COVERS AND THE CURVE COMPLEX

KASRA RAFI AND SAUL SCHLEIMER

ABSTRACT. The removal of a point from a closed surface induces a quasi-isometric embedding of the associated curve complexes. The same is true of finite-sheeted coverings.

1. INTRODUCTION

Suppose that Σ is a compact connected orientable surface. A simple closed curve $\alpha \subset \Sigma$ is *essential* if α does not bound a disk in Σ. The curve α is *non-peripheral* if α is not boundary parallel.

Definition 1.1 (Harvey [Har81])**.** The *complex of curves* $\mathcal{C}(\Sigma)$ has isotopy classes of essential, non-peripheral curves as its vertices. A collection of $k+1$ vertices spans a k–simplex if every pair of vertices has disjoint representatives.

Virtually all known information about the geometry of $\mathcal{C}(\Sigma)$ is due to Masur and Minsky [MM99] and [MM00]. Both of these, especially the first, are influenced by Gromov's notion of *quasi-isometry*. Recall that the most important invariant of quasi-isometry is the *growth*: for example, the increase of volume of metric balls. However, since the curve complex is locally infinite it is unclear how such invariants can be used in this context.

Thus, we concentrate on a different family of invariants: the metrically natural subspaces. Note that almost all of the well-known subspaces of the curve complex, such as the complex of separating curves, the disk complex, and so on, are *not* quasi-isometrically embedded and so do not obviously give invariants.

This paper discusses the first non-trivial examples of one curve complex being quasi-isometrically embedded is another. These arise in two ways: by puncturing a closed surface and from covering maps.

2. STATEMENTS

It will be enough to study only the one-skeleton of $\mathcal{C}(\Sigma)$, for which we use the same notation. This is because the one-skeleton and the

Date: January 15, 2007.

entire complex are quasi-isometric. Giving all edges of $\mathcal{C}(\Sigma)$ of length one, we denote the distance between vertices by $d_\Sigma(\cdot,\cdot)$.

Puncturing. Let S be the closed surface of genus g and Σ be the surface of genus g with one puncture. Inspired by Lemma 3.6 of Harer [Har86]:

Theorem 2.1. *$\mathcal{C}(S)$ embeds isometrically into $\mathcal{C}(\Sigma)$.*

In fact, as we shall see, there are many such embeddings.

Proof of Theorem 2.1. Pick a hyperbolic metric on S. By the Baire category theorem, the union of geodesic representatives of simple closed curves does not cover S. (In fact this union has Hausdorff dimension one, by work of Bonahon.) Let $*$ be a point in the complement of this union and identify Σ with $S \setminus \{*\}$. This defines an embedding $\Pi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ which depends on the choice of metric, point, and identification. Let $P\colon \mathcal{C}(\Sigma) \to \mathcal{C}(S)$ be the map obtained by filling the point $*$. Note that $P \circ \Pi$ is the identity map.

We observe, for $a, b \in \mathcal{C}(S)$ and $\alpha = \Pi(a)$, $\beta = \Pi(b)$ that

$$d_S(a,b) = d_\Sigma(\alpha,\beta).$$

This is because P and Π send disjoint curves to disjoint curves. Therefore, if $L \subset \mathcal{C}(S)$ is a geodesic connecting a and b, then $\Pi(L)$ is a path in $\mathcal{C}(\Sigma)$ of the same length connecting α to β. Conversely, if $\Lambda \subset \mathcal{C}(\Sigma)$ is a geodesic connecting α to β, then $P(\Lambda)$ is a path in $\mathcal{C}(S)$ of the same length connecting a to b. $\square$

Coverings. Let Σ and S be compact connected orientable surfaces and let $P\colon \Sigma \to S$ be a covering map. This defines a relation between the corresponding complexes of curves. That is, $a \in \mathcal{C}(S)$ is related to $\alpha \in \mathcal{C}(\Sigma)$ if $P(\alpha) = a$. Abusing notation, we write this relation as a map $\Pi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$.

As a bit of notation, if A, B, c are non-negative real numbers, with $\mathsf{c} \geq 1$, and if $A \leq \mathsf{c}B + \mathsf{c}$ then we write $A \prec_{\mathsf{c}} B$. If $A \prec_{\mathsf{c}} B$ and $B \prec_{\mathsf{c}} A$ then we write $A \asymp_{\mathsf{c}} B$. Our goal is:

Theorem 7.1. *The relation $\Pi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a quasi-isometric embedding. That is, if $P(\alpha) = a$ and $P(\beta) = b$, for $\alpha, \beta \in \mathcal{C}(\Sigma)$ and $a, b \in \mathcal{C}(S)$, then*

$$d_\Sigma(\alpha,\beta) \asymp_{\mathsf{Q}} d_S(a,b),$$

where the constant Q depends on the topology of S and the degree of the covering map only.

When Σ is the orientation double cover of a nonorientable S, Theorem 7.1 is due to Masur-Schleimer [MS07].

Note that one of the inequalities, namely $d_\Sigma(\alpha, \beta) \leq d_S(a, b)$, follows immediately because disjoint curves in S have disjoint lifts to Σ. The opposite inequality is harder to obtain and occupies the rest of the paper.

3. Subsurface projection

Suppose that Σ is a compact connected orientable surface. A subsurface Ψ is *cleanly embedded* if all components of $\partial\Psi$ are essential and whenever $\gamma \subset \partial\Psi$ is isotopic to $\delta \subset \partial\Sigma$ then $\gamma = \delta$. All subsurfaces we consider will be cleanly embedded.

We now recall the definition of the *subsurface projection* relation $\pi_\Psi \colon \mathcal{C}(\Sigma) \to \mathcal{C}(\Psi)$, supposing that Ψ is not an annulus. Fix a hyperbolic metric on the interior of Σ. Let Σ' be the Gromov compactification of the cover of Σ corresponding to the inclusion $\pi_1(\Psi) \to \pi_1(\Sigma)$. Thus Σ' is homeomorphic to Ψ; this gives a canonical identification of $\mathcal{C}(\Psi)$ with $\mathcal{C}(\Sigma')$. For any $\alpha \in \mathcal{C}(\Sigma)$ let α' be the closure of the preimage of α in Σ'. Let α'' be any component of α' which is not properly isotopic into the boundary. Note that if none exist then α is not related to any vertex of $\mathcal{C}(\Psi)$; here we write $\pi_\Psi(\alpha) = \emptyset$. Let N be a closed regular neighborhood of $\alpha'' \cup \partial\Sigma'$. Fix attention on α''', any boundary component of N which is essential and non-peripheral. Then $\alpha \in \mathcal{C}(\Sigma)$ is related to $\alpha''' \in \mathcal{C}(\Psi)$ and we write $\pi_\Psi(\alpha) = \alpha'''$.

If Ψ is an annulus, then the definition of $\mathcal{C}(\Psi)$ is altered. Vertices are proper isotopy classes of essential arcs in Ψ. Edges are placed between vertices with representatives having disjoint interiors. The projection map is defined as above, omitting the final steps involving the regular neighborhood N.

If Ψ is a four-holed sphere or a once-holed torus then the curve complex of Ψ is the well-known *Farey graph*: since all curves intersect, edges are instead placed between curves that intersect exactly twice or exactly once. The definition of π_Ψ is as in the non-annular case.

We say that $\alpha \in \mathcal{C}(\Sigma)$ *cuts* the subsurface Ψ if $\pi_\Psi(\alpha) \neq \emptyset$. Otherwise we say that α *misses* Ψ. Suppose now that $\alpha, \beta \in \mathcal{C}(\Sigma)$ both cut Ψ. We define the *projection distance* to be

$$d_\Psi(\alpha, \beta) = d_\Psi(\pi_\Psi(\alpha), \pi_\Psi(\beta)).$$

Now we may state the Bounded Geodesic Image Theorem:

Theorem 3.1 (Masur-Minsky [MM00]). *Fix a surface Σ. There is a constant $\mathsf{M} = \mathsf{M}(\Sigma)$ so that for any vertices $\alpha, \beta \in \mathcal{C}(\Sigma)$, for any geodesic $\Lambda \subset \mathcal{C}(\Sigma)$ connecting α to β, and for any $\Omega \subsetneq \Sigma$, if $d_\Omega(\alpha, \beta) \geq \mathsf{M}$ then there is a vertex of Λ which misses Ω.* $\square$

Fix α and β in $\mathcal{C}(\Sigma)$ and *thresholds* $\mathsf{T}_0 > 0$ and $\mathsf{T}_1 > 0$. We say that a set $\mathcal{J}$ of subsurfaces $\Omega \subsetneq \Sigma$, is an $(\mathsf{T}_0, \mathsf{T}_1)$–*antichain* for Σ, α and β if $\mathcal{J}$ satisfies the following properties.

- If $\Omega, \Omega' \in \mathcal{J}$ then Ω is not a strict subsurface of Ω'.
- If $\Omega \in \mathcal{J}$ then $d_\Omega(\alpha, \beta) \geq \mathsf{T}_0$.
- For any $\Psi \subsetneq \Sigma$, either Ψ is a subsurface of some element of $\mathcal{J}$ or $d_\Psi(\alpha, \beta) < \mathsf{T}_1$.

Notice that there may be many different antichains for the given data $(\Sigma, \alpha, \beta, \mathsf{T}_0, \mathsf{T}_1)$. One particularly nice example is when $\mathsf{T}_0 = \mathsf{T}_1 = \mathsf{T}$ and $\mathcal{J}$ is defined to be the maxima of the set

$$\{\Omega \subsetneq \Sigma \mid d_\Omega(\alpha, \beta) \geq \mathsf{T}\}$$

as ordered by inclusion. We call this the T–*antichain of maxima* for Σ, α and β. By $|\mathcal{J}|$ we mean the number of elements of $\mathcal{J}$.

We may now state and prove:

Lemma 3.2. *For every surface Σ and for every pair of sufficiently large thresholds $\mathsf{T}_0, \mathsf{T}_1$, there is an accumulation constant $\mathsf{A} = \mathsf{A}(\Sigma, \mathsf{T}_0, \mathsf{T}_1)$ so that, if $\mathcal{J}$ is an $(\mathsf{T}_0, \mathsf{T}_1)$–antichain for Σ, α and β then*

$$d_\Sigma(\alpha, \beta) \geq |\mathcal{J}|/\mathsf{A}.$$

Proof. We proceed via induction: for the Farey graph it suffices for both thresholds to be larger than 3 and then $\mathsf{A} = 1$ (see [Min99]). Let C be a constant so that: if $\Omega \subset \Psi \subset \Sigma$ and α', β' are the projections of α, β to Ψ then

$$|d_\Omega(\alpha, \beta) - d_\Omega(\alpha', \beta')| \leq \mathsf{C}.$$

In the general case, we take the thresholds large enough so that:

- the theorem still applies to any strict subsurface Ψ with thresholds $\mathsf{T}_0 - \mathsf{C}, \mathsf{T}_1 + \mathsf{C}$, and
- $\mathsf{T}_0 \geq \mathsf{M}(\Sigma)$; thus by Theorem 3.1 for any surface in $\Omega \in \mathcal{J}$ and any geodesic Λ in $\mathcal{C}(\Sigma)$ connecting α and β, there is a curve γ in Λ so that γ misses Ω.

Now fix such a Λ and γ. Let Ψ (and Ψ') be the component(s) of $\Sigma \smallsetminus \gamma$.

Claim. Let $\mathsf{A}_\Psi = \mathsf{A}(\Psi, \mathsf{T}_0 - \mathsf{C}, \mathsf{T}_1 + \mathsf{C})$. The number of elements of $\mathcal{J}_\Psi = \{\Omega \in \mathcal{J} \mid \Omega \subsetneq \Psi\}$ is at most

$$\mathsf{A}_\Psi \cdot (\mathsf{T}_1 + \mathsf{C}).$$

By the claim it will suffice to take $\mathsf{A}(\Sigma, \mathsf{T}_0, \mathsf{T}_1)$ equal to

$$(\mathsf{A}_\Psi + \mathsf{A}'_\Psi)(\mathsf{T}_1 + \mathsf{C}) + 3.$$

This is because any element of $\mathcal{J}$ which is disjoint from γ is either a strict subsurface of Ψ or Ψ', an annular neighborhood of γ, or Ψ or Ψ'

themselves. Since every surface in $\mathcal{J}$ is disjoint from some vertex of Λ, the theorem follows from the pigeonhole principle.

It remains to prove the claim. If Ψ is a subsurface of an element of $\mathcal{J}$ there is nothing to prove. Thus we may assume that

$$\mathsf{T}_1 \geq d_\Psi(\alpha, \beta).$$

Let α' and β' be the projections of α and β to Ψ. From the definition of C, $\mathcal{J}_\Psi$ is a $(\mathsf{T}_0 - \mathsf{C}, \mathsf{T}_1 + \mathsf{C})$–antichain for Ψ, α' and β'. Thus,

$$\mathsf{T}_1 \geq d_\Psi(\alpha, \beta) \geq d_\Psi(\alpha', \beta') - \mathsf{C} \geq |\mathcal{J}_\Psi|/\mathsf{A}_\Psi - \mathsf{C}$$

and so

$$\mathsf{T}_1 + \mathsf{C} \geq |\mathcal{J}_\Psi|/\mathsf{A}_\Psi. \qquad \square$$

4. Teichmüller space

We use $\mathcal{T}(\Sigma)$ to denote the Teichmüller space of Σ: the space of complete hyperbolic metrics on the interior of Σ up to isotopy.

There is a uniform bound for the length of the shortest closed curve in any hyperbolic metric on Σ. For any metric σ on Σ, a curve γ has *bounded length* in σ if the length of γ in σ is less than this constant. Let $\mathsf{e}_0 > 0$ be a constant such that, for curves γ and δ, if γ has bounded length in σ and δ has a length less than e_0 then γ and δ have intersection number zero.

Suppose that α and β are vertices of $\mathcal{C}(\Sigma)$. Fix metrics σ and τ in $\mathcal{T}(\Sigma)$ so that α and β have bounded length at σ and τ respectively. Let $\Gamma\colon [t_\sigma, t_\tau] \to \mathcal{T}(S)$ be a geodesic connecting σ to τ. For any curve γ let $l_t(\gamma)$ be the length of its geodesic representative in the hyperbolic metric $\Gamma(t)$. The following theorems are consequences of Theorem 6.2 and Lemma 7.3 in [Raf05].

Theorem 4.1 ([Raf05])**.** *For e_0 as above there exists a threshold $\mathsf{T}_{\min}$ such that, for a strict subsurface Ω of Σ, if $d_\Omega(\alpha, \beta) \geq \mathsf{T}_{\min}$ then there is a time t_Ω so that the length of each boundary component of Ω in $\Gamma(t_\Omega)$ is less than e_0.* $\square$

Theorem 4.2 ([Raf05])**.** *For every threshold T_1, there is a constant e_1 such that of $l_t(\gamma) \leq \mathsf{e}_1$, for some curve γ, then there exists a subsurface Ψ disjoint from γ such that $d_\Psi(\alpha, \beta) \geq \mathsf{T}_1$.* $\square$

The *shadow* of the Teichmüller geodesic Γ to $\mathcal{C}(\Sigma)$ is the set of curves γ, such that γ has bounded length in $\Gamma(t)$ for some $t \in [t_\sigma, t_\tau]$. The following is a consequence of the fact that the shadow is an unparameterized quasi-geodesic. (See Theorem 2.6 and then apply Theorem 2.3 in [MM99].)

Theorem 4.3 ([MM99])**.** *The shadow of a Teichmüller geodesic to $\mathcal{C}(\Sigma)$ does not backtrack and so satisfies the reverse triangle inequality. That is, there exists a backtracking constant $\mathsf{B} = \mathsf{B}(\Sigma)$ such that if $t_\sigma \le t_0 \le t_1 \le t_2 \le t_\tau$ and if γ_i has bounded length in $\Gamma(t_i)$, $i = 0, 1, 2$ then*

$$d_\Sigma(\gamma_0, \gamma_2) \ge d_\Sigma(\gamma_0, \gamma_1) + d_\Sigma(\gamma_1, \gamma_2) - \mathsf{B}.$$

We say that $\Gamma(t)$ is e–*thick* if the shortest closed geodesic γ in $\Gamma(t)$ has a length of at least e.

Lemma 4.4. *For any $\mathsf{e} > 0$ there is a progress constant $\mathsf{P} > 0$ so that if $t_\sigma \le t_0 \le t_1 \le t_\tau$, $\Gamma(t)$ is e–thick at every time $t \in [t_0, t_1]$, and if γ_i has bounded length in $\Gamma(t_i)$, $i = 0, 1$, then*

$$d_\Sigma(\gamma_0, \gamma_1) \asymp_{\mathsf{P}} t_1 - t_0.$$

Proof. Let μ_i be a short markings in $\Gamma(t_i)$, $i = 0, 1$. As above, using Theorem 6.2 and Lemma 7.3 in [Raf05], and the fact that $\Gamma(t)$ is e–thick at every time $t \in [t_0, t_1]$, we can conclude that $d_\Omega(\mu_0, \mu_1)$ is uniformly bounded for any strict subsurface of Ω of Σ. The lemma is then a consequence of Theorem 1.1 and Remark 5.5 in [Raf06]. (Referring to the statement and notation of [Raf06, Theorem 1.1]: take k large enough such that the only non-zero term in the right hand side of [Raf06, Equation (1)] is $d_\Sigma(\mu_0, \mu_1)$). □

In general Γ may stray into the thin part of $\mathcal{T}(S)$. We take $\Gamma^{\ge \mathsf{e}}$ to be the set of times in the domain of Γ which are e–thick. Notice that $\Gamma^{\ge \mathsf{e}}$ is a union of closed intervals. Let $\Gamma(\mathsf{e}, \mathsf{L})$ be the union of intervals of $\Gamma^{\ge \mathsf{e}}$ which have length at least L. We use $|\Gamma(\mathsf{e}, \mathsf{L})|$ to denote the sum of the lengths of the components of $\Gamma(\mathsf{e}, \mathsf{L})$.

Lemma 4.5. *For every e there exists L_0 such that if $\mathsf{L} \ge \mathsf{L}_0$, then*

$$d_\Sigma(\alpha, \beta) \ge |\Gamma(\mathsf{e}, \mathsf{L})|/2\mathsf{P}.$$

Proof. Pick L_0 large enough so that, for $\mathsf{L} \ge \mathsf{L}_0$,

$$(\mathsf{L}/2\mathsf{P}) \ge \mathsf{P} + 2\mathsf{B}.$$

Let $\Gamma(\mathsf{e}, \mathsf{L})$ be the union of intervals $[t_i, s_i]$, $i = 1, \ldots, m$. Let γ_i be a curve of bounded length in $\Gamma(t_i)$ and δ_i be a curve of bounded length in $\Gamma(s_i)$.

By Theorem 4.3 we have

$$d_\Sigma(\alpha, \beta) \ge \left(\sum_i d_\Sigma(\gamma_i, \delta_i) \right) - 2m\mathsf{B}.$$

By Lemma 4.4 we deduce

$$d_\Sigma(\alpha,\beta) \geq \left(\sum_i \frac{1}{\mathsf{P}}(s_i - t_i) - \mathsf{P} \right) - 2m\mathsf{B}.$$

Rearranging, we find

$$d_\Sigma(\alpha,\beta) \geq \frac{1}{\mathsf{P}} |\Gamma(\mathsf{e},\mathsf{L})| - m(\mathsf{P} + 2\mathsf{B}).$$

Thus, as desired:

$$d_\Sigma(\alpha,\beta) \geq \frac{1}{2\mathsf{P}} |\Gamma(\mathsf{e},\mathsf{L})|. \qquad \square$$

5. An estimate of distance

In this section we provide the main estimate for $d_\Sigma(\alpha,\beta)$. Let e_0 be as before. We choose thresholds $\mathsf{T}_0 \geq \mathsf{T}_{\min}$ (see Theorem 4.1) and T_1 so that Lemma 3.2 holds. Let e_1 be the constant provided in Lemma 4.4 and let $\mathsf{e} > 0$ be any constant smaller than $\min(\mathsf{e}_0, \mathsf{e}_1)$. Finally, we pick L_0 such that Lemma 4.5 holds and that $\mathsf{L}_0/2\mathsf{P} > 4$. Let L be any length larger than L_0.

Theorem 5.1. *Let T_0, T_1, e and L be constants chosen as above and assume that $\mathcal{J}$ is a $(\mathsf{T}_0,\mathsf{T}_1)$–antichain for Σ, α and β. Then, there is a constant $\mathsf{K} = \mathsf{K}(\Sigma, \mathsf{T}_0, \mathsf{T}_1, \mathsf{e}, \mathsf{L})$ such that*

$$d_\Sigma(\alpha,\beta) \asymp_{\mathsf{K}} |\mathcal{J}| + |\Gamma(\mathsf{e},\mathsf{L})|.$$

Proof. For $\mathsf{K} \geq \max(2\mathsf{P}, \mathsf{A})$, the inequality

$$d_\Sigma(\alpha,\beta) \succ_{\mathsf{K}} |\mathcal{J}| + |\Gamma(\mathsf{e},\mathsf{L})|$$

follows from Lemmas 3.2 and 4.5. It remains to show that

$$d_\Sigma(\alpha,\beta) \prec_{\mathsf{K}} |\mathcal{J}| + |\Gamma(\mathsf{e},\mathsf{L})|.$$

For each $\Omega \in \mathcal{J}$, fix a time $t_\Omega \in [t_\sigma, t_\tau]$ so that all boundary components of Ω are e_0–short in $\Gamma(t_\Omega)$ (see Theorem 4.1). Let $\mathcal{E}$ be the union:

$$\left\{ t_\Omega \;\middle|\; \Omega \in \mathcal{J},\; t_\Omega \notin \Gamma(\mathsf{e},\mathsf{L}) \right\} \cup \left\{ \partial I \;\middle|\; I \text{ a component of } \Gamma(\mathsf{e},\mathsf{L}) \right\}.$$

We write $\mathcal{E} = \{t_0, \ldots, t_n\}$, indexed so that $t_i < t_{i+1}$.

Claim. The number of intervals in $\Gamma(\mathsf{e},\mathsf{L})$ is at most $|\mathcal{J}| + 1$. Hence, $|\mathcal{E}| \leq 3|\mathcal{J}| + 1$.

Proof. At some time between any consecutive intervals I and J in $\Gamma(\mathsf{e}, \mathsf{L})$ some curve γ becomes e–short (and hence e_1–short). Therefore, by Theorem 4.2, γ is disjoint from a subsurface Ψ where $d_\Psi(\alpha, \beta) \geq \mathsf{T}_1$. Since $\mathcal{J}$ is an $(\mathsf{T}_0, \mathsf{T}_1)$–antichain, Ψ is a subsurface of some element $\Omega \in \mathcal{J}$. It follows that $d_\Sigma(\gamma, \partial\Omega) \leq 2$. This defines a one-to-one map from pairs of consecutive intervals to $\mathcal{J}$. To see the injectivity consider another such pair of consecutive intervals I' and J' and the corresponding curve γ' and subsurface Ω'. By Lemma 4.4, $d_\Sigma(\gamma, \gamma') \geq \mathsf{L}/2\mathsf{P} > 4$, and therefore Ω is not equal to Ω'. □

Let γ_i be a curve of bounded length in $\Gamma(t_i)$.

Claim.

$$d_\Sigma(\gamma_i, \gamma_{i+1}) \leq \begin{cases} \mathsf{P}(t_{i+1} - t_i) + \mathsf{P}, & \text{if } [t_i, t_{i+1}] \subset \Gamma(\mathsf{e}, \mathsf{L}) \\ 2\mathsf{B} + \mathsf{PL} + \mathsf{P} + 2, & \text{otherwise} \end{cases}$$

Proof. The first case follows from Lemma 4.4. So suppose that the interior of $[t_i, t_{i+1}]$ is disjoint from the interior of $\Gamma(\mathsf{e}, \mathsf{L})$.

We define sets $I_+, I_- \subset [t_i, t_{i+1}]$ as follows: A point $t \in [t_i, t_{i+1}]$ lies in I_- if

- there is a curve γ which is e–short in $\Gamma(t)$ and
- for some $\Omega \in \mathcal{J}$, so that $d_\Sigma(\partial\Omega, \gamma) \leq 2$, $t_\Omega \leq t_i$.

If instead $t_\Omega \geq t_{i+1}$ then we place t in I_+. Finally, we place t_i in I_- and t_{i+1} in I_+.

Notice that if $\Omega \in \mathcal{J}$ then t_Ω does not lie in the open interval (t_i, t_{i+1}). It follows that every e–thin point of $[t_i, t_{i+1}]$ lies in I_-, I_+, or both. If $t \in I_-$ and γ is the corresponding e–short curve then $d_\Sigma(\gamma_i, \gamma) \leq \mathsf{B} + 2$. This is because either $t = t_i$ and so γ and γ_i are in fact disjoint, or there is a surface $\Omega \in \mathcal{J}$ as above with

$$2 \geq d_\Sigma(\partial\Omega, \gamma) \geq d_\Sigma(\gamma_i, \gamma) - \mathsf{B},$$

Similarly if $t \in I_+$, $d_\Sigma(\gamma_{i+1}, \gamma) \leq \mathsf{B} + 2$.

If I_+ and I_- have non-empty intersection then $d_\Sigma(\gamma_i, \gamma_{i+1}) \leq 2B + 4$ by the triangle inequality.

Otherwise, there is an interval $[s, s']$ that is e–thick, has length less than L such that $s \in I_-$ and $s' \in I_+$. Let γ and γ' be the corresponding short curves in $\Gamma(s)$ and $\Gamma(s')$. Thus

$$d_\Sigma(\gamma_i, \gamma) \leq \mathsf{B} + 2 \quad \text{and} \quad d_\Sigma(\gamma', \gamma_{i+1}) \leq \mathsf{B} + 2.$$

We also know from Lemma 4.4 that

$$d_\Sigma(\gamma, \gamma') \leq \mathsf{PL} + \mathsf{P}.$$

This finishes the proof of our claim. □

It follows that

$$\begin{aligned} d_\Sigma(\alpha,\beta) &\le d_\Sigma(\gamma_0,\gamma_1)+\ldots+d_\Sigma(\gamma_{n-1},\gamma_n) \\ &\le |\mathcal{E}|(2\mathsf{B}+\mathsf{PL}+\mathsf{P}+2)+\mathsf{P}|\Gamma(\mathsf{e},\mathsf{L})|+|\mathcal{E}|\mathsf{P} \\ &\prec_{\mathsf{K}} |\mathcal{J}|+|\Gamma(\mathsf{e},\mathsf{L})|, \end{aligned}$$

for an appropriate choice of K. This finishes the proof of the theorem. □

6. Symmetric curves and surfaces

Recall that $P\colon \Sigma \to S$ is a covering map.

Definition 6.1. A curve $\alpha \subset \Sigma$ is *symmetric* if there is a curve $a \subset S$ so that $P(\alpha) = a$. We make the same definition for a subsurface $\Omega \subset \Sigma$ lying over a subsurface $Z \subset S$.

For the rest of the paper, fix symmetric curves α and β. Let $x, y \in \mathcal{T}(S)$ be points in the Teichmüller space of S such that $a = P(\alpha)$ has bounded length in x and $b = P(\beta)$ is bounded in y. Let σ and τ be the lifts of x and y respectively. Let G be the Teichmüller geodesic connecting x to y and Γ be the lift of G. Note that Γ is a geodesic in $\mathcal{T}(\Sigma)$. This is because a Teichmüller geodesic is an image of a quadratic differential under the action of

$$\left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Both quadratic differentials and the action of this group are defined locally and therefore lift to the cover.

Proposition 6.2 (Proposition 3.7 [Raf06])**.** *For any* e*, there is a constant* N *such that the following holds. Assume that, for all* $t \in [r,s]$*, there is a component of* $\partial\Omega$ *whose length in* $\Gamma(t)$ *is larger than* e*. Suppose* γ *has bounded length in* $\Gamma(r)$ *and* δ *has bounded length in* $\Gamma(s)$*. Then*

$$d_\Omega(\gamma,\delta) \le \mathsf{N}.$$

Lemma 6.3. *For* e *small enough,* N *as above and any subsurface* $\Omega \subset \Sigma$*, if* $d_\Omega(\alpha,\beta) \ge 2\mathsf{N}+1$*, then* Ω *is symmetric.*

Proof. Consider the first time t^- and last time t^+ that the boundary of Ω is e–short. Since the boundary of Ω is short in Γ at these times, so is its image $P(\partial\Omega)$ in G at the corresponding times. Therefore, all components of the image are simple. (This is a version of the Collar Lemma. See Theorem 4.2.2 of [Bus92].) It follows that the boundary of Ω is symmetric. This is because, choosing e small enough will ensure that curves in $P^{-1}(P(\Omega))$ have bounded length at both t^- and t^+. (The length of each is at most the degree of the covering map times e.) If any such curve γ intersects Ω we have $d_\Omega(\gamma, \alpha) \leq \mathsf{N}$ and $d_\Omega(\gamma, \beta) \leq \mathsf{N}$, contradicting the assumption $d_\Omega(\alpha, \beta) \geq 2\mathsf{N} + 1$. Thus, the subsurface Ω is symmetric. □

7. The quasi-isometric embedding

We now prove the main theorem:

Theorem 7.1. *The covering relation $\Pi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a quasi-isometric embedding. That is, if $P(\alpha) = a$ and $P(\beta) = b$, for $\alpha, \beta \in \mathcal{C}(\Sigma)$ and $a, b \in \mathcal{C}(S)$, then*

$$d_\Sigma(\alpha, \beta) \asymp_{\mathsf{Q}} d_S(a, b),$$

where Q depends on the topology of S and the degree of the covering map only.

Proof. As mentioned before, we only need to show that

$$d_\Sigma(\alpha, \beta) \succ_{\mathsf{Q}} d_S(a, b).$$

Suppose that d is the degree of the covering. We prove the theorem by induction on the complexity of S. In the case where S is an annulus, the cover Σ is also an annulus and the distances in $\mathcal{C}(\Sigma)$ and $\mathcal{C}(S)$ are equal to the intersection number plus one. But, in this case,

$$i(\alpha, \beta) \geq i(a, b)/\mathsf{d}.$$

Therefore, the theorem is true with $\mathsf{Q} = \mathsf{d}$.

Now assume the theorem is true for all subsurfaces of S with the quasi-isometric constant Q'. Choose the threshold T, constant e and length L such that Theorem 5.1 holds for both the data $(S, \mathsf{T}, \mathsf{T}, \mathsf{e}, \mathsf{L})$ as well as $(\Sigma, (\mathsf{T}/\mathsf{Q}') - \mathsf{Q}', \mathsf{T}, \mathsf{e}, \mathsf{L})$. We also assume that $\mathsf{T} \geq 2\mathsf{N} + 1$. Note that all these choices depend on the topology of S and the degree d only.

Let $\mathcal{J}_S$ be the T–antichain of maxima for S, a and b and let $\mathcal{J}_\Sigma$ be the set of lifts of elements of $\mathcal{J}_S$.

Claim. The set $\mathcal{J}_\Sigma$ is a $((\mathsf{T}/\mathsf{Q}') - \mathsf{Q}', \mathsf{T})$–antichain for Σ, α, and β.

We check the conditions for being an antichain. Since elements of $\mathcal{J}_S$ are not subsets of each other, the same holds for their lifts. The condition $d_\Omega(\alpha, \beta) \geq (\mathsf{T}/\mathsf{Q}') - \mathsf{Q}'$ is the induction hypothesis. Now suppose $\Psi \subset \Sigma$ with $d_\Psi(\alpha, \beta) \geq \mathsf{T}$. By Lemma 6.3, Ψ is symmetric. That is, it is a lift of a surface $Y \subset S$ and

$$d_Y(a, b) \geq d_\Psi(\alpha, \beta) \geq \mathsf{T}.$$

This implies that $Y \subset Z$ for some $Z \in \mathcal{J}_S$. Therefore, taking Ω to be the lift of Z, we have $\Psi \subset \Omega \in \mathcal{J}_\Sigma$. This proves the claim.

Hence, there are constants K and K' such that

$$d_S(a,b) \asymp_{\mathsf{K}} |\mathcal{J}_S| + |G(\mathsf{e},\mathsf{L})|,$$

and

$$d_\Sigma(\alpha,\beta) \asymp_{\mathsf{K}'} |\mathcal{J}_\Sigma| + |\Gamma(\mathsf{e},\mathsf{L})|.$$

Note that $|\mathcal{J}_S| \leq \mathsf{d}|\mathcal{J}_\Sigma|$ as a subsurface of S has at most d lifts. Note also that $|G(\mathsf{e},\mathsf{L})| \leq |\Gamma(\mathsf{e},\mathsf{L})|$ because $\Gamma(t)$ is at least as thick as $G(t)$. Therefore

$$d_S(a,b) \prec_{\mathsf{Q}} d_\Sigma(\alpha,\beta),$$

for $\mathsf{Q} = \mathsf{d}\,\mathsf{K}\,\mathsf{K}'$. This finishes the proof. $\square$

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E-mail address: `rafi@math.uconn.edu`
URL: `http://www.math.uconn.edu/~rafi`

E-mail address: `saulsch@math.rutgers.edu`
URL: `http://www.math.rutgers.edu/~saulsch`