# CURVE COMPLEXES WITH CONNECTED BOUNDARY ARE RIGID 

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#### Abstract

When the boundary of the curve complex is connected any quasi-isometry is bounded distance from a simplicial automorphism. As a consequence, when the boundary is connected the quasi-isometry type of the curve complex determines the homeomorphism type of the surface.


## 1. Introduction

The curve complex of a surface was introduced into the study of Teichmüller space by Harvey [5] as an analogue of the Tits building of a symmetric space. Since then the curve complex has played a key role in both the study of the cohomology of the mapping class group and also the classification of infinite volume hyperbolic three-manifolds.

Our motivation is the work of Masur and Minsky [11, 12], which focuses on the coarse geometric structure of the curve complex, the mapping class group, and other combinatorial moduli spaces. It is a sign of the richness of low-dimensional topology that the geometric structure of such objects is not well understood.

Suppose that $S=S_{g, n}$ is an orientable, connected, compact surface with genus $g$ and $n$ boundary components. Let $\mathcal{C}(S)$ denote the curve complex of $S$. When $S$ is a sphere, disk or pants then $\mathcal{C}(S)$ is empty and we disregard these cases. Here is the main theorem:
Theorem 6.1. Suppose that $\partial \mathcal{C}(S)$ is connected. Then every quasiisometry of $\mathcal{C}(S)$ is bounded distance from a simplical automorphism.
Remark 1.1. Leininger and the second author [10] have shown that the boundary of curve complex is connected if $S$ has genus at least four, or if the genus is at least two and $\partial S$ is non-empty.

Recall that $\mathrm{QI}(\mathcal{X})$ is the group of quasi-isometries of a geodesic metric space $\mathcal{X}$, modulo the following equivalence relation: quasi-isometries $f$ and $g$ are equivalent if and only if there is a constant C so that for

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every $x \in \mathcal{X}$ we have $d_{\mathcal{X}}(f(x), g(x)) \leq \mathrm{C}$. There is a homomorphism from the isometry group to the quasi-isometry group.
Corollary 1.2. Suppose that $\partial \mathcal{C}(S)$ is connected. Then $\operatorname{QI}(\mathcal{C}(S))$ is isomorphic to $\operatorname{Aut}(\mathcal{C}(S))$, the group of simplical automorphisms.

Proof. Recall that all elements of $\operatorname{Aut}(\mathcal{C}(S))$ are isometries. So we have a homomorphism $\operatorname{Aut}(\mathcal{C}(S)) \rightarrow \mathrm{QI}(\mathcal{C}(S))$.

This map is an injection: Suppose that $\phi \in \operatorname{Aut}(\mathcal{C}(S))$ is not the identity element. It follows that $\phi(\alpha) \neq \alpha$, for some curve $\alpha$.

Now, by Ivanov's Theorem [6], this $\phi$ is induced by some homeomorphism, again called $\phi$. Consider the action of $\phi$ on $\mathcal{P} \mathcal{M} \mathcal{F}(S)$. Then there is a small neighborhood $U \subset \mathcal{P} \mathcal{M F}(S)$ of $\alpha$ so that $\phi(U) \cap U=\emptyset$. Thus $\phi$ moves some ending lamination. By Klarreich's Theorem (see Theorem 2.3 below) we deduce that $\phi$ moves some point of $\partial \mathcal{C}(S)$. Finally, any isometry of a Gromov hyperbolic space moving a point of the boundary is nontrival in the quasi-isometry group.

On the other hand, when $\partial \mathcal{C}(S)$ is connected, Theorem 6.1 implies that the homomorphism $\operatorname{Aut}(\mathcal{C}(S)) \rightarrow \operatorname{QI}(\mathcal{C}(S))$ is a surjection.

Remark 1.3. If $\partial \mathcal{C}(S)$ is not connected then the conclusion of Corollary 1.2 may fail. For example, when $S$ is a four-holed sphere or onceholed torus the curve complex is a copy of the Farey graph. Thus $\mathcal{C}(S)$ is quasi-isometric to $T_{\infty}$, the countably infinite valence tree [1]. Hence $\operatorname{QI}(\mathcal{C}(S))$ is uncountable while $\operatorname{Aut}(\mathcal{C}(S))=\operatorname{PGL}(2, \mathbb{Z})$ is countable.
Theorem 1.4. Suppose that $S$ and $\Sigma$ are surfaces with $\mathcal{C}(S)$ quasiisometric to $\mathcal{C}(\Sigma)$. Suppose that $\partial \mathcal{C}(S)$ is connected. Suppose that neither $S$ nor $\Sigma$ is homeomorphic to $S_{2}$ or $S_{1,2}$. Then $S$ and $\Sigma$ are homeomorphic.

Proof. By Corollary 1.2 the automorphism groups of $\mathcal{C}(S)$ and $\mathcal{C}(\Sigma)$ are isomorphic. Ivanov's Theorem [6] tells us that, for these surfaces, the simplicial automorphism group is isomorphic to the mapping class group. Finally, it is well-known that surfaces are characterized, up to homeomorphism, by their mapping class groups [7].

The proof of Theorem 6.1 has the following ingredients: We begin by examining pairs of ending laminations. Such a pair is cobounded if all subsurface projections to strict subsurfaces of $S$ are uniformly bounded. We prove:

Theorem 4.2. Suppose that $\partial \mathcal{C}(S)$ is connected and $\phi: \mathcal{C}(S) \rightarrow \mathcal{C}(\Sigma)$ is a quasi-isometric embedding. Then the induced map on boundaries preserves the coboundedness of ending laminations.

This result is where the connectedness of the boundary is used in an essential fashion. Now recall that $\mathcal{M}(S)$ denotes the marking complex of the surface $S$. There is a natural projection map $p: \mathcal{M}(S) \rightarrow \mathcal{C}(S)$ which is mapping class group equivarient. We show:
Theorem 5.1. Suppose that $\partial \mathcal{C}(S)$ is connected and $\phi: \mathcal{C}(S) \rightarrow \mathcal{C}(\Sigma)$ is a q-quasi-isometric embedding. Then $\phi$ induces a map $\Phi: \mathcal{M}(S) \rightarrow$ $\mathcal{M}(\Sigma)$ so that the diagram

commutes up to an additive error. Furthermore, the map $\Phi$ is coarsely distance non-increasing: there is a constant Q so that for all markings $m, m^{\prime} \in \mathcal{M}(S)$ we have

$$
d_{\mathcal{M}}\left(\mu, \mu^{\prime}\right) \leq \mathrm{Q} \cdot d_{\mathcal{M}}\left(m, m^{\prime}\right)+\mathrm{Q} .
$$

where $\mu=\Phi(m)$ and $\mu^{\prime}=\Phi\left(m^{\prime}\right)$.
When $\phi: \mathcal{C}(S) \rightarrow \mathcal{C}(S)$ is a quasi-isometry we apply Theorem 5.1 in both directions. It follows that the induced map $\Phi$ is a quasi-isometry of marking complexes. We now turn to a recent claim of Behrstock, Kleiner, Minsky and Mosher as well as Hamenstädt [4]:
Theorem 1.5. Every quasi-isometry of $\mathcal{M}(S)$ is bounded distance from the action of a homeomorphism of $S$.

Our result, Theorem 6.1, now immediately follows from Theorem 5.1.
Acknowledgements. This paper was sparked by a question of Slava Matveyev, asking whether the intrinsic metric on the complement of a ball in the complex of curves is Gromov hyperbolic.

## 2. Background

Hyperbolic spaces. Suppose that $r, s, \mathrm{q} \in \mathbb{R}_{\geq 0}$. If $r \leq \mathrm{q} s+\mathrm{q}$ we write $r \prec_{\mathrm{q}} s$. If $r \prec_{\mathrm{q}} s$ and $s \prec_{\mathrm{q}} r$ we write $r \asymp_{\mathrm{q}} s$, omitting the subscript when clear from context.

A geodesic metric space $\mathcal{X}$ is Gromov hyperbolic if there is a hyperbolicity constant, $\delta_{\mathcal{X}}$, so that every triangle is $\delta_{\mathcal{X}}$-slim. For every triple of vertices $x, y, z \in \mathcal{X}$ and every triple of geodesics $[x, y],[y, z],[z, x]$ the $\delta_{\mathcal{X}}$ neighborhood of $[x, y] \cup[y, z]$ contains $[z, x]$.

Suppose that $\left(\mathcal{X}, d_{\mathcal{X}}\right)$ and $\left(\mathcal{Y}, d_{\mathcal{Y}}\right)$ are geodesic metric spaces and $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a map. Then $f$ is a q-quasi-isometric embedding if for all
$x, y \in \mathcal{X}$ we have

$$
d_{\mathcal{X}}(x, y) \asymp_{\mathrm{q}} d_{\mathcal{Y}}(f(x), f(y)) .
$$

Two maps $f, g: \mathcal{X} \rightarrow \mathcal{Y}$ are $\mathbf{C}$-close if for all $x \in \mathcal{X}$ we find

$$
d_{\mathcal{Y}}(f(x), g(x)) \leq \mathrm{C}
$$

If $f: \mathcal{X} \rightarrow \mathcal{Y}$ and $g: \mathcal{Y} \rightarrow \mathcal{X}$ are q-quasi-isometric embeddings so that $f \circ g$ and $g \circ f$ are q-close to identity maps then $f$ and $g$ are q-quasi-isometries.

A quasi-isometric embedding of an interval $[s, t] \subset \mathbb{Z}$, with the usual metric, is called a quasi-geodesic. In hyperbolic spaces quasi-geodesics are stable:
Lemma 2.1. Suppose that $\left(\mathcal{X}, d_{\mathcal{X}}\right)$ is $\delta_{\mathcal{X}}$-hyperbolic. Suppose that $f:[s, t] \rightarrow \mathcal{X}$ is a q-quasi-geodesic. Then there is a constant $\mathrm{M}=$ $\mathrm{M}\left(\delta_{\mathcal{X}}, \mathbf{q}\right)$ so that for any $[p, q] \subset[s, t]$ the image $f([p, q])$ and any geodesic $[f(p), f(q)]$ have Hausdorff distance at most M in $\mathcal{X}$.

See [2] for further background on hyperbolic spaces.
Curve Complexes. Let $S=S_{g, n}$ be a surface, as before. Define the vertex set of the curve complex, $\mathcal{C}(S)$, to be the set of simple closed curves in $S$ that are essential and non-peripherial, considered up to isotopy.

When the complexity $\xi(S)=3 g-3+n$ is at least two, distinct vertices $a, b \in \mathcal{C}(S)$ are connected by an edge if they have disjoint representatives.

When $\xi(S)=1$ vertices are connected by an edge if there are representatives with geometric intersection exactly one for the once-holed torus or exactly two for the four-holed sphere. This gives the Farey graph. When $S$ is an annulus the vertices are essential embedded arcs, considered up to isotopy fixing the boundry pointwise. Vertices are connected by an edge if there are representatives with disjoint interiors.

For any vertices $a, b \in \mathcal{C}(S)$ define the distance $d_{S}(a, b)$ to be the minimal number of edges appearing in an edge path between $a$ and $b$.
Theorem 2.2 (Masur-Minsky [11]). The complex of curves $\mathcal{C}(S)$ is Gromov hyperbolic.

We use $\delta_{S}$ to denote the hyperbolicity constant of $\mathcal{C}(S)$.
Boundary of the curve complex. Let $\partial \mathcal{C}(S)$ be the Gromov boundary of $\mathcal{C}(S)$. This is the space of quasi-geodesic rays in $\mathcal{C}(S)$ modulo the following equivalence relation: rays $f, g:[0, \infty) \rightarrow \mathcal{C}(S)$ are equivalent if and only if there is a constant C so that $f$ and $g$ are $\mathrm{C}-$ close.

Recall that $\mathcal{P} \mathcal{M} \mathcal{L}(S)$ is the projectivized space of measured laminations in $(S)$. A measured lamination $\ell$ is filling if every component $S \backslash \ell$. Take $\mathcal{F} \mathcal{L}(S) \subset \mathcal{P} \mathcal{M} \mathcal{L}(S)$ to be the set of filling laminations with the subspace topology. Define $\mathcal{E} \mathcal{L}(S)$, the space of ending laminations, to be the quotient of $\mathcal{F} \mathcal{L}(S)$ obtained by forgetting the measures. See [8] for an expansive discussion of laminations.
Theorem 2.3 (Klarrich [9]). There is a mapping class group equivariant homeomorphism between $\partial \mathcal{C}(S)$ and $\mathcal{E L}(S)$.

We define $\overline{\mathcal{C}(S)}=\mathcal{C}(S) \cup \partial \mathcal{C}(S)$. Note that $\partial \mathcal{C}(S)$ is not connected when $S$ is an annulus, once-holed torus or four-holed sphere. On the other hand, Remark 1.1 gives many examples where $\partial \mathcal{C}(S)$ is connected.

Subsurface projection. Suppose that $Z \subset S$ is an essential subsurface: $Z$ is embedded, every component of $\partial Z$ is essential in $S$ and $Z$ is not a boundary parallel annulus. An essential subsurface $Z \subset S$ is strict if $Z$ is not homeomorphic to $S$.

We say that a curve or lamination cuts the subsurface $Z$ if every representative intersects $Z$. If a curve $b$ does not cut $Z$ we say that $b$ misses $Z$.

Suppose now that $a, b \in \overline{\mathcal{C}(S)}$ both cut a strict subsurface $Z$. Define the subsurface projection distance $d_{Z}(a, b)$ as follows: tighten $a$ and $b$ with respect to $\partial Z$ to realize the intersection number. Surger the arcs of $a \cap Z$ to obtain $\pi_{Z}(a)$, a finite set of vertices in $\mathcal{C}(Z)$. Notice that $\pi_{Z}(a)$ has uniformly bounded diameter in $\mathcal{C}(Z)$, independent of $a, Z$ and $S$. Define

$$
d_{Z}(a, b)=\max \left\{d_{Z}\left(a^{\prime}, b^{\prime}\right) \mid a^{\prime} \in \pi_{Z}(a), b^{\prime} \in \pi_{Z}(b)\right\}
$$

We now recall the Lipschitz Projection Lemma [12, Lemma 2.3]:
Lemma 2.4 (Masur-Minsky). Suppose that $\left\{a_{i}\right\}_{i=0}^{N} \subset \mathcal{C}(S)$ is a path where every vertex cuts $Z \subset S$. Then $d_{Z}\left(a_{0}, a_{N}\right) \leq 2 N$.

For geodesics, more is true [12, Bounded Geodesic Image Theorem]:
Theorem 2.5 (Masur-Minsky). There is a constant $\mathrm{C}_{0}=\mathrm{C}_{0}(S)$ with the following property. For any strict subsurface $Z$ and any points $a, b \in \overline{\mathcal{C}(S)}$, if every vertex of the geodesic $[a, b]$ cuts $Z$ then $d_{Z}(a, b) \leq$ $\mathrm{C}_{0}(S)$.

Marking complex. We now discuss the marking complex: A marking $m$ is a pants decomposition base $(m)$ of $S$ together with a transversal $t_{a}$ for each element $a \in \operatorname{base}(m)$. To define $t_{a}$, let $X_{a}$ be the non-pants component of $S \backslash(\operatorname{base}(m) \backslash\{a\})$. Then any vertex of $\mathcal{C}\left(X_{a}\right)$ not equal to $a$ and meeting $a$ minimally can serve as the transversal $t_{a}$.

In [12], Masur and Minsky define elementary moves on markings. The set of markings and these moves define a locally finite graph, the marking complex, $\mathcal{M}(S)$. We recall that if $m$ and $m^{\prime}$ differ by an elementary move then the total intersection number $\iota\left(m, m^{\prime}\right)$ is uniformly bounded. It follows that there is a constant J so that for any subsurface $Z$ of $S$, we have $d_{Z}\left(m, m^{\prime}\right) \leq \mathrm{J}$. A converse also holds: for every constant C there is a bound B with the following property. If $d_{Z}\left(m, m^{\prime}\right) \leq \mathrm{C}$ for all $Z \subseteq S$ then $d_{\mathcal{M}}\left(m, m^{\prime}\right) \leq \mathrm{B}$; the markings $m, m^{\prime}$ differ by at most B elementary moves.
2.1. Cobounded. A pair of curves, markings or laminations $a, b$ are C-cobounded if $d_{Z}(a, b) \leq \mathrm{C}$ for all strict subsurfaces $Z \subset S$ cut by both $a$ and $b$.

Lemma 2.6. Suppose that $m$ is a marking and $a \in$ base( $m$ ). Suppose that $\ell$ is an ending lamination where ( $a, \ell$ ) is C -cobounded, for sufficiently large C . Then there is a mapping class $\phi$, supported in $S \backslash a$, so that $(m, \phi(k))$ are 2 C -cobounded.

Proof. Here.
Lemma 2.7. Suppose that $a \in \mathcal{C}(S)$ and $b \in \overline{\mathcal{C}(S)}$. Then there is a point $\ell \in \partial \mathcal{C}(S)$ so that the vertex a lies in the one-neighborhood of $[b, \ell]$.

Furthermore, if the pair $(a, b)$ is C -cobounded, for C sufficently larger than $\mathrm{C}_{0}$, then there is such an $\ell$ where the pairs $(b, \ell)$ and $(a, \ell)$ are $3 \mathrm{C}-$ cobounded.

Proof. Both claims are easy to obtain if $S$ is a once-holed torus or fourholed sphere. Likewise, if $a=b$ then the conclusion is clear. For the remainder of the proof we assume that $\xi(S)>1$ and $a \neq b$.

Let be $Y$ be a component of $S \backslash a$ which is not a pants and which meets $b$. Take $c$ to be a curve in $Y$ which is either non-separating or which cuts a pair of pants off of $S$. We may assume that $b$ and $c$ meet.

Now pick $k \in \partial \mathcal{C}(S)$. We now rotate $[c, k]$ : let $Z$ be the component of $S \backslash c$ which is not a pants. Apply a homeomorphism of $S$ with support in $Z$. Let $\ell$ be the image of $k$. We may arrange matters so that $d_{Z}(b, \ell)>C_{0}$. It follows from Theorem 2.5 that $c \in[b, \ell]$. Thus $d_{S}(a,[b, \ell]) \leq 1$ and we are done.

Now suppose that $(a, b)$ is C -cobounded. We follow the above construction, with more careful choices. Choose $c \subset Y$ as above so that

$$
\begin{equation*}
d_{Y}(b, c) \geq 3 \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { for all } W \subseteq Y, \quad d_{W}(b, c) \leq \mathrm{C} \tag{2.9}
\end{equation*}
$$

For $k$ we further assume that $(c, k)$ is C -cobounded. This is possible, for example, by choosing a subray of a quasi-axis for a pseudo-Anosov map and then moving the ray so that the initial point is $c$.

As above we rotate $[c, k]$ inside of $Z$, letting $\ell$ be the image of $k$. Thus $(c, \ell)$ is also C -cobounded. We may arrange matters so that

$$
\begin{equation*}
\mathrm{C}_{0}<d_{Z}(b, \ell) \leq \mathrm{C} \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { for all } W \subsetneq Z, \quad d_{W}(b, \ell) \leq \mathrm{C} \tag{2.11}
\end{equation*}
$$

To show that $(b, \ell)$ is cobounded, we must estimate $d_{W}(b, \ell)$ for every strict subsurface $W$ meeting $b$. If $W$ is disjoint from $c$ then $W \subseteq Z$ and we are done by Equation 2.10 or Equation 2.11. If $c$ cuts $W$ then

$$
d_{W}(b, \ell) \leq d_{W}(b, c)+d_{W}(c, \ell) \leq \mathrm{C}+\mathrm{C}=2 \mathrm{C},
$$

by Equation 2.9 and the fact that $[c, \ell]$ is C -cobounded. To show that $(a, \ell)$ is cobounded consider any $W$ cutting $a$. Thus $W$ meets $Y$. By Equation 2.8 one of $b$ or $c$ cuts $W$. If $W$ is cut by $b$ then

$$
d_{W}(a, \ell) \leq d_{W}(a, b)+d_{W}(b, \ell) \leq \mathrm{C}+2 \mathrm{C}=3 \mathrm{C},
$$

because $(a, b)$ is C -cobounded and, as shown above, $(b, \ell)$ is 2 C -cobounded. Now, if $W$ is cut by $c$ then

$$
d_{W}(a, \ell) \leq d_{W}(a, c)+d_{W}(c, \ell) \leq 2+\mathrm{C},
$$

since $a$ and $c$ are disjoint and $(c, \ell)$ is C -cobounded.
2.2. Tight geodesics. We will need [13, Lemma 5.14]:

Lemma 2.12 (Minsky). If $a, b \in \overline{\mathcal{C}(S)}$ then there is a tight geodesic $[a, b] \subset \mathcal{C}(S)$ connecting them.

Here, tight is technical hypothesis which provides a certain kind of local finiteness. The only property of tight geodesics used in this paper is:
Lemma 2.13 (Minsky). If $(a, b)$ is a C -cobounded pair of points in $\overline{\mathcal{C}(S)}$ and $c$ is a vertex of a tight geodesic connecting $a$ to $b$ then $(a, c)$ and $(c, b)$ are C -cobounded.

All geodesics from here on are assumed to be tight. We will need:
Lemma 2.14. Suppose that $a \in \mathcal{C}(S)$ and $\ell \in \partial \mathcal{C}(S)$. Suppose that $(a, \ell)$ is C -cobounded, where C is sufficiently large. Then there is a marking $\mu$ so that $a \in \operatorname{base}(\mu)$ and $(b, \mu)$ are 2 C -cobounded.

See [REF] for a proof.

## 3. The shell is connected

Let $\mathcal{B}(z, \mathrm{R})$ be the ball of radius R about $z \in \mathcal{C}(S)$. The difference of concentric balls is called a shell.

Proposition 3.1. Suppose that $\partial \mathcal{C}(S)$ is connected and $\mathrm{D} \geq \max \left\{\delta_{S}, 1\right\}$. Then for any $R \geq 0$ the shell

$$
\mathcal{B}(z, \mathrm{R}+2 \mathrm{D}) \backslash \mathcal{B}(z, \mathrm{R}-1)
$$

is connected.
One difficulty in the proof lies in pushing points of the inner boundary into the interior of the shell. To deal with this we use the fact that $\mathcal{C}(S)$ has no dead ends.
Lemma 3.2. Fix vertices $z, a \in \mathcal{C}(S)$. Suppose $d_{S}(z, a)=R$. Then there is a vertex $a^{\prime} \in \mathcal{C}(S)$ with $d_{S}\left(a, a^{\prime}\right) \leq 2$ and $d_{S}\left(z, a^{\prime}\right)=R+1$.

Note that this implies that any geodesic $\left[a, a^{\prime}\right]$ lies outside of $\mathcal{B}(z, R-$ 1). For a proof of Lemma 3.2, see Proposition 3.1 of [14].

Proof of Proposition 3.1. Fix $z \in \mathcal{C}(S)$. Define a norm on $\overline{\mathcal{C}(S)}$ by:

$$
\langle a, b\rangle_{z}=\inf \left\{d_{S}(z,[a, b])\right\}
$$

where the infimum ranges over all geodesics $[a, b]$. For every $k \in \partial \mathcal{C}(S)$ let

$$
U(k)=\left\{\ell \in \partial \mathcal{C}(S) \mid\langle k, \ell\rangle_{z}>\mathrm{R}+2 \mathrm{D}\right\} .
$$

The set $U(k)$ is a neighborhood of $k$, by the definition of the topology on the boundary [3]. Notice that if $\ell \in U(k)$ then $k \in U(\ell)$.

Consider the set $V(k)$ of all $\ell \in \partial \mathcal{C}(S)$ so that there is a finite sequence $k=k_{0}, k_{1}, \ldots, k_{N}=\ell$ with $k_{i+1} \in U\left(k_{i}\right)$ for all $i$. Now, if $\ell \in V(k)$ then $U(\ell) \subset V(k)$; thus $V(k)$ is open. If $\ell$ is a limit point of $V(k)$ then there is a sequence $\ell_{i} \in V(k)$ entering every neighborhood of $\ell$. So there is some $i$ where $\ell_{i} \in U(\ell)$. Thus $\ell \in U\left(\ell_{i}\right) \subset V(k)$ and we find that $V(k)$ is closed. Finally, as $\partial \mathcal{C}(S)$ is connected, $V(k)=\partial \mathcal{C}(S)$.

Let $a^{\prime}, b^{\prime}$ be any vertices in the shell $\mathcal{B}(z, \mathrm{R}+2 \mathrm{D}) \backslash \mathcal{B}(z, \mathrm{R}-1)$. We connect $a^{\prime}$, via a path in the shell, to a vertex $a$ so that $d_{S}(z, a)=\mathrm{R}+\mathrm{D}$. We do the same for $b^{\prime}$ and $b$. This is always possible: points far from $z$ may be pushed inward along geodesics and points near $z$ may be pushed outward by Lemma 3.2.

By Lemma 2.7 there are points $k, \ell \in \partial \mathcal{C}(S)$ so that there are geodesic rays $[z, k]$ and $[z, \ell]$ within distance one of $a$ and $b$ respectively. Connect $k$ to $\ell$ by a chain of points $\left\{k_{i}\right\}$ in $V(k)$, as above. Define $a_{i} \in\left[z, k_{i}\right]$ so that $d_{S}\left(z, a_{i}\right)=R+\mathrm{D}$. Connect $a$ to $a_{0}$ via a path of length at most 2 .

Notice that $d_{S}\left(a_{i},\left[k_{i}, k_{i+1}\right]\right)>\mathrm{D} \geq \delta$. By hyperbolicity, the vertiex $a_{i}$ is $\delta$-close to $\left[z, k_{i+1}\right]$. Thus $a_{i}$ and $a_{i+1}$ may be connected inside of the shell via a path of length at most $2 \delta$.

## 4. Image of a cobounded geodesic is cobounded

We begin with a simple lemma:
Lemma 4.1. For every C and R there is a constant K with the following property: Let $[a, b] \subset \mathcal{C}(S)$ be a geodesic segment of length 2 R with $(a, b)$ being C -cobounded. Let $z$ be the midpoint. Then there is a path $P$ of length at most K connecting $a$ to $b$ outside of $\mathcal{B}(z, \mathrm{R}-1)$.
Proof. There are only finitely many such triples $(a, z, b)$, up the action of the mapping class group. The conclusion now follows from the connectedness of the shell (Proposition 3.1).

Let $\phi: \mathcal{C}(S) \rightarrow \mathcal{C}(\Sigma)$ be a q-quasi-isometric embedding. Note that $\phi$ extends to a one-to-one continuous map from $\partial \mathcal{C}(S)$ to $\partial \mathcal{C}(\Sigma)$.
Theorem 4.2. Let $k$ and $\ell$ be a pair of C -cobounded laminations in $\partial \mathcal{C}(S)$. Then $\kappa=\phi(k)$ and $\lambda=\phi(\ell)$ are $\mathrm{C}^{\prime}$-cobounded, where $\mathrm{C}^{\prime}$ depends on C, q, $\xi(S)$ and $\xi(\Sigma)$ only.

Proof. For every strict subsurface $\Omega \subset \Sigma$ we must bound $d_{\Omega}(\kappa, \lambda)$ from above. Now, if $d_{\Sigma}(\partial \Omega,[\kappa, \lambda]) \geq 2$, then by Bounded Geodesic Image Theorem (2.5) we find $d_{\Omega}(\kappa, \lambda) \leq \mathrm{C}_{0}(\Sigma)$ and we are done.


Figure 1. The ball around $z$ has radius R while the ball around $\partial \Omega$ has radius ???

Now suppose $d_{\Sigma}(\partial \Omega,[\kappa, \lambda]) \leq 1$. Note that $[\kappa, \lambda]$ lies in the $\mathrm{M}-$ neighborhood of $\phi([k, \ell])$ by Lemma 2.1. So we can find a vertex $z \in$
$[k, \ell]$ so that $d_{\Sigma}(\phi(z), \partial \Omega) \leq \mathrm{M}+1$. Set $\mathrm{R}=\mathrm{q}(\mathrm{q}+2 \mathrm{M}+3)+\mathrm{q}$. Thus

$$
\begin{aligned}
d_{S}(y, z) \geq \mathrm{R} & \Longrightarrow d_{\Sigma}(\phi(y), \phi(z)) \geq \mathrm{q}+2 \mathrm{M}+3 \\
& \Longrightarrow d_{\Sigma}(\phi(y), \partial \Omega) \geq \mathrm{q}+\mathrm{M}+2
\end{aligned}
$$

Let $a$ and $b$ be the intersections of $[k, \ell]$ with $\partial \mathcal{B}(z, \mathrm{R})$, chosen so that $[k, a]$ and $[b, \ell]$ meet $\mathcal{B}(z, \mathrm{R})$ at the vertices $a$ and $b$ only. Connect $a$ to $b$ via a path $P$ of length K as provided by Lemma 4.1.

Let $\alpha=\phi(a)$ and $\beta=\phi(b)$. Now, any consecutive vertices of $P$ are mapped by $\phi$ to vertices of $\mathcal{C}(\sigma)$ that are at distance at most $2 \mathbf{q}$. Connecting these by geodesic segments gives a path $\Pi$ from $\alpha$ to $\beta$.

Note that $\Pi$ has length at most qK. Since every vertex of $\phi(P)$ is $\mathrm{q}+\mathrm{M}+2$-far from $\partial \Omega$ every vertex of $\Pi$ is $\mathrm{M}+2$-far from $\partial \Omega$. So every vertex of $\Pi$ cuts $\Omega$. It follows that $d_{\Omega}(\alpha, \beta) \leq 2 \mathrm{qK}$, by Lemma 2.4.

All that remains is to bound $d_{\Omega}(\kappa, \alpha)$ and $d_{\Omega}(\beta, \lambda)$. It suffices, by the Bounded Geodesic Image Theorem, to show that every vertex of $[\kappa, \alpha]$ cuts $\Omega$. The same will hold for $[\beta, \lambda]$.

Every vertex of $[\kappa, \alpha]$ is M -close to a vertex of $\phi([k, a])$. But each of these is $\mathbf{q}+\mathrm{M}+2$-far from $\partial \Omega$. This completes the proof.

## 5. The induced map on markings

In this section, for every a quasi-isometric embedding of curve complexes we construct a quasi-isometric embedding of marking complexes.

Let $\mathcal{M}(S)$ and $\mathcal{M}(\Sigma)$ be the marking complexes of $S$ and $\Sigma$ respectively. Let $p: \mathcal{M}(S) \rightarrow \mathcal{C}(S)$ and $\pi: \mathcal{M}(\Sigma) \rightarrow \mathcal{C}(\Sigma)$ be maps that sends a marking to some curve in that marking.
Theorem 5.1. Suppose that $\partial \mathcal{C}(S)$ is connected and $\phi: \mathcal{C}(S) \rightarrow \mathcal{C}(\Sigma)$ is a q-quasi-isometric embedding. Then $\phi$ induces a map $\Phi: \mathcal{M}(S) \rightarrow$ $\mathcal{M}(\Sigma)$ so that the diagram

commutes up to an additive error. Furthermore, the map $\Phi$ is coarsely distance non-increasing: there is a constant Q so that for all markings $m, m^{\prime} \in \mathcal{M}(S)$ we have

$$
d_{\mathcal{M}}\left(\mu, \mu^{\prime}\right) \leq \mathbf{Q} \cdot d_{\mathcal{M}}\left(m, m^{\prime}\right)+\mathbf{Q} .
$$

where $\mu=\Phi(m)$ and $\mu^{\prime}=\Phi\left(m^{\prime}\right)$.

Proof. Let $m$ be a marking on $S$. We construct $\Phi(m)$, a marking on $\Sigma$, as follows. Pick a curve $a \in \operatorname{base}(m)$
ack
and let $(k, \ell)$ be a C -cobounded pair of laminations such that $a$ lies in a one-neighborhood of the geodesic $[k, \ell]$. Let $\kappa=\phi(k), \lambda=\phi(\ell)$ and $\alpha=\phi(a)$. Note that $\alpha$ is at most $(2 q+\mathrm{M})$-away from $[\kappa, \lambda]$ and $(\kappa, \lambda)$ is a $C^{\prime}$-cobounded pair, by Theorem 4.2.

Let $\beta$ be a closest point projection of $\alpha$ to the geodesic $[\kappa, \lambda]$. Using Lemma 2.14, there is a marking $\mu$ so that $\beta \in \operatorname{base}(\mu)$ and $(\mu, \lambda)$ are $2 \mathrm{C}^{\prime}$-cobounded. We say $\mu$ is a marking obtained from the triple ( $a, k, \ell$ ) and we set $\Phi(m)=\mu$.

Let $\mu$ be the marking constructed in (??) from the initial data $\alpha, \kappa$ and $\lambda$. We recall that $d_{\Sigma}(\mu, \alpha) \leq \mathrm{P}$ and for every proper subsurface $\Omega \subsetneq \Sigma$,

$$
d_{\Omega}(\mu, \kappa) \leq \mathrm{P}
$$

We need to show that the map $\phi$ is well defined. That is, if $a^{\prime}$ is a different curve in $m,\left(k^{\prime}, l^{\prime}\right)$ is another C -cobounded pair of laminations whose D-neighborhood contains $a^{\prime}$, then $\mu^{\prime \prime \prime}$, the marking obtained from $\left(a^{\prime}, k^{\prime}, l^{\prime}\right)$, is bounded apart from $\mu$ in $\mathcal{M}(\Sigma)$.

Since $\mathcal{C}(S)$ is $\delta$-hyperbolic, $a$ is in a $(\delta+\mathrm{D})$-neighborhood of either [ $\left.k, l^{\prime}\right]$ or $\left[l, l^{\prime}\right]$. Without loss of generality we can assume that $a$ is in a bounded neighborhood $\left[k, l^{\prime}\right]$. Let $\mu^{\prime}$ be the marking obtained from $\left(a, k, l^{\prime}\right)$ and $\mu^{\prime \prime}$ be the marking obtained from $\left(a, k^{\prime}, l^{\prime}\right)$. We will show that pairs $(\mu, \mu)^{\prime},\left(\mu^{\prime}, \mu^{\prime \prime}\right)$ and $\left(\mu^{\prime \prime}, \mu^{\prime \prime \prime}\right)$ are at most bounded apart in the marking complex by showing that the projections of each pair of markings to any subsurface is uniformly bounded above.

Consider the pair $\left(\mu, \mu^{\prime}\right)$. When $\Omega=\Sigma$, we have

$$
d_{\Sigma}\left(\mu, \mu^{\prime}\right) \leq d_{\Sigma}(\mu, \alpha)+d_{\Sigma}\left(\alpha, \mu^{\prime}\right) \leq 2 \mathrm{P}
$$

If $\Omega$ is a proper subsurface of $\Sigma$ we have

$$
d_{\Omega}\left(\mu, \mu^{\prime}\right) \leq d_{\Omega}\left(\mu, \kappa^{\prime}\right)+d_{\Omega}\left(\kappa^{\prime}, \mu^{\prime}\right) \leq 2 \mathrm{P} .
$$

The proof for othe pairs is similar (each pair shares a lamination in the boundary) except for the pair ( $\mu^{\prime \prime}, \mu^{\prime \prime}$ ) we estimates theire distance in $\mathcal{C}(\Sigma)$ as

$$
d_{\Sigma}\left(\mu^{\prime \prime}, \mu^{\prime \prime \prime}\right) \leq d_{\Sigma}\left(\mu^{\prime \prime}, \alpha\right)+d_{\Sigma}\left(\alpha, \alpha^{\prime}\right)+d_{\Sigma}\left(\alpha^{\prime}, \mu^{\prime \prime \prime}\right) \leq 2(\mathrm{P}+\mathrm{q})
$$

This finishes the proof.

## 6. Rigidity of the curve complex

Theorem 6.1. Suppose that $\partial \mathcal{C}(S)$ is connected. Then every quasiisometry of $\mathcal{C}(S)$ is bounded distance from a simplical automorphism.


Figure 2. Markings $\mu$ and $\mu^{\prime \prime \prime}$ are bounded apart.

Proof. Let $f: \mathcal{C}(S) \rightarrow \mathcal{C}(S)$ be a q-quasi-isometry. We need to find $G \in \mathcal{M C G}(S)$ such that
???
Using Theorem 5.1 we can construct a map $F: \mathcal{M}(S) \rightarrow \mathcal{M}(S)$ such that for every marking $m \in \mathcal{M}(S)$,

$$
d_{\mathcal{C}}(p(F(m)), f(p(m))=O(1) .
$$

By the rigidity of the marking complex, $F$ is is within an additive error of the action of a mapping class on $\mathcal{M}(S)$ that is, there is $G \in \mathcal{M C \mathcal { G }}(S)$ such that

$$
\left.d_{\mathcal{M}}(F(m), G . m)\right)=O(1)
$$

## References

[1] Gregory Bell and Koji Fujiwara. The asymptotic dimension of a curve graph is finite. arXiv:math/0509216.
[2] Martin R. Bridson and André Haefliger. Metric spaces of non-positive curvature. Springer-Verlag, Berlin, 1999.
[3] Mikhael Gromov. Hyperbolic groups. In Essays in group theory, pages 75-263. Springer, New York, 1987.
[4] Ursula Hamenstaedt. Geometry of the mapping class groups III: Quasiisometric rigidity. arXiv:math/0512429.
[5] Willam J. Harvey. Boundary structure of the modular group. In Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pages 245-251, Princeton, N.J., 1981. Princeton Univ. Press.
[6] Nikolai V. Ivanov. Mapping class groups. In Handbook of geometric topology, pages 523-633. North-Holland, Amsterdam, 2002.
[7] Nikolai V. Ivanov. 2007. Personal communication.
[8] Michael Kapovich. Hyperbolic manifolds and discrete groups. Birkhäuser Boston Inc., Boston, MA, 2001.
[9] Erica Klarreich. The boundary at infinity of the curve complex and the relative Teichmüller space. http://nasw.org/users/klarreich/research.htm.
[10] Chris Leininger and Saul Schleimer. On the connectedness of the boundary of the curve complex. In preparation.
[11] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. Invent. Math., 138(1):103-149, 1999. arXiv:math/9804098.
[12] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. Geom. Funct. Anal., 10(4):902-974, 2000. arXiv:math/9807150.
[13] Yair N. Minsky. The classification of Kleinian surface groups, I: Models and bounds. arXiv:math/0302208.
[14] Saul Schleimer. The end of the curve complex. arXiv:math/0608505.
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