COVERS AND THE CURVE COMPLEX

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Abstract. The inclusion of curve complexes, induced by a covering map, is a quasi-isometric embedding.

1. Introduction

Suppose that Σ is a compact connected orientable surface. A simple closed curve $\alpha \subset \Sigma$ is *essential* if α does not bound a disk in Σ. The curve α is *non-peripheral* if α does not bound a once-punctured disk in Σ.

Definition 1.1 (Harvey [2])**.** The *complex of curves* $\mathcal{C}(\Sigma)$ has isotopy classes of essential, non-peripheral curves as its vertices. A collection of $k+1$ vertices spans a k–simplex if every pair of vertices has disjoint representatives.

We are interested in the coarse geometry of $\mathcal{C}(\Sigma)$, since this is closely related to the geometry of both Teichmüller space and the study of Kleinian groups. It will be enough to study only the 1–skeleton of $\mathcal{C}(\Sigma)$, for which we use the same notation. Giving all edges of $\mathcal{C}(\Sigma)$ length one, we denote the distance between vertices by $d_\Sigma(\cdot,\cdot)$.

As a bit of notation, if A, B, c are non-negative real numbers, with $c \geq 1$, and if

$$A \leq cB + c$$

then we write $A \prec B$. If $A \prec B$ and $B \prec A$ then we write $A \asymp B$. The number c is always some constant uniform over a family of (A, B) pairs.

Let Σ and S be two compact connected orientable and let $p\colon \Sigma \to S$ be a covering map. This defines a relation between the corresponding complexes of curves. That is, $a \in \mathcal{C}(S)$ is related to $\alpha \in \mathcal{C}(\Sigma)$ if $p(\alpha) = a$. Abusing notation, we write this as $p^*\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$. Our goal is:

Theorem 6.1. *The relation $p^*\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a quasi-isometric embedding. That is, if $p(\alpha) = a$ and $p(\beta) = b$, for $\alpha, \beta \in \mathcal{C}(\Sigma)$ and*

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$a, b \in \mathcal{C}(S)$, *then*

$$d_\Sigma(\alpha, \beta) \asymp d_S(a, b).$$

Note that one of the inequalities, namely $d_\Sigma(\alpha, \beta) \leq d_S(a, b)$, follows immediately because disjoint curves in S have disjoint lifts in Σ. The opposite inequality is harder to obtain and occupies the rest of the paper.

In a slightly different situation, where Σ is the orientation double cover of a nonorientable S, Theorem 6.1 is due to Masur-Schleimer [5].

2. Subsurface projection

Suppose that Σ is a compact connected orientable surface. A subsurface Ψ is *cleanly embedded* if all components of $\partial\Psi$ are essential and whenever $\gamma \subset \partial\Psi$ is istopic to $\delta \subset \partial\Sigma$ then $\gamma = \delta$. All subsurfaces we consider will be cleanly embedded.

We now recall the definition of the *subsurface projection* relation $\pi_\Psi : \mathcal{C}(\Sigma) \to \mathcal{C}(\Psi)$, supposing that Ψ is not an annulus. Fix a hyperbolic metric on the interior of Σ. Let Σ' be the Gromov compactification of the cover of Σ corresponding to the inclusion $\pi_1(\Psi) \to \pi_1(\Sigma)$. Thus $\Sigma' \cong \Psi$ and we identify $\mathcal{C}(\Sigma')$ with $\mathcal{C}(\Psi)$. For any $\alpha \in \mathcal{C}(\Sigma)$ let α' be the closure of the preimage of α in Σ'. Let α'' be any component of α' which is not properly isotopic into the boundary. (If none exist then α is not related to any vertex of $\mathcal{C}(\Psi)$ and we write $\pi_\Psi(\alpha) = \emptyset$.) Let N be a closed regular neighborhood of $\alpha'' \cup \partial\Sigma'$. Fix attention on α''', any boundary component of N which is essential and non-peripheral. Then $\alpha \in \mathcal{C}(\Sigma)$ is related to $\alpha''' \in \mathcal{C}(\Psi)$ and we write $\pi_\Psi(\alpha) = \alpha'''$.

If Ψ is an annulus, then the definition of $\mathcal{C}(\Psi)$ is altered. Vertices are proper isotopy classes of essential arcs in Ψ. Edges are placed between vertices with representatives having disjoint interiors. The projection map is defined as above, omitting the final steps involving the regular neighborhood N.

If Ψ is a four-holed sphere or once-holed torus then the curve complex of Ψ is the well-known *Farey graph*: since all curves intersect, edges are instead placed between curves that intersect exactly twice or exactly once. The definition of π_Ψ is as in the non-annular case.

We say that $\alpha \in \mathcal{C}(\Sigma)$ *cuts* the subsurface Ψ if $\pi_\Psi(\alpha) \neq \emptyset$. Otherwise we say that α *misses* Ψ. Suppose now that $\alpha, \beta \in \mathcal{C}(\Sigma)$ both cut Ψ. We define the *projection distance* to be

$$d_\Psi(\alpha, \beta) = d_\Psi(\pi_\Psi(\alpha), \pi_\Psi(\beta)).$$

Here is the Bounded Geodesic Image Theorem:

Theorem 2.1 (Masur-Minsky [4]). *Fix a surface Σ. There is a constant $\mathsf{B} = \mathsf{B}(\Sigma)$ so that for any vertices $\alpha, \beta \in \mathcal{C}(\Sigma)$, for any geodesic $\Gamma \subset \mathcal{C}(\Sigma)$ connecting α to β, and for any $\Omega \subsetneq \Sigma$, if $d_\Omega(\alpha, \beta) \geq \mathsf{B}$ then there is a vertex of Γ which misses Ω.*

Fix α and β in $\mathcal{C}(\Sigma)$. Fix *threshholds* $\mathsf{T}_0 > 0$ and $\mathsf{T}_1 > 0$. We say that

$$\mathcal{J} = \mathcal{J}(\Sigma, \alpha, \beta, \mathsf{T}_0, \mathsf{T}_1),$$

a set of subsurfaces $\Omega \subsetneq \Sigma$, is an *antichain* if $\mathcal{J}$ satisfies the following properties.

- If $\Omega, \Omega' \in \mathcal{J}$ then Ω is not a subsurface of Ω'.
- If $\Omega \in \mathcal{J}$ then $d_\Omega(\alpha, \beta) \geq \mathsf{T}_0$.
- If $\Omega \in \mathcal{J}$ and $\Omega \subsetneq \Psi \subsetneq \Sigma$, then $d_\Psi(\alpha, \beta) \leq \mathsf{T}_1$.

Notice that there may be many different antichains for the given data $(\Sigma, \alpha, \beta, \mathsf{T}_0, \mathsf{T}_1)$. One particularly nice example is when $\mathsf{T}_0 < \mathsf{T}_1$ and $\mathcal{J}$ is defined to be the maxima of the set

$$\mathcal{B} = \{\Omega \subsetneq \Sigma \mid d_\Omega(\alpha, \beta) \geq \mathsf{T}_0\}$$

as ordered by inclusion. We call this an *antichain of maxima*. By $|\mathcal{J}|$ we mean the number of elements of $\mathcal{J}$.

We may now state and prove our first result.

Theorem 2.2. *For every surface Σ and for every pair of sufficiently large threshholds $\mathsf{T}_0, \mathsf{T}_1$ there is an* accumulation constant $\mathsf{A} = \mathsf{A}(\Sigma, \mathsf{T}_0, \mathsf{T}_1)$ *so that, if $\mathcal{J}$ is an antichain then*

$$d_\Sigma(\alpha, \beta) \geq |\mathcal{J}|/\mathsf{A}.$$

Proof. We prove the theorem by induction: for the Farey graph it suffices for both threshholds to be larger than 3 and then $\mathsf{A} = 1$.

Let C be a constant so that the following holds: If $\Omega \subset \Psi \subset \Sigma$ and α', β' are the projections of α, β to Ψ then

$$|d_\Omega(\alpha, \beta) - d_\Omega(\alpha', \beta')| \leq C.$$

In the general case, we take the threshholds large enough so that:

- the theorem still applies to any strict subsurface Ψ with threshholds $\mathsf{T}_0 - C, \mathsf{T}_1 + C$, and
- $\mathsf{T}_0 \geq \mathsf{B}(\Sigma)$; thus by Theorem 2.1 any surface in $\mathcal{J}$ is disjoint from some curve γ of the geodesic in $\mathcal{C}(\Sigma)$ connecting α and β.

Now fix such a curve γ. Let Ψ (and Ψ') be the component(s) of $\Sigma \smallsetminus \gamma$.

Claim. Let $\mathsf{A}_\Psi = \mathsf{A}(\Psi, \mathsf{T}_0 - C, \mathsf{T}_1 + C)$. The number of elements of $\mathcal{J}_\Psi = \{\Omega \in \mathcal{J} \mid \Omega \subsetneq \Psi\}$ is at most

$$\mathsf{A}_\Psi \cdot (\mathsf{T}_1 + C).$$

By the claim it will suffice to take $\mathsf{A}(\Sigma, \mathsf{T}_0, \mathsf{T}_1)$ equal to

$$(\mathsf{A}_\Psi + \mathsf{A}'_\Psi)(\mathsf{T}_1 + C) + 3.$$

This is because any element of $\mathcal{J}$ which is disjoint from γ is either a strict subsurface of Ψ or Ψ', an annular neighborhood of γ, or Ψ or Ψ' themselves. Since every surface in $\mathcal{J}$ is disjoint from some curve γ of the geodesic connecting α and β, the theorem follows from the pigeonhole principle.

It remains to prove the claim. If Ψ is a subsurface of an element of $\mathcal{J}$ there is nothing to prove. Likewise, if Ψ contains no elements of $\mathcal{J}$ then there is nothing to prove.

Thus we may assume that

$$\mathsf{T}_1 \geq d_\Psi(\alpha, \beta).$$

Let α' and β' be the projections of α and β to Ψ. From the definition of C, $\mathcal{J}_\Psi$ satisfies the conditions of the theorem with the data $(\Psi, \alpha', \beta', \mathsf{T}_0 - C, \mathsf{T}_1 + C)$. Thus,

$$\mathsf{T}_1 \geq d_\Psi(\alpha, \beta) \geq d_\Psi(\alpha', \beta') - C \geq |\mathcal{J}_\Psi|/\mathsf{A}_\Psi - C$$

and so

$$\mathsf{T}_1 + C \geq |\mathcal{J}_\Psi|/\mathsf{A}_\Psi.$$

□

3. Teichmüller space

We also need several definitions involving $\mathcal{T}(\Sigma)$, the Teichmüller space of interior(Σ). Suppose that α, β are vertices of $\mathcal{C}(\Sigma)$. Fix σ, τ points of $\mathcal{T}(\Sigma)$ so that α and β have bounded length at σ and τ respectively. Let $\Gamma\colon [t_\sigma, t_\tau] \to \mathcal{T}(S)$ be a geodesic connecting σ to τ. For any curve γ let $l_t(\gamma)$ be the length of the geodesic representative of γ in the surface Γ_t.

By the work of the first author, there are constants $\epsilon_1 \geq \epsilon_2$, both smaller than the Margulis constant and depending only on Σ, so that for any subsurface Ω there is a (possibly empty) *accessible* interval $I_\Omega \subset \text{domain}(\Gamma)$ with the following properties:

- For all $t \in I_\Omega$, every component $\gamma \subset \partial\Omega \setminus \partial\Sigma$ satisfies $l_t(\gamma) \leq \epsilon_2$.
- For all $t \notin I_\Omega$ there is a component $\gamma \subset \partial\Omega \setminus \partial\Sigma$ which satisfies $l_t(\gamma) \geq \epsilon_1$.
- If γ is essential and non-peripheral in Ω then there is a $t \in I_\Omega$ so that $l_t(\gamma) \geq \epsilon_1$.
- Furthermore, there is a constant D depending only on Σ so that if $d_\Omega(\alpha, \beta) \geq \mathsf{D}$ then I_Ω is non-empty.

We say that Γ_t is ϵ*–thick* if the shortest closed geodesic γ in Γ_t has length at least ϵ.

Theorem 3.1. *Fix Σ. For any $\epsilon > 0$ there is a constant $\mathsf{D} > 0$ so that for any α, β and $\Gamma \colon [t_\sigma, t_\tau] \to \mathcal{T}(S)$ as above, if Γ_t is ϵ–thick at every time t then*

$$\mathsf{D}(t_\tau - t_\sigma) + \mathsf{D} \geq d_\Sigma(\alpha, \beta) \geq \frac{1}{\mathsf{D}}(t_\tau - t_\sigma) - \mathsf{D}.$$

□

In general Γ may stray into the thin part of $\mathcal{T}(S)$. We take $\Gamma^{\geq \epsilon}$ to be the set of times in the domain of Γ which are ϵ–thick. Notice that $\Gamma^{\geq \epsilon}$ is a union of closed intervals. Let $\Gamma(\epsilon, L)$ be the union of intervals of $\Gamma^{\geq \epsilon}$ which have length at least L. We use $|\Gamma(\epsilon, L)|$ to denote the sum of the lengths of the components of $\Gamma(\epsilon, L)$.

Theorem 3.2.

$$d_\Sigma(\alpha, \beta) \succ |\Gamma(\epsilon, L)|.$$

Proof. Sketch: Pick an interval $[s, t] \in \Gamma(\epsilon, L)$. Let γ_s, γ_t be a pair of bounded length curves in Γ_s and Γ_t. Note that $d_\Sigma(\gamma_s, \gamma_t) \asymp t - s$, by Theorem 3.1. Also, by Masur-Minsky [3], the bounded length curves of Γ_t, as t varies, forms a unparametrized quasi-geodesic in $\mathcal{C}(\Sigma)$. In particular, these curves do not backtrack. It follow that

$$d_\Sigma(\alpha, \beta) \succ \sum_{[s,t] \in \Gamma(\epsilon, L)} t - s.$$

□

4. An estimate of distance

Theorem 4.1. *For any surface Σ, $\exists \mathsf{T}_0 > 0$ $\forall \mathsf{T} \geq \mathsf{T}_0$ $\exists \epsilon_0 > 0$ $\forall \epsilon \in (0, \epsilon_0]$ $\exists L_0 > 0$ $\forall L \geq L_0$ so that for any α, β, and Teichmüller geodesic Γ as above we have*

$$d_\Sigma(\alpha, \beta) \asymp |\mathcal{J}| + |\Gamma(\epsilon, L)|.$$

Here $\mathcal{J}$ is the antichain of maxima with both threshholds equal to T.

Proof. The inequality

$$d_\Sigma(\alpha, \beta) \succ |\mathcal{J}| + |\Gamma(\epsilon, L)|$$

follows Theorems 2.2 and 3.2. This completes the proof in one direction. It remains to show that

$$d_\Sigma(\alpha, \beta) \prec |\mathcal{J}| + |\Gamma(\epsilon, L)|.$$

Let $\mathcal{E}$ be the set of endpoints of all intervals in $\Gamma(\epsilon, L)$ and of all intervals I_Ω where $\Omega \in \mathcal{J}$. Write $\mathcal{E} = \{t_0, \ldots, t_n\}$, indexed so that $t_i < t_{i+1}$. Note that $|\mathcal{E}| \prec |\mathcal{J}|$.

Remark 4.2. To see this: we choose ϵ_0 sufficently small so that if $l_t(\gamma) \leq \epsilon_0$ then there exists a subsurface Ψ disjoint from γ where $d_\Psi(\alpha, \beta) \geq \mathsf{T}_0$. This Ψ is contained in some element $\Omega \in \mathcal{J}$. We then choose L_0 large enough so that any $\Omega \in \mathcal{J}$ is so associated to at most two endpoints of intervals in $\Gamma(\epsilon, L)$.

Let γ_i be a curve of bounded length in Γ_{t_i}.

Claim 4.3.

$$d_\Sigma(\gamma_i, \gamma_{i+1}) \asymp \begin{cases} t_{i+1} - t_i, & \text{if } [t_i, t_{i+1}] \subset I \in \Gamma(\epsilon, L) \\ 1, & \text{otherwise.} \end{cases}$$

Proof of Claim. Sketch: The first case follows from Theorem 3.1. So suppose that the interior of $I = [t_i, t_{i+1}]$ is disjoint from the interior of $\Gamma(\epsilon, L)$. If I is contained in an interval I_Ω, for $\Omega \in \mathcal{J}$ then $d_\Sigma(\gamma_i, \gamma_{i+1}) \leq 2$ and we are done. So suppose that I is also disjoint from the interior of all I_Ω, $\Omega \in \mathcal{J}$.

Then I is a union of intervals

$$I_1 \cup I_{\Psi_1} \cup \ldots \cup I_{\Psi_{n-1}} \cup I_n$$

where the I_k lie in the ϵ–thick part of $\mathcal{T}(\Sigma)$ and the I_{Ψ_k} are accessible intervals for the subsurface Ψ_k. These intervals may overlap (and some of the I_k may be empty) but they do cover I. This is essentially due to Remark 4.2. Let δ_k be a boundary component of Ψ_k.

By assumption $\Psi_k \notin \mathcal{J}$ for all k. Since $\mathcal{J}$ is an antichain of maxima, every Ψ_k is a subsurface of some $\Omega_k \in \mathcal{J}$. Let k be the last index so that $\sup I_{\Omega_k} \leq t_i$. Since $\pi_\Sigma(\Gamma)$ does not backtrack, it follows that γ_i lies close to the geodesic (in $\mathcal{C}(\Sigma)$) between $\partial\Omega_k$ and δ_k. But these are disjoint and we conclude that $d_\Sigma(\gamma_i, \delta_k) = O(1)$. Now, the same reasoning applies to δ_{k+1}, a boundary component of Ψ_{k+1}, except this time we find that $\inf I_{\Omega_{k+1}} \geq t_{i+1}$. Thus $d_\Sigma(\delta_{k+1}, \gamma_{i+1}) = O(1)$.

Finally, either I_{Ψ_k} and I_{Ψ_k} overlap, and so $d_\Sigma(\delta_k, \delta_{k+1}) = 1$, or the two intervals are separated by I_{k+1} lying in the ϵ–thick part of $\mathcal{T}(\Sigma)$. However, we have assumed that I_k does not lie in $\Gamma(\epsilon, L)$ and so $|I_k| \leq L$. Thus $d_\Sigma(\delta_k, \delta_{k+1}) = O(1)$ by Theorem 3.1. This completes the proof of the claim. □

It follows that

$$\begin{aligned} d_\Sigma(\alpha, \beta) &\leq d_\Sigma(\gamma_0, \gamma_1) + \ldots + d_\Sigma(\gamma_{n-1}, \gamma_n) \\ &\prec 2n + |\Gamma(\epsilon, L)| \\ &\prec |\mathcal{J}| + |\Gamma(\epsilon, L)| \end{aligned}$$

This completes the proof of Theorem 4.1. □

5. Symmetric curves and surfaces

Definition 5.1. A curve $\alpha \subset \Sigma$ is *symmetric* if there is a curve $a \subset S$ so that $p(\alpha) = a$. We make the same definition for a subsurface $\Omega \subset \Sigma$ lying over a subsurface $Z \subset S$.

For the rest of the paper, fix symmetric curves α and β. Let $x, y \in \mathcal{T}(S)$ be points in the Teichmüller space of S such that a has bounded length in x and b is bounded in y. Let σ and τ be the lifts of x and y respectively. Let G be the Teichmüller geodesic connecting x to y and Γ be the lift.

Theorem 5.2. *There is a constant K such that, for any subsurface $\Omega \subset \Sigma$, if $d_\Omega(\alpha, \beta) \geq K$, then Ω is symmetric.*

Proof. Consider the interval $I = I_\Omega$ given above. Since the boundary of Ω is short, so is its image $p(\partial\Omega)$ in S during the corresponding interval. Therefore, all components of the image are simple. (This is a version of the "Collar Lemma". See Theorem 4.2.2 of [1].) It follows that the boundary of Ω is symmetric. Curves in $p^{-1}(p(\Omega))$ can not intersect Ω, because of the third property of I_Ω. Thus, the subsurface Ω is symmetric. □

6. The quasi-isometric embedding

We are now equipped to prove:

Theorem 6.1. *The relation $p^* \colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a quasi-isometric embedding. That is, if $p(\alpha) = a$ and $p(\beta) = b$, for $\alpha, \beta \in \mathcal{C}(\Sigma)$ and $a, b \in \mathcal{C}(S)$, then*

$$d_\Sigma(\alpha, \beta) \asymp d_S(a, b).$$

Proof. Here is a rough sketch.

Suppose that D is the degree of the covering. $d_S(a, b)$ is less than $|\mathcal{J}| + |G(\epsilon, L)|$ where $\mathcal{J}$ is the antichain of maxima with a large enough threshhold. Let $\mathcal{J}'$ be the set of components of covers of elements of $\mathcal{J}$. (So $\mathcal{J}'$ is at most D times larger that $\mathcal{J}$.) The projection distance of an element $\Omega \in \mathcal{J}'$ may be smaller than that of the surface, Z, which Ω covers, but applying the theorem inductively to Z there is a multiplicative bound on how much the projection distance drops. So, we may take the lower and upper threshholds for $\mathcal{J}$ to be at least this multiplicative constant times the lower bound for all threshholds for Σ. Taking the upper and lower threshholds of $\mathcal{J}'$ to be equal, we find that $\mathcal{J}'$ is an antichain for Σ and so $d_\Sigma(\alpha, \beta) \succ |\mathcal{J}'|$.

Notive that Γ_t is at least as thick as G_t. It follows that $|\Gamma(\epsilon, L)| \geq |G(\epsilon, L)|$. Applying the theorems above completes the proof. $\square$

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