Sculptures in S^3

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Abstract

We describe the construction of a number of sculptures. Each is based on a geometric design native to the three-sphere: the unit sphere in four-dimensional space. Via stereographic projection, we transfer the design to three-dimensional space. All of the sculptures are then fabricated by the 3D printing service Shapeways.

1 Introduction

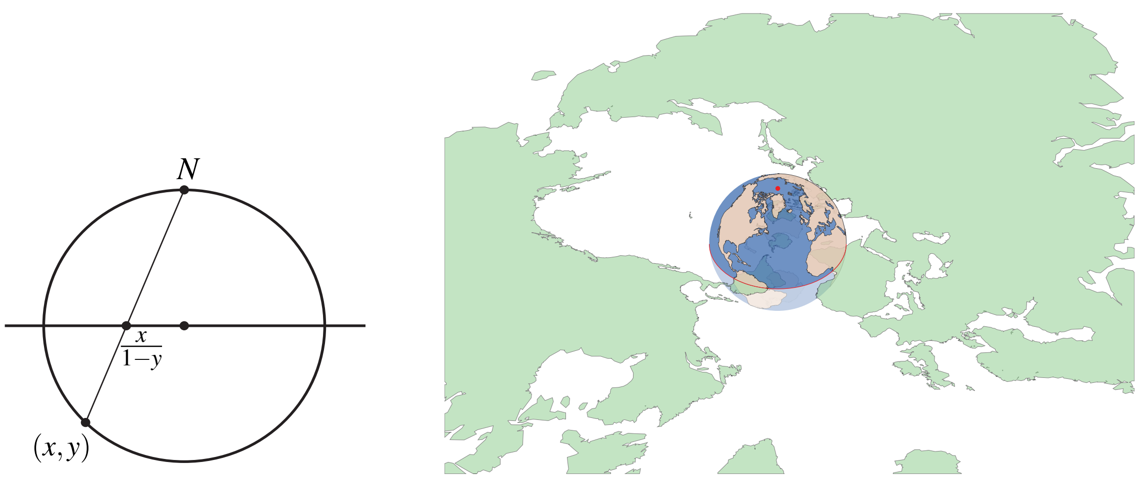
The three-sphere, denoted S^3, is a three-dimensional analog of the ordinary two-dimensional sphere, S^2. In general, the n–dimensional sphere is a subset of $\mathbb{R}^{n+1}$ as follows:

$$S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

Thus S^2 can be seen as the usual unit sphere in $\mathbb{R}^3$. Visualising objects in dimensions higher than three is non-trivial. However for S^3 we can use stereographic projection to reduce the dimension from four to three. Let $N = (0, \ldots, 0, 1)$ be the north pole of S^n. We define **stereographic projection** $\rho : S^n - \{N\} \to \mathbb{R}^n$ by

$$\rho(x_0, x_1, \ldots, x_n) = \left(\frac{x_0}{1 - x_n}, \frac{x_1}{1 - x_n}, \ldots, \frac{x_{n-1}}{1 - x_n} \right).$$

See [1, page 27]. Figure 1a displays the one-dimensional case; this is also a cross-section of the n–dimensional case. For any point $(x, y) \in S^1 - \{N\}$ draw the straight line L between N and (x, y). Then L meets $\mathbb{R}^1$ at a single point; this is $\rho(x, y)$.



(a) Stereographic projection from $S^1 - \{N\}$ to $\mathbb{R}^1$.

(b) Two-dimensional stereographic projection applied to the Earth. Notice that features near the north pole are very large in the image.

Figure 1: Stereographic projection.

Stereographic projection is a bijection and, by adding in a point at infinity corresponding to the north pole, it extends to a homeomorphism from S^n to $\mathbb{R}^n \cup \{\infty\}$. In this way we may think of S^3 as $\mathbb{R}^3$ plus a single "point at infinity". So we may use stereographic projection to represent, in $\mathbb{R}^3$, objects that live in S^3. Note that there is nothing special about the choice of $N = (0,\dots,0,1)$. We can alter the formula so that any point in S^3 becomes the **projection point**.

2 The geometry of S^3

Circles and lines A generic plane in $\mathbb{R}^4$, if it meets S^3, meets S^3 in a circle. The following fact is fundamental: stereographic projection of any circle in S^3 is a circle or line in $\mathbb{R}^3$. We will refer to this very useful property as the **circline property**: here a **circline** is either a circle or a line in $\mathbb{R}^3$. See [1, Section 3.2] for a more general discussion. A circle $C \subset S^3$ maps to a line if and only if C meets the projection point.

Great circles Any plane in $\mathbb{R}^4$ through the origin cuts through S^3 in a **great circle**. The great circles are the geodesics, or locally shortest paths, in the geometry on S^3. Just as for the usual sphere, S^2, two distinct great circles meet at two points: say at $x \in \mathbb{R}^4$ and also at the antipodal point $-x$.

Conformality Stereographic projection is **conformal**: if two circles in S^3 intersect at a given angle then the corresponding circlines in $\mathbb{R}^3$ meet at the same angle. So stereographic projection preserves angles [1, Section 3.2]. Note that lengths are not preserved; as shown in Figure 1b the distortion of length becomes infinite as we approach the projection point. However, this defect is unavoidable; there is no isometric embedding of any open subset of the three-sphere into $\mathbb{R}^3$.

The quaternionic picture of S^3 In order to get a sense of the shape of S^3, it is useful to have some landmarks. A good way to do this is to view S^3 in terms of the unit quaternions [2]. The quaternions are an extension of the complex numbers, from two dimensions to four. A quaternion is a formal sum $a+bi+cj+dk$ where $a,b,c,d \in \mathbb{R}$ and where i,j,k are non-commuting symbols satisfying

$$i^2 = j^2 = k^2 = ijk = -1.$$

The set of quaternions is called $\mathbb{H}$ in honour of Hamilton, its discoverer. There is a natural bijection between $\mathbb{R}^4$ and $\mathbb{H}$ via $(a,b,c,d) \mapsto a+bi+cj+dk$. So we may view S^3 as the set of **unit quaternions**: those with length $|a+bi+cj+dk| = \sqrt{a^2+b^2+c^2+d^2}$ equal to one. Once this is established the points $\pm 1, \pm i, \pm j, \pm k$ serve as our landmarks. See Figure 2. All of the circlines shown correspond to great circles in S^3 with particularly nice quaternionic expressions.

The isometries of S^3 The isometries of S^2 are the rigid motions of $\mathbb{R}^3$ that fix the origin. Under composition these form the orthogonal group $O(3)$. Analogously, the isometries of S^3 form the group $O(4)$. The unit quaternions can be realised as a subgroup of $O(4)$ in the following manner. As above we identify $\mathbb{H}$ and $\mathbb{R}^4$. For $q \in \mathbb{H}$ with $|q| = 1$, the map $f_q : \mathbb{H} \to \mathbb{H}$ given by $f_q(x) = q \cdot x$ is an element of $O(4)$. This serves our purpose to move objects around in S^3. For example, if we want to move point a to point b, then an easy way to achieve this is to apply the isometry corresponding to the quaternion $b \cdot a^{-1}$.

One application of this technique is to adjust the stereographic projection of a subset of S^3. For example, if $F \subset S^3$ is a surface then as q varies the image of $q \cdot F$ in $\mathbb{R}^3$ becomes more or less symmetric as $q \cdot F$ moves with respect to the projection point. Equivalently we can think of moving the projection point to a more convenient location. We do this several times below.

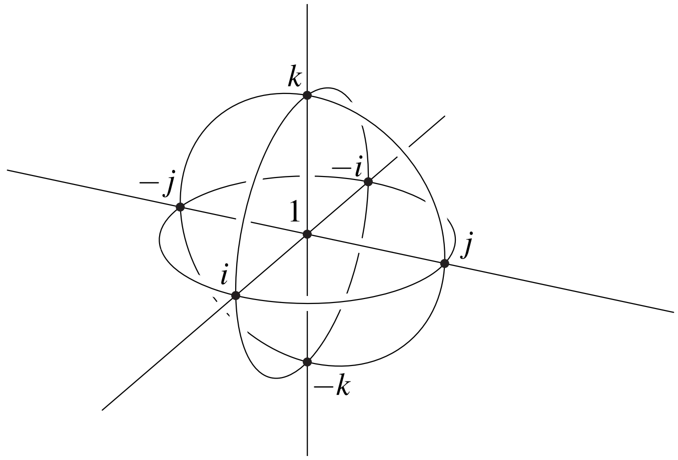


Figure 2: The unit quaternions in S^3 stereographically projected to $\mathbb{R}^3$ from the projection point -1.

3 Objects native to S^3

3.1 Radial projections of four dimensional polytopes

Suppose that $\sigma \subset \mathbb{R}^n$ is a finite set that does not lie in a hyperplane. Then $P(\sigma)$, the convex hull of Σ, is a n**–polytope** [7, page 4]. Suppose that $\tau \subset \sigma$. If $P(\tau)$ lies in the boundary of $P(\sigma)$ and if for all $\tau \subsetneq \mu \subset \sigma$ we have $\dim(P(\tau)) < \dim(P(\mu))$ then we say $P(\tau)$ is a **face** of $P(\sigma)$. Any maximal ascending chain of faces

$$P(\tau_0) \subset P(\tau_1) \subset \ldots \subset P(\tau_n) = P(\sigma)$$

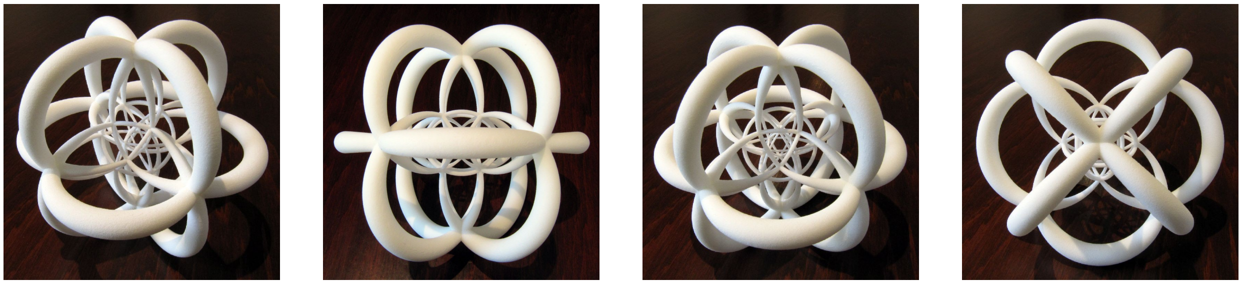
is called a **flag**. Then $P = P(\sigma)$ is **regular** if for any two flags F, G of P there is an isometry of $\mathbb{R}^n$ that preserves P and sends F to G.

In dimensions one, two and three the regular polytopes are known of old. These are the interval, the regular k–gons and the **Platonic solids**: the tetrahedron (simplex), the cube, the octahedron (cross-polytope), the dodecahedron, and the icosahedron. In all higher dimensions there are versions of the simplex, cube, and cross-polytope. Surprisingly, the only remaining regular polyhedra appear in dimension four! There are only three of them: the 24–cell, the 120–cell, and the 600–cell [3, page 136].

Suppose P is an regular n–polytope. The extreme symmetry of P implies that we can choose coordinates for $\mathbb{R}^n$ so that all vertices of P lie on the unit sphere, S^{n-1}. Projecting radially from the origin transfers P from $\mathbb{R}^n$ into S^{n-1}. Stereographic projection then places P into $\mathbb{R}^{n-1}$.

Applied to a four-polytope, these projections turn the Euclidean geometry of P first into a design of arcs of great circles in S^3 and then into a design of circline arcs in $\mathbb{R}^3$. If P meets the projection point then the design includes line segments running off to infinity. In order to produce such a design as a physical object, we need to thicken the circline arcs to have non-zero volume. One possible approach would be to use the Euclidean geometry of $\mathbb{R}^3$: one would thicken all arcs of the design to get tubular neighbourhoods of constant radius. However, the result is not satisfactory: near the origin in $\mathbb{R}^3$, the tubes are much too thick compared to their separation.

A better solution is to use tubular neighbourhoods in the intermediate S^3 geometry. For this we must parameterise the image of such a tube under stereographic projection. Again, the circline property is very useful. The boundary of a tubular neighbourhood of a geodesic in S^3 can be made as a union of small circles in $\mathbb{R}^4$. (These circles lie in S^3, but are not great.) The small circles map to circlines in $\mathbb{R}^3$, which can be directly parameterised. Computer visualisation of stereographic projections of 4-dimensional polytopes, in this style, have been beautifully implemented in the program Jenn3d [6]. In Figure 3 we show four views of a 3D printed copy of the 24–cell.



(a) A generic view. (b) A 2-fold symmetry axis. (c) A 3-fold symmetry axis. (d) A 4-fold symmetry axis.

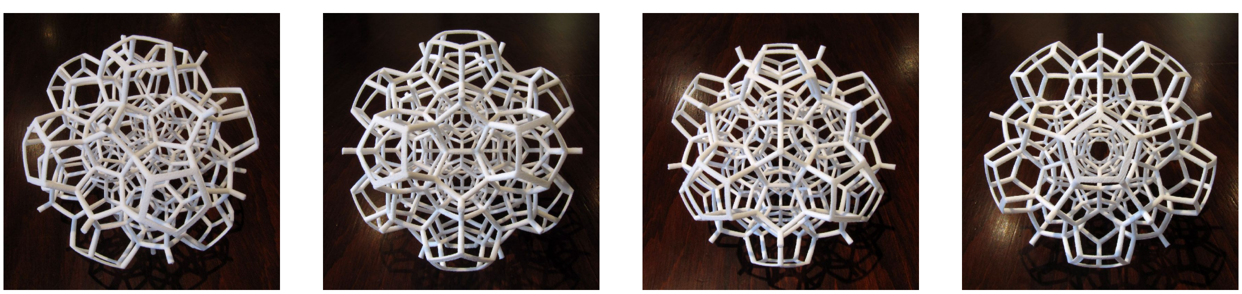
Figure 3: 24–Cell, 2011, $9.0 \times 9.0 \times 9.0$ cm.

The sculpture shown in Figure 3 illustrates a problem inherent in 3D printing of stereographic projections. Depending on the design, parts that began near the projection point can be unboundedly larger (in $\mathbb{R}^3$) than parts that began near the antipodal point. There is a minimal feature size required to make 3D printing possible; with current technology this minimal size is approximately 1mm. These two properties combine to force a lower bound for the size of the design. This size is feasible for the 24–cell. However, the 120–cell would have to be printed on a very large scale. If we rotate the 120–cell so that the projection point lies at the center of a dodecahedral face, then the furthest point of the projection of the 1-skeleton from the origin is around 29.4 times further out than the closest point, and a neighbourhood at that point is around the same number of times thicker than at the closest point. This means that the final design would be very large and thus too expensive to print.

One solution to this problem, as employed by Hart [5], is use a **projective transformation** instead of stereographic projection. This takes a 4–polytope to its **Schlegel diagram** [7, page 133]. This is typically much more compact. However, conformality is lost; the resulting figure distorts both lengths and angles.

Our alternative, shown in Figure 4, is to only print half of the object. We cut S^3 along the equatorial S^2; the sphere equidistant from the north and south poles. Choosing the north pole as the projection point, we project the half of the design in the southern hemihypersphere. The image is contained in the unit ball $\mathbb{B}^3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$.

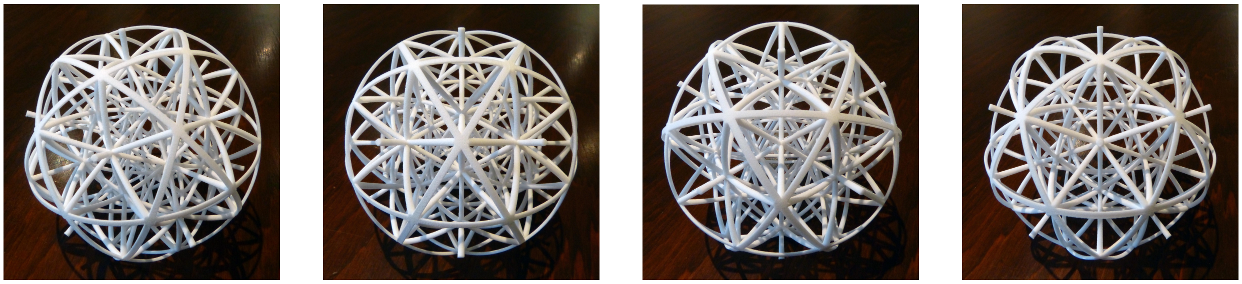
This done, the thinnest and thickest parts differ only by a factor of two, at the most. For stereographic projection, parts of the design near the projection point are the real problem, in terms of size. Eliminating the half nearest the projection point eliminates the problem.



(a) A generic viewpoint. (b) A 2-fold symmetry axis. (c) A 3-fold symmetry axis. (d) A 5-fold symmetry axis.

Figure 4: Half of a 120-Cell, 2011, $9.9 \times 9.9 \times 9.9$ cm.

Note that half of the 120–cell is still very complicated! However, one can understand the whole of the 120–cell by imagining reflecting the object across the cutting S^2. Note as well, that printing only the southern half of a design allows us to print objects that pass through the north pole, which ordinarily would be infinitely expensive. For example, in Figure 5 we show one-half of the stereographic projection of the vertex centered 600–cell. This version of the 600–cell is positioned so as to be dual to the facet-centered 120-cell shown in Figure 4. The other half of the vertex-centered 600–cell cannot be printed because the vertex antipodal to the origin meets the projection point.



(a) A generic viewpoint. (b) A 2-fold symmetry axis. (c) A 3-fold symmetry axis. (d) A 5-fold symmetry axis.

Figure 5: Half of a 600-Cell, 2011, $9.9 \times 9.9 \times 9.9$ cm.

3.2 Parameterisations of surfaces and torus knots

The geometry of S^3 lends itself particularly well to the representation of tori and torus knots. There seem to be two reasons for this. First, in its natural position certain geodesics in the torus are great circles in S^3. Second, quaternionic multiplication and its relatives directly parametrise torus knots.

When representing a surface as a 3D printed object, it is often a good idea to drill holes in the surface, both to save on material used and so the viewer can see, partly, through the surface to what is behind. In our approach, the pattern of holes shows the parameterisation, by realising the surface as a grid with grid lines in the direction of the parameters.

Clifford torus Recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ parametrises a great circle S^1. The same formula holds replacing i everywhere by j or by k. A Clifford torus is foremost a torus, and so can be parameterised as a product [4, page 139] via

$$\begin{aligned}\mathbb{T} = S^1 \times S^1 &= \left\{ \frac{1}{\sqrt{2}}\big(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta)\big) \;\middle|\; 0 \le \alpha < 2\pi, 0 \le \beta < 2\pi \right\} \\ &= \left\{ \frac{1}{\sqrt{2}}\big(e^{i\alpha} + e^{i\beta} \cdot j\big) \;\middle|\; 0 \le \alpha < 2\pi, 0 \le \beta < 2\pi \right\}.\end{aligned}$$

The factor of $1/\sqrt{2}$ rescales the torus to lie inside of the unit sphere $S^3 \subset \mathbb{R}^4$. Note that if we vary α while fixing β the point traces out a $(1,0)$ curve on $\mathbb{T}$. Conversely varying β while fixing α yields a $(0,1)$ curve. Unfortunately none of these curves are great circles in S^3.

On the other hand, if we vary α and β simultaneously, at the same (respectively, opposite) velocity the the point traces out a $(1,1)$ (respectively $(1,-1)$) curve. As we shall see, these are great circles.

We also want to rotate the torus $\mathbb{T}$ to ensure the most symmetric outcome. To arrange this, note that $\mathbb{T}$ divides S^3 into a pair of **solid tori**: copies of $S^1 \times D^2$. Thus the optimal position is when $\mathbb{T}$ meets the projection point; after stereographic projection the two solid tori are interchangeable by an isometry of $\mathbb{R}^3$.

We can use quaternions to both fix the parameterisation, giving us the desired $(1,1)$ and $(1,-1)$ curves, and also to move $\mathbb{T}$ as to meet the desired projection point $1 \in S^3 \subset \mathbb{H}$. Solving the second problem first, note that $\frac{1}{\sqrt{2}}(1+j)$ lies in $\mathbb{T}$. Suppose that q is a quaternion satisfying $\frac{1}{\sqrt{2}}(1+j)q = 1$. Thus $q = \frac{1}{\sqrt{2}}(1-j)$. The new parameterisation of the torus is given by post-multiplication by q:

$$\frac{1}{\sqrt{2}}\big(e^{i\alpha} + e^{i\beta} \cdot j\big) \cdot \frac{1}{\sqrt{2}}(1-j) = \frac{1}{2}\big(e^{i\alpha} + e^{i\beta} + (e^{i\beta} - e^{i\alpha}) \cdot j\big).$$

The torus meets the desired projection point when $\alpha = \beta = 0$.

We now solve the second problem, by rotating the coordinates through 45°. Take new coordinates θ, ϕ satisfying $\theta = (\alpha+\beta)/2$ and $\phi = (\alpha-\beta)/2$. So $\alpha = \theta+\phi$ and $\beta = \theta-\phi$. Plugging in and simplifying, the above parametrisation becomes $e^{i\theta}e^{-k\phi}$. Keeping ϕ fixed and varying θ now gives a $(1,1)$ curve, which is also a great circle. Note that we only need $0 \le \theta < 2\pi, 0 \le \phi < \pi$ to cover the whole torus. We permute coordinates and change a sign to get a slightly neater form:

$$e^{i\phi}e^{j\theta} = \big(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi)\big)$$

for $0 \le \theta < 2\pi, 0 \le \phi < \pi$. The operations of permuting the coordinates and changing the sign are symmetries of S^3, so the geometry is unchanged and the surface $\mathbb{T}$ still meets the desired projection point, 1.

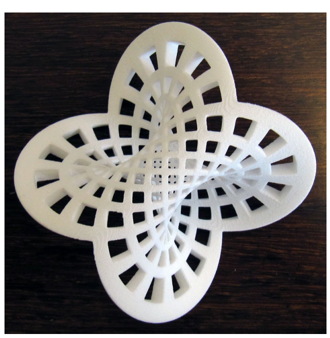
Finding the normal After stereographic projection, we get a 2-dimensional surface in $\mathbb{R}^3 \cup \{\infty\}$. As in Section 3.1, for 3D printing we must thicken the torus to have positive volume. Our plan is to additionally parametrise the **normal** (that is, perpendicular) to the surface, and then thicken in that direction.

Suppose that F is any surface in S^3, with parametrisation $p(\theta,\phi) \in S^3 \subset \mathbb{R}^4$. Compute the tangent vectors $\frac{\partial}{\partial\theta}p(\theta,\phi)$ and $\frac{\partial}{\partial\phi}p(\theta,\phi) \in \mathbb{R}^4$. Since F lies in S^3, these vectors are tangent to S^3 and so perpendicular to $p(\theta,\phi)$, thought of as a vector from the origin. The desired normal vector $n(\theta,\phi)$ is a unit vector that is perpendicular to the three given vectors p, $\frac{\partial}{\partial\theta}p$, and $\frac{\partial}{\partial\phi}p$. This determines n up to sign. Thus finding n amounts to computing the kernel of the matrix with rows p, $\frac{\partial}{\partial\theta}p$ and $\frac{\partial}{\partial\phi}p$. As these vectors vary with the parameters θ and ϕ it is most convenient to compute n via an application of Cramer's rule: n is the determinant of the matrix with first three rows p, $\frac{\partial}{\partial\theta}p$, $\frac{\partial}{\partial\phi}p$, and fourth row the vector $(1,i,j,k)$.

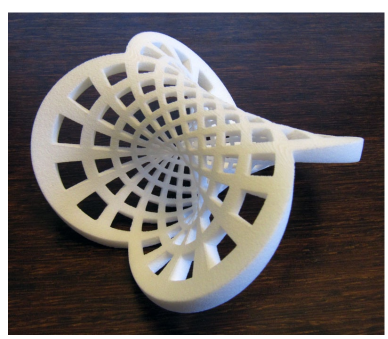
For the above parametrisation of the Clifford torus we find:

$$\begin{array}{rcllllll}
p(\theta,\phi) & = & (& \cos(\theta)\cos(\phi), & \cos(\theta)\sin(\phi), & \sin(\theta)\cos(\phi), & \sin(\theta)\sin(\phi) &) \\
\frac{\partial}{\partial\theta}p(\theta,\phi) & = & (& -\sin(\theta)\cos(\phi), & -\sin(\theta)\sin(\phi), & \cos(\theta)\cos(\phi), & \cos(\theta)\sin(\phi) &) \\
\frac{\partial}{\partial\phi}p(\theta,\phi) & = & (& -\cos(\theta)\sin(\phi), & \cos(\theta)\cos(\phi), & -\sin(\theta)\sin(\phi), & \sin(\theta)\cos(\phi) &) \\
n(\theta,\phi) & = & (& -\sin(\theta)\sin(\phi), & \sin(\theta)\cos(\phi), & \cos(\theta)\sin(\phi), & -\cos(\theta)\cos(\phi) &)
\end{array}$$

We introduce the parameter ε for the thickness of the surface. To move from $p(\theta,\phi)$ in the direction of $n(\theta,\phi)$ we take a "trigonometric average", namely $q(\theta,\phi,\varepsilon) = \cos(\varepsilon)p(\theta,\phi) + \sin(\varepsilon)n(\theta,\phi)$. Since p and n are perpendicular unit vectors, q lies in S^3.



(a) A 2-fold symmetry axis.



(b) A generic viewpoint.

Figure 6: Hopf Fibration Grid, 2011, $10.8 \times 10.8 \times 3.4$ cm.

Since the surface $\mathbb{T}$ passes through the projection point we would require an infinite amount of plastic to print the stereographic projection. We therefore puncture $\mathbb{T}$ near the projection point, cutting a square hole in the grid pattern. This square is visible around the outside of the sculpture shown in Figure 6.

Möbius strip and Klein Bottle A slight variant of the torus gives a Möbius strip:

$$\left\{\left(\cos(\theta)\cos(\phi),\cos(\theta)\sin(\phi),\sin(\theta)\cos(2\phi),\sin(\theta)\sin(2\phi)\right) \mid 0 \le \theta < \pi, 0 \le \phi < \pi\right\}$$

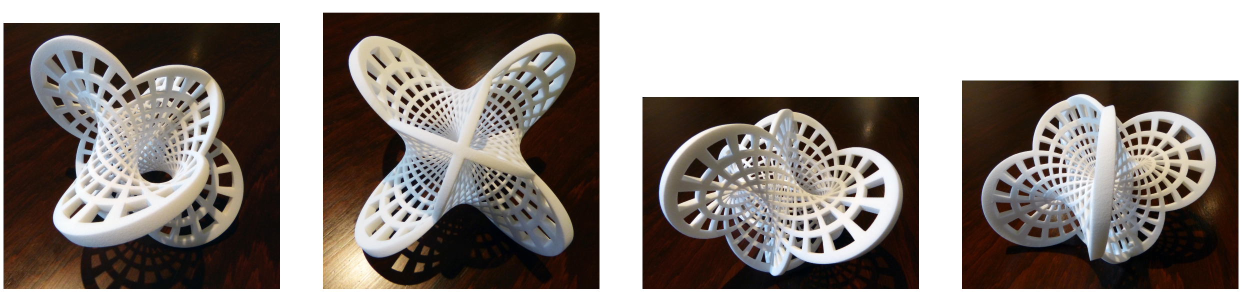
The border of the Möbius strip is given by the points for which θ is 0 or π. Since these points form a geodesic in S^3, the boundary is a circline in $\mathbb{R}^3$ by the circline property. With the given parameterisation, stereographic projection from $(0,0,-1,0)$ gives a circular boundary as in Figure 7. The normal vector is calculated analogously to the torus case:

$$\begin{array}{rclrrrrr}
p(\theta,\phi) & = & & (& \cos(\theta)\cos(\phi), & \cos(\theta)\sin(\phi), & \sin(\theta)\cos(2\phi), & \sin(\theta)\sin(2\phi) &) \\
\frac{\partial}{\partial\theta}p(\theta,\phi) & = & & (& -\sin(\theta)\cos(\phi), & -\sin(\theta)\sin(\phi), & \cos(\theta)\cos(2\phi), & \cos(\theta)\sin(2\phi) &) \\
\frac{\partial}{\partial\phi}p(\theta,\phi) & = & & (& -\cos(\theta)\sin(\phi), & \cos(\theta)\cos(\phi), & -2\sin(\theta)\sin(2\phi), & 2\sin(\theta)\cos(2\phi) &) \\
n(\theta,\phi) & = & \frac{1}{\sqrt{1+\sin^2(\theta)}} & (& -2\sin(\theta)\sin(\phi), & 2\sin(\theta)\cos(\phi), & \cos(\theta)\sin(2\phi), & -\cos(\theta)\cos(2\phi) &)
\end{array}$$

Again the surface is punctured with a square hole in the grid pattern. If we extend the surface beyond its boundary, taking $0 \le \theta < 2\pi$, we get the union of two punctured Möbius strips along their boundaries, which is a twice punctured Klein bottle as shown in Figure 8.

(a) A 2-fold symmetry axis. (b) A generic viewpoint.

Figure 7: Round Möbius Strip, 2011, $15.2 \times 10.9 \times 6.2$ cm.



(a) A generic viewpoint. (b) The 4-fold symmetry axis. (c) One of the 2-fold symmetry axes. (d) The other 2-fold symmetry axis.

Figure 8: Round Klein Bottle, 2011, $15.2 \times 15.2 \times 10.9$ cm.

Torus knot A further variant gives a parameterisation of a torus knot, in this case the trefoil knot:

$$\left\{\left(\cos(\theta)\cos(\phi),\cos(\theta)\sin(\phi),\sin(\theta)\cos((3/2)\phi),\sin(\theta)\sin((3/2)\phi)\right) \mid 0 \le \phi < 4\pi\right\}$$

Here θ has a fixed value, greater than 0 and smaller than $\pi/2$. Altering the fraction $3/2$ will produce other torus knots. The normal vector can be defined in an analogous way to previously, although for this model we used an "alternative" to the normal vector, choosing

$$n(\theta,\phi) = \big(-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), \cos(\theta)\sin((3/2)\phi), -\cos(\theta)\cos((3/2)\phi)\big).$$

Using the local coordinates $(\theta,\phi,\varepsilon)$, we can add small features to the sculpture, using any shape we could define in ordinary 3-dimensional space. In the case shown in Figure 9, we add cog teeth, which are simply truncated pyramids in $(\theta,\phi,\varepsilon)$ coordinates. The alternative normal vector adds a slight shear slope to the teeth, which we feel is aesthetically preferable.

Figure 9: Knotted Cog, 2011, $3.8 \times 3.4 \times 1.3$ cm.

4 Future directions

Our sculptures are tangible representives of topological and geometric abstractions. In order to do this, we naturally must construct designs that occur in $\mathbb{R}^3$: that is, actual space. In each case we attempted to choose the most canonical such geometries available and then the most faithful projections.

There is a wild array of further topological and combinatorial objects. For example, there is a rich theory of knots and surfaces and their interrelations. We have not yet found (or perhaps better, understood) satisfactory geometric representations, or at least representatives which map to $\mathbb{R}^3$ in satisfactory ways. An example of the latter problem would be surfaces of genus at least two. These have nice hyperbolic structures, but they cannot be mapped into $\mathbb{R}^3$ in a very satisfying way.

References

[1] Alan F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York, 1983.

[2] John H. Conway and Derek A. Smith, *On quaternions and octonions: their geometry, arithmetic, and symmetry*, A K Peters Ltd., Natick, MA, 2003.

[3] H. S. M. Coxeter, *Regular polytopes*, third ed., Dover Publications Inc., New York, 1973.

[4] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty.

[5] George W. Hart, *4d polytope projection models by 3d printing*, to appear in Hyperspace.

[6] Fritz H. Obermeyer, *Jenn3d*, a computer program for visualizing Coxeter polytopes, available from http://www.math.cmu.edu/ fho/jenn/.

[7] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.