SEP(S_2) IS HYPERBOLIC

Abstract. We prove that the complex of of separating curves in the closed genus two surface is Gromov hyperbolic. We indicate how the techniques generalize to other complexes.

1. Notation and outline

We assume that the reader is familar with the papers of Masur and Minsky [2] and [1]. If not, the following will not make a lot of sense.

Let $S = S_{g,n}$ be the compact orientable connected surface with genus g and n boundary components. The subscript n is omitted if S is closed. In any case, define the *complexity* of S to be $\zeta(S) = 3g - 3 + n$. For any separating curve $\alpha \subset S$ define the *complexity* of α to be $\zeta(\alpha) = \min\{\zeta(S'), \zeta(S'')\}$ where S' and S'' are the components of $S \smallsetminus \alpha$.

Remark 1.1. If S is closed then α is sometimes called a *genus k curve*, where $k = \min\{g(S'), g(S'')\}$.

Now to define the graph $\mathrm{Sep}(S_2)$: vertices are isotopy classes of separating curves in S_2, edges connect vertices α, β if and only if $\iota(\alpha, \beta) = 4$. Define $d_{\mathrm{Sep}}(\cdot, \cdot)$ to be the edge metric in $\mathrm{Sep}(S_2)$.

Suppose that $X \subset S$ is an essential subsurface. Recall the *subsurface projection* map $\pi_X \colon \mathrm{Sep}(S_2) \to \mathcal{C}(X)$. Define $d_X(\alpha, \beta) = d_X(\pi_X(\alpha, \beta))$ to be the *projection distance* in $\mathcal{C}(X)$ between α and β. We say that X is a *hole* for $\mathrm{Sep}(S_2)$ if every vertex of $\mathrm{Sep}(S_2)$ cuts X, or, equivalently, $\pi_X(\alpha)$ is defined for every $\alpha \in \mathrm{Sep}(S_2)$.

Our main goal is to show:

Theorem 1.2. *Sep(S_2) is Gromov hyperbolic.*

To prove this we produce a family of uniform quasi-geodesics in $\mathrm{Sep}(S_2)$, one for each pair of vertices. As a corollary we obtain the *distance estimate*:

Theorem 1.3. *There is a constant $C_0 \geq 0$ so that, for any $C \geq C_0$ there are constants $K \geq 1, E \geq 0$ where*

$$d_{\mathrm{Sep}}(\alpha, \beta) \stackrel{K,E}{=} \sum [d_X(\alpha, \beta)]_C.$$

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This holds for any separating curves α and β. The right-hand sum is over all holes for $\mathrm{Sep}(S_2)$.

We use this to prove that any triangle, made of three quasi-geodesics of the family, is δ-slim for a uniform δ. This implies hyperbolicity.

2. Basics

Lemma 2.1. $\mathrm{Sep}(S_2)$ *is connected.* □

Lemma 2.2. *The surface $X \subset S$ is a hole for* $\mathrm{Sep}(S_2)$ *if and only if X is homeomorphic to S_2, $S_{1,2}$, or $S_{0,4}$.* □

Lemma 2.3. *All holes for* $\mathrm{Sep}(S_2)$ *intersect.* □

Lemma 2.4. *There is a constant K_1 so that, for any hole X and for any pair of separating curves α and β we find*

$$d_X(\alpha, \beta) \leq K_1 \cdot d_{\mathrm{Sep}}(\alpha, \beta).$$

□

3. Lower bound

4. Innermost holes

Suppose that $X \subset S$ is homeomorphic to $S_{0,4}$. Write $\partial X = A \cup A' \cup B \cup B'$ where A and A' cobound an annulus in S as do B and B'.

The curve complex of X is $\mathcal{F}(X)$: a copy of the Farey graph. The vertices of $\mathcal{F}(X)$ fall into three types depending on how they partition the components of ∂X. Curves giving the partition $AA'|BB'$ are of *type one*. The partition $AB|A'B'$ gives *type two* curves. The partition $AB'|A'B$ gives *type three* curves. Every triangle of $\mathcal{F}(X)$ contains one curve of each type. Curves of type one are separating in S_2 and thus give vertices of $\mathrm{Sep}(S_2)$.

Lemma 4.1. *The curves of type one in $\mathcal{F}(X)$ span an infinite valence tree T_∞ in* $\mathrm{Sep}(S_2)$.

Proof. Fix an edge e in $\mathcal{F}(X)$ with endpoints γ and δ of type two and three respectively. Let f and f' be the two triangles adjacent to e. Let α and α' be the remaining two vertices of f and f'. Both α and α' are separating in S. Also, $\iota(\alpha, \alpha') = 4$ so α and α' are connected by an edge E in $\mathrm{Sep}(S_2)$. The action of the mapping class group $\mathcal{MCG}(X)$ moves E around, giving a tree $T_\infty \subset \mathrm{Sep}(S_2)$ of infinite valence.

Now let Γ be the subgraph of $\mathrm{Sep}(S_2)$ spanned by all of the type one vertices of $\mathcal{F}(X)$. Note that T_∞ is contained in Γ. Suppose that β and β' are vertices of T_∞, separated in $\mathcal{F}(X)$ by the edge e. Suppose that $\{\alpha, \alpha'\} \neq \{\beta, \beta'\}$. It follows that $\iota(\beta, \beta') > 4$ and so β and β' are not connected by an edge in $\mathrm{Sep}(S_2)$. Thus the edge E is not contained in any cycle in Γ. Thus Γ is a tree, and so equals T_∞. □

We note that the tree T_∞ is a very inefficient way to move around in $\mathcal{F}(X)$: Suppose that α and γ are adjacent vertices of $\mathcal{F}(X)$ of types one and two. Let $\alpha_n = D^n_\gamma(\alpha)$ be the separating curve obtained by Dehn twisting α exactly n times about γ. Then the distance between α_i and α_j is exactly $|j - i|$ in T_∞, but is only two in $\mathcal{F}(X)$. To remedy this problem we will add a vertex $\bar{\gamma}$ to T_∞. Note that $S \setminus \gamma \cong S_{1,2}$. The remains of α in $S \setminus \gamma$ are a pair of arcs, as shown on the right of Figure 1.

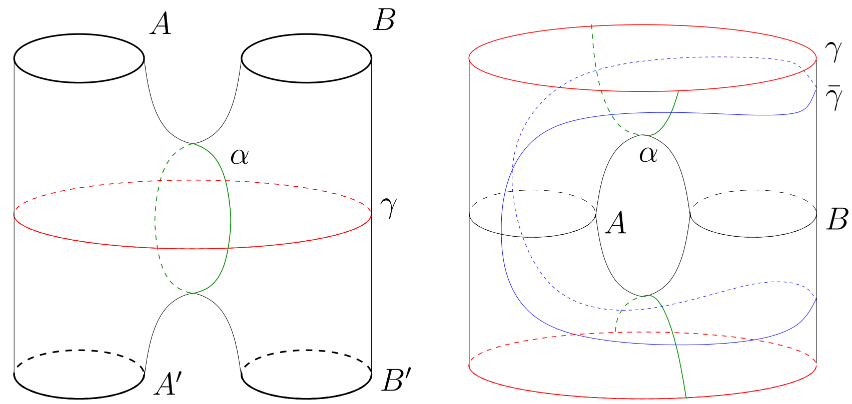

FIGURE 1. On the left we see X containing α and γ. On the right we have $S \setminus \gamma$. Note that $\bar{\gamma}$ is separating in S and meets α_n four times, regardless of n.

To be precise, $\bar{\gamma}$ is a separating curve in S, is disjoint from $B \subset \partial X$, meets $A \subset \partial X$ twice, and meets α_n four times, regardless of n. (These properties determine the curve $\bar{\gamma}$ up to Dehn twists about A.) It follows that $d_{\mathrm{Sep}}(\alpha_i, \alpha_j) \leq 2$ independent of i and j.

In a similar fashion, for every type two and three vertex of $\mathcal{F}(X)$ we further augment T_∞. Denote the resulting subgraph of $\mathrm{Sep}(S_2)$ by $T(X)$. We shall see below that $T(X)$ is quasi-isometric to the Farey graph $\mathcal{F}(X)$.

Fix α and β a pair of type one vertices in $\mathcal{F}(X)$. Let $k = d_X(\alpha, \beta)$, the distance between α and β measured in $\mathcal{F}(X)$.

Lemma 4.2. *There is a path $g = \{\alpha_j\}$ in $\mathrm{Sep}(S_2)$ from α to β of length at most $2k$. Furthermore, for all j we have $\iota(\alpha_j, \partial X) \leq 4$.*

Proof. Let $h = \{\gamma_j\}$ be a path of length k connecting α to β, in the Farey graph $\mathcal{F}(X)$. For indices j where γ_j is of type one, let $\alpha_j = \gamma_j$.

For all other indices let α_j be the vertex $\bar{\gamma}_j$ in the augmented graph $T(X)$.

If γ_j is of type one and γ_{j+1} is of type two or three then α_j is connected to α_{j+1} via an edge in $\mathrm{Sep}(S_2)$. This follows from the construction of $T(X)$.

If γ_j is of type two and γ_{j+1} is of type three then α_j may *not* be connected to α_{j+1} via an edge in $\mathrm{Sep}(S_2)$. However, γ_j and γ_{j+1} are adjacent via an edge in $\mathcal{F}(X)$. It follows that there are two type one vertices adjacent to γ_j and γ_{j+1} in $\mathcal{F}(X)$. Let α'_j be either of these. Again, by the construction of $T(X)$ the separating curve α'_j is adjacent to both of α_j and α_{j+1} in $\mathrm{Sep}(S_2)$.

The situation is similar if γ_j is of type three and γ_{j+1} is of type two. It follows that $\{\alpha_j\} \cup \{\alpha'_j\}$ is a path in $\mathrm{Sep}(S_2)$ of length at most $2k$. Every vertex of this path is either inside of X or meets ∂X at most twice. $\square$

References

[1] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999. arXiv:math.GT/9804098.

[2] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000. arXiv:math.GT/9807150.