UNKNOTTING TUNNELS FOR $P(-2,3,7)$

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Abstract. There are exactly four mutually non-isotopic unknotting tunnels for the pretzel knot $P(-2,3,7)$. We also classify two tunnel unknotting systems.

1. Preliminaries

The main results of this paper, proved in Sections 2 and 3, are as follows.

Theorem 1.1. *There are exactly four mutually non-isotopic unknotting tunnels, τ_i $(i=1,\dots,4)$ as shown in Figure 1.1, for the pretzel knot $P(-2,3,7)$.*

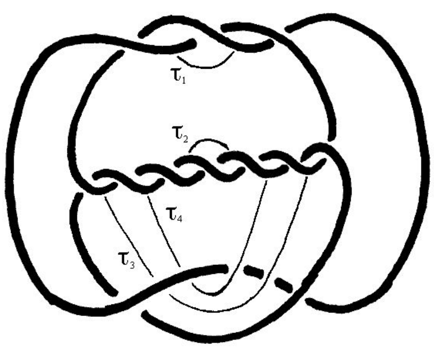

Figure 1.1

Theorem 1.2. *Let (σ,τ) be a two-tunnel unknotting system for the pretzel knot $P(-2,3,7)$. Then σ can be isotoped to a trivial arc in the complement of $P(-2,3,7)\cup\tau$, and τ is isotopic to one of the τ_i of Theorem* 1.1.

We begin with some preliminaries. For the definitions of handlebody, compression body, and ideal polyhedral decomposition (IPD) we refer to [2]. The notation $N(*)$ shall refer to a regular neighborhood of $*$, and $\sharp(*)$ refers to the number of components of $*$. We use the convention that the genus of a disconnected surface is the sum of genii of its connected components.

Let M be a compact 3-manifold. A *Heegaard splitting* of M is a 3-tuple $(H_0,H_1;F)$ such that H_0,H_1 is a pair of compression bodies with the property that $M=H_0\cup H_1$ and $H_0\cap H_1=\partial_+H_0=\partial_+H_1=F$, for some closed connected surface F embedded in M. The surface F is called the *splitting surface* of the Heegaard splitting. Two Heegaard splittings of M are considered *equivalent* if their splitting surfaces

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are isotopic. A Heegaard splitting is *stabilized* if it can be obtained from another splitting by taking the connected sum of pairs (M, F) and (S^3, T^2), where T^2 is the standard unknotted torus in S^3.

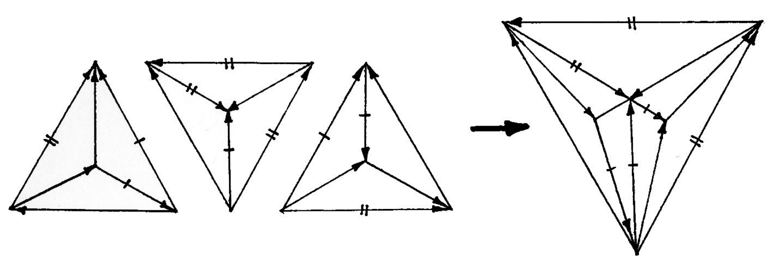
A *spine*, τ, of a compression body H is a properly embedded 1-complex such that $H = N(\partial_- H \cup \tau)$. Let K be a knot, $(H_0, H_1; F)$ be a genus 2 Heegaard splitting of its complement, and assume that $\partial_- H_0 = \partial N(K)$, that is, H_0 is the compression body and H_1 the handlebody of the splitting. Let τ be a spine for H_0. It is clear that τ can be chosen to be an arc. Then τ is called an *unknotting tunnel* for K, and K is called a *tunnel-1 knot*.

A knot K in the 3-sphere S^3 is said to admit a *(1,1)-decomposition* if (S^3, K) is decomposed into a union $(V_1, t_1) \cup (V_2, t_2)$ where t_i is a trivial arc in a solid torus $V_i (i = 1, 2)$. A knot which admits a (1,1)-decomposition is called a *(1,1)-knot*. It is easy to see that a core of each solid torus in a (1,1)-decomposition contributes an unknotting tunnel , which is called a *(1,1)-tunnel*. In [6], by using the dihedral symmetry of the Brieskorn homology sphere $\Sigma(2, 3, 7)$, it was shown that $p(-2, 3, 7)$ admits two non-homeomorphic (1,1)-decompositions which are doubly covered by the vertical and horizontal Heegaard decompositions of $\Sigma(2, 3, 7)$ respectively. Here, the vertical and horizontal Heegaard decompositions are two well-known types of irredicible Heegaard decompositions of Seifert fibered spaces, see [5]. For this reason, we called the two (1,1)-decompositions of $p(-2, 3, 7)$ *vertical* and *horizontal* respectively. From the vertical (resp. horizontal) (1,1)-decomposition of $p(-2, 3, 7)$, we obtain the pair of (1,1)-tunnels τ_1, τ_2 (resp. τ_3, τ_4) described in Figure 1.1.

Let $\theta = K \cup \tau$, where K is a knot and τ an unknotting tunnel for K. Note that K can be divided into two arcs α_1 and α_2 by $\partial\tau$. Then $K_1 = \alpha_1 \cup \tau$ and $K_2 = \alpha_2 \cup \tau$ are called the *constituent knots* of θ with respect to K.

Let T be an IPD (ideal polyhedral decomposition) of M. We denote the k-skeleton of T by T_k. As the 1-skeleton will be of importance to us, we denote the separate elements of the 1-skeleton as T_1^j for appropriate j.

If G is a 2-sided surface which intersects T_3 in discs and T_2 in normal arcs, then G is called a *normalized surface*. A normalized surface G for which $\sharp(G \cap T_1^j) > 2$ for at least one j is called a *Gabai surface*. We say that a set $G_1, \ldots, G_n$ of connected, normalized surfaces is a *base* for (M, T) if any connected, normalized surface in (M, T) is represented (up to isotopy preserving the number of intersections with each T_1^j) by one of the G_i. If M has a base with respect to T, then T is said to be *good*, otherwise it is *bad*.



ideal triangulation
Figure 1.2A

IPD
Figure 1.2B

Henceforth we let $K = P(-2, 3, 7)$, and $E(K) = S^3 - N(K)$. From [1] or [9] we take a minimal ideal triangulation of $E(K)$, which can be seen in Figure 1.2A. We glue these together to obtain the IPD, denoted T, of Figure 1.2B. We have denoted the three elements of the 1-skeleton by T_1^0, T_1^1, and T_1^2, and have marked them in the diagram with zero, one, or two marks respectively.

Following the notation of [2], we denote any normalized surface G by a triple of even integers (a_0, a_1, a_2) where $a_j = \sharp(G \cap T_1^j)$. It is elementary to check that $\{(2,2,2),\ (2,2,4),\ (2,4,2),\ (2,4,4),\ (4,2,2),\ (4,2,4),\ (4,2,6),\ (4,2,8),\ (4,4,2),\ (4,6,2),\ (4,8,2)\}$ is a base for $\big(E(K), T\big)$, so that T is good; this will allow us to use [2]. It is also elementary to calculate the genus of the elements of the base, they are: 1, 2, 2, 3, 4, 3, 3, 4, 3, 3, and 4 respectively. (For example, surface $(2,2,4)$ shown in Figure 2.1A has 5 faces (discs in T_3), 15 edges (normal arcs), and 8 vertices, hence $\chi(2,2,4) = -2$.)

Remark 1. We note that only three of the surfaces in the base above have genus 2 or less. They are: $(2,2,2)$, $(2,2,4)$, and $(2,4,2)$. Of those, only the latter two are Gabai surfaces.

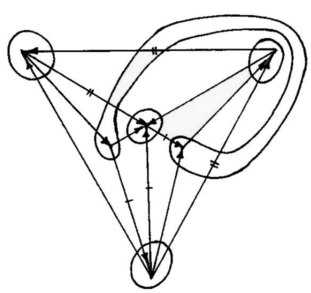
2. Proof of Theorem 1.1

We shall prove Theorem 1.1 by proving the following 2 lemmas.

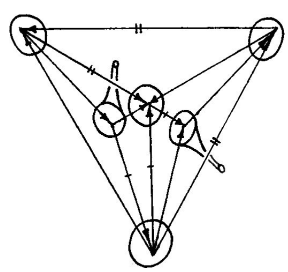
Lemma 2.1. *There are at most four distinct genus 2 Heegaard splittings for $E(K)$.*

Proof: Applying the Main Theorem of [2] to $\big(E(K), T\big)$ (see Figure 1.2), we see that any genus 2 Heegaard splitting is induced either by an element of the 1-skeleton of T or by a genus 2 Gabai surface.

There are 3 elements of the 1-skeleton, hence at most three inequivalent genus 2 Heegaard splittings are induced by the 1-skeleton.



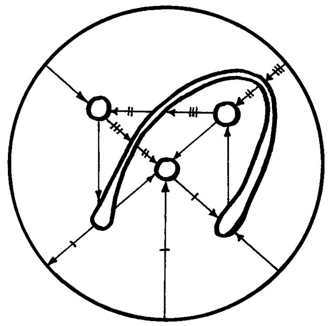
$(2,2,4)$ before isotopy
Figure 2.1A



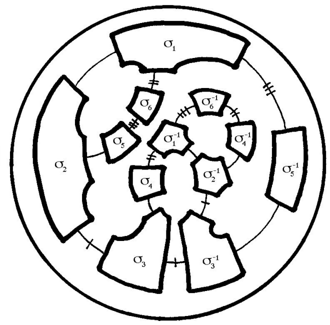
$(2,2,4)$ after isotopy
Figure 2.1B

By Remark 1, there are two Gabai surfaces of genus 2. (The Gabai surface $(4,4,4) = (2,2,2) \cup (2,2,2)$, is eliminated in [2].) Hence there are at most two inequivalent genus 2 Heegaard splittings induced by Gabai surfaces.

Figure 2.1A shows Gabai surface $(2,2,4)$, hereafter denoted G. Figure 2.1B shows the same surface after an isotopy; the tube that appears to leave the polyhedron represents a tube exiting the back face. We claim that G does not induce a Heegaard splitting. To see this, note that G clearly splits $E(K)$ into two pieces, call them $M_0^{T(2,2,4)}$ and $M_1^{T(2,2,4)}$. Let α be a core of the tube shown in Figure 2.1B, and let $M_0^{T(2,2,4)} = N(\partial E(K) \cup \alpha)$. Then $M_1^{T(2,2,4)}$ is the closure of $E(K) - M_0^{T(2,2,4)}$.



$M_1^{T(2,2,4)}$
Figure 2.2A



$M_1^{T(2,2,4)} - N(\text{edges})$
Figure 2.2B

It is clear that $M_0^{T(2,2,4)}$ is a compression body. We show that $M_1^{T(2,2,4)}$ is not a handlebody.

Beginning with Figure 2.1A, we eliminate $M_0^{T(2,2,4)}$, stretching the bottom disc behind the ball, and splitting the twice marked edge into two edges, marked twice and thrice in Figure 2.2A. This is a polyhedral decomposition of $M_1^{T(2,2,4)}$. Following Section 3.3 of [8], we remove a neighborhood of each of the four edges, leaving behind a mark to tell us how to glue them back in, obtaining the decomposition in Figure 2.2B. We note that the identifications create a genus 6 handlebody, and the marks describe how to attach four 2-handles to the surface of the handlebody. This gives us 6 generators $(\sigma_1, \ldots, \sigma_6)$ and 4 relations

(1) $\sigma_1\sigma_2^{-1}\sigma_5\sigma_3^{-1}\sigma_4\sigma_2^{-1}$,
(2) $\sigma_3^2\sigma_2^{-1}$,
(3) $\sigma_1\sigma_4\sigma_6^{-1}$, and
(4) $\sigma_1^{-1}\sigma_5^{-1}\sigma_6$.

Relations 2, 3, and 4 allow us to replace σ_2, σ_5, and σ_6 with σ_3^2, $\sigma_1\sigma_4\sigma_1^{-1}$, and $\sigma_5\sigma_1$, respectively. Then $\pi_1(M_1^{T(2,2,4)}) =< \sigma_1, \sigma_3, \sigma_4 : \sigma_1\sigma_3^{-2}\sigma_1\sigma_4\sigma_1^{-1}\sigma_3^{-1}\sigma_4\sigma_3^{-2} >$, which is not free. Hence $M_1^{T(2,2,4)}$ is not a handlebody, and G does not induce a Heegaard splitting. This finishes the proof of Lemma 2.1.

Remark 2. There are at least two minimal triangulations of $E(K)$. The second can be used to form another good IPD, call it T', see Figure 2.4. We note that ${T'}_1^0 = T_1^0$, ${T'}_1^2 = T_1^2$, and ${T'}_1^1$ corresponds to a spine of the compression body of Gabai surface (2,4,2) of T. (Similarly, T_1^1 corresponds to a spine of the compression body of Gabai surface (2,4,2) of T'.) Then Lemma 2.1 can be proven by applying [2] to $\big(E(K), T'\big)$. The IPD T', like T, has a genus 2 Gabai surface that does not induce a Heegaard splitting. However, the first homology group of either $M_1^{T(2,2,4)}$ or $M_1^{T'(2,4,4)}$ (the analog for $M_1^{T(2,2,4)}$ in T') is $\mathbf{Z^2}$, so that the second genus 2 Gabai surface for either IPD induces a "homological Heegaard splitting."

Lemma 2.2. *The unknotting tunnels* τ_i, $(i = 1, \ldots, 4)$ *for* $P(-2, 3, 7)$, *shown in Figure 1.1, are mutually non-isotopic.*

Proof: By the main theorem of [4], if two equivalent unknotting tunnels σ_1 and σ_2 for K can be embedded in an incompressible Seifert surface for K, then the constituent knots of $K \cup \sigma_1$ and $K \cup \sigma_2$ with respect to K are identical. Hence we consider the theta curves $\theta_i = K \cup \tau_i$, $(i = 1, \ldots, 4)$.

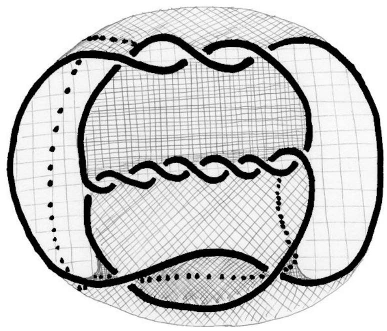


Figure 2.3: S with τ_3

Figure 2.3 shows the unique incompressible (see [3]) Seifert surface S of K. Note τ_i, $(i \neq 4)$ can be isotoped to lie in S; this has been shown for τ_3 by the dotted arc, and we leave the easier cases of τ_1 and τ_2 to the reader. We have not found an isotopy of τ_4 into S, but note that if it cannot be embedded in S, then it is clearly non-isotopic to the τ_i that can be. Hence we may assume that τ_4 can also be embedded in S, and thus that we may apply the main theorem of [4].

Each of the θ_i has the unknot as one of its constituent knots; the other constituent knots are the trefoil, 7_1, 5_1, and 8_{19} respectively. Thus, as constituent knots are distinct, τ_i, $(i = 1, \ldots, 4)$ must be mutually non-isotopic. This completes the proof of Lemma 2.2 and of Theorem 1.1.

3. Proof of Theorem 1.2

We note that a stabilized genus 3 Heegaard splitting of $E(K)$ corresponds to a splitting in which one of the arcs can be isotoped into $\partial E(K)$, hence we need only show that any genus 3 Heegaard splitting is stabilized.

Applying [2] to $\big(E(K), T\big)$, we find that any genus 3 Heegaard splitting must be induced by one of the following:

(1) a 2-edge subset of the 1-skeleton of T, that is $T_1^0 \cup T_1^1$, $T_1^0 \cup T_1^2$, or $T_1^1 \cup T_1^2$,
(2) a genus 3 Gabai surface of T with no more than 2 parallel copies of any base surface, that is $(2,4,4)$, $(4,2,4)$, $(4,2,6)$, $(4,4,2)$, and $(4,6,2)$, $(4,4,6) = (2,2,2) \cup (2,2,4)$, or $(4,6,4) = (2,2,2) \cup (2,4,2)$,
(3) any genus 3 Heegaard splitting of $E(K) - N(T_1^i)$, for $i = 0, 1$ or 2.
(4) any genus 3 Heegaard splitting of $M_1^{T(2,4,2)}$,
(5) any genus 3 Heegaard splitting of $M_1^{T(2,2,4)}$.

In the first case, it is easy to check that each induce stabilized Heegaard splittings.

In the second case we can use arguments similar to those at the end of Lemma 2.1 to eliminate all Gabai surfaces except for $(4,6,4)$ as not inducing Heegaard splittings (though each is, again, homologically equivalent to a Heegaard splitting). Gabai surface $(4,6,4) = (2,2,2) \cup (2,4,2)$ does induce a Heegaard splitting which is an amalgamation of a standard Heegaard splitting of $\big(\partial E(K)\big) \times I$ with the Heegaard splitting of $E(K)$ induced by Gabai surface $(2,4,2)$. It is elementary to check that this splitting is stabilized.

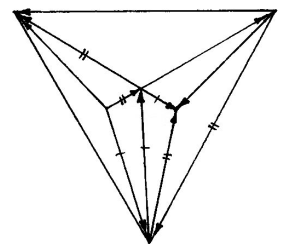


Figure 2.4: Alternate IPD T'

In cases 3 and 4 we merely note that as the subspace of $E(K)$ we are considering is a genus 2 handlebody, and any genus 3 Heegaard splitting of a genus 2 handlebody is stabilized by [7], we are done.

In the fifth case we can apply [2] to the IPD T'' of $M_1^{T(2,2,4)}$ obtained from Figure 2.2A by shrinking the boundary circles to vertices. Then any genus 3 Heegaard splitting of $M_1^{T(2,2,4)}$ is induced by (a) an element of the 1-skeleton of T'' or by (b) a Gabai surface for T'' of genus 3.

In sub-case (a), it is easy to check that an unknotting tunnel for $M_1^{T(2,2,4)}$ must be ${T''}_1^0 = T_1^0$, ${T''}_1^1 = T_1^1$, ${T''}_1^2 \sim T_1^0$, or ${T''}_1^3 \sim T_1^0$, where "$\sim$" means "isotopic in $E(K)$ to," remembering that $M_1^{T(2,2,4)} \subset E(K)$. Again checking fundamental group, we find that ${T''}_1^1$ does not induce a Heegaard splitting. Using the same argument as at the end of Lemma 2.1, but instead actually doing the gluing of faces and 2-handles, we find that ${T''}_1^0$ forms a Heegaard splitting. Then we note that a spine of this Heegaard splitting can be taken to be two disjoint arcs: a core of Gabai surface $(2,2,4)$ of T and T_1^0. Considering the same arcs in reverse order we obtain case 3 above, which we already know to be stabilized.

In the second sub-case, a base for T'' consists of only three surfaces $\{(2,2,2,2), (2,4,2,2), (4,2,4,4)\}$, with genus 2, 3, and 4 respectively, so that only $(2,4,2,2)$ could induce a genus 3 Heegaard splitting. We can either use calculation of fundamental group or notice that $(2,4,2,2)$ of T'' is isotopic in $E(K)$ to $(2,4,4)$ of T, which we eliminated in case 2. This completes the proof of Theorem 1.2.

We note that it may be possible to continue these arguments to classify all unknotting tunnel systems; as [2] eliminates any Gabai surfaces with a parallel trio of base surfaces, the number of calculations is theoretically finite (in this case we only need to go to genus 14). However, the arguments become quite tedious in the larger genus cases, so we shall content ourselves with the genus 2 and 3 cases, and leave the rest to conjecture.

Conjecture 3.1. *Any n-unknotting tunnel system for $P(-2,3,7)$ is isotopic to one in which $n-1$ tunnels can be isotoped into $\partial E(K)$, and the remaining tunnel is one of the τ_i of Figure 1.1.*

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