# Trees and convex cocompactness 

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## 1 Introduction

If $S$ is a closed surface with genus at least two, and $S^{\circ}$ is the surface $S$ punctured once, then filling in the puncture will often induce a map from a geometric space associated to $S^{\circ}$ to a corresponding space associated to $S$. The situation we consider in this paper is that of the curve complex, and the filling map $\mathcal{C}\left(S^{\circ}\right) \rightarrow \mathcal{C}(S)$. The fibers of this map are connected, and as we will see, they are familiar objects.

Recall that $\pi_{1}(S)$ embeds into $\operatorname{Mod}\left(S^{\circ}\right)$, the mapping class group of $S^{\circ}$, according to the Birman Exact Sequence (see §2). In particular, the fibers of $\mathcal{C}\left(S^{\circ}\right) \rightarrow \mathcal{C}(S)$ are invariant by $\pi_{1}(S)$. Any point of $\mathcal{C}(S)$ determines a multicurve, which itself can be used to be used to define an action of $\pi_{1}(S)$ on a tree (see $\S 4$ ). Our first theorem relates this tree to the map $\mathcal{C}\left(S^{\circ}\right) \rightarrow \mathcal{C}(S)$.

Theorem 1.1. The fiber over $v$ of the $\operatorname{map} \mathcal{C}\left(S^{\circ}\right) \rightarrow \mathcal{C}(S)$ is $\pi_{1}(S)$-equivariantly homeomorphic to the tree determined by $v$.

When $S$ is not closed, there is no natural map $\mathcal{C}\left(S^{\circ}\right) \rightarrow \mathcal{C}(S)$. However, we define an augmented curve complex (see §2.1) and prove an analogous theorem in that setting (assuming $\xi(S) \geq 4$; see $\S 2$ ). This theorem follows from an alternative description of $\mathcal{C}\left(S^{\circ}\right)$ which mixes algebra and combinatorial topologysee Theorem 5.4. This sheds some light on the space $\mathcal{C}\left(S^{\circ}\right)$, and in particular provides a useful tool for studying the subgroup $\pi_{1}(S)<\operatorname{Mod}\left(S^{\circ}\right)$. As an application of the ideas, we prove the following theorem which answers Question 6 of [6].

Theorem 1.2. If $\xi(S) \geq 4$, and $G<\pi_{1}(S)$ is finitely generated and purely pseudo-Anosov as a subgroup of $\operatorname{Mod}\left(S^{\circ}\right)$, then $G$ is convex cocompact.

Convex cocompactness for subgroups of the mapping class group was invented by Farb and Mosher in [4] and is intimately related to the geometry of the associated surface group extension. It was further studied by the first two authors in [6] and by Hamenstädt in [5]. As convex cocompact groups are

[^0]finitely generated and purely pseudo-Anosov, it is of interest to decide whether or not this is also a sufficient condition. Farb and Mosher first asked this in [4], as well as for some special "test" cases (see below). In particular, as a special case, Theorem 1.2 provides the answer for one of their test questions, Question 6.1 of [4].

Corollary 1.3. Whittlesey's groups are locally convex cocompact.
Recall that Whittlesey's groups are normal, purely pseudo-Anosov subgroups of the mapping class groups of a genus 2 surface and of the sphere with $n \geq 5$ punctures. The genus 2 case is equivalent to that of the sphere with 6 punctures as the associated Teichmüller spaces are isometric in a mapping class groups equivariant way - the mapping class groups are virtually isomorphic.

Proof of Corollary 1.3 assuming Theorem 1.2. It suffices to prove the theorem for Whittlesey's groups in the $n$-punctured sphere mapping class groups $\operatorname{Mod}\left(S_{0, n}\right)$. The $n$-punctures define $n$-maps $\operatorname{Mod}\left(S_{0, n}\right) \rightarrow \operatorname{Mod}\left(S_{0, n-1}\right)$ by filling in the puncture. The intersection of the kernels of all of these maps is Whittlesey's group, and hence lies in $\pi_{1}\left(S_{0, n-1}\right)<\operatorname{Mod}\left(S_{0, n}\right)$. Any finitely generated subgroup of Whittlesey's group is thus also a finitely generated purely pseudoAnosov subgroup of $\pi_{1}\left(S_{0, n-1}\right)$. Since $n \geq 5$, Theorem 1.2 implies that the group is convex cocompact.

Here we give a sketch of the contents of the paper.
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## 2 Notation and background

Let $S=S_{g, m}$ denote a surface with genus $g$ and $m$ marked points (or punctures). When $m=0$, we will also write $S_{g, 0}=S_{g}$. We use the complexity $\xi(S)=3 g+m$ and will assume throughout that $\xi(S) \geq 4$. We will let $S^{\circ}$ denote $S$ with an additional puncture, and $S^{\bullet}$, the surface $S$ with an additional marked point which we refer to as $\bullet$. We will frequently regard $\bullet$ as a base point for $S$.

For clarity, we note that

$$
\xi\left(S^{\circ}\right)=\xi\left(S^{\bullet}\right)=\xi(S)+1
$$

need to be more careful with this throughout

The distinction between $S^{\circ}$ and $S^{\bullet}$ is generally unimportant, but to make definitions precise, this distinction will be made without mention.

We fix a hyperbolic metric on $S$. Let $p: \widetilde{S} \rightarrow S$ denote the universal covering. The hyperbolic metric on $S$ pulls back to one on $\widetilde{S}$ thus making $\widetilde{S}$ isometric to the hyperbolic plane.

The Birman Exact Sequence [2] relates the mapping class group of $S$ with that of $S^{\bullet}$ and $\pi_{1}(S, \bullet)$. Namely, we have

$$
1 \rightarrow \pi_{1}(S) \rightarrow \operatorname{Mod}\left(S^{\bullet}\right) \rightarrow \operatorname{Mod}(S) \rightarrow 1
$$

To make this correct, we must assume, as we will, that a mapping class fixes each of the punctures (or marked points).

The embedding of $\pi_{1}(S, \bullet)$ into $\operatorname{Mod}\left(S^{\bullet}\right)=\operatorname{Mod}\left(S^{\circ}\right)$ can be described as follows. We represent an element $[\gamma] \in \pi_{1}(S, \bullet)$, by a loop $\gamma$ based at • and we "push" • around $\gamma$. More precisely, if we write $\gamma:[0,1] \rightarrow S$ with $\gamma(0)=$ $\gamma(1)=\bullet$, then there is an isotopy $h_{t}: S \rightarrow S$, for $t \in[0,1]$ such that $h_{0}=i d_{S}$, $h_{1}(\bullet)=\bullet$, and so that $\gamma(t)=h_{t}(\bullet)$. The mapping class in $\operatorname{Mod}\left(S^{\bullet}\right)$ associated to $[\gamma]$ is the mapping class of $h_{1}$.

An explicit construction of $h_{t}$ can obtained by assuming $\gamma$ is a smooth immersion (as we may), and flowing along a time dependent vector field supported in the image of a tubular neighborhood of $\gamma$ that pushes • along $\gamma$. As such, we can assume that for any neighborhood of the image of $\gamma, h_{t}$ is supported in that neighborhood.

The following result classifying those elements in $\pi_{1}(S, \bullet)$ which represent pseudo-Anosov mapping classes in $\operatorname{Mod}(S)$ is due to Kra [8] and will play an crucial role in our proof of Theorem 1.2.

Theorem 2.1 (Kra). An element $[\gamma] \in \pi_{1}(S, \bullet)$ is pseudo-Anosov in $\operatorname{Mod}\left(S^{\circ}\right)$ if and only if the (free) homotopy class determined by $[\gamma]$ fills $S$.

Here, a homotopy class of curves fills $S$ if every representative of the homotopy class intersects every other essential curve on $S$.

We view $\pi_{1}(S)$ as the group of covering transformations of the universal covering $p: \widetilde{S} \rightarrow S$ and fix this action once and for all. Given any point $\widetilde{\bullet}$ of $\widetilde{S}$, this determines an isomorphism of $\pi_{1}(S)$ with the fundamental group $\pi_{1}(S, \bullet)$, for $\bullet=p(\widetilde{\bullet})$. When describing elements of $\pi_{1}(S)$ as homotopy classes of loops, we will assume a fixed basepoint $\widetilde{\bullet}$, and hence fixed isomorphism $\pi_{1}(S) \cong$ $\pi_{1}(S, \bullet)$.

### 2.1 Curve complexes

The curve complex of $S$ will be denoted $\mathcal{C}(S)$ (and similarly for the surface $S^{\circ}$ or $S^{\bullet}$ ). The $k$-simplices of the simplicial complex are sets $v=\left\{v_{0}, \ldots, v_{k}\right\}$ of $k+1$ distinct isotopy classes of pairwise disjoint essential simple closed curves (here, essential means homotopically nontrivial and nonperipheral). We confuse simplices with the multicurves they determine.

We make $\mathcal{C}(S)$ into a geodesic metric space by declaring each simplex to be a regular euclidean simplex with all side lengths equal to 1 -see [3]. Masur and Minsky [9] prove that $\mathcal{C}(S)$ is in fact a $\delta$-hyperbolic metric space.

The 1-skeleton $\mathcal{C}^{1}(S)$ is itself a metric space (with the path metric), and the inclusion is a quasi-isometric embedding into $\mathcal{C}(S)$. Because geodesics in $\mathcal{C}^{1}(S)$ between vertices have a combinatorial description simply as a sequence of (adjacent) vertices, we can mix combinatorial and geometric arguments in the metric space $\mathcal{C}^{1}(S)$. We will therefore work with the metric on $\mathcal{C}^{0}(S)$ induced by the inclusion into $\mathcal{C}^{1}(S)$, which therefore takes on only integer values.

When $S$ has punctures, we will also have use for an enlargement of the space $\mathcal{C}(S)$. We define the augmented curve complex of $S$, denoted $\mathcal{C}_{a}(S)$, as
follows. The vertices of $\mathcal{C}_{a}(S)$ are isotopy classes of non-null-homotopic simple closed curves. The set $\mathcal{C}_{a}^{0}(S)$ thus contains $\mathcal{C}^{0}(S)$ plus a finite set of peripheral simple closed curves; since $\xi(S) \geq 4$, we see that $\mathcal{C}_{a}^{0}\left(S_{g, m}\right) \backslash \mathcal{C}^{0}\left(S_{g, m}\right)$ consists of exactly $m$ peripheral curves, one surrounding each puncture, and we denote these $\mathcal{C}_{a}^{0}\left(S_{g, m}\right) \backslash \mathcal{C}^{0}\left(S_{g, m}\right)=\left\{\zeta_{1}, \ldots, \zeta_{m}\right\}$.

A $k$-simplex in $\mathcal{C}_{a}(S)$ is a set $v=\left\{v_{0}, \ldots, v_{k}\right\}$ of $k+1$ pairwise disjoint curves in $\mathcal{C}_{a}^{0}(S)$ such that at most one vertex lies in $\mathcal{C}_{a}^{0}(S) \backslash \mathcal{C}^{0}(S)$. We also view $\mathcal{C}(S) \subset \mathcal{C}_{a}(S)$. We continue to confuse simplices in $\mathcal{C}_{a}(S)$ with the multicurves they define.

When $S$ is closed, to avoid making special cases, we define $\mathcal{C}_{a}(S)=\mathcal{C}(S)$.
We will typically denote simplices of $\mathcal{C}\left(S^{\circ}\right)$ by $u=\left\{u_{0}, \ldots, u_{k}\right\}$ and simplices of $\mathcal{C}_{a}(S)$ by $v=\left\{v_{0}, \ldots, v_{k}\right\}$.

As $\mathcal{C}(S), \mathcal{C}_{a}(S)$, and $\mathcal{C}\left(S^{\circ}\right)$ are metric spaces, it will at times be convenient to have some notation for the combinatorial simplicial complex, i.e. the set of simplices together with the partial ordering induced by the face relation. For this, we add a subscript $\Delta$; e.g. $\mathcal{C}_{\Delta}(S)$ is the set of simplices of $\mathcal{C}(S)$ with the partial order induced by faces.

### 2.2 Filling projection

The coarse metric geometry of $\mathcal{C}_{a}(S)$ is not terribly interesting (when $S$ has punctures). Indeed, it is quasi-isometric to a point since it has diameter two. The reason for defining $\mathcal{C}_{a}(S)$ is the following.

First, note that since $S^{\circ} \subset S$, any curve in $S^{\circ}$ can also be viewed as a curve in $S$, though an essential curve in $S^{\circ}$ may become peripheral in $S$. If $v$ is a simplex in $\mathcal{C}\left(S^{\circ}\right)$, then let $\Pi(v)$ denote this corresponding multicurve in $\mathcal{C}_{a}^{0}(S)$. We claim that this determines a simplicial map

$$
\Pi: \mathcal{C}\left(S^{\circ}\right) \rightarrow \mathcal{C}_{a}(S)
$$

which we refer to as the filling projection-it is obtained by filling in the puncture.

We must only verify that the map is well defined (it is clearly simplicial if it is well defined). The point we must check is that at most one vertex of any given simplex in $\mathcal{C}\left(S^{\circ}\right)$ becomes peripheral in $S$. The following lemma establishes this, along with the basic nature of the projection $\Pi$.

Lemma 2.2. Suppose $u=\left\{u_{0}, \ldots, u_{k}\right\}$ is any simplex in $\mathcal{C}\left(S^{\circ}\right)$. Then

$$
\left|\left\{\Pi\left(u_{0}\right), \ldots, \Pi\left(u_{k}\right)\right\}\right| \geq k
$$

$\Pi\left(u_{i}\right)$ is peripheral for at most one $i=0, \ldots, k$. Moreover, if there is peripheral curve, say $\Pi\left(u_{0}\right)$, then the restriction $\left.\Pi\right|_{u}$ is injective and $\bullet$ is contained in a once-punctured disk bounded by $u_{0}$ in $S^{\bullet}$.

If $\left.\Pi\right|_{u}$ is noninjective, and $\Pi\left(u_{0}\right)=\Pi\left(u_{1}\right)$, then $u_{0}$ and $u_{1}$ cobound an annulus containing • in $S^{\bullet}$.
maybe split this into a couple lemmas?

The lemma tells us that $\Pi$ is well defined, and moreover, the restriction to a $k$-simplex has rank at least $k-1$ (as a linear map).

Proof. Note that it suffices to prove the lemma when $u=\left\{u_{0}, \ldots, u_{\xi(S)}\right\}$ is a maximal simplex since any simplex is contained in a maximal simplex. That is, $u$ is a pants decomposition of $S^{\circ}$. Let $P$ denote the pair of pants containing the puncture $\circ$, which we replace by $\bullet$

If the other two cuffs of $P$ are curves in $u$, say $u_{0}$ and $u_{1}$, then $u_{0}$ and $u_{1}$ cobound an annulus containing •. Moreover, $\Pi\left(u_{1}\right), \ldots, \Pi\left(u_{\xi\left(S^{\circ}\right)}\right)$ gives a pants decomposition for $S$, and hence has $\xi(S)=\xi\left(S^{\circ}\right)-1$ distinct elements, as required. Note that it also follows that no curve has become peripheral in this case.

If one of the other two cuffs of $P$ is a puncture, then $P$ has just one curve cuff, say $u_{0}$, and $\Pi\left(u_{0}\right)$ is peripheral in $S$. That is, $u_{0}$ bounds a once punctured disk in $\S \bullet$ containing •. In this case, $\Pi\left(u_{1}\right), \ldots, \Pi\left(u_{\xi\left(S^{\circ}\right)}\right)$ is a pants decomposition for $S$, has no peripheral curves and $\Pi\left(u_{0}\right), \ldots, \Pi\left(u_{\xi\left(S^{\circ}\right)}\right)$ are all distinct vertices of $\mathcal{C}_{a}^{0}(S)$.

We will say that a simplex $u$ of $\mathcal{C}\left(S^{\circ}\right)$ is injective if $\left.\Pi\right|_{u}$ is injective and noninjective otherwise. Furthermore, we say that an injective simplex is nonperipheral if its $\Pi$-image is contained in $\mathcal{C}(S)$, and peripheral otherwise.

### 2.3 Fibers of $\Pi$

As $\Pi$ is simplicial, it is determined by a map $\Pi_{\Delta}: \mathcal{C}_{\Delta}\left(S^{\circ}\right) \rightarrow \mathcal{C}_{a, \Delta}(S)$. We [?note/recall?] the elementary fact that for any $x \in \mathcal{C}_{a}(S)$, the fiber $\Pi^{-1}(x)$ is naturally a simplicial complex affinely embedded in $\mathcal{C}\left(S^{\circ}\right)$. The combinatorial simplicial complex $\left(\Pi^{-1}(x)\right)_{\Delta}$ is order isomorphic to $\Pi_{\Delta}^{-1}(v)$ where $v$ is the unique simplex of $\mathcal{C}_{a}(S)$ containing $x$ in its interior, and where the partial order is the restriction of that on $\mathcal{C}_{\Delta}\left(S^{\circ}\right)$ [obvious, or give a proof or a ref?]. We note that $\Pi_{\Delta}^{-1}(v)$ is not a combinatorial simplicial subcomplex of $\mathcal{C}_{\Delta}\left(S^{\circ}\right)$, rather just a partially ordered set.

The action of $\pi_{1}(S)$ on $\left(\Pi^{-1}(x)\right)_{\Delta}$ is equivalent to the action of $\pi_{1}(S)$ on $\Pi_{\Delta}^{-1}(v)$. There are only finitely many $\pi_{1}(S)$-orbits for the action of $\Pi_{\Delta}^{-1}(v)$. The orbit of $u \in \Pi_{\Delta}^{-1}(v)$ is determined by $v=\Pi(u)$ and (1) if $u$ is noninjective, the vertex of $v$ in the image of two vertices of $u$, or (2) if $u$ is injective, the component of $S \backslash u$ containing the basepoint •. [obvious or give a proof?]

## 3 Regions

We define some specific realizations of curves and subsurfaces in $S$ and $\widetilde{S}$. Given a simplex $v$ in $\mathcal{C}(S)$, we let $[v]$ denote the geodesic representative.

Lemma 3.1. There exists positive numbers $\{\epsilon(v)\}_{v \in \mathcal{C}^{\circ}(S)}$ so that the (closed) neighborhoods $N_{\epsilon(v)}([v])$ are homeomorphic to annuli, and which satisfy

$$
N_{\epsilon(v)}([v]) \cap N_{\epsilon\left(v^{\prime}\right)}\left(\left[v^{\prime}\right]\right) \neq \emptyset \Leftrightarrow i\left(v, v^{\prime}\right) \neq 0
$$

Give a proof. This is a hyperbolic trig exercise...
We will assume in what follows that we have chosen the numbers $\{\epsilon(v)\}_{v \in \mathcal{C}^{0}(S)}$ as given by the Lemma. Because there is a lower bound to the length of any geodesic, an area argument implies that $\{\epsilon(v)\}_{v \in \mathcal{C}^{0}(S)}$ is a bounded set of numbers.

For each of the peripheral curves $\zeta_{1}, \ldots, \zeta_{m}$ in $\mathcal{C}_{a}^{0}(S)$, we fix a representative horocyclic curves $\left[\zeta_{1}\right], \ldots,\left[\zeta_{m}\right]$ (i.e. quotients of horocycles). We do this so that these are pairwise disjoint, and so that $\left[\zeta_{j}\right] \cap N_{\epsilon(v)}([v])=\emptyset$ for all $j=1, \ldots, m$ and all $v \in \mathcal{C}^{0}(S)$.

### 3.1 Subgroups compatible with simplices

Given a simplex $v=\left\{v_{0}, \ldots, v_{k}\right\}$ of $\mathcal{C}(S)$, we write

$$
N(v)=N_{\epsilon\left(v_{0}\right)}\left(\left[v_{0}\right]\right) \cup \ldots N_{\epsilon\left(v_{k}\right)}\left(\left[v_{k}\right]\right)
$$

By Lemma 3.1, this is a union of pairwise disjoint annuli. As a special case, note that when $v$ is a vertex, $N(v)=N_{\epsilon(v)}([v])$.

The complementary components will also be useful, and we name them as well. The complement of $N(v)$ may have several components, and to keep track of them, we can record the fundamental group of the component as a subgroup of $\pi_{1}(S)$. However, this is only well defined up to conjugation, so we keep track of only the conjugacy class of the fundamental group.

We define $U([\Gamma])$ to be the component of $\overline{S \backslash N([v])}$ for which $\pi_{1}(U([\Gamma]))<$ $\pi_{1}(S)$ is in the conjugacy class of $\Gamma$. In this setting we say that $\Gamma$ is vertex compatible with $v$. For reasons that will become apparent shortly, if $\Gamma$ is conjugate to a subgroup $\pi_{1}\left(N\left(v_{j}\right)\right)$ where $v_{j}$ is a vertex of $v$ we say that $\Gamma$ is edge compatible with $v$.

The subsurface $U([\Gamma])$ is not locally convex, which will be the cause minor complications later. For this reason, we also define the open subsurface $\hat{U}([\Gamma])$ to be the component of $S \backslash[v]$ containing $U([\Gamma])$. Observe that $\hat{U}([\Gamma])$ is locally
can probably get $\operatorname{rid}$ of $U$ and just work with $\hat{U}$ convex and strong deformation retracts onto $U([\Gamma])$ (thus $\pi_{1}(\hat{U}([\Gamma])$ is also conjugate to $\Gamma$ ).

We extend these definitions to incorporate simplices in the augmented curve complex. The horocyclic curves are the boundaries of horocyclic cusps (i.e. quotients of horoballs) which we denote $U(\zeta)$ for each peripheral curve $\zeta \in$ $\mathcal{C}_{a}^{0}(S)$.

It will be convenient to deal with all cases simultaneously, and for this reason we make the following definitions. If $v$ is a peripheral simplex in $\mathcal{C}_{a}(S)$ with peripheral vertex $\zeta$ we define $U([\Gamma])=U(\zeta)$ where $[\Gamma]$ is the conjugacy class of $\pi_{1}(U([\Gamma]))=\pi_{1}(U(\zeta))<\pi_{1}(S)$. In this situation, we define $v$ and $\Gamma$ to be peripherally compatible. We further define $\hat{U}([\Gamma])=U([\Gamma])$.

If $v$ and $\Gamma$ are vertex compatible, edge compatible, or peripherally compatible, then we simply say that $v$ and $\Gamma$ are compatible. We denote the set of subgroups compatible with $v$ by $\mathcal{D}_{v}$ and $\mathcal{D}=\left\{\Gamma \mid \Gamma \in \mathcal{D}_{v}\right.$ for some $\left.v\right\}$. We give $\mathcal{D}$ a partial order defined by inclusion. This is also a partial order on each $\mathcal{D}_{v}$.

### 3.2 Regions in $\widetilde{S}$

We now pull this entire picture back to $\widetilde{S}$. Recall that we have fixed an action of $\pi_{1}(S)$ on $\widetilde{S}$.

Fix a simplex $v=\left\{v_{0}, \ldots, v_{k}\right\}$ in $\mathcal{C}(S)$ and a compatible subgroup $\Gamma<\pi_{1}(S)$. In the universal cover, edge compatibility of $v$ and $\Gamma$ is equivalent to saying that $\Gamma$ is the stabilizer of some component of $p^{-1}(N(v))$, while vertex compatibility says that $\Gamma$ is the stabilizer of some component of $\widetilde{S} \backslash p^{-1}(N(v))$.

If $\Gamma$ is edge compatible with $v$ then we define $N(\Gamma)$ to be the component of $p^{-1}(N(v))$ stabilized by $\Gamma$. It will be convenient to also use the notation $\hat{U}(\Gamma)=N(\Gamma)$. We note that $N(\Gamma) / \Gamma=N\left(v_{0}\right)$ for some vertex $v_{0}<v$.

If $\Gamma$ is vertex compatible with $v$, then we define $U(\Gamma)$ to be the component of $\widetilde{\widetilde{S} \backslash p^{-1}(N(v))}$ stabilized by $\Gamma$. We also define $\hat{U}(\Gamma)$ to be the component of $\widetilde{S} \backslash p^{-1}([v])$ which contains $U(\Gamma)$. Again, $U(\Gamma) / \Gamma=U([\Gamma])$ and $\hat{U}(\Gamma) / \Gamma=$ $\hat{U}([\Gamma])$.

Similarly, if $v$ is a peripheral simplex in $C_{a}(S)$ and $\Gamma$ a peripherally compatible subgroup, we define $U(\Gamma)=\hat{U}(\Gamma)$ to be the component of $p^{-1}(U([\Gamma]))$ stabilized by $\Gamma$. Notice that $U(\Gamma)$ is a horoball, $\Gamma$ is the parabolic subgroup stabilizing it, and $U(\Gamma) / \Gamma=U([\Gamma])$.

The next proposition follows immediately from the definitions we have given, but we record it here for reference.

Proposition 3.2. For any $\Gamma \in \mathcal{D}, \hat{U}(\Gamma)$ is a convex set stabilized by $\Gamma$. If $\Gamma<\Gamma^{\prime}$, then $\hat{U}(\Gamma) \cap \hat{U}\left(\Gamma^{\prime}\right) \neq \emptyset$.

## 4 Trees

For any simplex $v$ of $\mathcal{C}(S)$, we can obtain an action of $\pi_{1}(S)$ on a tree denoted $T_{v}$; see [10] for a general introduction to actions on trees associated to hypersurfaces. There are a number of ways of constructing this tree, but given our setup so far, there is a simple and useful way of describing $T_{v}$.

Namely, we take a vertex for each subgroup of $\mathcal{D}_{v}$ vertex compatible with $v$, and an edge for every subgroup of $\mathcal{D}_{v}$ edge compatible with $v$, then declare a vertex $\Gamma$ to be a vertex of $\Gamma^{\prime}$ if $\Gamma>\Gamma^{\prime}$. That is, the abstract simplicial complex $T_{v, \Delta}$ is reverse-order isomorphic to $\mathcal{D}_{v}$.

Said differently, $v$ provides a graph of groups decomposition of $\pi_{1}(S)$ in which the vertex stabilizers are subgroups vertex compatible with $v$ and edge stabilizers are groups edge compatible with $v$.

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## 5 Algebra vs. Combinatorics

In this section we give an alternative description of $\mathcal{C}\left(S^{\circ}\right)$ which mixes algebraic and combinatorial information. We fix a basepoint $\widetilde{\bullet} \in \widetilde{S}$ and thus a basepoint $\bullet=p(\widetilde{\bullet})$ in $S$ and an isomorphism $\pi_{1}(S, \bullet) \cong \pi_{1}(S)$.

The following lemma describes the stabilizers of simplices of $\mathcal{C}\left(S^{\circ}\right)$ in $\pi_{1}(S)$. The key ingredient is Kra's Theorem 2.1. We assume that any simplex $u$ in $\mathcal{C}\left(S^{\circ}\right)$ is realized as a multicurve in $S^{\circ}$.

Lemma 5.1. A simplex $u$ of $\mathcal{C}\left(S^{\bullet}\right)$ is fixed by $[\gamma] \in \pi_{1}(S, \bullet)<\operatorname{Mod}\left(S^{\circ}\right)$ if and only if $[\gamma]$ is represented by a loop $\gamma$ which is disjoint from $u$.

Note that the stabilizer of a simplex $u$ fixes the simplex pointwise since $\pi_{1}(S, \bullet)$ consists of pure mapping classes in $\operatorname{Mod}\left(S^{\circ}\right)$ as it acts trivially on $H_{1}\left(S^{\circ}\right)$-see [7].

Proof. Suppose first that $[\gamma]$ has a representative $\gamma$ which is disjoint from $u$. As mentioned in $\S 2$, for any neighborhood of $\gamma$, there is a representative $h$ of $[\gamma]$ in $\operatorname{Mod}\left(S^{\circ}\right)$ supported on that neighborhood. Choosing a neighborhood disjoint from $u$, we obtain a representative of $h$ which leaves $u$ fixed, as required.

To prove the reverse implication, we prove its contrapositive. It suffices to assume $[\gamma]$ is primitive by the purity statement above. First, choose a representative $\gamma$ of $[\gamma]$ which has the minimal number of self intersections. This can be done so that the based loop actually has the minimal number of self intersections of the free homotopy class. To see this, let $\gamma_{t}: S^{1} \rightarrow S$ for $t \in[0,1]$ be the free homotopy of some representative $\gamma_{0}$ of $[\gamma]$ (with $\gamma_{0}(1)=\bullet$ with $1 \in S^{1}$ ) to a loop $\gamma_{1}$ with minimal self intersection. The path $\gamma_{t}(1)$ for $t \in[0,1]$ runs from - to some other point of $S$. In a similar manner as described in $\S 2$ we can find an isotopy $h_{t}: S \rightarrow S$ for $t \in[0,1]$ with $h_{0}=i d_{S}$, so that $h_{t}(\bullet)=\gamma_{t}(1)$. Now $h_{t}^{-1} \circ \gamma_{t}: S^{1} \rightarrow S$ is a based homotopy starting at $\gamma_{0}$ and ending at a representative differing from $\gamma_{1}$ by a homeomorphism (namely, $h_{1}^{-1}$ ). In particular, $\gamma=h_{1}^{-1} \circ \gamma_{1}$ is a representative of $[\gamma]$ minimizing self intersection number in the free homotopy class.

Let $\Sigma$ be the supporting subsurface of $S^{\circ}$ containing $\gamma\left(S^{1}\right) \backslash\{\bullet\}$. Choose a representative of $u$ having minimal intersection with $\Sigma$. Because $[\gamma]$ has no representative disjoint from $u$, we see that the intersection of $\Sigma$ with $u$ is nonempty and essential.

If $\gamma$ is a simple closed loop, then $\Sigma$ is a punctured annulus and $[\gamma] \in \operatorname{Mod}\left(S^{\circ}\right)$ is simply a Dehn twist in one boundary component, and an inverse Dehn twist in the other. Since $u$ nontrivially intersects $\Sigma$, it follows that $[\gamma] u \neq u$ as required. If $\gamma$ is not a simple closed loop, then note that $\xi(\Sigma) \geq 4$ and $\gamma$ is filling on $\Sigma$. According to Kra's Theorem 2.1, $[\gamma] \in \operatorname{Mod}\left(S^{\circ}\right)$ is a pseudo-Anosov mapping class when restricted to $\Sigma$. It again follows from the fact that $u$ nontrivially intersects $\Sigma$ that $[\gamma](u) \neq u$. This completes the proof.

Suppose now that $u \in \mathcal{C}\left(S^{\circ}\right)$ is given, and realized by some multicurve on $S^{\bullet}$. Let $V(u)$ be the component of the complement of $u$ containing •

Because any other representative of $u$ differs by an isotopy supported in the complement of •, we see that the subgroup (not just the conjugacy class) $\pi_{1}(V(u))=\pi_{1}(V(u), \bullet)<\pi_{1}(S, \bullet)=\pi_{1}(S)$ is a well defined subgroup depending only on $u$.

Note that $\pi_{1}(V(u))$ consists of precisely those elements of $\pi_{1}(S)$ which can be realized disjoint from $u$. Thus, by Lemma 5.1 we immediately obtain

Corollary 5.2. $\operatorname{Stab}_{\pi_{1}(S)}(u)=\pi_{1}(V(u))$
We next list some elementary observations about $\pi_{1}(V(u))$.
Lemma 5.3. If $u \in \mathcal{C}\left(S^{\circ}\right)$, then

1. if $u$ is noninjective then $\pi_{1}(V(u))$ is edge compatible with $\Pi(u)$.
2. if $u$ is injective and nonperipheral then $\pi_{1}(V(u))$ is vertex compatible with $\Pi(u)$.
3. if $u$ is peripheral then $\pi_{1}(V(u))$ is peripherally compatible with $\Pi(u)$.
4. if $u^{\prime}<u$ is a face, then $\pi_{1}\left(V\left(u^{\prime}\right)\right)>\pi_{1}(V(u))$.

Proof. Let $u=\left\{u_{0}, \ldots, u_{k}\right\}$ and $\Pi(u)=v=\left\{v_{0}, \ldots, v_{k}\right\}$ with $v_{j}=\Pi\left(u_{j}\right)$ for all $j=0, \ldots, k$.

To prove (1), we suppose that $u$ is noninjective. By Lemma 3.1, by relabeling if necessary, we assume that $v_{0}=v_{1}=\Pi\left(u_{0}\right)=\Pi\left(u_{1}\right) \in \mathcal{C}^{0}(S)$ and so that $\overline{V(u)}$ is an annulus bounded by $u_{0}$ and $u_{1}$ containing $\bullet$. Therefore, $\overline{V(u)}$ in $S$ is homotopic to $N\left(v_{0}\right)$, and hence $\pi_{1}(V(u))=\pi_{1}(\overline{V(u)})$ is edge compatible with $v=\Pi(u)$.

For (2), we suppose that that $u$ is injective and nonperipheral. Note that our realization of $u$ in $S^{\bullet}$ is also a realization of $v$ once we forget $\bullet$. Since this multicurve is isotopic to $[v]$, we see that $V(u)$ is homotopic to some component of $S \backslash[v]$. It follows that $\pi_{1}(V(u))$ is vertex compatible with $v=\Pi(u)$.

In order to verify (3), we assume $u$ is peripheral and let $\zeta$ be the peripheral vertex of $v$. We again appeal to Lemma 3.1 which says that in $S, V(u)$ is a once punctured disk with boundary homotopic to $\zeta$ in $S$. As any punctured disk neighborhood of a cusp is homotopic to any embedded horoball neighborhood, we see that $V(u)$ is homotopic to the horoball neighborhood bounded by $[\zeta]$. Hence $\pi_{1}(V(u))$ is peripherally compatible with $v=\Pi(u)$.

All that remains is to prove (4). This follows from the trivial observation that if $u^{\prime}<u$, then $V\left(u^{\prime}\right) \supset V(u)$, and hence $\pi_{1}\left(V\left(u^{\prime}\right)\right)>\pi_{1}(V(u))$.

Define a partially ordered set

$$
\mathcal{G}=\left\{(\Gamma, v) \mid \Gamma \in \mathcal{D}_{v}\right\}
$$

with the partial order defined by

$$
\left(\Gamma^{\prime}, v^{\prime}\right) \preceq(\Gamma, v) \Leftrightarrow v^{\prime}<v \text { and } \Gamma^{\prime}>\Gamma
$$

There is an obvious order preserving $\pi_{1}(S)$ action on $\mathcal{G}$ by conjugation on the first factor.

Lemma 5.3 and the fact that $\Pi$ is a simplicial map implies that the map

$$
\Phi: \mathcal{C}_{\Delta}\left(S^{\circ}\right) \rightarrow \mathcal{G}
$$

given by $\Phi(u)=\left(\pi_{1}(V(u)), \Pi(u)\right)$ is an order preserving map. If we let $\Upsilon: \mathcal{G} \rightarrow$ $\mathcal{C}_{a}(S)$ denote the projection onto the second coordinate, then by construction $\Pi(u)=\Upsilon(\Phi(u))$.

Theorem 5.4. $\Phi$ is a $\pi_{1}(S)$-equivariant order isomorphism.
Proof. We first prove that $\Phi$ is $\pi_{1}(S)$-equivariant. Fix a simplex $u$ in $\mathcal{C}\left(S^{\circ}\right)$ and a realization in $S^{\bullet}$. We must prove

Claim. For any $[\gamma] \in \pi_{1}(S), \pi_{1}(V([\gamma](u)))=[\gamma] \pi_{1}(V(u))[\gamma]^{-1}$.

Proof of claim. As in $\S 2$, given a loop $\gamma:[0,1] \rightarrow S$ representing $[\gamma]$, we construct an isotopy $h_{t}: S \rightarrow S, t \in[0,1]$, with the property that $h_{t}(\bullet)=\gamma(t)$. The mapping class $\left[h_{1}\right] \in \operatorname{Mod}\left(S^{\circ}\right)$ of $h_{1}$ is the mapping class associated to $[\gamma] \in \pi_{1}(S, \bullet)$. Now consider an element $[\sigma] \in \pi_{1}(V(u), \bullet)$, represented by a loop $\sigma:[0,1] \rightarrow V(u) \subset S$. In the same way, we construct an isotopy $k_{t}: S \rightarrow S$ so that $k_{t}(\bullet)=\sigma(t)$.

The element $[\gamma][\sigma][\gamma]^{-1} \in[\gamma] \pi_{1}(V(u))[\gamma]^{-1}$ is represented represented in $\operatorname{Mod}\left(S^{\circ}\right)$ by the homeomorphism $h_{1} \circ k_{1} \circ h_{1}^{-1}$. This is isotopic to the identity in $S$ by $h_{1} \circ k_{t} \circ h_{1}^{-1}$, for $t \in[0,1]$. Thus $h_{1} \circ k_{1} \circ h_{1}^{-1}$ represents the loop $h_{1}(\sigma(t))=h_{1} \circ k_{t} \circ h_{1}^{-1}(\bullet)$, which is contained in $V\left(\left[h_{1}\right](u)\right)=V([\gamma](u))$.

Therefore, $[\gamma] \pi_{1}(V(u))[\gamma]^{-1}<\pi_{1}(V([\gamma](u)))$. The same argument applied to $[\gamma](u)$ and $[\gamma]^{-1}$ implies

$$
[\gamma]^{-1} \pi_{1}(V([\gamma](u)))[\gamma]<\pi_{1}\left(V\left([\gamma]^{-1}[\gamma](u)\right)\right)=\pi_{1}(V(u))
$$

Conjugating by $[\gamma]$, we obtain $\pi_{1}(V([\gamma](u)))<[\gamma] \pi_{1}(V(u))[\gamma]^{-1}$ from which the desired equality follows.

So, we have a $\pi_{1}(S)$-equivariant, order preserving map. To prove that the map is a bijection, it suffices to show that (1) $\Phi$ induces a bijection of the orbit spaces, and (2) the stabilizer of a simplex is equal to the stabilizer of the image of the simplex.

To prove (1), let $v$ be a simplex in $\mathcal{C}_{a}(S)$ and $\Gamma$ compatible with $v$. By an isotopy of $N(v)$, we can assume that the component of the image of $N(v)$ or of $\overline{S \backslash N(v)}$ that contains our basepoint $\bullet$ is conjugate to $\Gamma$. Taking $\partial N(v)$, and identifying components which are homotopic in $S^{\circ}$, we see that $\Phi(u)=(\Gamma, v)$. So $\Phi$ induces a surjection on orbit spaces. To see that the map is injective, we check that the finitely many orbits in $\Pi_{\Delta}^{-1}(v)$ are sent to distinct orbits of $\mathcal{G}$ [more details here...].

The statement (2) follows easily from Corollary 5.2 and the fact that the subgroups $\pi_{1}(V(u))$ are malnormal [give a ref?].

To see that $\Phi$ is an isomorphism, we need only show that if $\Phi\left(u^{\prime}\right) \preceq \Phi(u)$, then $u^{\prime}<u$. If $\Pi\left(u^{\prime}\right)<\Pi(u)$ and $\pi_{1}\left(V\left(u^{\prime}\right)\right)>\pi_{1}(V(u))$, then after applying a homotopy in $S^{\circ}$, we have $V\left(u^{\prime}\right) \supset V(u)$, and it follows that $u^{\prime}<u$ as required.

We denote the projection onto the first factor of $\mathcal{G}$ by

$$
\rho: \mathcal{G} \rightarrow \mathcal{D}
$$

and write

$$
\eta=\rho \circ \Phi: \mathcal{C}_{\Delta}\left(S^{\circ}\right) \rightarrow \mathcal{D}
$$

Fixing $v \in \mathcal{C}_{a}(S)$, we have $\mathcal{D}_{v}=\Upsilon^{-1}(v)$ and $\Pi_{\Delta}^{-1}(v)=\eta^{-1}\left(\mathcal{D}_{v}\right)$. Restricting $\Phi$ to this last space we obtain
Corollary 5.5. $\left.\Phi\right|_{\Pi_{\Delta}^{-1}(v)}: \Pi_{\Delta}^{-1}(v) \rightarrow \mathcal{D}_{v}$ is a reverse-order isomorphism.
We now see that Theorem 1.1 follows: Given any $v \in \mathcal{C}(S), \mathcal{D}_{v}$ is reverseorder isomorphic to $T_{v, \Delta}$, and $\Pi_{\Delta}^{-1}(v)$ is order isomorphic to $\left(\Pi^{-1}(x)\right)_{\Delta}$.

The following is essentially a restatement of Proposition 3.2, appealing to Lemma 5.3 or Theorem 5.4.

Proposition 5.6. If $u^{\prime}<u$ is a face, then

$$
\hat{U}\left(\eta\left(u^{\prime}\right)\right) \cap \hat{U}(\eta(u)) \neq \emptyset
$$

## 6 Distance and piecewise geodesic paths

Consider any pair of points $x, y \in \widetilde{S}$ with $x \in U\left(\eta\left(u_{x}\right)\right)$ and $y \in U\left(\eta\left(u_{y}\right)\right)$ for some $u_{x}, u_{y} \in \mathcal{C}^{0}\left(S^{\circ}\right)$. Let $\left[u_{0}, \ldots, u_{n}\right]$ denote an edge path connecting $u_{0}=u_{x}$ to $u_{n}=u_{y}$. Of particular interest to us is the case that, $\left[u_{0}, \ldots, u_{n}\right]$ is the geodesic in $\mathcal{C}^{1}\left(S^{\circ}\right)$, so that $n=d_{\mathcal{C}^{1}\left(S^{\circ}\right)}\left(u_{x}, u_{y}\right)$.
Proposition 6.1. There exists a path

$$
\gamma:[0,2 n+1] \rightarrow \widetilde{S}
$$

connecting $x$ to $y$ with the property that

- $\gamma([2 j, 2 j+1])$ is a geodesic segment contained in $\hat{U}\left(\eta\left(u_{j}\right)\right)$ for each $j=$ $0, \ldots, n$, and
- $\gamma([2 j+1,2 j+2])$ is a geodesic segment contained in $\hat{U}\left(\eta\left(\left\{u_{j}, u_{j+1}\right\}\right)\right)$ for each $j=0, \ldots, n-1$.
Proof. According to Proposition 5.6, the sets $\hat{U}\left(\eta\left(u_{j}\right)\right)$ and $\hat{U}\left(\eta\left(\left\{u_{j}, u_{j+1}\right\}\right)\right.$ are convex, and $\hat{U}\left(\eta\left(\left\{u_{j}, u_{j+1}\right\}\right)\right)$ nontrivially intersects both $\hat{U}\left(\eta\left(u_{j}\right)\right)$ and $\hat{U}\left(\eta\left(u_{j+1}\right)\right)$ for each $j=0, \ldots, n-1$. From these two observations, the proposition easily follows.


## 7 Purely pseudo-Anosov subgroups

Let $G<\pi_{1}(S)$ be a finitely generated purely pseudo-Anosov subgroup. By Kra's Theorem 2.1, every element of $G$ represents a curve in $S$ which fills $S$. Let $\widetilde{\Sigma}$ denote the convex hull of the limit set of $G$ acting on $\widetilde{S}$ and

$$
p_{0}: \widetilde{\Sigma} \rightarrow \Sigma=\widetilde{\Sigma} / G
$$

the quotient compact hyperbolic surface with geodesic boundary. That $\widetilde{\Sigma}$ is compact follows from the fact that finite generation for Fuchsian groups is equivalent to geometric finiteness (see [1]) and the fact that every element is hyperbolic since the loop defining it fills $S$. The inclusion $\widetilde{\Sigma} \rightarrow \widetilde{S}$ induces an immersion $f: \Sigma \rightarrow S$ and $f_{*}\left(\pi_{1}(\Sigma)\right)=G$.


By the filling property of every conjugacy class in $G$, we see that if $[v]$ is a geodesic on $S$, then $f^{-1}([v])$ cuts $\Sigma$ into disks. Moreover, as we will see, the family of arcs of $f^{-1}([v])$ as $v$ ranges over all of $\mathcal{C}^{0}(S)$ is a precompact family.

It will be convenient to prove this statement for a slightly larger surface. Namely, let $\Sigma_{1}=N_{1}(\widetilde{\Sigma}) / G$ denote the quotient of the 1-neighborhood of $\widetilde{\Sigma}$ by $G$. This adds a width-1 collar to each boundary component of $\Sigma$. There is an obvious extension of $f$, to $\Sigma_{1} \supset \Sigma$, which we still denote $f: \Sigma_{1} \rightarrow S$. Let $\mathcal{A}$ denote the set of all arcs of $f^{-1}([v])$ in $\Sigma_{1}$ as $v$ ranges over all of $\mathcal{C}^{0}(S)$.

Proposition 7.1. The family $\mathcal{A}$ is precompact in the space of all proper geodesic arcs in $\Sigma_{1}$. In particular, there are only finitely many isotopy classes in $\mathcal{A}$ and there is a uniform bound on the length of any arc in $\mathcal{A}$.

Proof. By the Arzela-Ascoli Theorem, it suffices to prove that there is a uniform bound to the length of any arc in $\mathcal{A}$.

Suppose to the contrary that there exists a sequence $\left\{v_{n}\right\} \subset \mathcal{C}^{0}(S)$ and components $L_{n} \subset f^{-1}\left(\left[v_{n}\right]\right)$ so that $\ell\left(L_{n}\right) \rightarrow \infty$ as $n \rightarrow \infty$. We may also assume that $\left[v_{n}\right]$ has a Hausdorff limit $\lambda$, which is a geodesic lamination on $S$. Let $\lambda^{\prime}$ be the maximal measurable sublamination of $\lambda$, i.e., $\lambda^{\prime}$ is obtained from $\lambda$ by throwing away all non-closed isolated leaves.

Note that the Hausdorff limit of $L_{n}$ is contained in $f^{-1}(\lambda)$. Because $\ell\left(L_{n}\right) \rightarrow$ $\infty$, it follows that there is a geodesic lamination $\lambda_{0}$ contained in this Hausdorff limit. The sublamination $f\left(\lambda_{0}\right)$ is a component of $\lambda^{\prime}$. Let $\alpha$ be a (geodesic) boundary component of the boundary of the supporting subsurface of $\lambda_{0}$ (I think this is the complement of what Casson-Bleiler call the "core"). The image of this subsurface by $f$ is the supporting subsurface for $f\left(\lambda_{0}\right)$ and as such, its boundary is disjoint from $\lambda^{\prime}$. It follows that $f(\alpha)$ is disjoint from $\lambda^{\prime}$. This is impossible, since every curve representing a conjugacy class of $G$ intersects every lamination in $S$, and $f(\alpha)$ represents a conjugacy class in $G$.

Note that each component of $\Sigma_{1} \backslash f^{-1}([v])$ is a not only a disk, but a disk with uniformly bounded diameter: it is convex, and has uniformly bounded circumference. This implies the same statement for the disks $N_{1}(\widetilde{\Sigma}) \backslash p^{-1}([v])$ which are just the disks of $p^{-1}\left(\widetilde{\Sigma}_{1} \backslash f^{-1}([v])\right)$. Moreover, since $\epsilon(v)$ is uniformly bounded over all $v \in \mathcal{C}^{0}(S)$, we see that the diameter of any component of $N_{1}(\widetilde{\Sigma}) \cap N\left(p^{-1}([v])\right)$. We record this as

Corollary 7.2. There exists $D>0$ so that for any nonperipheral simplex $u$ in $\mathcal{C}\left(S^{\circ}\right)$, any component of $\hat{U}(\eta(u)) \cap N_{1}(\widetilde{\Sigma})$ has diameter at most $D$.

## 8 Convex cocompactness

We now fix a purely pseudo-Anosov subgroup $G<\pi_{1}(S)<\operatorname{Mod}\left(S^{\circ}\right)$ and let $\widetilde{\Sigma} \subset N_{1}(\widetilde{\Sigma}) \subset \widetilde{S}$ and $f: \Sigma_{1} \rightarrow S$ be as in the previous section. We assume that we have chosen our horoball cusps $U(\zeta)$ sufficiently small to lie outside $f\left(\Sigma_{1}\right)$, and hence $\hat{U}(\Gamma) \cap N_{1}(\widetilde{\Sigma})=\emptyset$ for any $\Gamma \in \mathcal{D}_{\zeta}$ and any peripheral vertex $\zeta \in \mathcal{C}_{a}^{0}(S)$.

Fix a vertex $u \in \mathcal{C}^{0}\left(S^{\circ}\right)$ and $x \in U(\eta(u)) \cap \widetilde{\Sigma}$.
We could choose some generating set for $G$ to define a metric, though we have a natural (and useful) $G$-invariant metric in the q.i. class given to us.

$$
d_{G}(g, h):=d_{\widetilde{\Sigma}}(g(x), h(x))=d_{\widetilde{S}}(g(x), h(x))
$$

The next theorem implies Theorem 1.2.
Theorem 8.1. The orbit map

$$
G \rightarrow G \cdot u
$$

given by $g \mapsto g(u)$ is a quasi-isometric embedding into $\mathcal{C}\left(S^{\circ}\right)$. In particular, $G$ is convex cocompact.

The last statement follows from the first and Theorem 1.3 of [6] or Theorem 2.9 of [5].

Proof. We must find $K \geq 1$ and $C \geq 0$ so that for any $g \in G$, we have

$$
\frac{d_{G}(\mathbf{1}, g)}{K}-C \leq d_{\mathcal{C}^{1}\left(S^{\circ}\right)}(u, g(u)) \leq K \mathrm{~d}_{G}(\mathbf{1}, g)+C
$$

Fixing a generating set $\left\{g_{1}, \ldots, g_{r}\right\}$ for $G$, taking any $C \geq 0$ and

$$
K \geq \max \left\{1,\left\{\frac{\mathrm{~d}_{\mathcal{C}^{1}\left(S^{\circ}\right)}\left(u, g_{i}(u)\right)}{\mathrm{d}_{G}\left(\mathbf{1}, g_{i}\right)}\right\}_{i=1}^{r}\right\}
$$

the required upper bound on $d_{\mathcal{C}^{1}\left(S^{\circ}\right)}(u, g(u))$ easily follows from the triangle inequality. We assume that the $K$ and $C$ we produce for the lower bound also
satisfies these two inequalities.

We now proceed to the proof of the lower bound.
Let $\tau: \widetilde{S} \rightarrow \widetilde{\Sigma}$ denote the closest point projection. This is well known to be give a ref a contraction. In fact, there exists $R>0$, so that if $\sigma$ is any geodesic segment outside $N_{1}(\widetilde{\Sigma})$ has length $\ell(\tau(\sigma)) \leq R$.

Next, suppose $u^{\prime}$ is a simplex in $\mathcal{C}\left(S^{\circ}\right)$ and $\sigma$ is a geodesic segment contained in $\hat{U}\left(\eta\left(u^{\prime}\right)\right)$. Since $\hat{U}\left(\eta\left(u^{\prime}\right)\right) \cap N_{1}(\widetilde{\Sigma})$ is convex, $\sigma$ is cut into at most three geodesic segments by this set, at most one of which is contained in $\hat{U}\left(\eta\left(u^{\prime}\right)\right) \cap N_{1}(\widetilde{\Sigma})$. It follows that

$$
\ell(\tau(\sigma)) \leq 2 R+D
$$

Now suppose $\mathrm{d}_{\mathcal{C}^{1}\left(S^{\circ}\right)}(u, g(u))=n$ and connect $u$ to $g(u)$ by a geodesic edge path $\left[u_{0}, \ldots, u_{n}\right]$. Let

$$
\gamma:[0,2 n+1] \rightarrow \widetilde{S}
$$

be a path connecting $x \in U(\eta(u))$ to $g(x) \in g(U(\eta(u)))=U(\eta(g(u)))$ given by Proposition 6.1. Since each geodesic segment $\gamma([i, i+1])$ is contained in some set of the form $\hat{U}(\eta(u))$ for some simplex $u$, we see that for every $i=0, \ldots, 2 n$

$$
\ell(\tau(\gamma([i, i+1]))) \leq 2 R+D
$$

Since $\gamma$ connects $x$ to $g(x)$, so does $\tau(\gamma)$, and its length bounds the distance from $x$ to $g(x)$. Therefore we obtain

$$
\mathrm{d}_{G}(\mathbf{1}, g)=\mathrm{d}_{\widetilde{\Sigma}}(x, g(x)) \leq \ell(\tau(\gamma)) \leq(2 n+1)(2 R+D)
$$

Isolating $n=\mathrm{d}_{\mathcal{C}^{1}\left(S^{\circ}\right)}(u, g(u))$ in this inequality, we obtain

$$
\mathrm{d}_{\mathcal{C}^{1}\left(S^{\circ}\right)}(u, g(u))=n \geq \frac{\mathrm{d}_{G}(\mathbf{1}, g)}{2(2 R+D)}-\frac{1}{2}
$$

Taking any $K \geq 2(2 R+D)$ and $C=1 / 2$, completes the proof.

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