RESEARCH PROPOSAL

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Abstract. The object of this proposal is to study the structure of Heegaard splittings of three-manifolds. Among others we hope to answer the question: Do non-Haken manfolds contain only finitely many strongly irreducible Heegaard splittings?

1. Introduction

Three-dimensional manifolds have been systemically studied ever since the late 1800's and count Poincaré [**?**] among its founding fathers. One of the earliest modes of presentation for three-manifolds was the *Heegaard splitting* named after Poul Heegaard who introduced the concept in his thesis [**?**].

Heegaard splittings are closely related to triangulations in their extreme flexibility. Just as every compact three-manifold may be triangulated [**?**] so may every manifold be cut along a surface to obtain a Heegaard splitting [**?**]. This contrasts with geometric structures (after Thurston [**?**]) which, as yet, are only conjectured to exist for every closed three-manifold.

The object, then, of this proposal is to study the structure of Heegaard splittings of three-manifolds. We will pay particular attention to the splittings of non-Haken manifolds. Here a valuable tool is available — Casson and Gordon's concept of a *strongly irreducible* splitting [**?**]. This has been extensively studied in [Cite lots of papers here – suggestions?]

Nonetheless many fascinating questions remain. In the remainder of this section we outline our research goals. The following sections lay out our methods in detail.

We are first interested in the collection of strongly irreducible splittings of non-Haken manifolds.

Question. Do non-Haken manfolds contain only finitely many strongly irreducible Heegaard splittings?

An affirmative answer would indicate a certain amount of rigidity for non-Haken manifolds — not as strong as the topological rigidity of

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Haken (or more generally, geometric) manifolds but a definite improvement over the Jaco-Rubinstein solution to the Waldhausen conjecture.

A halfway point to an answer for the above is:

Question. Suppose H is strongly irreducible and has high genus. Show that after stablizing to obtain H' the splitting H' destablizes twice. More generally, find *any* condition at all on a splitting H which insures that the stablization H' will destablize twice.

As a related issue we are also interested in classification theorems for Heegaard splittings. Precisely we have:

Question. Suppose that $K \subset M$ is a knot and M' is a double cover of M, branched along K. Suppose further that the Heegaard splittings of M' are classified. Using this classify the splittings of (M, K).

This line of thought is inspired Kobayashi's complete answer to the classification question for two-bridge knot complements [**?**]. Other large families which may succumb to such analysis are the Montesinos knots and knots with a Solv structure on their double-branched cover.

2. Non-Haken manifolds

Recall that a *non-Haken* manifold is a closed, orientable, connected, irreducible three-manifold which does not contain a two-sided incompressible surface. Let $H \subset M$ be a splitting surface for the non-Haken manifold M. As a bit of notation we will refer to the two handlebody components of $M - H$ as V and W.

Recall also that H is strongly irreducible exactly when every essential disk in V meets every essential disk of W. We have then our first question:

Question. Do non-Haken manfolds contain only finitely many strongly irreducible Heegaard splittings?

Already there has been a great deal of work approaching this question

- Non-Haken manifolds contain only finitely many strongly irreducible splittings? Does this bear on topological rigidity for these manifolds?
- Suppose H is an almost normal strongly irreducible splitting and K is a normal incompressible surface and the Haken sum $H + nK$ is also strongly irreducible, for all n. What can be said?
- Suppose H is strongly irreducible and has high genus. Show that after stablizing to obtain H' the splitting H' destablizes

twice. (This has a nice consequence for the tree of Heegaard splittings.) More generally, can you find *any* condition at all on a splitting H which insures that the stablization H' will destablize twice?

- Weirdness about p.A. maps and your method for proving minimal genus..
- cf Kobayashi's classification of Heegaard splittings of two-bridge knot complememts — Give classification of splittings of certain knot complements in S^3, using the fact that the double branched cover is "nice" (eg, a lens space) and its Heegaard splittings are understood. Thought that just this moment occured to me: Cooper and Scharlemann classify all splittings of (almost) all Solv manifolds. Which knots in S^3 have Solv double branched covers?

  Also mentioned in my notes: Punctured torus bundles over the circle.
- My odd varient of the above: If the ideal triangulation or polyhedrization of the knot complement is "nice" (cf figure eight knot) then you might be able to do a bit of easy linear algebra and apply Stocking's theorem to find all strongly irreducible Heegaard splittings. I believe that Jaco and Sedgwick have discussed doing exactly this for lens spaces... Perhaps we could find all Heegaard splittings of the Borromean Rings?

[**?**]

References

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