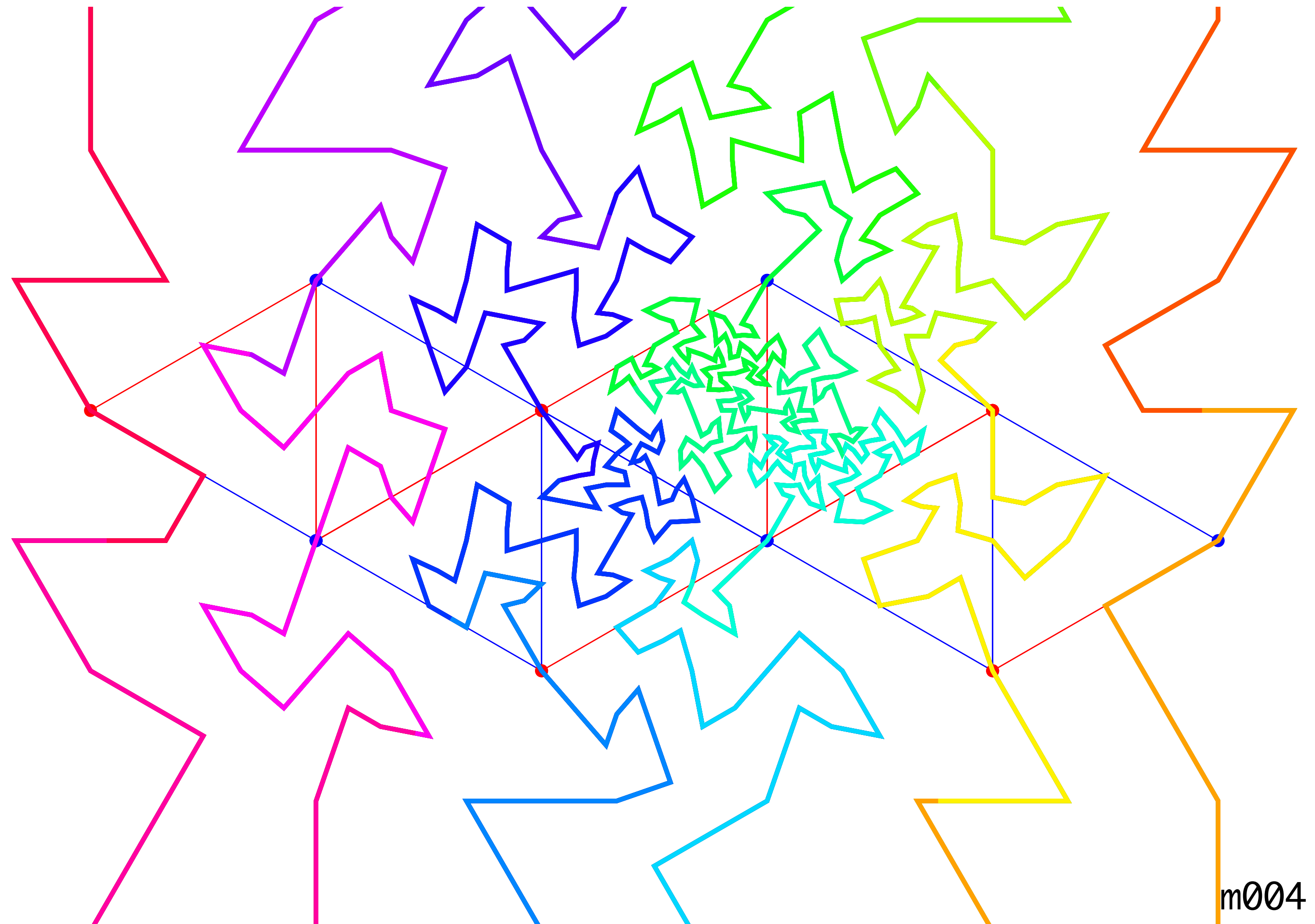
Focus in on one fundamental domain of the boundary torus.



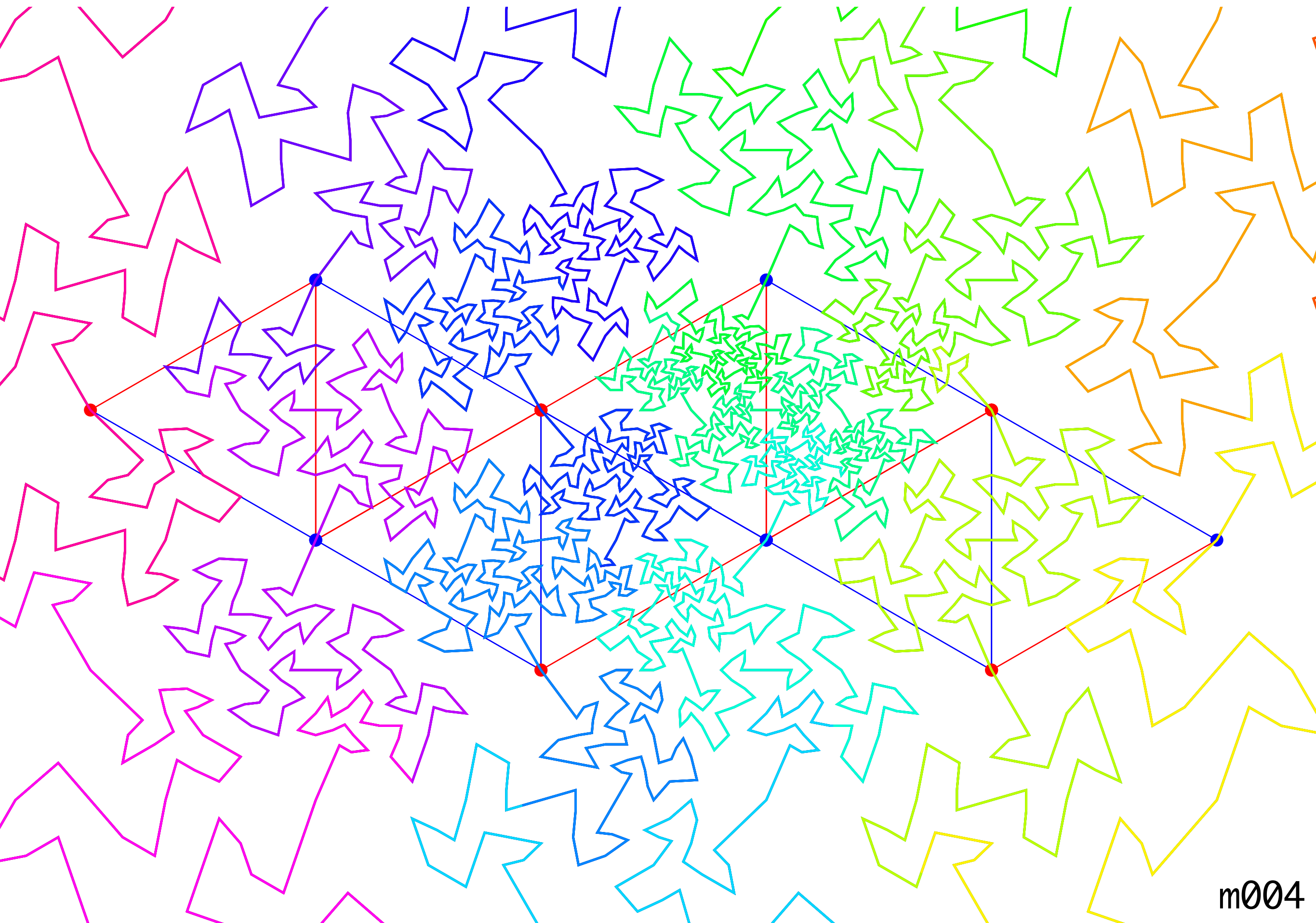
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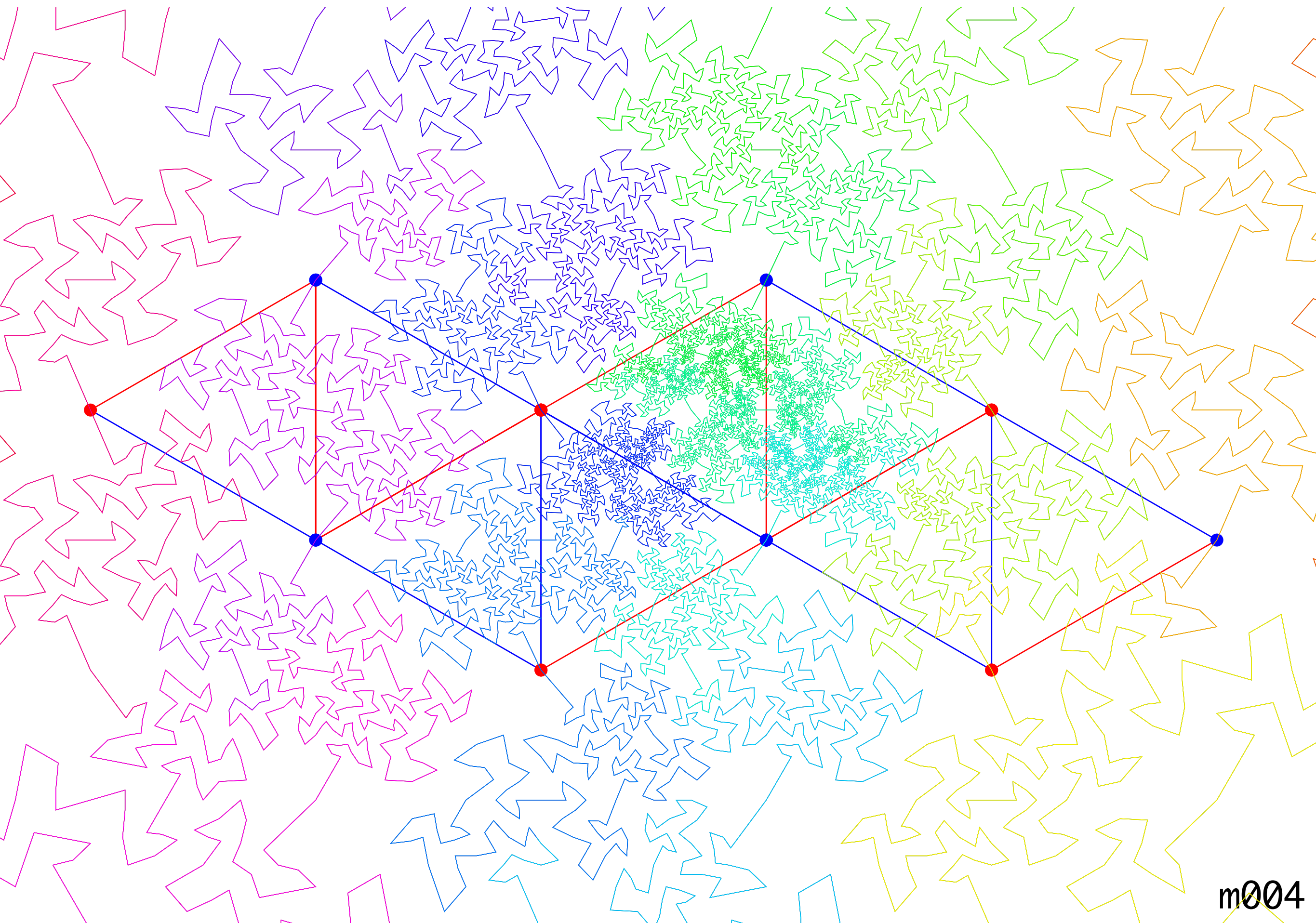
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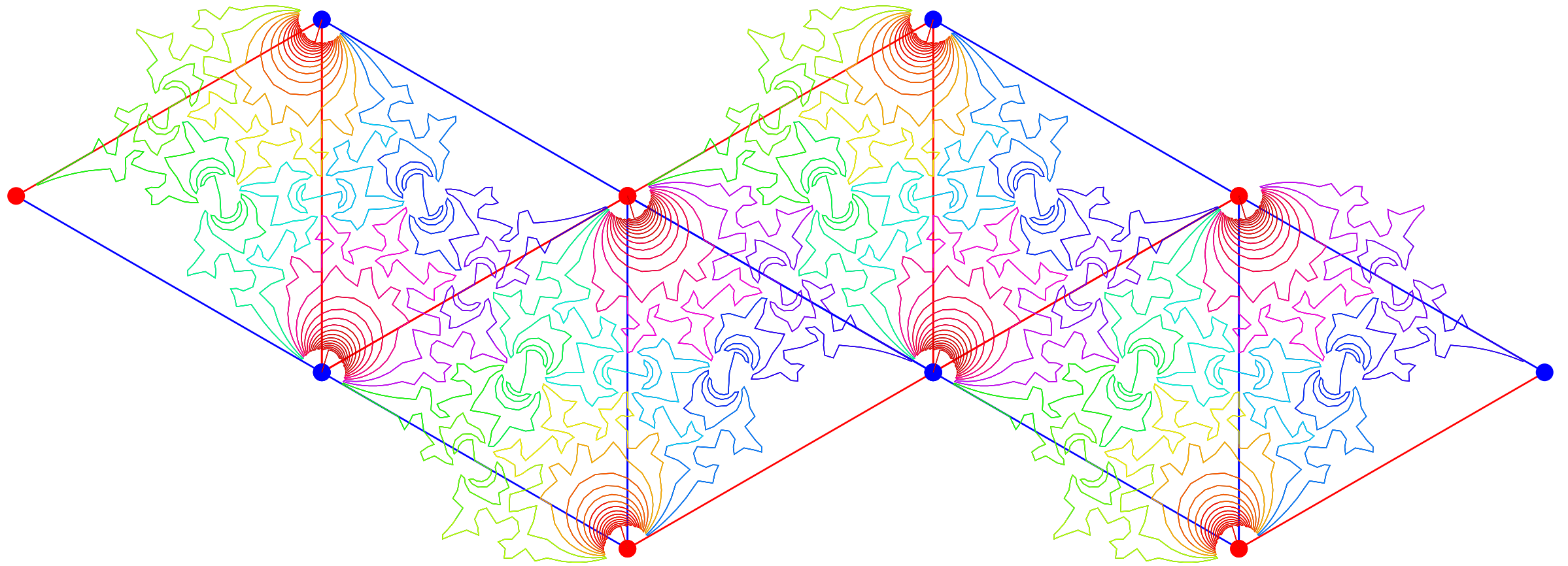
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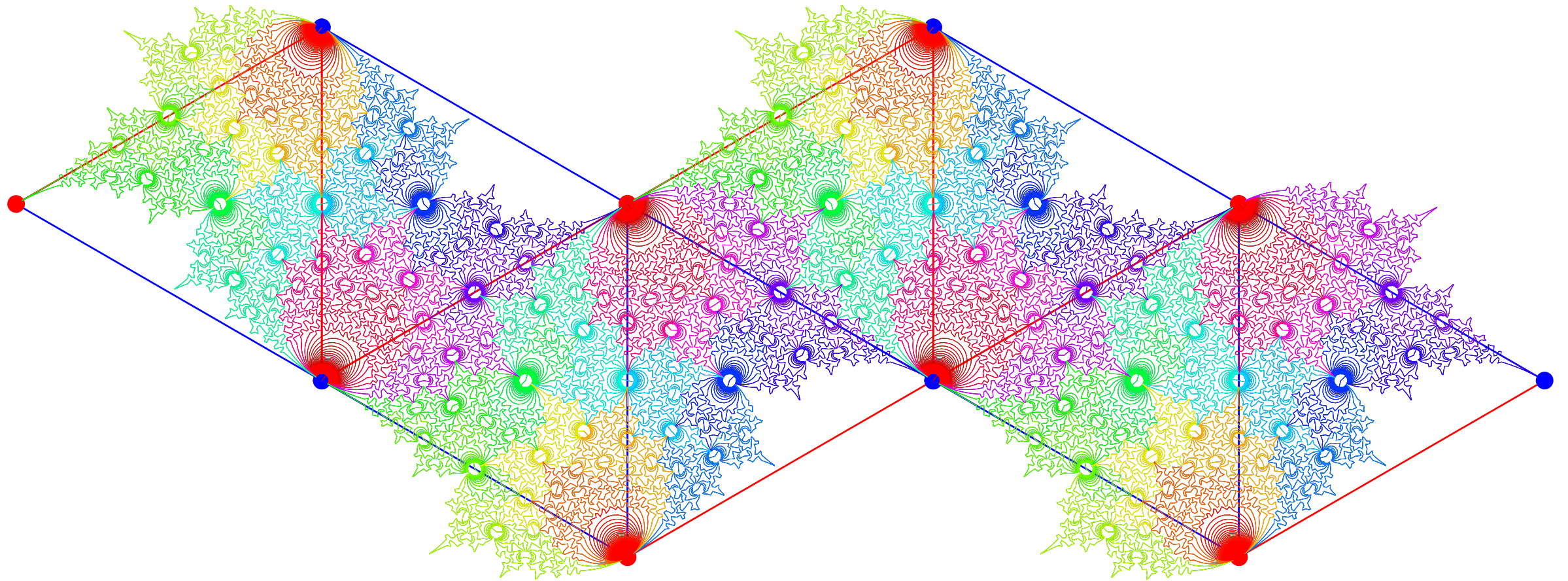
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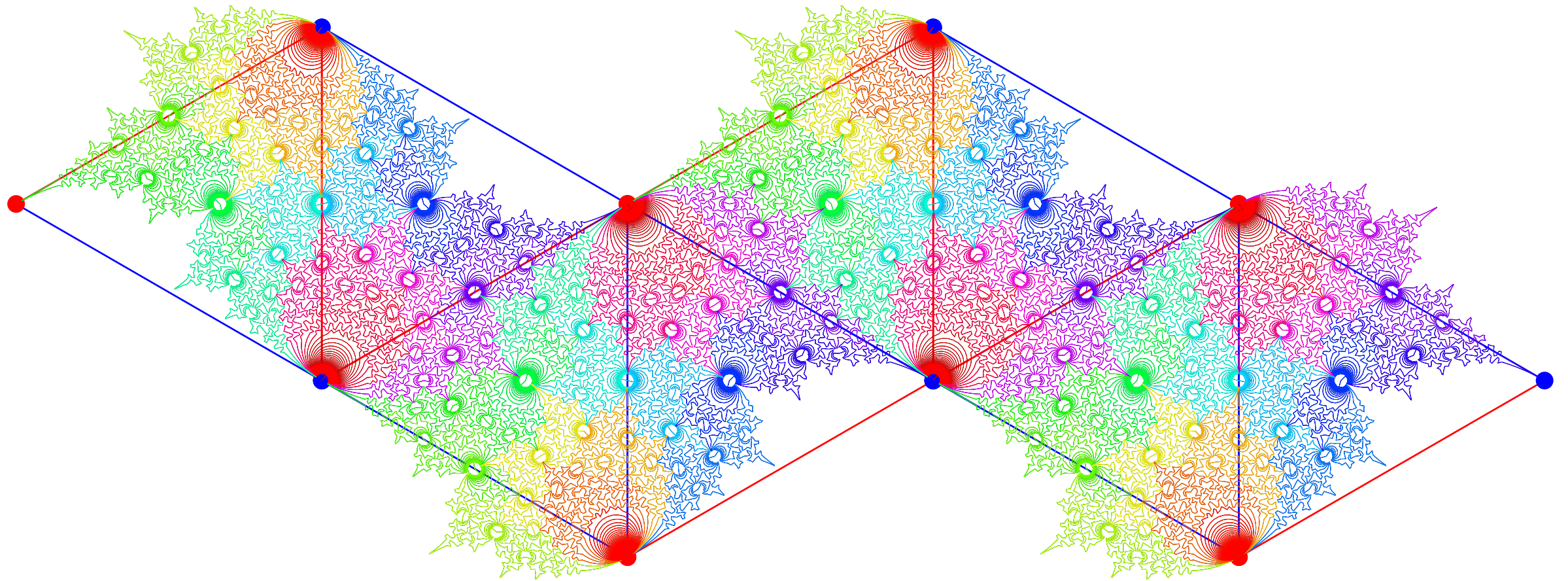
With less naive algorithms we can draw just this fundamental domain, with a more uniform density.



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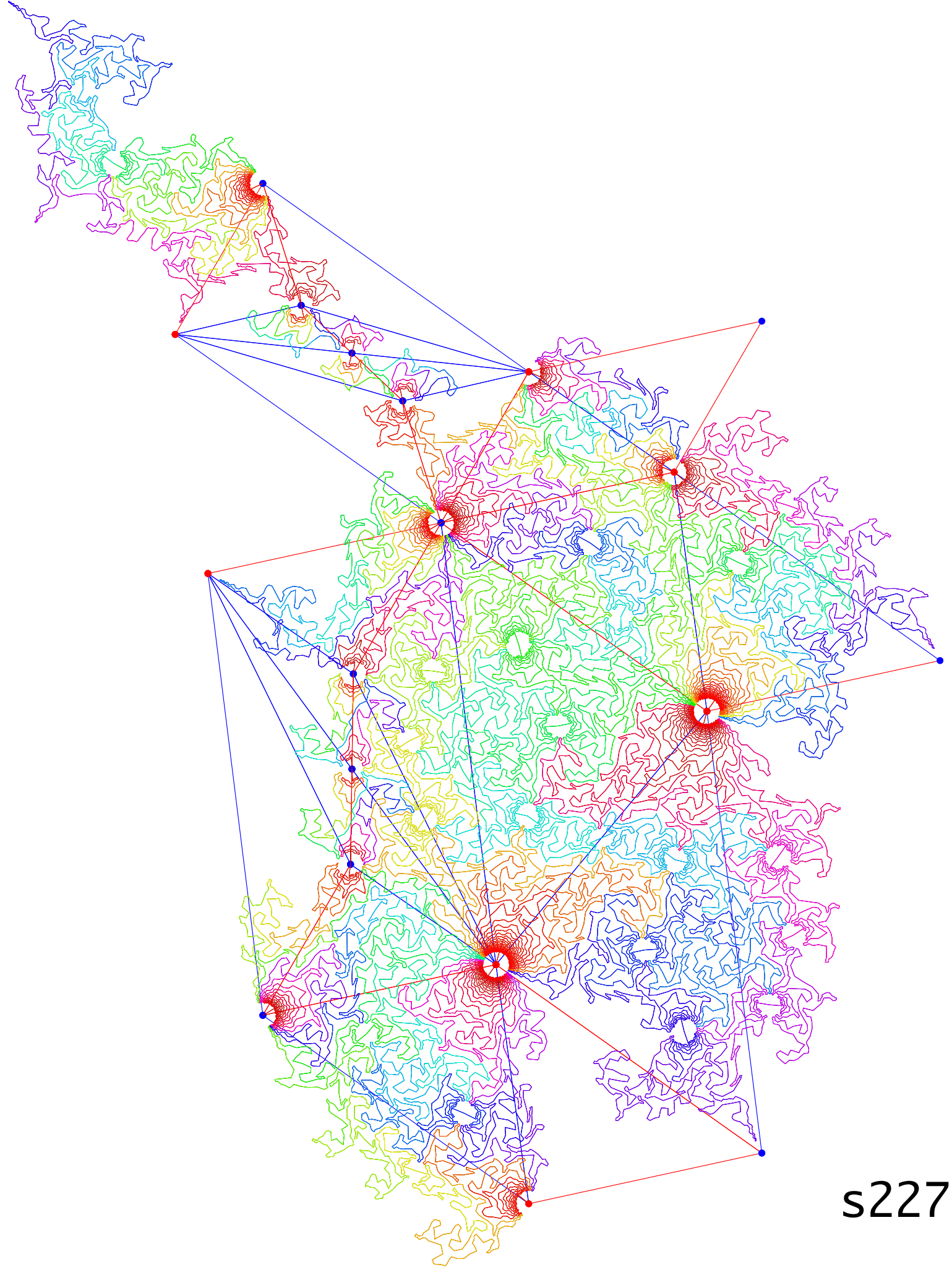
With less naive algorithms we can draw just this fundamental domain, with a more uniform density.



For a fibered manifold, we recover the Cannon-Thurston map built in earlier constructions.

But for non-fibered
manifolds, we get
something new.

We prove that these
Cannon-Thurston maps
cannot come from
surface subgroups, even
virtually.



Proof:

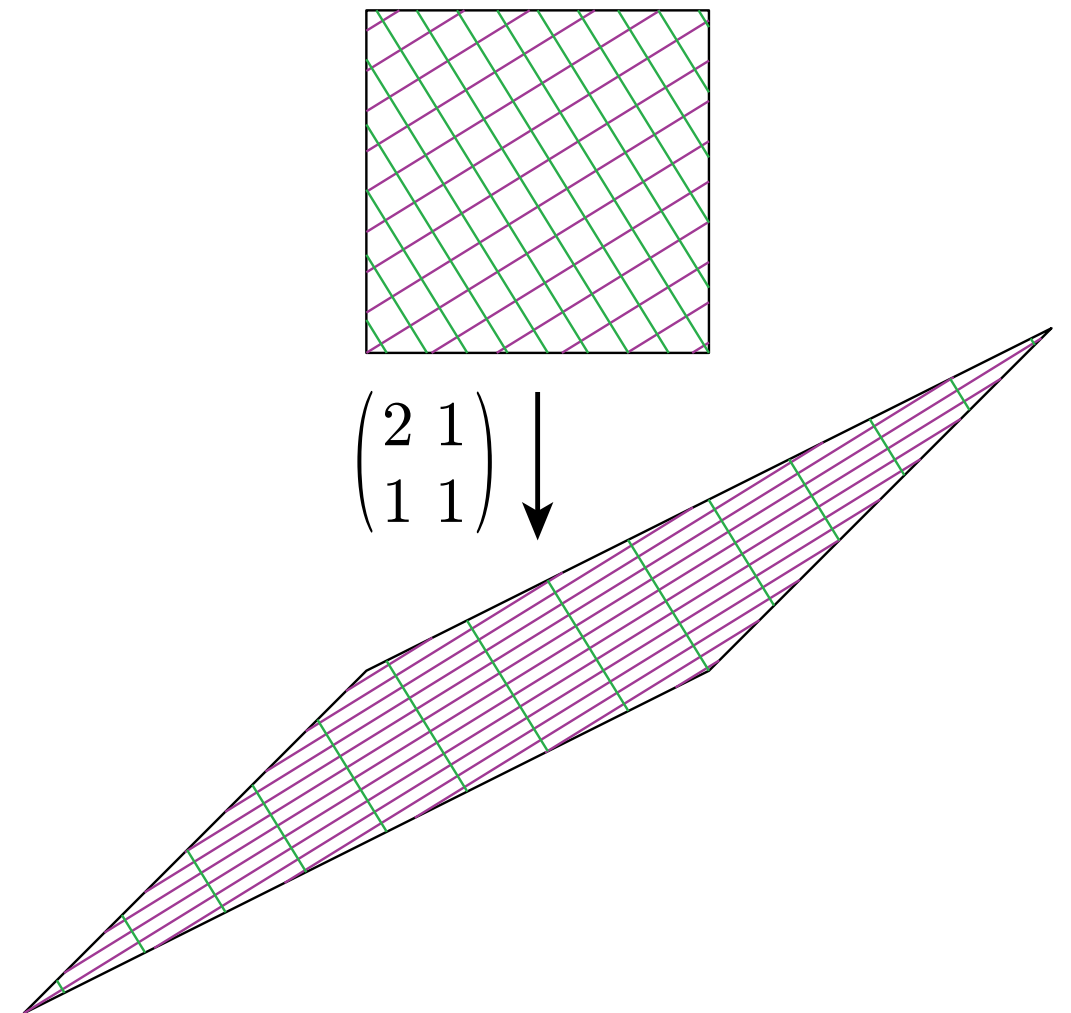
Suppose we have $\phi : S^1 \rightarrow S^2$, $\psi : S^1 \rightarrow S^2$, a pair of Cannon-Thurston maps. We say that these are *equivalent* if there are homeomorphisms before and after making the diagram commute. From ϕ , we can recover the laminations Λ^ϕ and Λ_ϕ in S^1 ; they are exactly the non-injectivity loci of ϕ (with some work and a binary choice). Thus we can recover the link space (in our first paper), and so the veering triangulation. This means that the given three-manifolds have commensurable veering triangulations (more work to get from infinite sheeted to finite sheeted here), so one is fibered if and only if the other is.

Q.E.D.

Laminations in the veering circle

We build our map $S^1 \rightarrow S^2$ by collapsing the *upper and lower laminations* in the veering circle.

These are related to the *stable* and *unstable* foliations for a pseudo-Anosov map.

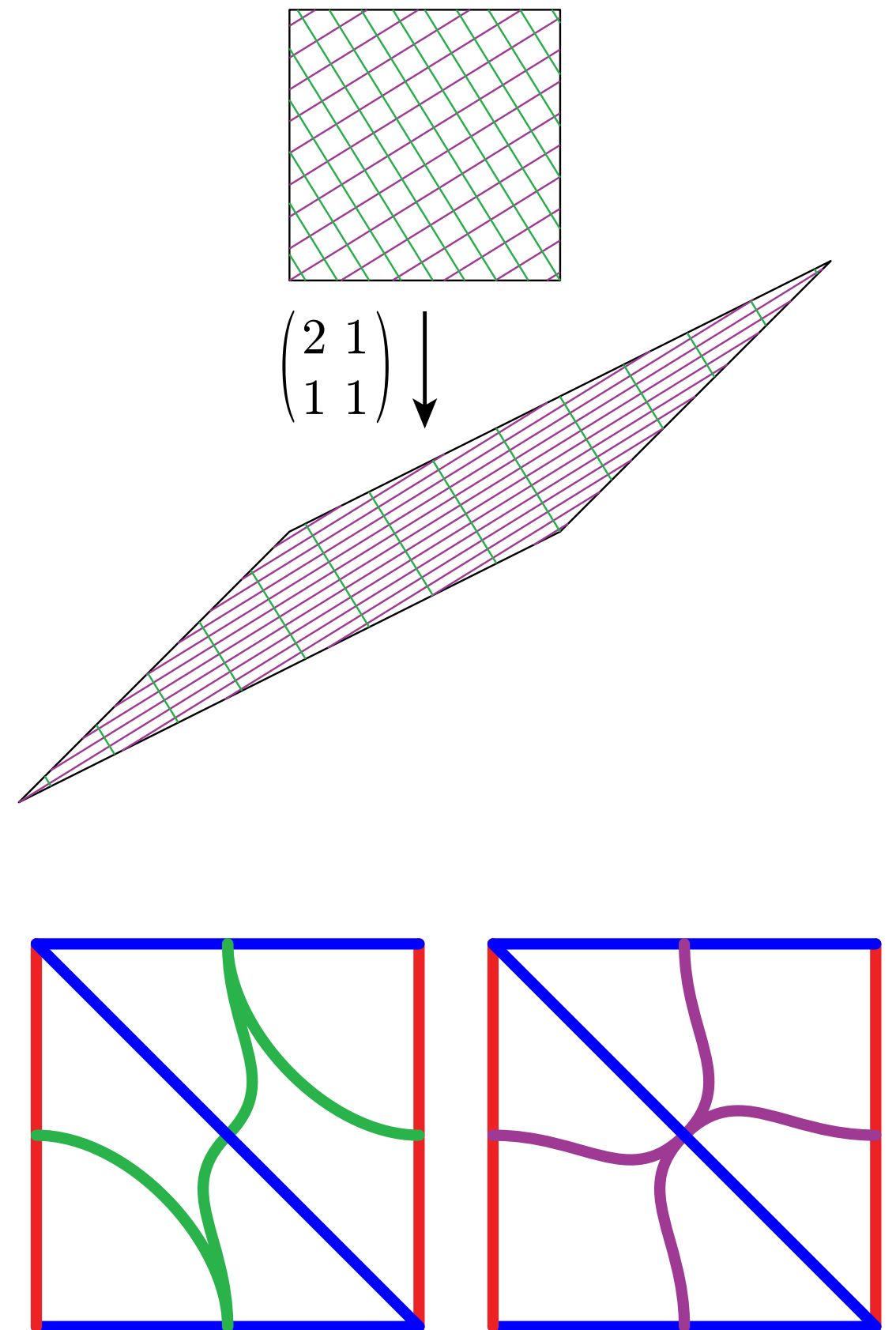


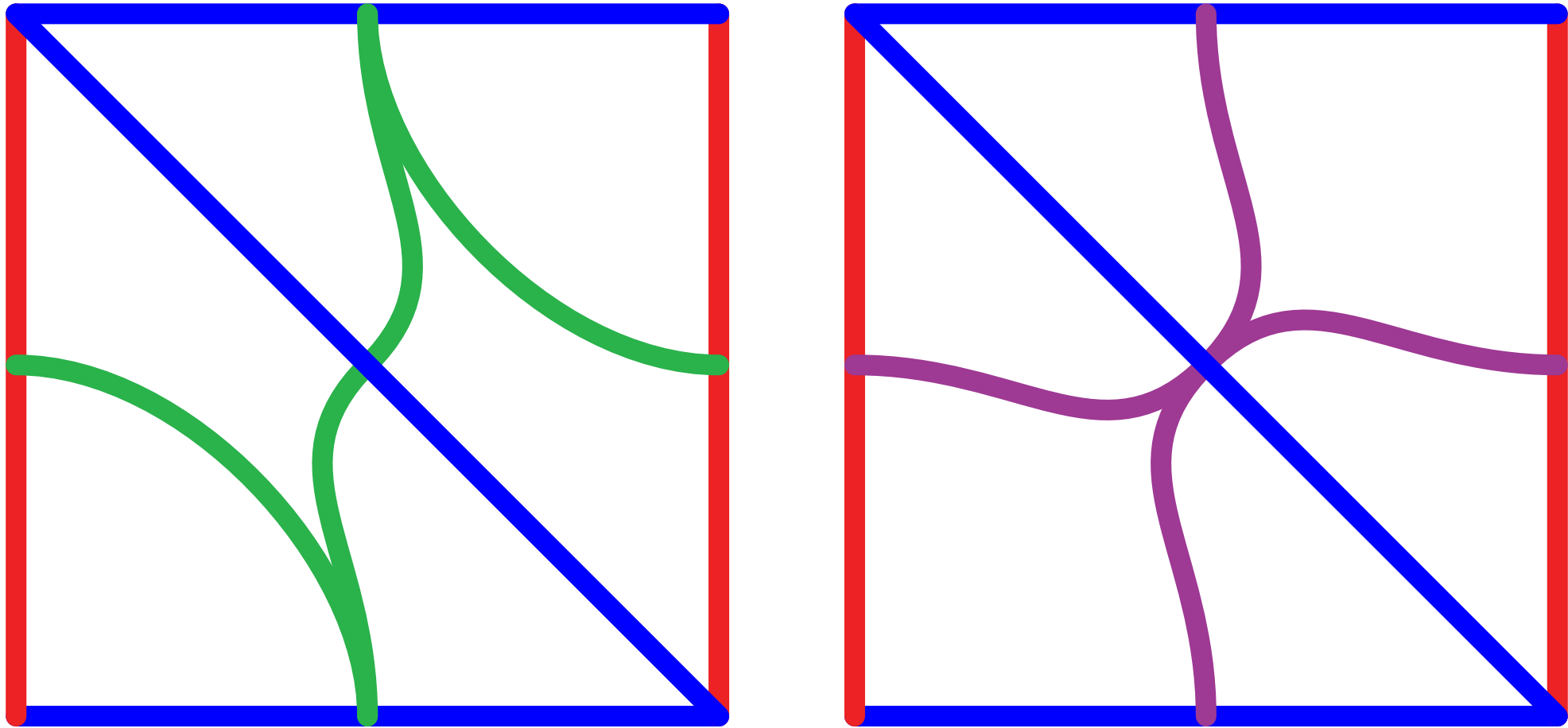
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We build our map $S^1 \rightarrow S^2$ by collapsing the *upper and lower laminations* in the veering circle.

These are related to the *stable* and *unstable* foliations for a pseudo-Anosov map.

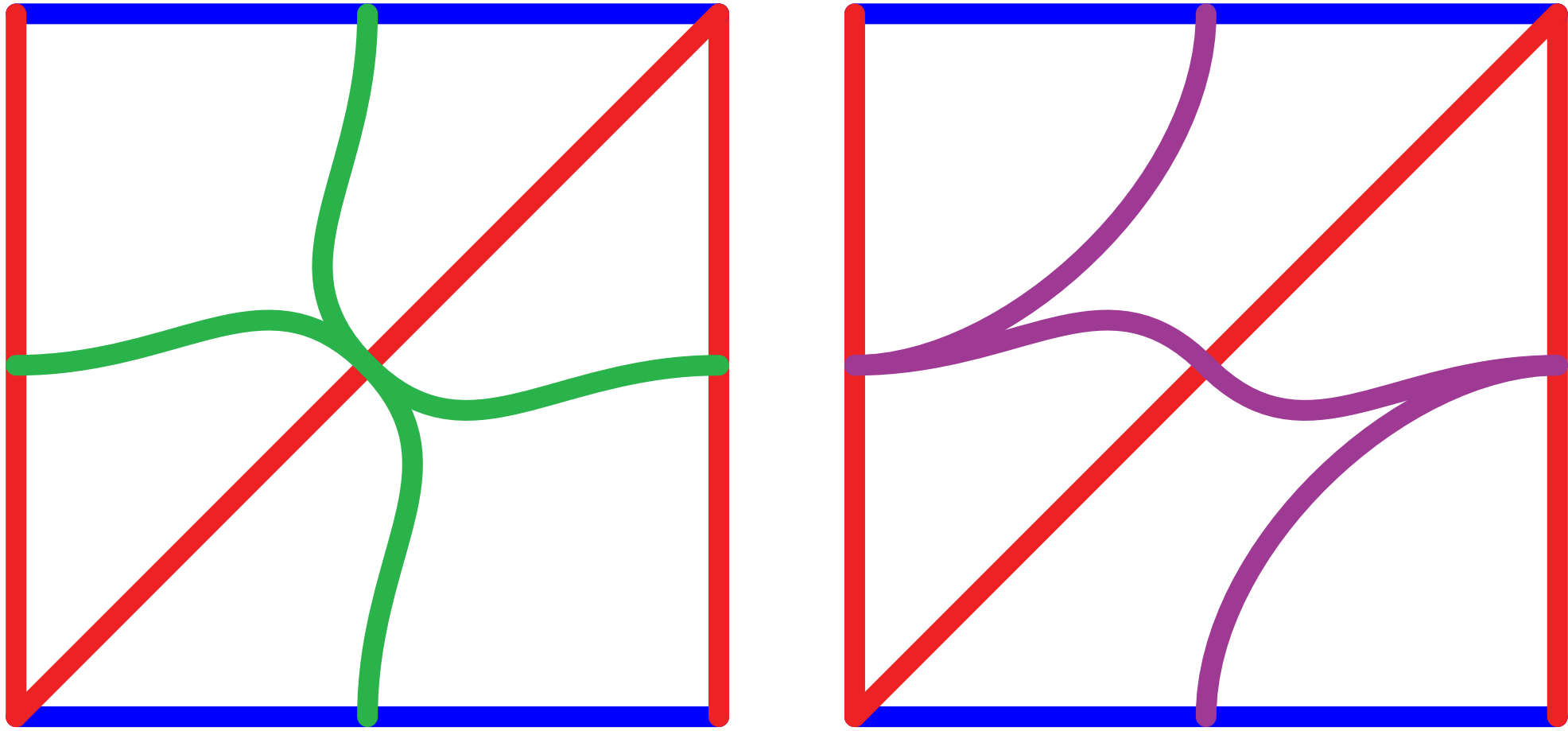
Agol's layered construction uses splitting sequences of train tracks. These give us a hint for where these laminations come from.





upper faces

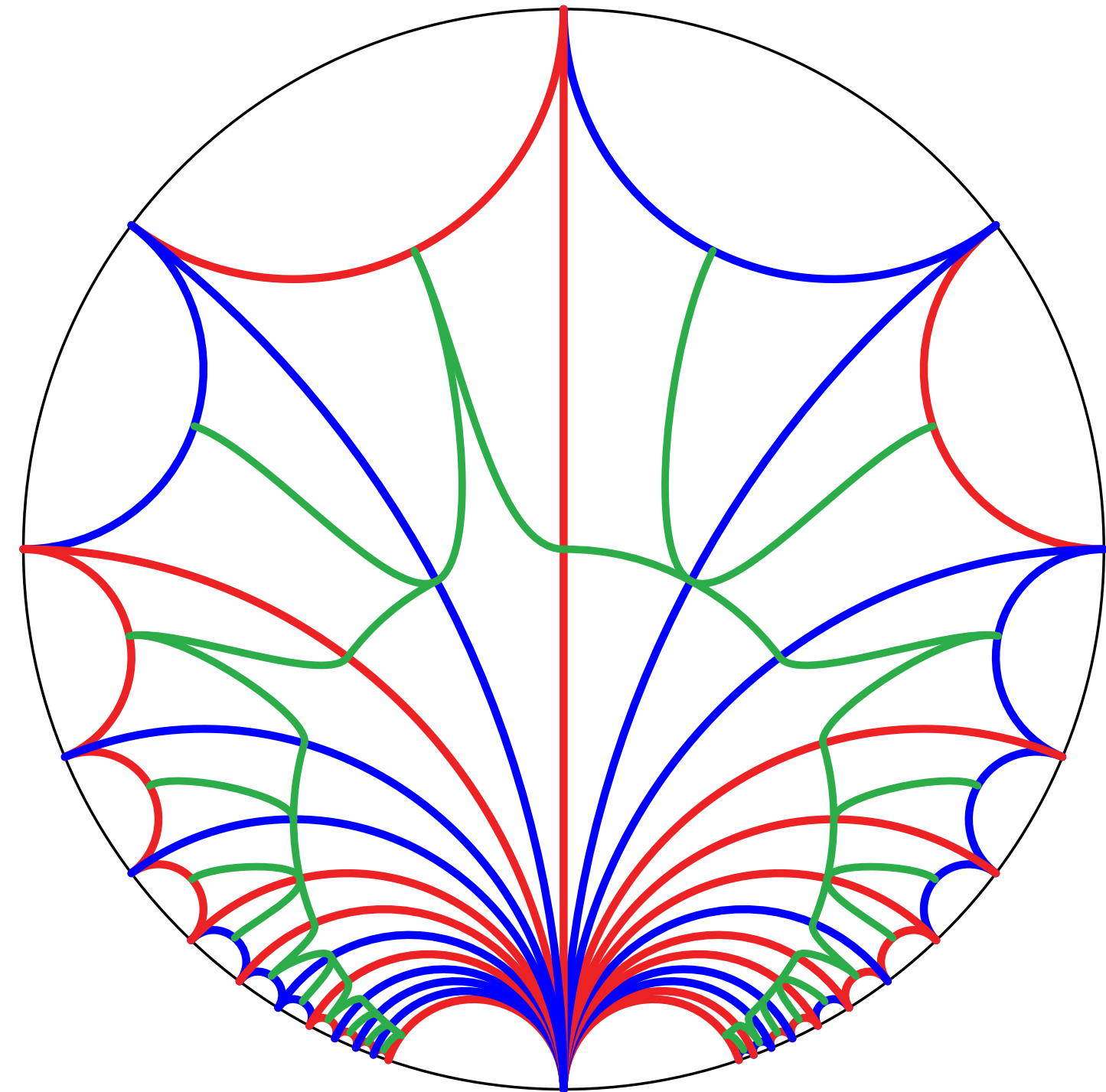
Moving down through a tetrahedron folds the *upper track* (green) and splits the *lower track* (purple).



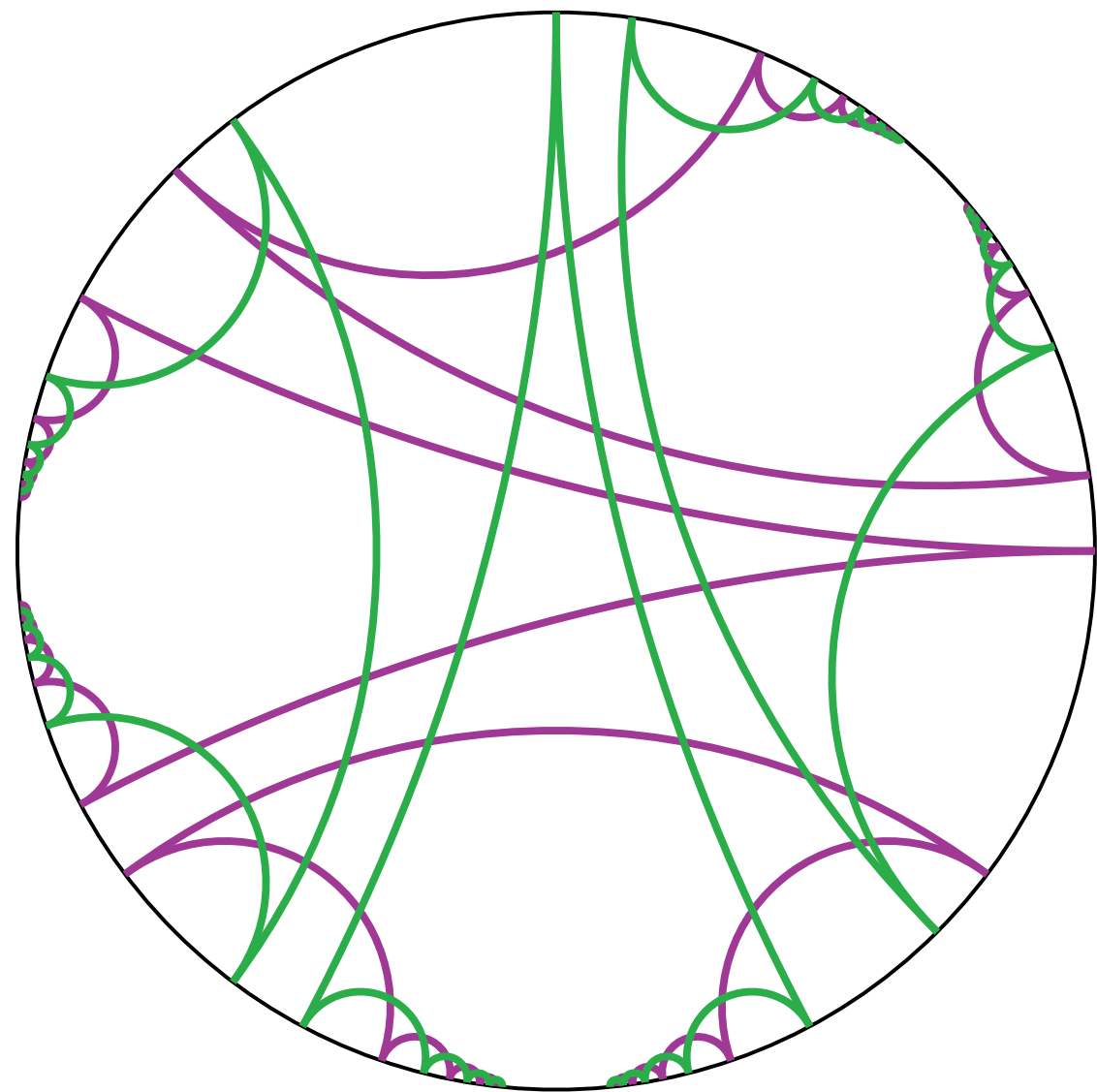
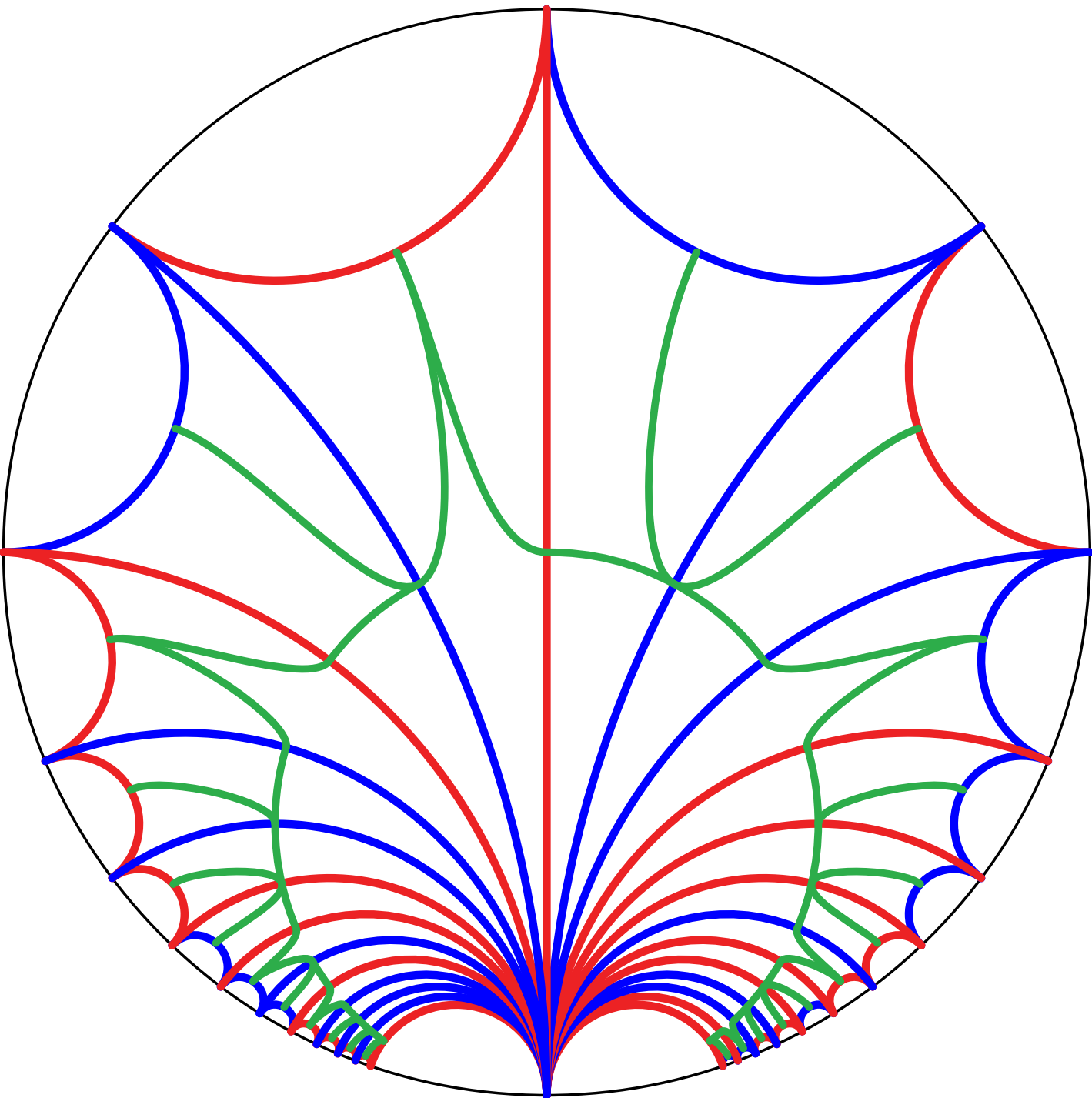
lower faces

Moving down through a tetrahedron folds the *upper track* (green) and splits the *lower track* (purple).

We showed that for any veering triangulation (\mathcal{T}, α) the universal cover $(\widetilde{\mathcal{T}}, \widetilde{\alpha})$ is layered.



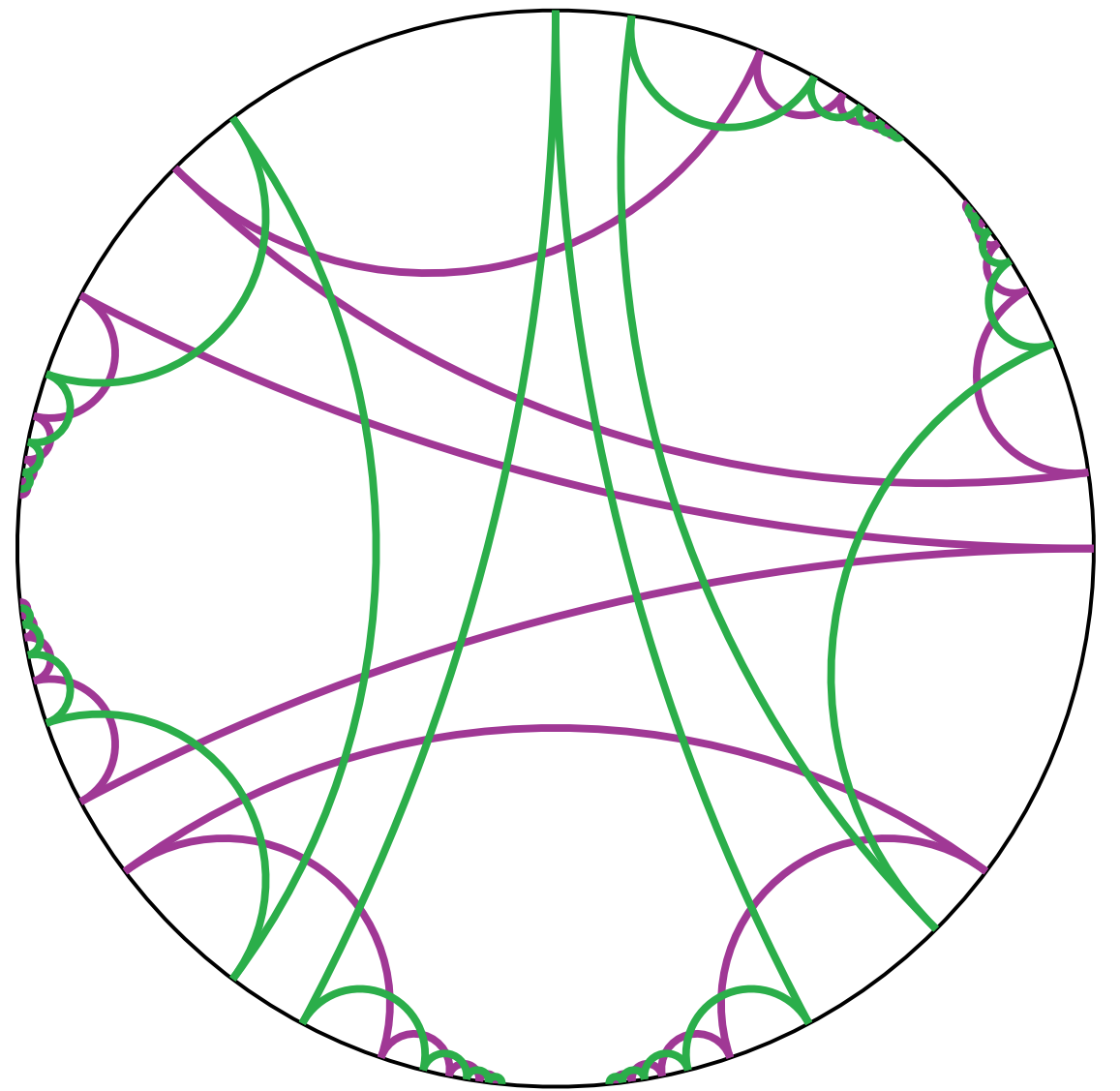
We showed that for any veering triangulation (\mathcal{T}, α) the universal cover $(\tilde{\mathcal{T}}, \tilde{\alpha})$ is layered.



We split train tracks through all layers to generate the upper and lower laminations Λ^α , Λ_α in the veering circle.

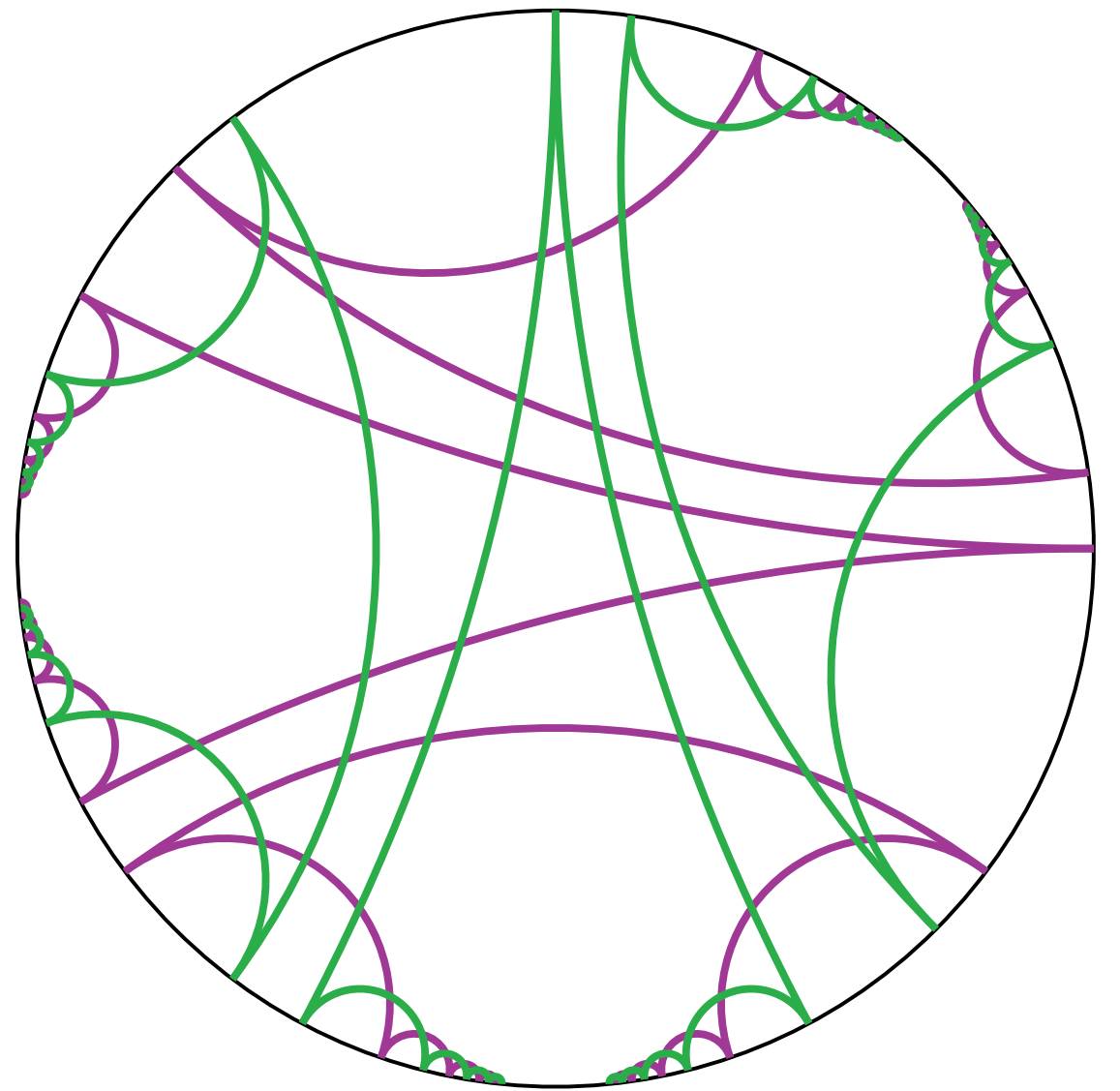
Properties of these laminations:

- Leaves of Λ^α share endpoints only around *crowns*, associated to cusps of $\widetilde{\mathcal{T}}$. (Same for Λ_α .)
- Leaves of Λ^α share no endpoints with leaves of Λ_α .
- No isolated leaves in Λ^α or Λ_α .



Properties of these laminations:

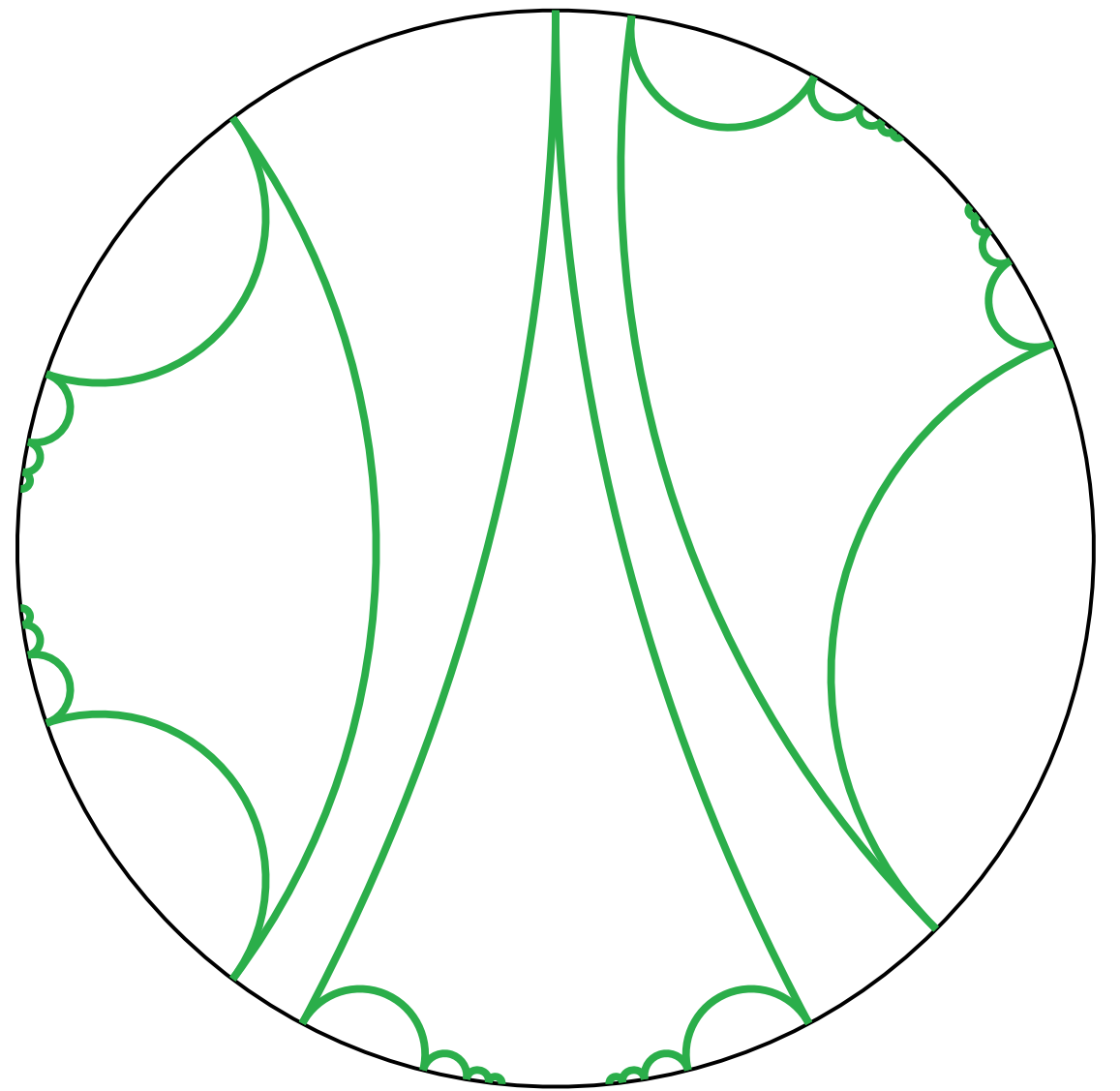
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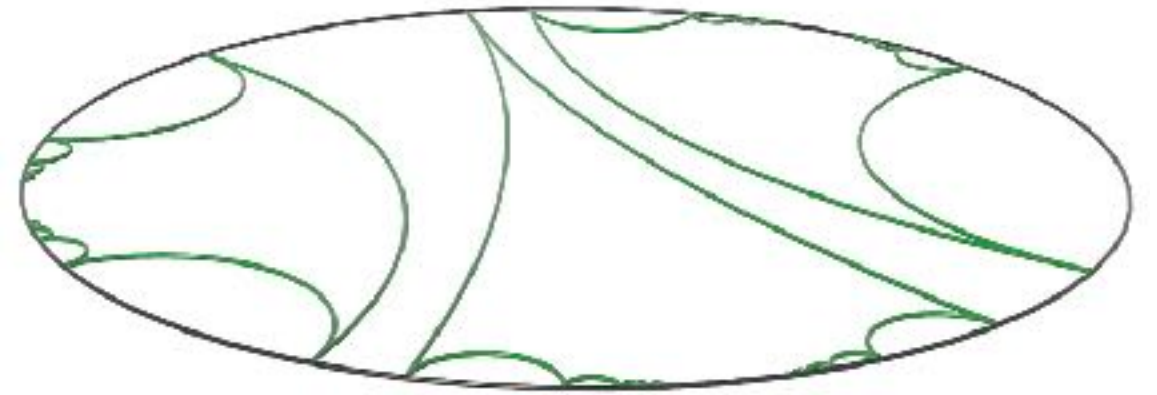
We decompose the veering circle into a collection of disjoint *decomposition elements*. There are three types of these:

- A cusp and the tips of its upper and lower crowns.
- The two endpoints of a non-crown leaf.
- A singleton. (Every other point of the veering circle.)

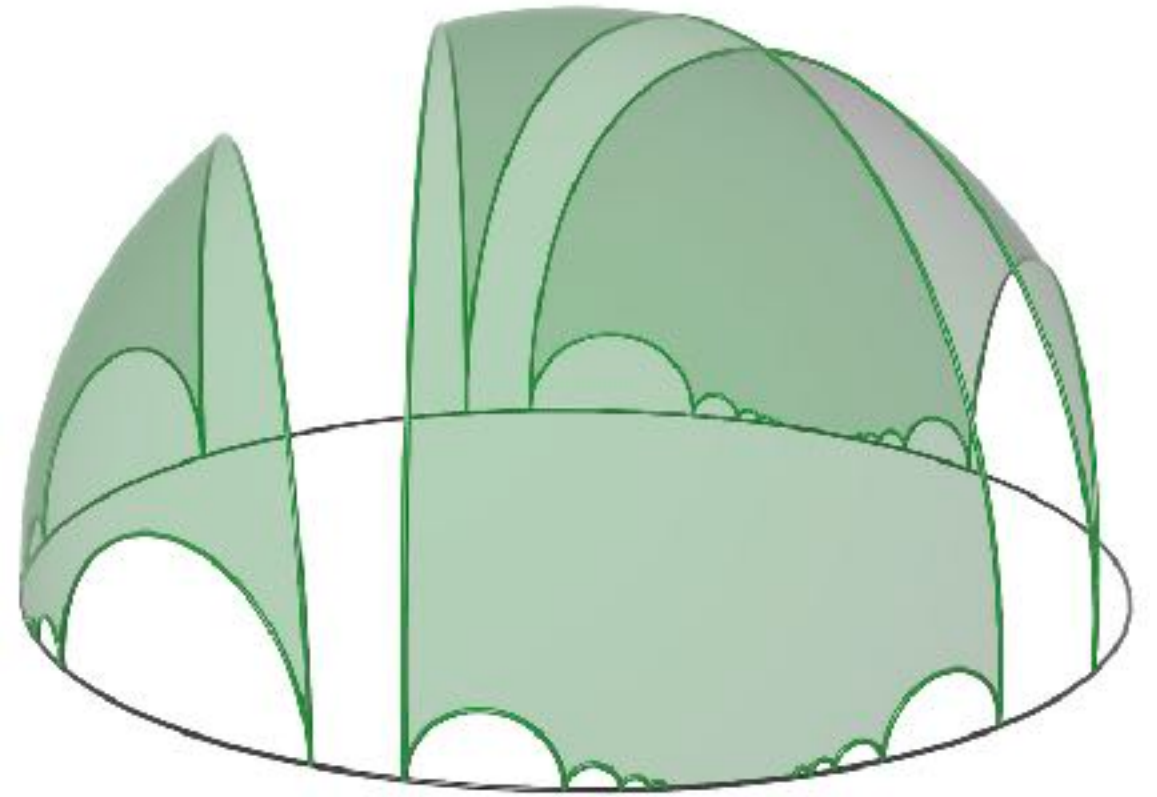
Project the laminations Λ^α , Λ_α to the upper and lower hemispheres of a sphere with equator the veering circle.



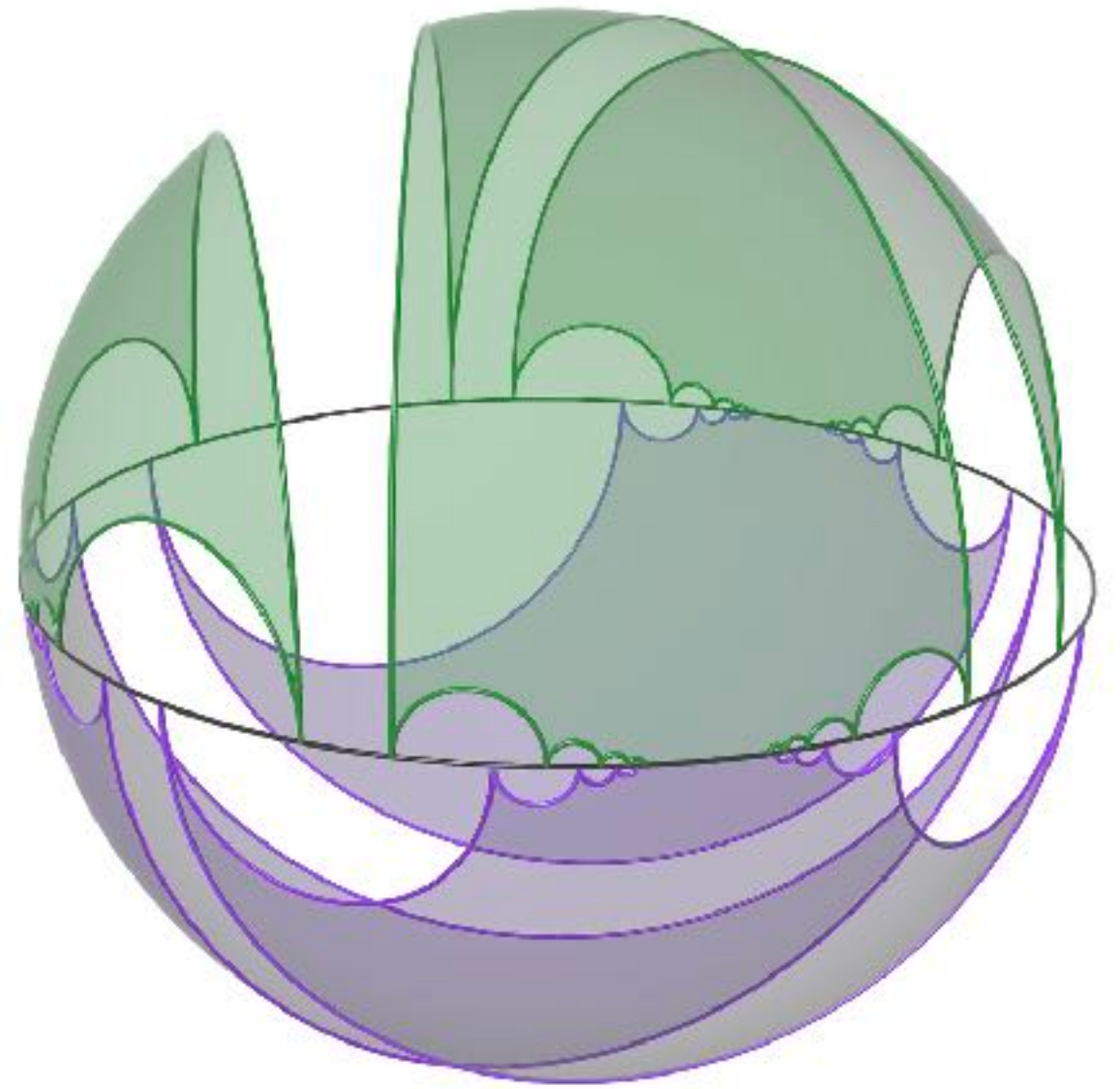
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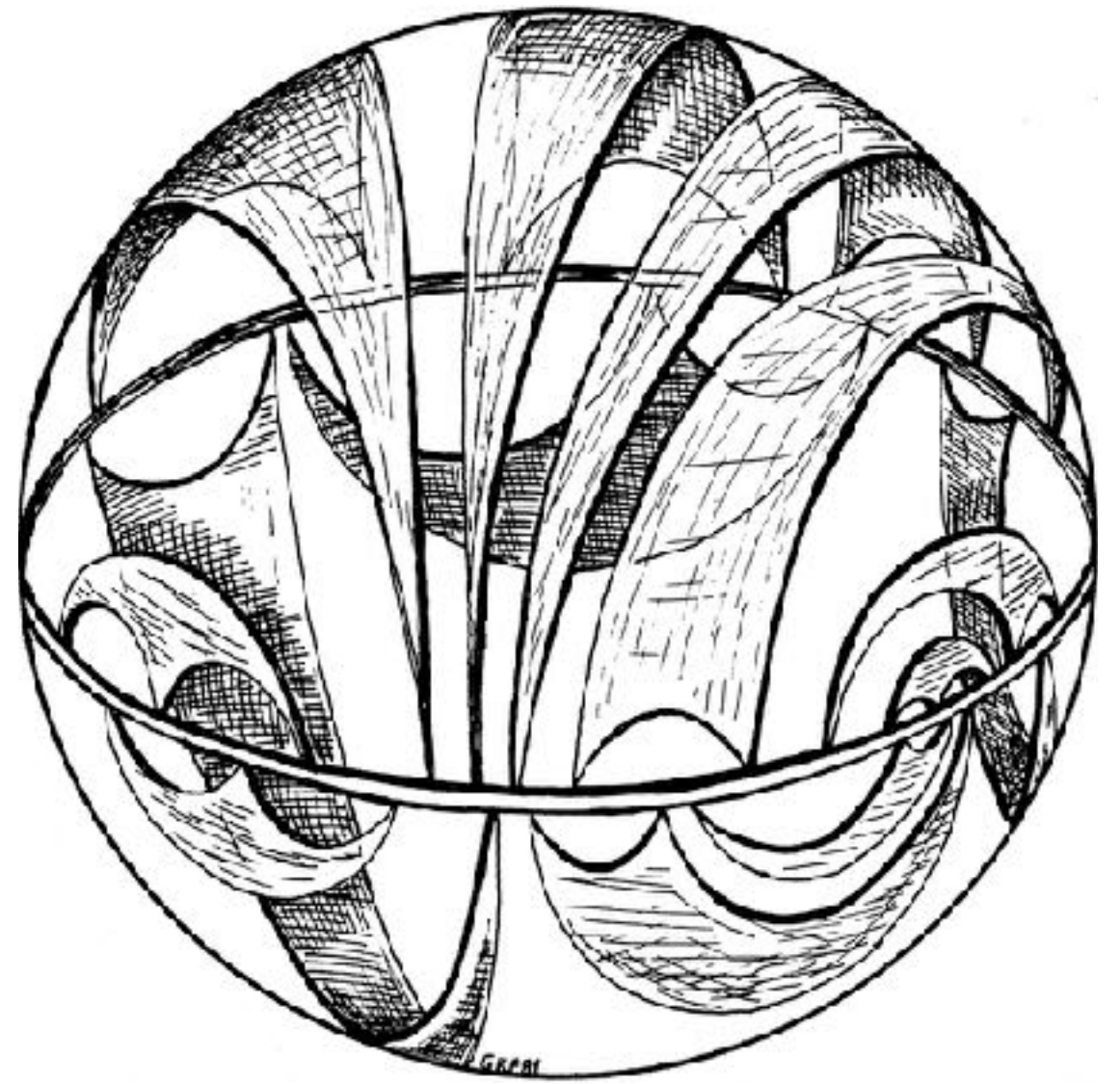
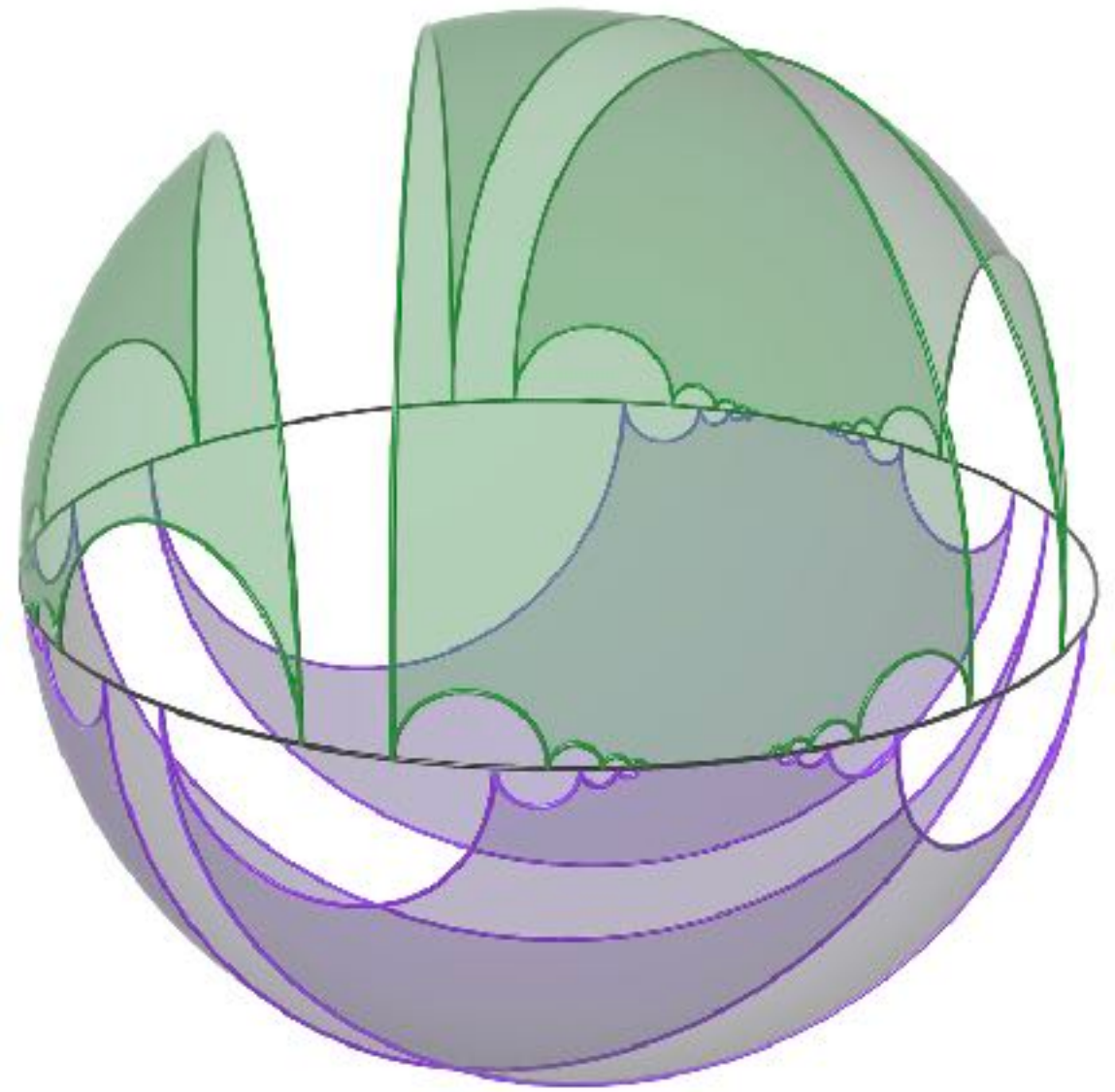


Figure by George Francis, from *Three dimensional manifolds, kleinian groups and hyperbolic geometry* by Bill Thurston

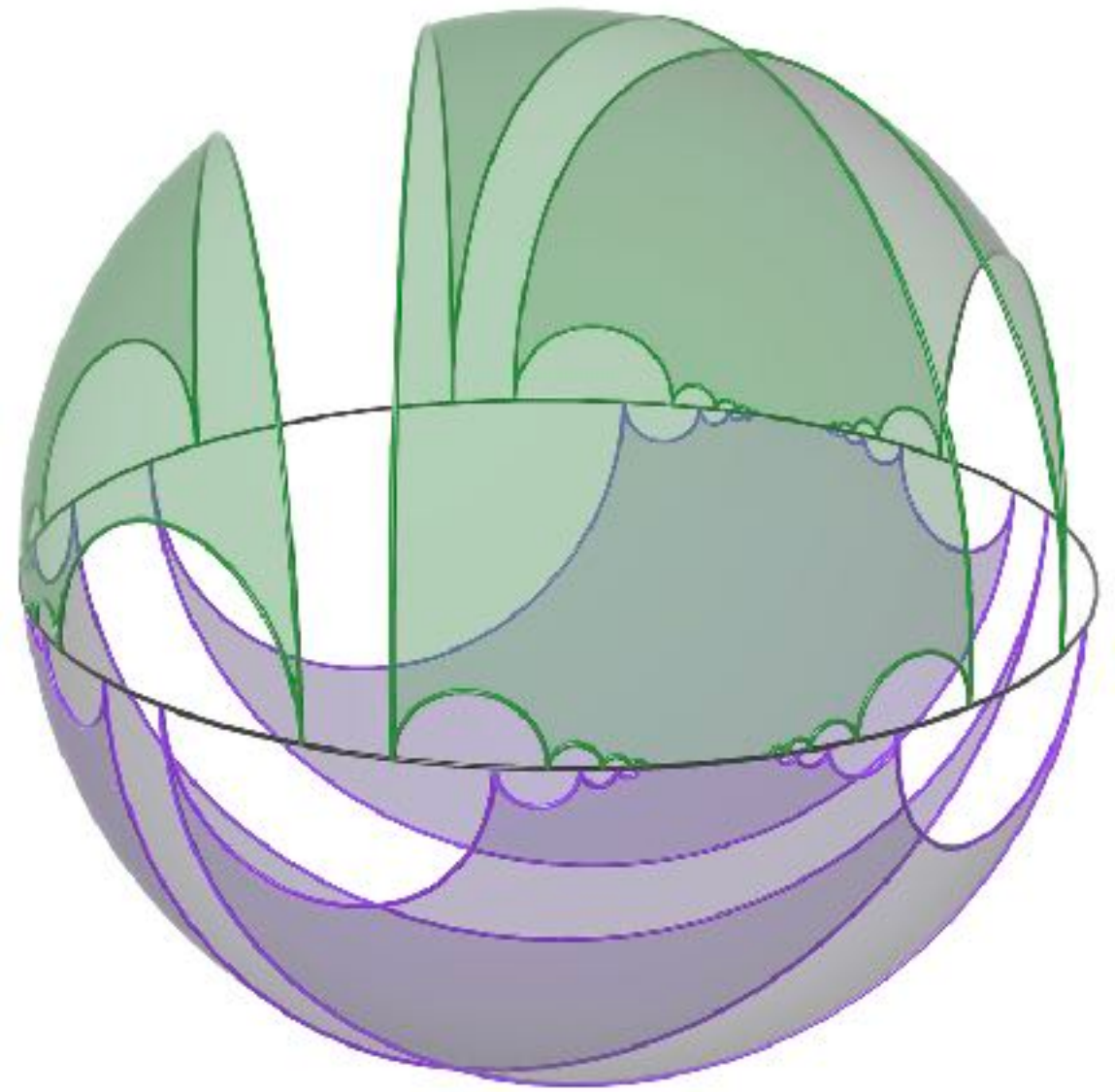
For a closed manifold, the decomposition elements have finite size.

Project the laminations Λ^α , Λ_α to the upper and lower hemispheres of a sphere with equator the veering circle.



Project the laminations Λ^α , Λ_α to the upper and lower hemispheres of a sphere with equator the veering circle.

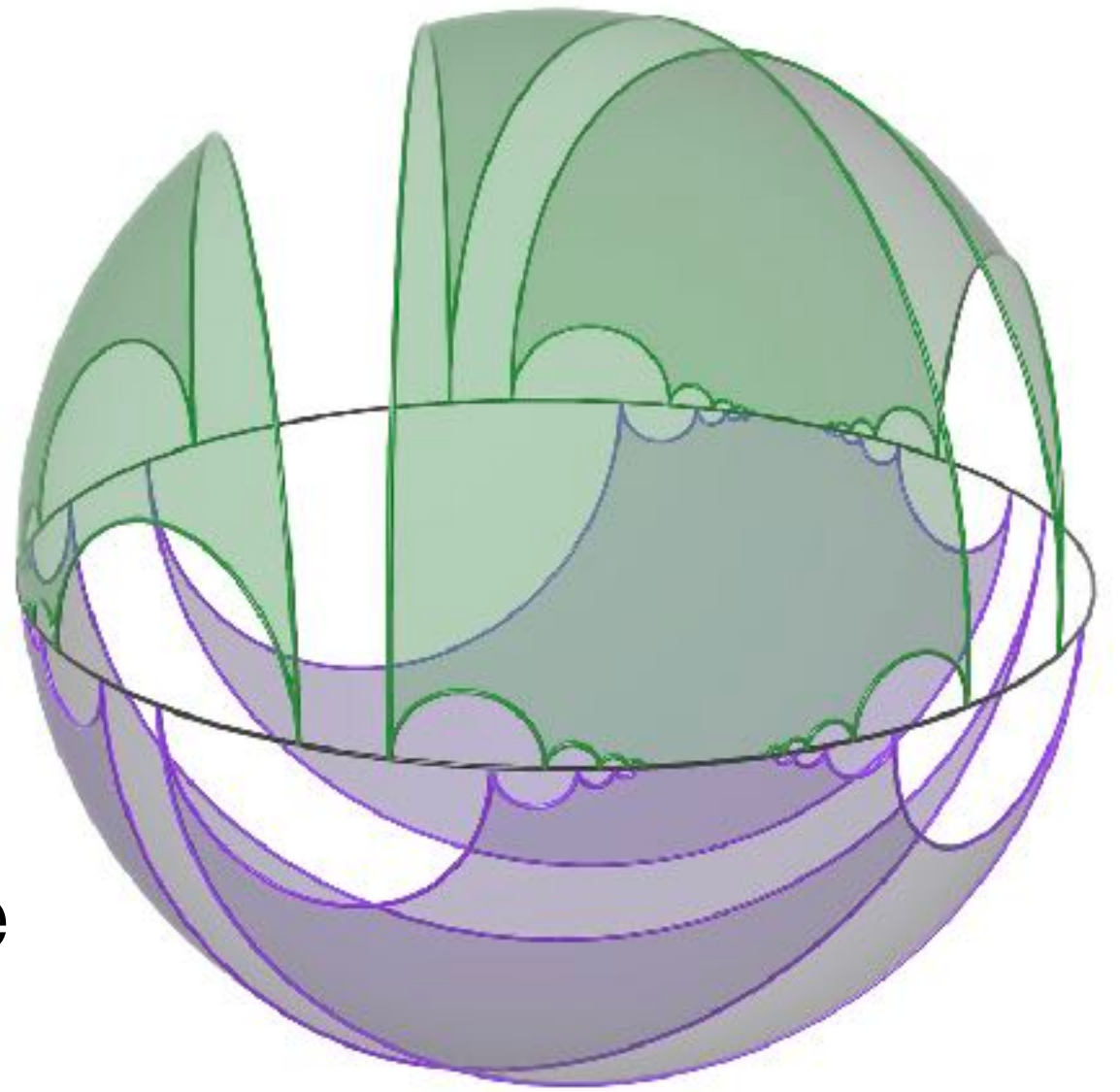
Collapse each decomposition element to a point.



Project the laminations $\Lambda^\alpha, \Lambda_\alpha$ to the upper and lower hemispheres of a sphere with equator the veering circle.

Collapse each decomposition element to a point.

A theorem of Moore shows that the quotient is homeomorphic to S^2 . We call this the *veering two-sphere*.

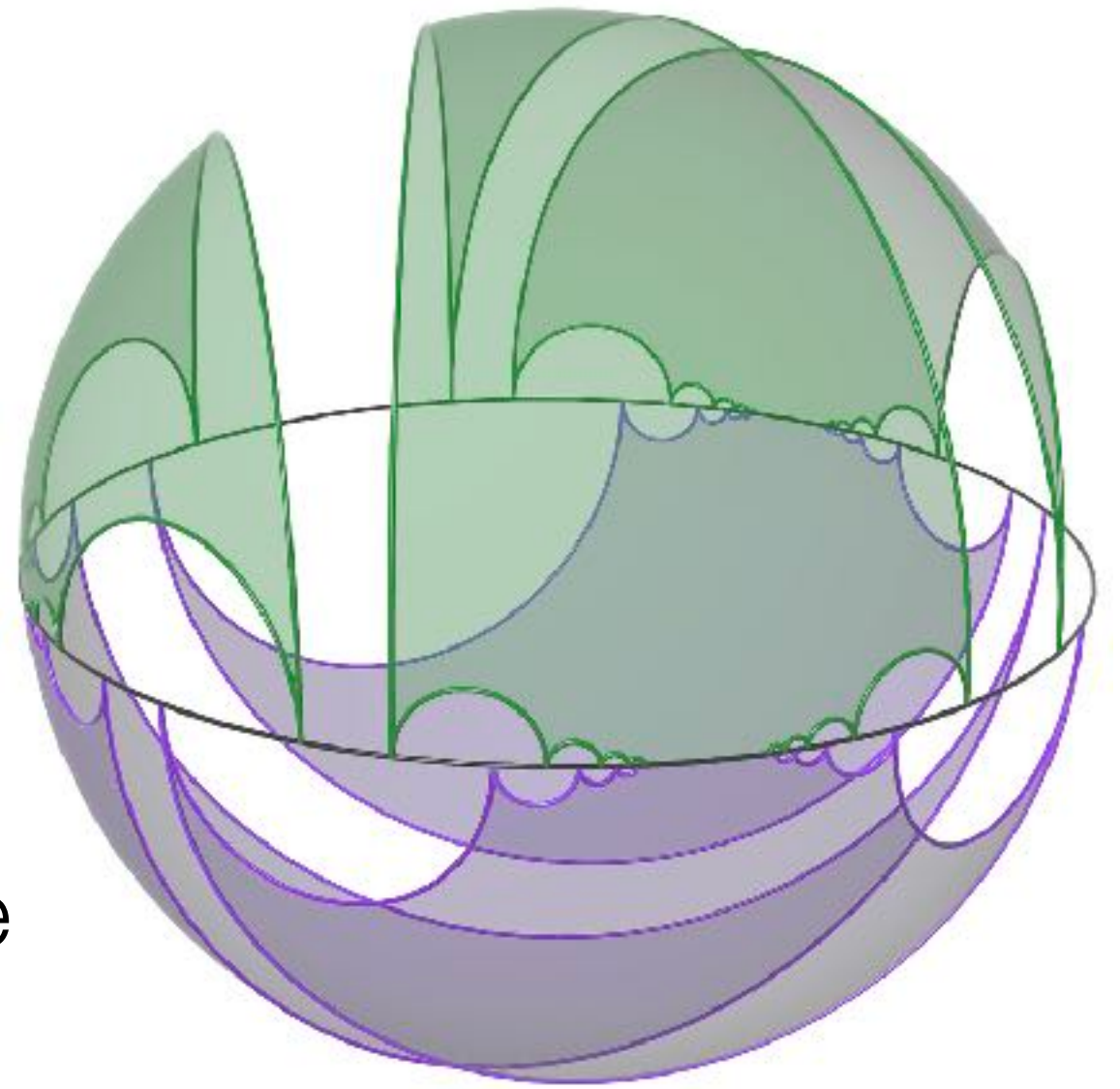


Project the laminations $\Lambda^\alpha, \Lambda_\alpha$ to the upper and lower hemispheres of a sphere with equator the veering circle.

Collapse each decomposition element to a point.

A theorem of Moore shows that the quotient is homeomorphic to S^2 .
We call this the *veering two-sphere*.

Now we prove that the action of the fundamental group on the veering two-sphere is a convergence group action*.
Finally we apply a theorem of Yaman to show that the veering two-sphere is equivariantly homeomorphic to $\partial\mathbb{H}^3$.



Thanks!

