WALDHAUSEN'S THEOREM

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ABSTRACT. This note is an exposition of Waldhausen's proof of Waldhausen's Theorem: the three-sphere has a single Heegaard splitting, up to isotopy, in every genus. As a necessary step we also give a sketch of the Reidemeister-Singer Theorem.

1. INTRODUCTION

Waldhausen's Theorem [Wal68] tells us that Heegaard splittings of the three-sphere are unique up to isotopy. This is an important tool in low-dimensional topology and there are several modern proofs [ST94, RS96, JR06, Rie06]. Additionally, at least two survey articles on Heegaard splittings [Sch02, Joh07] include proofs of Waldhausen's Theorem.

This note is intended as an exposition of Waldhausen's original proof, as his techniques are still of interest. See, for example, Bartolini and Rubinstein's [BR06] classification of one-sided splittings of $\mathbb{RP}^3$.

In Section 2 we recall foundational material, set out the necessary definitions and give a precise statement of Waldhausen's Theorem. Section 3 is devoted to *stable equivalence* of splittings and a proof of the Reidemeister-Singer Theorem. In Section 4 we discuss Waldhausen's *good* and *great* systems of meridian disks. Section 5 gives the proof of Waldhausen's Theorem. Finally, Section 6 is a brief account of the work-to-date on the questions raised by Waldhausen in Section 4 of his paper.

Acknowledgments. The material in this paper's Sections 4 and 5 do not pretend to any originality: often I simply translate Waldhausen's paper directly. I also closely follow, in places word for word, a copy of handwritten and typed notes from the a seminar on Waldhausen's proof, held at the University of Wisconsin at Madison, Fall 1967. The notes were written by Ric Ancel, Russ McMillan and Jonathan Simon. The speakers at the seminar included Ancel and Simon.

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2. Foundations

The books by Hempel [Hem76] and Rolfsen [Rol76] and also Hatcher's notes [Hat01] are excellent references on three-manifolds. Moise's book [Moi77] additionally covers foundational issues in PL topology, as does the book by Rourke and Sanderson [RS72].

We will use M to represent a connected compact orientable three-manifold. We say M is *closed* if the boundary ∂M is empty. A *triangulation* of M is a simplicial complex $\mathcal{K}$ so that the underlying space $||\mathcal{K}||$ is homeomorphic to M. When no confusion can arise, we will regard the cells of $||\mathcal{K}||$ as being subsets of M.

Example 2.1. The three-sphere is given by
$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}.$$
The boundary of the four-simplex gives a five-tetrahedron triangulation of S^3.

Requiring that M be given with a triangulation is not a restriction:

Theorem 2.2 (Triangulation)**.** *Every compact three-manifold M admits a triangulation.* □

Furthermore, in dimension three there is only one PL structure:

Theorem 2.3 (Hauptvermutung)**.** *Any two triangulations of M are related by a PL homeomorphism that is isotopic to the identity in M.* □

These theorems are due to Moise [Moi52, Moi77]. (See also [Sha84].) An alternative proof is given by Bing [Bin59]. Our version of the Hauptvermutung may be found in Hamilton [Ham76].

We now return to notational issues. We will use F to represent a closed connected orientable surface embedded in M. A simple closed curve $\alpha \subset F$ is *essential* if α does not bound a disk in F.

For any $X \subset M$ we use $U(X)$ to denote a regular open neighborhood of X, taken in M. This neighborhood is assumed to be small with respect to everything relevant. If X is a topological space, we use $|X|$ to denote the number of components of X.

A *handlebody*, usually denoted by V or W, is a homeomorph of a closed regular neighborhood of a finite, connected graph embedded in

$\mathbb{R}^3$. The *genus* of V agrees with the genus of ∂V. Notice that if $\mathcal{K}$ is a triangulation of M then a closed regular neighborhood of the one-skeleton of $||\mathcal{K}||$ is a handlebody embedded in M.

A disk v_0 is *properly embedded* in a handlebody V if $v_0 \cap \partial V = \partial v_0$; this definition generalizes naturally to surfaces and arcs contained in bounded three-manifolds and also to arcs contained in bounded surfaces.

A *Heegaard splitting* is a pair (M, F) where M is a closed oriented three-manifold, F is an oriented closed surface embedded in M, and $M \smallsetminus U(F)$ is a disjoint union of handlebodies.

Example 2.4. There is an equatorial two-sphere $S^2 \subset S^3$:
$$S^2 = \{(z, w) \in S^3 \mid \mathrm{Im}(w) = 0\}.$$
Note that S^2 bounds a three-ball on each side. We call (S^3, S^2) the *standard* splitting of genus zero.

The Alexander trick proves that any three-manifold with a splitting of genus zero is homeomorphic to standard splitting of genus zero. Furthermore, we have:

Theorem 2.5 (Alexander [Ale24])**.** *Every PL two-sphere in S^3 bounds three-balls on both sides.* $\qquad\square$

See [Hat01] for a detailed and accessible proof in the differentiable category. It follows that every PL two-sphere gives a Heegaard splitting of S^3.

Example 2.6. There is a torus $T \subset S^3$:
$$T = \{(z, w) \in S^3 \mid |z| = |w| = 1\}.$$
It is an exercise to check that T bounds a solid torus $(D^2 \times S^1)$ on each side. We call (S^3, T) the standard splitting of S^3 of genus one.

The three-manifolds admitting splittings of genus one are S^3, $S^2 \times S^1$ and the lens spaces. As an easy exercise from the definitions we have:

Lemma 2.7. *Suppose $\mathcal{K}$ is a triangulation of a closed orientable manifold M. Suppose that F is the boundary of a closed regular neighborhood of the one-skeleton of $||\mathcal{K}||$. Then (M, F) is a Heegaard splitting.* $\quad\square$

See, for example, page 241 of [Rol76]. The splitting (M, F) so given is the splitting *associated* to the triangulation $\mathcal{K}$. As an immediate consequence of the Triangulation Theorem (2.2) and Lemma 2.7 we find that every closed three-manifold has infinitely many Heegaard splittings. To control this extravagance of examples we make:

Definition 2.8. A pair of Heegaard splittings (M, F) and (M, F') are *equivalent*, written $(M, F) \approx (M, F')$, if there is a homeomorphism $h \colon M \to M$ such that

- h is isotopic to the identity and
- $h|F$ is an orientation preserving homeomorphism from F to F'.

It is an important visualization exercise to show that (S^3, T) is equivalent to $(S^3, -T)$. Here $-T$ is the torus T equipped with the opposite orientation. In general, (M, F) is *not* equivalent to $(M, -F)$; for examples see the discussion of lens spaces at the end of Section 6. We now have another foundational theorem:

Theorem 2.9 (Gugenheim [Gug53]). *If B and B' are PL three-balls in a three-manifold M then there is an isotopy of M carrying B to B'.* $\qquad\square$

See Theorem 3.34 of Rourke and Sanderson [RS72] for a discussion. They also give as Theorem 4.20 a relative version. In any case, it follows that all genus zero splittings of S^3 are equivalent to the standard one, so justifying the name.

Exercise 2.10. Show that any genus one splitting of S^3 is isotopic to the standard one. (Corollary 4.16 of [RS72] may be useful.)

Waldhausen's Theorem generalizes this result to every genus:

Theorem 5.1. *If (S^3, F) and (S^3, F') are Heegaard splittings of equal genus then (S^3, F) is equivalent to (S^3, F').*

Remark 2.11. Waldhausen's original statement is even simpler:

  Wir zeigen, daß es nur die bekannten gibt.

That is: "We show that only the well-known [splittings of S^3] exist."

3. Stabilization and the Reidemeister-Singer Theorem

A key step in Waldhausen's proof is the Reidemeister-Singer Theorem (Theorem 3.6, below). In this section we lay out the necessary definitions and sketch a proof of the Reidemeister-Singer Theorem. Most approaches to Reidemeister-Singer, including ours, are via piecewise linear topology. Bonahon in an unpublished manuscript has given a proof relying on Morse theory.

For further details and the history of the problem we refer the reader to the original papers of Reidemeister and Singer [Rei33, Sin33] as well as the more modern treatment by Craggs [Cra76]. A version of Craggs' proof is also given in [FM97, Theorem 5.2]. Note also that Lei [Lei00],

in an amusing reversal, gives a very short proof of the Reidemeister-Singer Theorem by assuming Waldhausen's Theorem.

We begin by stating the basic definitions and then the theorem.

Definition 3.1. Suppose that V is a handlebody. A properly embedded arc $\alpha \subset V$ is *unknotted* if there is an arc $\beta \subset \partial V$ and an embedded disk $B \subset V$ so that $\partial \alpha = \partial \beta$ and $\partial B = \alpha \cup \beta$.

Definition 3.2. Suppose that (M, F) is a Heegaard splitting with handlebodies V and W. Let α be an unknotted arc in V. Let $F' = \partial(V \smallsetminus U(\alpha)) = (V \smallsetminus U(\alpha)) \cap (W \cup \overline{U(\alpha)})$. Then the pair (M, F') is a *stabilization* of F in M. Also, the pair (M, F) is a *destabilization* of (M, F').

Observe that (S^3, T) is isotopic to a stabilization of (S^3, S^2). It is an exercise to prove, using the relative version of Theorem 2.9 and Exercise 2.10, that if (M, F') and (M, F'') are stabilizations of (M, F), then $(M, F') \approx (M, F'')$. On the other hand, as discussed below, destabilization need *not* be a unique operation.

Recall that the *connect sum* $M \# N$ is obtained by removing the interior of a ball from each of M and N and then identifying the resulting boundary components via an orientation reversal.

Definition 3.3. Let (M, F) be a Heegaard splitting. Let (S^3, T) be the standard genus one splitting of S^3. Pick embedded three-balls meeting $F \subset M$ and $T \subset S^3$ in disks. The *connect sum* of the splittings is the connect sum of pairs: $(M, F) \# (S^3, T) = (M \# S^3, F \# T)$.

Again, this operation is unique and the proof is similar to that of uniqueness of stabilization. This is not a surprise, as stabilization and connect sum with (S^3, T) produce equivalent splittings. Thus we do not distinguish between them notationally.

Remark 3.4. Fix a manifold M. We may construct a graph $\Sigma(M)$ where vertices are equivalence classes of splittings and edges correspond to stabilizations. From Theorem 2.2 it follows that $\Sigma(M)$ is nonempty. The uniqueness of stabilization implies that $\Sigma(M)$ has no cycles and so is a forest. Finally, $\Sigma(M)$ is infinite because splittings of differing genera cannot be isotopic.

Define $(M, F) \#_n (S^3, T) = ((M, F) \#_{n-1} (S^3, T)) \# (S^3, T)$.

Definition 3.5. Two splittings, (M, F) and (M, F'), are *stably equivalent* if there are $m, n \in \mathbb{N}$ so that $(M, F) \#_m (S^3, T) \approx (M, F') \#_n (S^3, T)$.

We now may state:

Theorem 3.6 (Reidemeister-Singer)**.** *Suppose that M is a closed, connected, orientable three-manifold. Then any two Heegaard splittings of M are stably equivalent.*

Remark 3.7. The theorem may be restated as follows: $\Sigma(M)$ is connected. Since Remark 3.4 shows that $\Sigma(M)$ is a forest, it is a tree.

We say that (M, F) is *unstabilized* if it is not equivalent to a stabilized splitting. Waldhausen calls such splittings "minimal". However modern authors reserve "minimal" to mean *minimal genus*. This is because there are manifolds containing unstabilized splittings that are not of minimal genus. For examples, see Sedgwick's discussion of splittings of Seifert fibered spaces [Sed99]. Note that unstabilized splittings correspond to leaves of the tree $\Sigma(M)$.

Finally, there are fixed manifolds that contain unstabilized splittings of arbitrarily large genus. The first such examples are due to Casson and Gordon [CG85]. The papers [Kob92, LM00, MSS06] contain generalizations.

We now set out the tools necessary for our proof of Theorem 3.6. A *pseudo-triangulation* $\mathcal{T} = \{\Delta_i\}$ of a three-manifold M is a collection of tetrahedra together with face identifications. We require that the resulting quotient space $||\mathcal{T}||$ be homeomorphic to M and that every open cell of $\mathcal{T}$ embeds. We do not require that $\mathcal{T}$ be a simplicial complex. It is a pleasant exercise to find all pseudo-triangulations of S^3 consisting of a single tetrahedron.

As with triangulations, if $\mathcal{T}$ is a pseudo-triangulation of M then the boundary of a closed regular neighborhood of the one-skeleton of $||\mathcal{T}||$ is a Heegaard splitting of M. Notice that the second barycentric subdivision of $\mathcal{T}$ is a triangulation of M.

Lemma 3.8. *For any splitting (M, F) there is an $n \in \mathbb{N}$ and a triangulation $\mathcal{K}$ of M so that $(M, F)\#_n(S^3, T)$ is associated to $\mathcal{K}$.*

Proof. We may assume, stabilizing if necessary, that F has genus at least one. Now, F cuts M into a pair of handlebodies V and W, both of genus g. Choose g disks $\{v_i\}$ properly embedded in V so that the v_i cut V into a ball. Choose $\{w_j\}$ in W similarly. After a proper isotopy of the v_i inside of V, increasing $|(\cup v_i) \cap (\cup w_j)|$ as necessary, we may assume that all components of $F \smallsetminus \Gamma$ are disks. Here $\Gamma = F \cap ((\cup v_i) \cup (\cup w_j))$ is the *Heegaard diagram* of (F, v_i, w_j).

We build a pseudo-triangulation $\mathcal{T}$ of M, with exactly two vertices, by taking the dual of the two-dimensional CW complex $F \cup (\cup v_i) \cup (\cup w_j)$. Since every vertex of Γ is a transverse intersection of $\cup \partial v_i$ and $\cup \partial w_j$ in F every three-cell of $\mathcal{T}$ is a tetrahedron. Since every edge of Γ

has degree three in $F \cup (\cup v_i) \cup (\cup w_j)$ every two-cell of $\mathcal{T}$ is a triangle. Finally, we have edges of $\mathcal{T}$ for every face of Γ and for each of the $2g$ disks. Also there are exactly two vertices.

Let $\mathcal{T}_V$ be the union of the edges of $\mathcal{T}$ dual to the disks v_i. Define $\mathcal{T}_W$ similarly. Let e be any edge of $\mathcal{T}$ connecting the two vertices of $\mathcal{T}^0$. Notice that F is isotopic to the boundary of a regular neighborhood of $\mathcal{T}_V$. After g stabilizations of F we obtain a surface F' that is isotopic to the boundary of a regular neighborhood of $\mathcal{T}_V \cup e \cup \mathcal{T}_W$. Now a further sequence of stabilizations of F' gives the splitting associated to $\mathcal{T}$. We end with an easy exercise: if a splitting (M, G) is associated to a pseudo-triangulation $\mathcal{T}$ then some stabilization of G is associated to the second barycentric subdivision of $\mathcal{T}$. $\qquad\square$

We now describe the 1/4 and 2/3 bistellar flips in dimension three. These are also often called *Pachner moves*. In any triangulation, the 1/4 flip replaces one tetrahedron by four; add a vertex at the center of the chosen tetrahedron and cone to the faces. Similarly the 2/3 flip replaces a pair of distinct tetrahedra, adjacent along a face, by three; remove the face, replace it by a dual edge, and add three faces. The 4/1 and 3/2 flips are the reverses. See Figure 1 for illustrations of the 1/3 and 2/2 flips in dimension two.
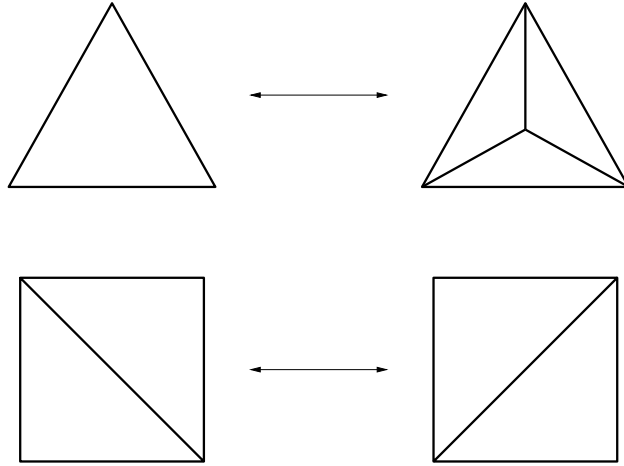

FIGURE 1. The 1/3 and 2/2 bistellar flips.

Suppose that (M, F) and (M, F') are associated to triangulations $\mathcal{K}$ and $\mathcal{K}'$. Now, if $\mathcal{K}'$ is obtained from $\mathcal{K}$ via a 2/3 bistellar flip then (M, F') is the stabilization of (M, F). When a 1/4 flip is used then (M, F') is the third stabilization of (M, F).

We may now state an important corollary of the Hauptvermutung (2.3), due to Pachner [Pac91].

Theorem 3.9. *Suppose that M is a closed three-manifold and $\mathcal{K}, \mathcal{K}'$ are triangulations of M. Then there is a sequence of isotopies and bistellar flips that transforms $\mathcal{K}$ into $\mathcal{K}'$.*   □

Lickorish's article [Lic99] gives a discussion of Pachner's Theorem and its application to the construction of three-manifold invariants. Now we have:

Proof of Theorem 3.6. Suppose that (M, F) and (M, F') are a pair of splittings. Using Lemma 3.8 stabilize each to obtain splittings, again called F and F', which are associated to triangulations. By Pachner's Theorem (3.9) these triangulations are related by a sequence of bistellar flips and isotopy. Consecutive splittings along the sequence are related by stabilization or destabilization. The uniqueness of stabilization now implies that (M, F) and (M, F') are stably equivalent.   □

4. Meridian disks

We carefully study meridian disks of handlebodies before diving into the proof proper of Waldhausen's Theorem (5.1).

Meridional pairs. If V is a handlebody and $v_0 \subset V$ is a properly embedded disk, with ∂v_0 essential in ∂V, then we call v_0 a *meridional* disk of V. Fix now a splitting (M, F). Let V and W be the handlebodies that are the closures of the components of $M \smallsetminus F$. So $V \cap W = F$.

Definition 4.1. Suppose that v_0 and w_0 are meridional disks of V and W. Suppose that ∂v_0 and ∂w_0 meet exactly once, transversely. Then we call $\{v_0, w_0\}$ a *meridional pair* for (M, F).

Note that $\{v_0, w_0\}$ is often called a *destabilizing pair*. To explain this terminology, one must check that $V' = V \smallsetminus U(v_0)$ and $W' = W \cup \overline{U(v_0)}$ are both handlebodies. Thus, taking $F' = \partial V' = \partial W'$, we find that (M, F') is a Heegaard splitting and that $(M, F) \approx (M, F') \# (S^3, T)$.

Remark 4.2. If $\{v_1, w_1\}, \ldots, \{v_n, w_n\}$ are pairwise disjoint meridional pairs then $V' = V \smallsetminus U(\cup_i v_i)$ is ambient isotopic to $V'' = V \cup U(\cup_i w_i)$. When $n = 1$ this is a pleasant exercise and the general case then follows from disjointness.

Furthermore, in this situation $V' = V \smallsetminus U(\cup_i v_i)$ and $W' = W \cup \overline{U(\cup_i v_i)}$ are handlebodies. So $F' = \partial V' = \partial W'$ gives a splitting (M, F') and we have $(M, F) = (M, F') \#_n (S^3, T)$.

Conversely, fix a splitting equivalent to $(M, F) \#_n (S^3, T)$. There is a natural choice of pairwise disjoint meridional pairs $\{v_1, w_1\}, \ldots, \{v_n, w_n\}$ so that the above construction recovers (M, F). As we shall see, the choice of pairs is not unique. This leads to the non-uniqueness of destabilization.

Suppose now that we have two splittings (M, F) and (M, G) that we must show are equivalent. By the Reidemeister-Singer Theorem above we may stabilize to obtain equivalent splittings $(M, F') \approx (M, G')$. So (M, F') admits two collections of pairwise disjoint meridional pairs. These record the handles of F' that must be cut to recover F or G. If, under suitable conditions, we can make our collections similar enough then we can deduce that the original splittings (M, F) and (M, G) are equivalent. Unfortunately, our process for modifying collections of meridional pairs does not preserve pairwise disjointness. To deal with this Waldhausen introduces the notions of *good* and *great* systems of meridional disks.

Good and great systems. Fix a splitting (M, F) with handlebodies V and W. Fix an ordered collection $v = \{v_1, \ldots, v_n\}$ of disjoint meridian disks of V.

Definition 4.3. We say v is a *good system* if there is an ordered collection $w = \{w_1, \ldots, w_n\}$ of disjoint meridian disks of W so that

- $\{v_i, w_i\}$ is a meridional pair for all i and
- $v_i \cap w_j = \emptyset$ whenever $i > j$.

If the latter condition holds whenever $i \neq j$ then we call v a *great system*. In either case we call w a *v-determined* system.

Both conditions can be understood via the intersection matrix $A = |v_i \cap w_j|$. For v to be a good system we must find a system w so that A is upper-triangular, with ones on the diagonal. For v to be great A must be the identity matrix.

Lemma 4.4 (Waldhausen 2.2, part 1). *Every good system is great.*

Proof. Suppose that $v = \{v_1, \ldots, v_n\}$ is good and $w = \{w_1, \ldots, w_n\}$ is the given v-determined system. We may assume that w has been isotoped to minimize $|v \cap w|$. If v is also great with respect to w then we are done.

Supposing otherwise, let k be the smallest index so that $v_k \cap w$ is not a single point. It follows that $v \cap w_k$ is a single point. Let α be a subarc of ∂v_k so that $\partial \alpha$ is contained in ∂w, one point of $\partial \alpha$ lies in ∂w_k, and the interior of α is disjoint from w.

It follows that the other endpoint of α lies in ∂w_l for some $l > k$. Let $N = \overline{U(w_k \cup \alpha \cup w_l)}$ be a closed regular neighborhood of the indicated union. Then $\partial N \cap W$ consists of three essential disks, two of which are parallel to w_k and w_l. Let w_l' be the remaining disk. Let $w' = (w \smallsetminus \{w_l\}) \cup \{w_l'\}$. It follows that v is still good with respect to w' and the total intersection number has been decreased. By induction, we are done. $\qquad\square$

Remark 4.5. The last step of the proof may be phrased as follows: obtain a new disk w_l' via a *handle-slide* of w_l over w_k along the arc α. The hypotheses tell us that the chosen slide does not destroy "goodness."

Lemma 4.6 (Waldhausen 2.2, part 2). *Suppose that v is a good system with respect to w. Then $V \smallsetminus U(v)$ and $V \cup \overline{U(w)}$ are ambient isotopic in M.*

Proof. By Remark 4.2 the lemma holds when w makes v a great system. Thus, by the proof of Lemma 4.4 all we need check is that $V \cup \overline{U(w)}$ is isotopic to $V \cup \overline{U(w')}$, where w and w' are assumed to differ by a single handle-slide. This verification is an easy exercise. $\qquad\square$

Reduction of (M, F) by v. Let v be a good system with respect to w. Since v does not separate V the difference $V \smallsetminus U(v)$ is a handlebody, as is $W \smallsetminus U(w)$. By Lemma 4.6 the unions $V \cup \overline{U(w)}$ and $W \cup \overline{U(v)}$ are also handlebodies. Let $F(v)$ be the boundary of $V \smallsetminus U(v)$. It follows that $(M, F(v))$ is a Heegaard splitting. We will call this the *reduction* of (M, F) along v. Taking $F(w)$ equal to the boundary of $W \smallsetminus U(w)$ we likewise find that $(M, F(w))$ is a splitting. With the induced orientations, we find that $(M, F(v)) \approx (M, F(w))$. We immediately deduce:

Lemma 4.7 (Waldhausen 2.4). *If v and v' are both good systems with respect to w then $(M, F(v)) \approx (M, F(v'))$.* $\qquad\square$

From the Reidemeister-Singer Theorem and the definitions we have:

Lemma 4.8 (Waldhausen 2.5, part 1). *Suppose that (M, F_1) and (M, F_2) have a common stabilization (M, F). Then there is a system $v \subset V$ good with respect to $w \subset W$ and a system $x \subset V$ good with respect to $y \subset W$ so that $(M, F(v)) \approx (M, F_1)$ and $(M, F(x)) \approx (M, F_2)$.* $\qquad\square$

Remark 4.9. We now have one decomposition and two sets of instructions for reducing (cutting open trivial handles). If we knew, for example, that y was a v–determined system then we would be done; but this is more than we actually need.

Getting along with your neighbors.

Lemma 4.10 (Waldhausen 2.5, part 2). *In the preceding lemma, F, v, x, w, y can be chosen so that $v \cap x = w \cap y = \emptyset$.*

Proof. We proceed in several steps.

Step 1: Apply a small isotopy to ensure:

- $(x \cap y) \cap (v \cup w) = \emptyset = (v \cap w) \cap (x \cup y)$.
- $v \cap x$ and $w \cap y$ are collections of pairwise disjoint simple closed curves and arcs.
- $v \cap x \cap F = \partial(v \cap x)$ and $w \cap y \cap F = \partial(w \cap y)$.

Step 2: Now we eliminate all simple closed curves of intersection between v and x. Suppose that $v \cap x$ contains a simple closed curve. Then there is an *innermost* disk $D \subset v$ so that $D \cap x = \partial D$. Use D to perform a *disk surgery* on x: since x is a union of disks, ∂D bounds a disk, say $D' \subset x$. Let x' be a copy of $(x \smallsetminus D') \cup D$, after a small isotopy supported in $U(D)$. Arrange matters so that $|v \cap x'| \le |v \cap x|$. By Lemma 4.7, $(M, F(x)) \approx (M, F(x'))$. Proceeding in this fashion, remove all simple closed curves of $v \cap x$. Apply the same procedure to remove all simple closed curves of $w \cap y$.

Step 3: Now we eliminate all arcs of intersection between v and x. To do this, we will replace F, and the various systems, by highly stabilized versions. Let k be an arc of $v_i \cap x_j$. Let v_i' and v_i'' be the two components of $v_i \smallsetminus U(k)$. These are both disks. Similarly, let x_j' and x_j'' be the two components of $x_j \smallsetminus U(k)$. Choose notation so that $|v_i' \cap w_i| = 1$, $|v_i'' \cap w_i| = 0$, and similarly for x_j' and x_j''. Let $\overline{w}$ and $\overline{y}$ be disjoint spanning disks of the cylinder $U(k) \cap V$. Take $F' = \partial(V \smallsetminus U(k))$.

Observe that

- (M, F') is a Heegaard splitting and is a stabilization of (M, F).
- The system

$$v' = \{v_1, \ldots, v_{i-1}, v_i', v_i'', v_{i+1}, \ldots, v_n\}$$

is good with respect to the system

$$w' = \{w_1, \ldots, w_i, \overline{w}, w_{i+1}, \ldots, w_n\}.$$

- The same holds for x' and y'.
- $(M, F'(w')) \approx (M, F(w))$ and $(M, F'(y')) \approx (M, F(y))$.
- $|v' \cap x'| < |v \cap x|$ and $|w' \cap y'| = |w \cap y|$.

Repeated stabilization in this fashion removes all arcs of intersection and so proves the lemma. $\square$

5. The proof of Waldhausen's Theorem

We may now begin the proof of:

Theorem 5.1 (Waldhausen 3.1). *Suppose that (S^3, G) is an unstabilized Heegaard splitting. Then $(S^3, G) \approx (S^3, S^2)$.*

This, and the uniqueness of stabilization, immediately implies our earlier version of the theorem: up to isotopy, the three-sphere has a unique splitting of every genus.

Let (S^3, G) be an unstabilized splitting. By Lemmas 4.8 and 4.10 there is a splitting (S^3, F) that is a common stabilization of (S^3, G) and (S^3, S^2) with several useful properties. First, let V, W denote handle-bodies so that $V \cup W = S^3$, $V \cap W = F$. Next, note that $\text{genus}(F) \geq \text{genus}(G)$. Letting $n = \text{genus}(F)$ and $m = \text{genus}(F) - \text{genus}(G)$ we assume that;

- There are good systems $v = \{v_1, \ldots, v_n\}$ and $x = \{x_1, \ldots, x_m\}$ in V.
- There is a v–determined system $w = \{w_1, \ldots, w_n\}$ and an x–determined system $y = \{y_1, \ldots, y_m\}$ in W.
- $(S^3, S^2) \approx (S^3, F(v))$ and $(S^3, G) \approx (S^3, F(x))$.
- $x \cap v = \emptyset = y \cap w$.

Suppose that the surface F is also chosen with minimal possible genus. We shall show, via contradiction, that $\text{genus}(F) = 0$. Since F was a stabilization of G it will follow that $\text{genus}(G) = 0$, as desired. So assume for the remainder of the proof that $n > 0$.

Lemma 5.2 (Waldhausen 3.2). *Altering y only we can ensure that $|y \cap v_n| \leq 1$.*

Proof. There are two possible cases.

Case 1: Suppose some element of y hits v_n in at least two points. Let $C = W \smallsetminus U(w)$. (This is a three-ball with *spots*.) Note that y is a collection of disjoint disks in C. Thus the disks y cut C into a collection of three-balls. Note that $w \cap \partial v_n$ is a single point. Hence $\gamma = \partial v_n \cap \partial C$ is a single arc with interior disjoint from the spots of ∂C. Since some element of y hits ∂v_n twice there is an element $y_j \in y$ and a subarc α contained in the interior of γ so that $\alpha \cap U(w) = \emptyset$, $\partial \alpha \subset y_j$, and interior$(\alpha) \cap y = \emptyset$.

Choose an arc β, properly embedded in y_j, so that $\partial \beta = \partial \alpha$. Then $\alpha \cup \beta$ bounds a disk $D \subset C$ so that $D \cap \partial C = \alpha$ and $D \cap y = \beta$. Again, this is true because $C \smallsetminus y$ is a collection of three-balls. (The disk D is called a *bigon*.) Let E be the component of $y_j \smallsetminus \beta$ that meets x_j exactly

once. Let $y'_j = D \cup E$. (The modern language is that y'_j is obtained from y_j via *bigon surgery* along D.)

Since $v \cap x = \emptyset$ it follows that $\alpha \cap x_j = \emptyset$. Thus y'_j meets x_j exactly once, $x_i \cap y'_j = \emptyset$ for all $i > j$, and $y_i \cap y'_j = \emptyset$ for all $i \neq j$. Thus $y' = (y \smallsetminus \{y_j\}) \cup \{y'_j\}$ is an x–determined system. Furthermore $y' \cap w = \emptyset$ and y' meets v_n fewer times than y does.

Case 2: Suppose every disk in y meets v_n in at most one point, and $|y \cap v_n| \geq 2$. Define $C = W \smallsetminus U(w)$ as above. There is an arc $\alpha \subset (\partial v_n) \cap \partial C$ so that $\alpha \cap y = \partial \alpha$. We may assume that one point of $\partial \alpha$ lies in y_i while the other lies in y_j, for $i < j$. Let y'_j be the disk obtained by doing a handle-slide of y_j over y_i along the arc α. As indicated in Remark 4.5, the system $y' = (y \smallsetminus \{y_j\}) \cup \{y'_j\}$ has all of the desired properties, and also reduces intersection with v_n.

Finally, iterating Case 1 and then Case 2 proves the lemma. $\square$

Proof of Theorem 5.1. (Waldhausen 3.3)

Case 1: If $y \cap v_n \neq \emptyset$ then by the above lemma we can assume that $y \cap v_n$ is a single point. Suppose that y_j meets v_n.

Define
$$x' = \{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m, v_n\},$$
$$y' = \{y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m, y_j\},$$
and notice that x' is good with respect to y'. Lemma 4.7 implies that $(S^3, F(y')) \approx (S^3, F(x'))$ and $(S^3, F(y)) \approx (S^3, G)$. Since y and y' are equal as sets $(S^3, F(y')) \approx (S^3, F(y))$. So $(S^3, F(x')) \approx (S^3, G)$.

Now we replace y' by another x'–determined system y'' by replacing y'_m by w_n. That is, define
$$y'' = \{y'_1, \ldots, y'_{m-1}, w_n\}.$$
The meridional pair $(v_n, w_n) = (x'_m, y''_m)$ represents the first trivial handle cut off in the process of transforming (S^3, F) into $(S^3, F(v))$ or $(S^3, F(x'))$. So the first step in the process of transforming (S^3, F) into $(S^3, F(x')) \approx (S^3, G)$ is the same as the first step in going from (S^3, F) to $(S^3, F(v)) \approx (S^3, S^2)$. Let (S^3, F') be the Heegaard decomposition obtained from (S^3, F) by cutting off this trivial handle. Then (S^3, F') has the same properties as (S^3, F) but F' has lower genus than F. This contradicts the minimality of F.

Case 2: If $y \cap v_n = \emptyset$ then we enlarge x and y to x^* and y^* by adding v_n and w_n. That is, we define
$$x^* = \{x_1, \ldots, x_m, v_n\}$$
and
$$y^* = \{y_1, \ldots, y_m, w_n\}.$$

Suppose in (S^3, F) we cut off the trivial handles of (x^*, y^*), obtaining $(S^3, F(x^*))$. Then we effectively cut off all the trivial handles of (x, y), obtaining $(S^3, F(x)) \approx (S^3, G)$ and additionally cut off the trivial handle represented by (v_n, w_n). So $(S^3, F(x^*))$ is obtained from (S^3, G) by removing a trivial handle. That is, $(S^3, G) \approx (S^3, F(x)) \approx (S^3, F(x^*)) \# (S^3, T)$. Thus G is a stabilized splitting. This is a contradiction. □

6. Remarks

Doubling a handlebody. Suppose that $T \subset S^2 \times S^1$ is the torus obtained by taking the product of the equator of the two-sphere and the S^1 factor. Let $(M_g, F_g) = \#_g(S^2 \times S^1, T)$. Notice that M_g may also be obtained by doubling a genus g handlebody across its boundary.

Waldhausen appears to claim the following:

Theorem 6.1 (Waldhausen 4.1). *F_g is the unique unstabilized splitting of M_g, up to isotopy.*

His actual sentence is:

> Hieraus und aus [Theorem 5.1] folgt, daß auch die Mannigfaltigkeiten $[M_g]$ nur die bekannten Heegaard-Zerlegungen besitzen.

(Brackets added.) This indicates that Theorem 6.1 follows from Theorem 5.1 and Haken's Lemma [Hak68]:

Lemma 6.2 (Haken). *Suppose that (M, F) is a Heegaard splitting and $S \subset M$ is a two-sphere which does not bound a three-ball in M. Then there is another such two-sphere $S' \subset M$ so that $S \cap F$ is a single curve.* □

The lemma can be used to prove that F_g is unique up to homeomorphism. It is not clear to this writer how to obtain Theorem 6.1 by following Waldhausen's remark.

It seems that no proof of Theorem 6.1 appears in the literature until the recent work of Carvalho and Oertel on automorphisms of handlebodies. See Theorem 1.10 of their paper [CO05]. A similar proof may be given using Hatcher's normal form for sphere systems (Proposition 1.1 of [Hat95]). Carvalho and Oertel also give an alternative proof, deducing Theorem 6.1 from work of Laudenbach [Lau73].

Compression bodies.

Definition 6.3 (Waldhausen 4.2). Suppose that V is a handlebody and D is a (perhaps empty) system of meridional disks properly embedded

in V. Let N be a closed regular neighborhood of $D \cup \partial V$, taken in V. Then N is a *compression body*.

Note that ∂N is disconnected and contains ∂V as a component. This component is called the *positive* boundary of N and is denoted by $\partial_+ N$. The *negative* boundary is $\partial_- N = \partial N \smallsetminus \partial_+ N$. Most modern authors disallow copies of S^2 appearing in $\partial_- N$.

Now suppose M is an orientable three-manifold and $F \subset M$ is a orientable closed surface in the interior of M. If F cuts M into two pieces V and W, where each of V and W is a handlebody or a compression body, and where $F = \partial_+ V = \partial_+ W$ then we say that (M, F) is a Heegaard splitting of M with respect to the *partition* $(V \cap \partial M, W \cap \partial M)$. Equivalence (up to isotopy), stabilization, and stable equivalence with respect to a fixed partition may all be defined as above. The Reidemeister-Singer Theorem can then be extended: any two Heegaard splittings of M giving the same partition of ∂M are stably equivalent.

Haken's Lemma in compression bodies. Haken's Lemma also applies to Heegaard splittings respecting a partition. Similarly, suppose that (M, F) is a Heegaard splitting respecting a partition and $D \subset M$ is a properly embedded disk so that ∂D is essential in ∂M. Then there is another such disk meeting F is a single curve. Using this and Theorem 5.1 we have:

Theorem 6.4 (Waldhausen 4.3). *If V is a handlebody and (V, F) is an unstabilized splitting then F is parallel to ∂V.* □

Lens spaces. As noted above, in addition to equivalence up to isotopy, we may define another equivalence relation on splittings (M, F); namely equivalence up to orientation-preserving homeomorphism of pairs. If $\partial M \neq \emptyset$ then we also require that the partition of ∂M be respected. Notice that these two equivalence relations do not generally agree, for example in the presence of *incompressible* tori. For a modern discussion, with references, see [BDT06].

Waldhausen notes that connect sum makes either set of equivalence classes into a commutative and associative monoid. This monoid is *not* cancellative. Suppose that (M, F) is a genus one splitting of a lens space, not equal to the three-sphere. Then (M, F) is characterized, up to homeomorphism, by a pair of relatively prime integers (p, q) with $0 < q < p$. Now, letting $-F$ represent F with the opposite orientation, we find that $(M, -F)$ is characterized by (p, q') where

$$q \cdot q' = 1 \,(\mathrm{mod}\ p).$$

It follows that (M, F) and $(M, -F)$ are generally not equivalent. On the other hand, $(M, F)\#(S^3, T)$ and $(M, -F)\#(S^3, T)$ are always equivalent. For suppose that $D \subset F$ is a small disk, N is a closed regular neighborhood of $F \smallsetminus \text{interior}(D)$, and $G = \partial N$. Then G is the desired common stabilization.

Waldhausen ends by suggesting that the pairs (M, F) characterized by $(5, 2)$ and $(7, 2)$, and their orientation reverses (namely $(5, 3)$ and $(7, 4)$), have interesting connect sums. He wonders how many distinct equivalence classes, up to isotopy or up to homeomorphism, are represented by the four sums

$$(5, 2)\#(7, 2), \ (5, 2)\#(7, 4), \ (5, 3)\#(7, 2), \ (5, 3)\#(7, 4).$$

This question was answered by Engmann [Eng70]; no pair of the suggested genus two splittings are homeomorphic.

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