OBTAINING SLIM TRIANGLES

0.1. **Triangles in hyperbolic space are thin.** Suppose that X is a δ–hyperbolic graph, choose x, y, z vertices, and fix geodesics h, k, l connecting these vertices cyclically. Recall that $\rho_h \colon X \to h$ is the closest points projection map.

Lemma 0.1. *There is a point on k within distance 2δ of $\rho_h(z)$.* $\square$

Lemma 0.2. *The diameter of $\{\rho_h(z), \rho_k(x), \rho_l(y)\}$ is at most 6δ.* $\square$

Lemma 0.3. *Suppose that z' is another point in X so that $d_X(z, z') \leq R$. Then $d_X(\rho_h(z), \rho_h(z')) \leq 2R + 4\delta$.* $\square$

Lemma 0.4. *Suppose that h' is another geodesic in X so that the endpoints of h' are within distance R of the points x and y. Then $d_X(\rho_h(z), \rho_{h'}(z)) \leq R + 12\delta$.* $\square$

0.2. **Index in a hole.** Fix $\mathcal{G}(S)$ a "combinatorial complex." For the following definitions, we assume that α and β are fixed vertices of $\mathcal{G}$.

For any hole X and for any geodesic $h \in \mathcal{C}(X)$ connecting a point of $\pi_X(\alpha)$ to a point of $\pi_X(\beta)$ we also define $\rho_h \colon \mathcal{G} \to h$ to be the map $\pi_X|\mathcal{G} \colon \mathcal{G} \to \mathcal{C}(X)$ followed by closest points projection to h. Define $\mathrm{index}^h_X \colon \mathcal{G} \to \mathbb{N}$ to be the *index* in X:

$$\mathrm{index}^h_X(\sigma) = d_X(\alpha, \rho_h(\sigma)).$$

Remark 0.5. Suppose that h' is a different geodesic connecting $\pi_X(\alpha)$ to $\pi_X(\beta)$. Then

$$|\mathrm{index}^h_X(\sigma) - \mathrm{index}^{h'}_X(\sigma)| \leq 12\delta + 2$$

by Lemma 0.4. Thus, if we are willing to accept a small additive error, the choice of geodesic h is irrelevant. Accordingly we will supress the superscript whenever possible.

0.3. **Projection control.** We say domains $X, Y \subset S$ *overlap* if X and Y intersect but are not nested. The following theorem (see Theorem 4.2.1 of Behrstock's thesis [1]) follows from Masur and Minsky's idea (see [2]) of *time ordered* domains in S:

Theorem 0.6. *There is a constant $M_1 = M_1(S)$ with the following property. Suppose that X, Y are overlapping non-simple domains. If $\gamma \in \mathcal{AC}(S)$ cuts both X and Y then either $d_X(\gamma, \partial Y) < M_1$ or $d_Y(\partial X, \gamma) < M_1$.* $\square$

We also require a more specialized version of Theorem 0.6 for the case where X and Y are nested.

Lemma 0.7. *There is a constant $M_2 = M_2(S)$ with the following property. Suppose that $X \subset Y$ are nested non-simple domains. Fix $\alpha, \beta, \gamma \in \mathcal{AC}(S)$ which cut both X and Y. Fix $k = [\alpha', \beta'] \subset \mathcal{C}(Y)$, a geodesic connecting a point of $\pi_Y(\alpha)$ to a point of $\pi_Y(\beta)$. Assume that $d_X(\alpha, \beta) \geq M_0$, the constant given by the Bounded Image Lemma.*

If $d_X(\alpha, \gamma) \geq M_2$ then

$$\mathrm{index}_Y^k(\partial X) - 4 \leq \mathrm{index}_Y^k(\gamma).$$

Symmetrically, we have

$$\mathrm{index}_Y^k(\gamma) \leq \mathrm{index}_Y^k(\partial X) + 4$$

if $d_X(\gamma, \beta) \geq M_2$. □

0.4. **Back and sidetracking.** Fix $\sigma, \tau \in \mathcal{G}$. We say σ *precedes* τ *by at least* K in X if

$$\mathrm{index}_X(\sigma) + K \leq \mathrm{index}_X(\tau).$$

We say σ precedes τ *by at most* K if the inequality is reversed. If σ precedes τ then we say τ *succeeds* σ.

Now take $\mathcal{P} = \sigma_i$ to be a path in $\mathcal{G}$ connecting α to β. We assume that σ_i and σ_{i+1} are disjoint.

We now formalize a pair of properties enjoyed by unparametrized quasi-geodesics to the situation at hand. The path $\mathcal{P}$ *backtracks* at most K if for every X and all indices $i < j$ we find that σ_j precedes σ_i by at most K. The path $\mathcal{P}$ *sidetracks* at most K if for every hole X and every index i we find that

$$d_X(\sigma_i, \rho_h(\sigma_i)) \leq K,$$

for some geodesic h connecting a point of $\pi_X(\alpha)$ to a point of $\pi_X(\beta)$.

Remark 0.8. As in Remark 0.5, allowing a small additive error makes irrelevant the choice of geodesic in the definition of sidetracking. We note that, if $\mathcal{P}$ has bounded sidetracking, one may freely use in calculation whichever of σ_i or $\rho_h(\sigma_i)$ is more convenient.

We now state our current goal:

Theorem 0.9. *Fix $\mathcal{G}$, a combinatorial complex. There is a constant K_0 so that, for every $K_1 \geq K_0$ there is a δ with the following property: If T is a triangle in $\mathcal{G}$ whose sides backtrack and sidetrack at most K_1, then T is δ–slim.*

We will prove this via a sequence of claims.

0.5. **Finding the midpoint of a side.** Let $K_0 = \max\{M_0, 4M_1, M_2, 8\} + 6\delta$. Fix $K_1 \geq K_0$. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be the sides of a triangle in $\mathcal{G}$ with vertices at α, β, γ. We may assume that each of $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ back and sidetracks at most K_1 in every hole.

Claim 0.10. If σ_i precedes γ in X and σ_j succeeds γ in Y, both by at least $2K_1$, then $i < j$.

Proof. To begin, as X and Y are holes, we need not consider the possibility that $X \cap Y = \emptyset$. If $X = Y$ we immediately deduce that

$$\mathrm{index}_X(\sigma_i) + 2K_1 \leq \mathrm{index}_X(\gamma) \leq \mathrm{index}_X(\sigma_j) - 2K_1.$$

Thus $\mathrm{index}_X(\sigma_i) + 4K_1 \leq \mathrm{index}_X(\sigma_j)$. Since $\mathcal{P}$ backtracks at most K_1 we have $i < j$, as desired.

Suppose instead that $X \subset Y$. Since σ_i precedes γ in X we immediately find $d_X(\alpha, \beta) \geq 2K_1 \geq M_0$ and $d_X(\alpha, \gamma) \geq 2K_1 - 2\delta \geq M_2$. Apply Lemma 0.7 to deduce $\mathrm{index}_Y(\partial X) - 4 \leq \mathrm{index}_Y(\gamma)$. Since σ_j succeeds γ in Y it follows that $\mathrm{index}_Y(\partial X) - 4 + 2K_1 \leq \mathrm{index}_Y(\sigma_j)$. Again using the fact that σ_i precedes γ in X we have that $d_X(\sigma_i, \beta) \geq M_2$. We deduce from Lemma 0.7 that $\mathrm{index}_Y(\sigma_i) \leq \mathrm{index}_Y(\partial X) + 4$. Thus

$$\mathrm{index}_Y(\sigma_i) - 8 + 2K_1 \leq \mathrm{index}_Y(\sigma_j).$$

Since $\mathcal{P}$ backtracks at most K_1 in Y we again deduce that $i < j$. The case where $Y \subset X$ is handled in symmetric fashion.

Suppose now that X and Y overlap. Applying Theorem 0.6 and breaking symmetry, we may assume that $d_X(\gamma, \partial Y) < M_1$. Since σ_i precedes γ we have $\mathrm{index}_X(\gamma) \geq 2K_1$. Thus, it follows that $\mathrm{index}_X(\partial Y) \geq 2K_1 - 2M_1 - 4\delta$ and so

$$d_X(\alpha, \partial Y) \geq 2K_1 - 2M_1 - 6\delta \geq M_1.$$

Applying Theorem 0.6 again, we find that $d_Y(\alpha, \partial X) < M_1$. Now, since σ_j succeeds γ in Y, we deduce that $\mathrm{index}_Y(\sigma_j) \geq 2K_1$. Similar considerations to the above show that

$$d_Y(\partial X, \sigma_j) \geq 2K_1 - M_1 - 2\delta \geq M_1.$$

Applying Theorem 0.6 one last time, we find that $d_X(\partial Y, \sigma_j) < M_1$. Thus $d_X(\gamma, \sigma_j) \leq 2M_1$. Finally, we deduce that the difference in index (in X) between σ_i and σ_j is at least $2K_1 - 4M_1 - 4\delta$. Since this is again greater than K_1, it follows that $i < j$. $\square$

Let $\sigma_\alpha \in \mathcal{P}$ be the *last* vertex of $\mathcal{P}$ preceding γ by at least $2K_1$ in some hole. If no such vertex of $\mathcal{P}$ exists then take $\sigma_\alpha = \alpha$.

Claim 0.11. There is a constant $N_1 = N_1(S)$ with the following property. For every hole X and geodesic h connecting $\pi_X(\alpha)$ to $\pi_X(\beta)$:

$$d_X(\sigma_\alpha, \rho_h(\gamma)) \leq N_1.$$

Proof. Since σ_i and σ_{i+1} are disjoint we have

$$|\mathrm{index}_X(\sigma_{i+1}) - \mathrm{index}_X(\sigma_i)| \leq 4\delta + 2.$$

Since $\mathcal{P}$ is a path connecting α to β the image $\rho_h(\mathcal{P})$ is $4\delta + 2$–dense in h. Thus, if $\mathrm{index}_X(\sigma_\alpha) + 2K_1 + 4\delta + 2 < \mathrm{index}_X(\gamma)$ then we have a contradiction to the definition of σ_α.

On the other hand, if $\mathrm{index}_X(\sigma_\alpha) \geq \mathrm{index}_X(\gamma) + K_1$ then σ_α succeeds γ. This directly contradicts Claim 0.10.

We deduce that the difference in index between σ_α and γ in X is at most $2K_1 + 4\delta + 2$. Finally, as $\mathcal{P}$ sidetracks by at most K_1 we have

$$d_X(\sigma_\alpha, \rho_h(\gamma)) \leq 3K_1 + 4\delta + 2$$

as desired. □

We define σ_β to be the first σ_i to succeed γ by at least $2K_1$ — if no such vertex of $\mathcal{P}$ exists take $\sigma_\beta = \beta$. If $\alpha = \beta$ then $\sigma_\alpha = \sigma_\beta$. Otherwise, from Claim 0.10, we immediately deduce that σ_α comes before σ_β in $\mathcal{P}$. A symmetric version of Claim 0.11 applies to σ_β: for every hole X

$$d_X(\rho_h(\gamma), \sigma_\beta) \leq N_1.$$

0.6. **Another side of the triangle.** Recall now that we are also given a path $\mathcal{R} = \{\tau_i\}$ connecting α to γ in $\mathcal{G}$. As before, $\mathcal{R}$ has bounded back and sidetracking. Thus we again find vertices τ_α and τ_γ the last/first to precede/succeed β by at least $2K_1$. Again, this is defined in terms the closest points projection of β to geodesics geodesics of the form $l = [\pi_X(\alpha), \pi_X(\gamma)]$. By Claim 0.11, for every hole X, τ_α and τ_γ are close to $\rho_l(\beta)$.

By Lemma 0.2, if $h = [\pi_X(\alpha), \pi_X(\beta)]$, then $d_X(\rho_h(\gamma), \rho_l(\beta)) \leq 6\delta$. We deduce:

Claim 0.12. $d_X(\sigma_\alpha, \tau_\alpha) \leq 2N_1 + 6\delta$. □

We now prove:

Claim 0.13. There is a constant $N_2 = N_2(S)$ with the following property. For every $\sigma_i \leq \sigma_\alpha$ in $\mathcal{P}$ there is a $\tau_j \leq \tau_\alpha$ in $\mathcal{R}$ so that

$$d_X(\sigma_i, \tau_j) \leq N_2$$

for every hole X.

Proof. We only sketch the proof, as the details are similar to the discussion above. Fix $\sigma_i \leq \sigma_\alpha$. Fix a hole X and geodesics $h = [\pi_X(\alpha), \pi_X(\beta)]$ and $l = [\pi_X(\alpha), \pi_X(\gamma)]$.

Suppose first that no vertex of $\mathcal{R}$ precedes σ_i by more than $2K_1$. Then $\rho_l(\sigma_i)$ in within distance $2K_1$ of $\pi_X(\alpha)$. Since $d_X(\sigma_\alpha, \tau_\alpha) \leq 2N_1 + 6\delta$, by Claim 0.12, the initial segments of h and l fellow travel. Because of bounded backtracking along $\mathcal{P}$, $\rho_h(\sigma_i)$ lies on, or at least near, this initial segment of h. Thus $\rho_l(\sigma_i)$ is close to $\rho_h(\sigma_i)$ which in turn is close to σ_i, because $\mathcal{P}$ has bounded sidetracking. Thus we may take $\tau_j = \tau_0 = \alpha$ and we are done.

Now suppose that some vertex of $\mathcal{R}$ precedes σ_i by at least $2K_1$. Take τ_j to be the last such vertex in $\mathcal{R}$. Following the proof of Claim 0.10 shows that τ_j comes before τ_α in $\mathcal{R}$. The argument now required to bound $d_X(\sigma_i, \tau_j)$ is essentially identical to the proof of Claim 0.11. □

By the distance estimate, we find that there is a uniform neighborhood of $[\sigma_0, \sigma_\alpha] \subset \mathcal{P}$, taken in $\mathcal{G}$, which contains $[\tau_0, \tau_\alpha] \subset \mathcal{P}$. The slimness of $\mathcal{PQR}$ follows directly. This completes the proof of Theorem 0.9. □

References

[1] Jason Behrstock. *Asymptotic geometry of the mapping class group and Teichmüller space.* PhD thesis, SUNY Stony Brook, 2004. http://www.math.columbia.edu/~jason/thesis.pdf.

[2] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000. arXiv:math.GT/9807150.