CURVE COMPLEXES ARE RIGID

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Abstract. Any quasi-isometry of the complex of curves is bounded distance from a simplicial automorphism. As a consequence, the quasi-isometry type of the curve complex determines the homeomorphism type of the surface.

1. Introduction

The *curve complex* of a surface was introduced into the study of Teichmüller space by Harvey [11] as an analogue of the Tits building of a symmetric space. Since then the curve complex has played a key role in many areas of geometric topology such as the classification of infinite volume hyperbolic three-manifolds, the study of the cohomology of mapping class groups, the geometry of Teichmüller space, and the combinatorics of Heegaard splittings.

Our motivation is the work of Masur and Minsky [22, 23] which focuses on the coarse geometric structure of the curve complex, the mapping class group, and other combinatorial moduli spaces. It is a sign of the richness of low-dimensional topology that the geometric structure of such objects is not well understood.

Suppose that $S = S_{\mathsf{g},\mathsf{b}}$ is an orientable, connected, compact surface with genus g and b boundary components. Define the *complexity* of S to be $\xi(S) = 3\mathsf{g} - 3 + \mathsf{b}$. Let $\mathcal{C}(S)$ be the curve complex of S. Our main theorem is:

Theorem 7.1. *Suppose that $\xi(S) \geq 2$. Then every quasi-isometry of $\mathcal{C}(S)$ is bounded distance from a simplicial automorphism of $\mathcal{C}(S)$.*

Before discussing the sharpness of Theorem 7.1 recall the definition of $\mathrm{QI}(\mathcal{X})$. This is the group of quasi-isometries of a geodesic metric space $\mathcal{X}$, modulo an equivalence relation; quasi-isometries f and g are equivalent if and only if there is a constant d so that for every $x \in \mathcal{X}$ we have $d_{\mathcal{X}}(f(x), g(x)) \leq \mathsf{d}$. Define $\mathrm{Aut}(\mathcal{C}(S))$ to be the group of simplicial automorphisms of $\mathcal{C}(S)$; notice that these are always isometries. From Theorem 7.1 deduce:

Date: May 11, 2010.

Corollary 1.1. *Suppose that $\xi(S) \geq 2$. Then the natural map*

$$\mathrm{Aut}(\mathcal{C}(S)) \to \mathrm{QI}(\mathcal{C}(S))$$

is an isomorphism.

Proof. The map is always an injection. To see this recall Ivanov's Theorem [13, 19, 21]: if $\xi(S) \geq 2$ then every $f \in \mathrm{Aut}(\mathcal{C}(S))$ is induced by some homeomorphism of S, called f_S. (When $\xi(S) = 2$ every automorphism of $\mathcal{C}(S)$ is induced by some homeomorphism of $S_{0,5}$; see [21].) Suppose that $f \in \mathrm{Aut}(\mathcal{C}(S))$ is not the identity element. Then there is some curve a with $f_S(a)$ not isotopic to a. Consider the action of f_S on $\mathcal{PML}(S)$. There is a small neighborhood of a in $\mathcal{PML}(S)$, say U, so that $f_S(U) \cap U = \emptyset$. Since ending laminations are dense, f_S moves some ending lamination of S. By Klarreich's Theorem (see Theorem 2.3 below), we deduce that f moves some point of $\partial\mathcal{C}(S)$. Finally, any isometry of a Gromov hyperbolic space moving a point of the boundary is nontrivial in the quasi-isometry group.

On the other hand, Theorem 7.1 implies that the homomorphism $\mathrm{Aut}(\mathcal{C}(S)) \to \mathrm{QI}(\mathcal{C}(S))$ is a surjection. □

Note that Theorem 7.1 and Corollary 1.1 are sharp. If S is a sphere, disk, or pair of pants then the complex of curves is empty. If S is an annulus then, following [23], the complex $\mathcal{C}(S)$ is quasi-isometric to $\mathbb{Z}$ (see below) and the conclusion of Theorem 7.1 does not hold. If S is a torus, four-holed sphere or once-holed torus then the curve complex is a copy of the Farey graph. Thus $\mathcal{C}(S)$ is quasi-isometric to T_∞, the countably infinite valence tree [4]. Hence $\mathrm{QI}(\mathcal{C}(S))$ is uncountable while $\mathrm{Aut}(\mathcal{C}(S)) = \mathrm{PGL}(2, \mathbb{Z})$ is countable. Thus, for these surfaces the conclusion of Theorem 7.1 does not hold.

We now give an application of Corollary 1.1:

Theorem 1.2. *Suppose that S and Σ are surfaces with $\mathcal{C}(S)$ quasi-isometric to $\mathcal{C}(\Sigma)$. Then either*

- *S and Σ are homeomorphic,*
- $\{S, \Sigma\} = \{S_{0,6}, S_2\}$,
- $\{S, \Sigma\} = \{S_{0,5}, S_{1,2}\}$,
- $\{S, \Sigma\} \subset \{S_{0,4}, S_1, S_{1,1}\}$, *or*
- $\{S, \Sigma\} \subset \{\mathbb{S}, \mathbb{D}, S_{0,3}\}$.

Thus, two curve complexes are quasi-isometric if and only if they are isomorphic.

To prove Theorem 1.2 we require Theorem 7.1 and the following folk theorem:

Theorem A.1. *Suppose that S and Σ are compact, connected, orientable surfaces with $\mathcal{MCG}(S)$ isomorphic to $\mathcal{MCG}(\Sigma)$. Then either*

- *S and Σ are homeomorphic,*
- *$\{S, \Sigma\} = \{S_1, S_{1,1}\}$, or*
- *$\{S, \Sigma\} = \{\mathbb{S}, \mathbb{D}\}$.*

Apparently no proof of Theorem A.1 appears in the literature. In Appendix A we discuss previous work (Remark A.2) and, for completeness, give a proof of Theorem A.1.

Proof of Theorem 1.2. For brevity, we restrict to the case where $\xi(S)$ and $\xi(\Sigma)$ are at least four. By Corollary 1.1 the automorphism groups of $\mathcal{C}(S)$ and $\mathcal{C}(\Sigma)$ are isomorphic. Ivanov's Theorem [13, 19, 21] tells us that the simplicial automorphism group is isomorphic to the mapping class group. Finally, it follows from Theorem A.1 that such surfaces are characterized, up to homeomorphism, by their mapping class groups. □

Outline of the paper. The proof of Theorem 7.1 has the following ingredients. A pair of ending laminations is *cobounded* if the projections of this pair to any strict subsurface of S are uniformly close to each other in the complex of curves of that subsurface (see Definition 2.8).

Theorem 5.2. *Suppose that $\xi(S) \geq 2$ and suppose that $\phi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a quasi-isometric embedding. Then the induced map on boundaries preserves the coboundedness of ending laminations.*

Theorem 5.2 is important in its own right and may have other applications. For example, it may be helpful in classifying quasi-isometric embeddings of one curve complex into another. (See [26].) The proof of Theorem 5.2 uses the following theorem in an essential way:

Theorem 1.3 (Gabai [8])**.** *Suppose that $\xi(S) \geq 2$. Then $\partial\mathcal{C}(S)$ is connected.* □

Remark 1.4. Leininger and the second author [20] previously gave a quite different proof of Theorem 1.3 in the cases where S has genus at least four, or where S has genus at least two and non-empty boundary. Note that Gabai's Theorem is sharp; $\partial\mathcal{C}(S)$ is not connected when S is an annulus, torus, once-holed torus or four-holed sphere.

Let $\mathcal{M}(S)$ denote the *marking complex* of the surface S. We show that a marking on S can be coarsely described by a pair of cobounded ending laminations and a curve in $\mathcal{C}(S)$. Theorem 5.2 implies that a quasi-isometric embedding of $\mathcal{C}(S)$ into $\mathcal{C}(\Sigma)$ induces a map from $\mathcal{M}(S)$ to $\mathcal{M}(\Sigma)$.

Theorem 6.1. *Suppose that $\xi(S) \geq 2$ and $\phi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a q–quasi-isometric embedding. Then ϕ induces a coarse Lipschitz map $\Phi\colon \mathcal{M}(S) \to \mathcal{M}(\Sigma)$ so that the diagram*

$$\begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{\Phi} & \mathcal{M}(\Sigma) \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{C}(S) & \xrightarrow{\phi} & \mathcal{C}(\Sigma) \end{array}$$

commutes up to an additive error. Furthermore, if ϕ is a quasi-isometry then so is Φ.

As the final step of the proof of Theorem 7.1 we turn to a recent theorem of Behrstock, Kleiner, Minsky and Mosher [1]. See also [10].

Theorem 1.5. *Suppose that $\xi(S) \geq 2$ and $S \neq S_{1,2}$. Then every quasi-isometry of $\mathcal{M}(S)$ is bounded distance from the action of a homeomorphism of S.* □

So, if $f\colon \mathcal{C}(S) \to \mathcal{C}(S)$ is a quasi-isometry then Theorem 6.1 gives a quasi-isometry F of marking complexes. This and Theorem 1.5 imply Theorem 7.1 except when $S = S_{1,2}$. But the curve complexes $\mathcal{C}(S_{0,5})$ and $\mathcal{C}(S_{1,2})$ are identical. Therefore to prove Theorem 7.1 for $\mathcal{C}(S_{1,2})$ it suffices to prove it for $\mathcal{C}(S_{0,5})$.

Acknowledgements. This paper was sparked by a question of Slava Matveyev. We thank Dan Margalit for useful conversations.

2. Background

Hyperbolic spaces. A geodesic metric space $\mathcal{X}$ is *Gromov hyperbolic* if there is a *hyperbolicity constant* $\delta \geq 0$ so that every triangle is *δ–slim*: for every triple of vertices $x, y, z \in \mathcal{X}$ and every triple of geodesics $[x, y], [y, z], [z, x]$ the δ–neighborhood of $[x, y] \cup [y, z]$ contains $[z, x]$.

Suppose that $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ are geodesic metric spaces and $f\colon \mathcal{X} \to \mathcal{Y}$ is a map. Then f is q–*coarsely Lipschitz* if for all $x, y \in \mathcal{X}$ we have

$$d_\mathcal{Y}(x', y') \leq \mathsf{q}\, d_\mathcal{X}(x, y) + \mathsf{q}$$

where $x' = f(x)$ and $y' = f(y)$. If, in addition,

$$d_\mathcal{X}(x, y) \leq \mathsf{q}\, d_\mathcal{Y}(x', y') + \mathsf{q}$$

then f is a q–*quasi-isometric embedding*. Two maps $f, g\colon \mathcal{X} \to \mathcal{Y}$ are d–*close* if for all $x \in \mathcal{X}$ we have

$$d_\mathcal{Y}(f(x), g(x)) \leq \mathsf{d}.$$

If $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{X}$ are q–coarsely Lipschitz and also $f \circ g$ and $g \circ f$ are q–close to identity maps then f and g are q–*quasi-isometries*.

A quasi-isometric embedding of an interval $[s,t] \subset \mathbb{Z}$, with the usual metric, is called a *quasi-geodesic*. In hyperbolic spaces quasi-geodesics are *stable*:

Lemma 2.1. *Suppose that $(\mathcal{X}, d_{\mathcal{X}})$ has hyperbolicity constant δ and that $f\colon [s,t] \to \mathcal{X}$ is a q–quasi-geodesic. Then there is a constant $\mathsf{M}_{\mathcal{X}} = \mathsf{M}(\delta, \mathsf{q})$ so that for any $[p,q] \subset [s,t]$ the image $f([p,q])$ and any geodesic $[f(p), f(q)]$ have Hausdorff distance at most $\mathsf{M}_{\mathcal{X}}$ in $\mathcal{X}$.* □

See [6] for further background on hyperbolic spaces.

Curve Complexes. Let $S = S_{\mathsf{g},\mathsf{b}}$, as before. Define the vertex set of the curve complex, $\mathcal{C}(S)$, to be the set of simple closed curves in S that are essential and non-peripheral, considered up to isotopy.

When the complexity $\xi(S)$ is at least two, distinct vertices $a, b \in \mathcal{C}(S)$ are connected by an edge if they have disjoint representatives.

When $\xi(S) = 1$ vertices are connected by an edge if there are representatives with geometric intersection exactly one for the torus and once-holed torus or exactly two for the four-holed sphere. This gives the *Farey graph*. When S is an annulus the vertices are essential embedded arcs, considered up to isotopy fixing the boundary pointwise. Vertices are connected by an edge if there are representatives with disjoint interiors.

For any vertices $a, b \in \mathcal{C}(S)$ define the distance $d_S(a,b)$ to be the minimal number of edges appearing in an edge path between a and b.

Theorem 2.2 (Masur-Minsky [22])**.** *The complex of curves $\mathcal{C}(S)$ is Gromov hyperbolic.* □

We use δ_S to denote the hyperbolicity constant of $\mathcal{C}(S)$.

Boundary of the curve complex. Let $\partial\mathcal{C}(S)$ be the Gromov boundary of $\mathcal{C}(S)$. This is the space of quasi-geodesic rays in $\mathcal{C}(S)$ modulo equivalence: two rays are equivalent if and only if their images have bounded Hausdorff distance.

Recall that $\mathcal{PML}(S)$ is the projectivized space of measured laminations on S. A measured lamination ℓ is *filling* if every component $S \smallsetminus \ell$ is a disk or a boundary-parallel annulus. Take $\mathcal{FL}(S) \subset \mathcal{PML}(S)$ to be the set of filling laminations with the subspace topology. Define $\mathcal{EL}(S)$, the space of ending laminations, to be the quotient of $\mathcal{FL}(S)$ obtained by forgetting the measures. See [17] for an expansive discussion of laminations.

Theorem 2.3 (Klarreich [18])**.** *There is a mapping class group equivariant homeomorphism between $\partial\mathcal{C}(S)$ and $\mathcal{EL}(S)$.* □

We define $\overline{\mathcal{C}(S)} = \mathcal{C}(S) \cup \partial\mathcal{C}(S)$.

Subsurface projection. Suppose that $Z \subset S$ is an *essential* subsurface: Z is embedded, every component of ∂Z is essential in S, and Z is not a boundary-parallel annulus nor a pair of pants. An essential subsurface $Z \subset S$ is *strict* if Z is not homeomorphic to S.

A lamination b *cuts* a subsurface Z if every isotopy representative of b intersects Z. If b does not cut Z then b *misses* Z.

Suppose now that $a, b \in \overline{\mathcal{C}(S)}$ both cut a strict subsurface Z. Define the *subsurface projection distance* $d_Z(a, b)$ as follows: isotope a with respect to ∂Z to realize the geometric intersection number. Surger the arcs of $a \cap Z$ to obtain $\pi_Z(a)$, a finite set of vertices in $\mathcal{C}(Z)$. Notice that $\pi_Z(a)$ has uniformly bounded diameter in $\mathcal{C}(Z)$ independent of a, Z or S. Define

$$d_Z(a, b) = \operatorname{diam}_Z \big(\pi_Z(a) \cup \pi_Z(b)\big).$$

We now recall the Lipschitz Projection Lemma [23, Lemma 2.3]:

Lemma 2.4 (Masur-Minsky)**.** *Suppose that $\{a_i\}_{i=0}^N \subset \mathcal{C}(S)$ is a path where every vertex cuts $Z \subset S$. Then $d_Z(a_0, a_N) \leq 2N$.* □

For geodesics, the much stronger Bounded Geodesic Image Theorem holds [23, 25]:

Theorem 2.5. *There is a constant $\mathsf{c}_0 = \mathsf{c}_0(S)$ with the following property. For any strict subsurface Z and any points $a, b \in \overline{\mathcal{C}(S)}$, if every vertex of the geodesic $[a, b]$ cuts Z then $d_Z(a, b) < \mathsf{c}_0$.* □

Marking complex. We now discuss the *marking complex*, following Masur and Minsky [23]. A *complete clean marking* m is a pants decomposition $\operatorname{base}(m)$ of S together with a *transversal* t_a for each element $a \in \operatorname{base}(m)$. To define t_a, let X_a be the non-pants component of $S \smallsetminus (\operatorname{base}(m) \smallsetminus \{a\})$. Then any vertex of $\mathcal{C}(X_a)$ not equal to a and meeting a minimally may serve as a transversal t_a. Notice that diameter of m in $\mathcal{C}(S)$ is at most 2.

Masur and Minsky also define *elementary moves* on markings. The set of markings and these moves define the *marking complex*, $\mathcal{M}(S)$: a locally finite graph quasi-isometric to the mapping class group. The projection map $p\colon \mathcal{M}(S) \to \mathcal{C}(S)$, sending m to any element of $\operatorname{base}(m)$, is coarsely mapping class group equivariant. We now record, from [23], the Elementary Move Projection Lemma:

Lemma 2.6. *If m and m' differ by an elementary move then for any essential subsurface $Z \subseteq S$, we have $d_Z(m, m') \leq 4$.* $\square$

A converse follows from the *distance estimate* [23].

Lemma 2.7. *For every constant c there is a bound $\mathsf{e} = \mathsf{e}(\mathsf{c}, S)$ with the following property. If $d_Z(m, m') \leq \mathsf{c}$ for every essential subsurface $Z \subseteq S$ then $d_{\mathcal{M}}(m, m') \leq \mathsf{e}$.* $\square$

2.1. **Tight geodesics.** The curve complex is locally infinite. Generally, there are infinitely many geodesics connecting a given pair of points in $\mathcal{C}(S)$. In [23] the notion of a *tight* geodesic is introduced. This is a technical hypothesis which provides a certain kind of local finiteness. Lemma 2.9 below is the only property of tight geodesics used in this paper.

Definition 2.8. A pair of curves, markings or laminations a, b are c–*cobounded* if $d_Z(a, b) \leq \mathsf{c}$ for all strict subsurfaces $Z \subset S$ cut by both a and b.

Minsky shows [25, Lemma 5.14] that if $a, b \in \overline{\mathcal{C}(S)}$ then there is a tight geodesic $[a, b] \subset \mathcal{C}(S)$ connecting them. All geodesics from here on are assumed to be tight.

Lemma 2.9 (Minsky)**.** *There is a constant $\mathsf{c}_1 = \mathsf{c}_1(S)$ with the following property. Suppose that (a, b) is a c–cobounded pair in $\overline{\mathcal{C}(S)}$ and $c \in [a, b]$ is a vertex of a tight geodesic. Then the pairs (a, c) and (c, b) are $(\mathsf{c} + \mathsf{c}_1)$–cobounded.* $\square$

3. Extension Lemmas

We now examine how points of $\mathcal{C}(S)$ can be connected to infinity.

Lemma 3.1 (Completion)**.** *There is a constant $\mathsf{c}_2 = \mathsf{c}_2(S)$ with the following property. Suppose that $b \in \mathcal{C}(S)$ and $\ell \in \overline{\mathcal{C}(S)}$. Suppose that the pair (b, ℓ) is c–cobounded. Then there is a marking m so that $b \in \operatorname{base}(m)$ and (m, ℓ) are $(\mathsf{c} + \mathsf{c}_2)$–cobounded.* $\square$

The existence of the marking m follows from the construction preceding [3, Lemma 6.1].

Lemma 3.2 (Extension past a point)**.** *Suppose that $a, z \in \mathcal{C}(S)$ with $a \neq z$. Then there is a point $\ell \in \partial\mathcal{C}(S)$ so that the vertex a lies in the one-neighborhood of $[z, \ell]$.*

Proof. Let $k \in \partial\mathcal{C}(S)$ be any lamination. Let Y be a component of $S \setminus a$ that meets z. Pick any mapping class ϕ with support in Y and with translation distance at least $(2\mathsf{c}_0 + 2)$ in $\mathcal{C}(Y)$. We have either

$$d_Y(z, k) \geq \mathsf{c}_0 \quad \text{or} \quad d_Y(z, \phi(k)) \geq \mathsf{c}_0.$$

By Theorem 2.5, at least one of the geodesics $[z, k]$ or $[z, \phi(k)]$ passes through the one-neighborhood of a. ☐

Proposition 3.3 (Extension past a marking)**.** *There is a constant $\mathsf{c}_3 = \mathsf{c}_3(S)$ such that if m is a marking on S, then there are laminations k and ℓ such that the pairs (k, ℓ), (k, m) and (m, ℓ) are c_3–cobounded and $[k, \ell]$ passes through the one-neighborhood of m.*

Proof. There are only finitely many markings up to the action of the mapping class group. Fix a class of markings and pick a representative m. We will find a pseudo-Anosov map with stable and unstable laminations k and ℓ such that $[k, \ell]$ passes through the one-neighborhood of m. This suffices to prove the proposition: there is a constant $\mathsf{c}_3(m)$ large enough so that the pairs (k, ℓ), (k, m) and (m, ℓ) are $\mathsf{c}_3(m)$–cobounded. The same constant works for every marking in the orbit $\mathcal{MCG}(S) \cdot m$, by conjugation. We can now take c_3 to be the maximum of the $\mathsf{c}_3(m)$ as m ranges over the finitely many points of the quotient $\mathcal{M}(S)/\mathcal{MCG}(S)$.

So choose any pseudo-Anosov map ϕ' with stable and unstable laminations k' and ℓ'. Choose any point $b' \in [k', \ell']$. We may conjugate ϕ' to ϕ, sending (k', ℓ', b') to (k, ℓ, b), so that b is disjoint from some curve $a \in \operatorname{base}(m)$. This finishes the proof. ☐

4. The shell is connected

Let $\mathcal{B}(z, \mathsf{r})$ be the ball of radius r about $z \in \mathcal{C}(S)$. The difference of concentric balls is called a *shell*.

Proposition 4.1. *Suppose that $\xi(S) \geq 2$ and $\mathsf{d} \geq \max\{\delta_S, 1\}$. Then, for any $\mathsf{r} \geq 0$, the shell*

$$\mathcal{B}(z, \mathsf{r} + 2\mathsf{d}) \smallsetminus \mathcal{B}(z, \mathsf{r} - 1)$$

is connected.

Below we will only need the corollary that $\mathcal{C}(S) \smallsetminus \mathcal{B}(z, \mathsf{r} - 1)$ is connected. However, the shell has other interesting geometric properties. We hope to return to this subject in a future paper.

One difficulty in the proof of Proposition 4.1 lies in pushing points of the inner boundary into the interior of the shell. To deal with this we use the fact that $\mathcal{C}(S)$ has no *dead ends*.

Lemma 4.2. *Fix vertices $z, a \in \mathcal{C}(S)$. Suppose $d_S(z, a) = \mathsf{r}$. Then there is a vertex $a' \in \mathcal{C}(S)$ with $d_S(a, a') \leq 2$ and $d_S(z, a') = \mathsf{r} + 1$.* ☐

Note that this implies that any geodesic $[a, a']$ lies outside of $\mathcal{B}(z, \mathsf{r}-1)$. For a proof of Lemma 4.2, see Proposition 3.1 of [27].

Proof of Proposition 4.1. For any $z \in \mathcal{C}(S)$ and any geodesic or geodesic segement $[a, b] \subset \overline{\mathcal{C}(S)}$ define $d_S(z, [a, b]) = \min\{d_S(z, c) \mid c \in [a, b]\}$. Define a product on $\overline{\mathcal{C}(S)}$ by:

$$\langle a, b\rangle_z = \inf \left\{d_S(z, [a, b])\right\}$$

where the infimum ranges over all geodesics $[a, b]$. For every $k \in \partial\mathcal{C}(S)$ let

$$U(k) = \{\ell \in \partial\mathcal{C}(S) \mid \langle k, \ell\rangle_z > \mathsf{r} + 2\mathsf{d}\}.$$

The set $U(k)$ is a neighborhood of k by the definition of the topology on the boundary [9]. Notice that if $\ell \in U(k)$ then $k \in U(\ell)$.

Consider the set $V(k)$ of all $\ell \in \partial\mathcal{C}(S)$ so that there is a finite sequence $k = k_0, k_1, \ldots, k_N = \ell$ with $k_{i+1} \in U(k_i)$ for all i. Now, if $\ell \in V(k)$ then $U(\ell) \subset V(k)$; thus $V(k)$ is open. If ℓ is a limit point of $V(k)$ then there is a sequence $\ell_i \in V(k)$ entering every neighborhood of ℓ. So there is some i where $\ell_i \in U(\ell)$. Thus $\ell \in U(\ell_i) \subset V(k)$ and we find that $V(k)$ is closed. Finally, as $\partial\mathcal{C}(S)$ is connected (Theorem 1.3), $V(k) = \partial\mathcal{C}(S)$.

Let a', b' be any vertices in the shell $\mathcal{B}(z, \mathsf{r} + 2\mathsf{d}) \smallsetminus \mathcal{B}(z, \mathsf{r} - 1)$. We connect a', via a path in the shell, to a vertex a so that $d_S(z, a) = \mathsf{r} + \mathsf{d}$. This is always possible: points far from z may be pushed inward along geodesics and points near z may be pushed outward by Lemma 4.2. Similarly connect b' to b.

By Lemma 3.2 there are points $k, \ell \in \partial\mathcal{C}(S)$ so that there are geodesic rays $[z, k]$ and $[z, \ell]$ within distance one of a and b respectively. Connect k to ℓ by a chain of points $\{k_i\}$ in $V(k)$, as above. Define $a_i \in [z, k_i]$ so that $d_S(z, a_i) = \mathsf{r} + \mathsf{d}$. Connect a to a_0 via a path of length at most 2.

Notice that $d_S(a_i, [k_i, k_{i+1}]) > \mathsf{d} \geq \delta$. As triangles are slim, the vertex a_i is δ–close to $[z, k_{i+1}]$. Thus a_i and a_{i+1} may be connected inside of the shell via a path of length at most 2δ. $\square$

5. Image of a cobounded geodesic is cobounded

We begin with a simple lemma:

Lemma 5.1. *For every c and r there is a constant K with the following property. Let $[a, b] \subset \mathcal{C}(S)$ be a geodesic segment of length $2\mathsf{r}$ with (a, b) being c–cobounded. Let z be the midpoint. Then there is a path P of length at most K connecting a to b outside of $\mathcal{B}(z, \mathsf{r} - 1)$.*

Proof. There are only finitely many such triples (a, z, b), up the action of the mapping class group. (This is because there are only finitely many hierarchies having total length less than a given upper bound;

see [23]). The conclusion now follows from the connectedness of the shell (Proposition 4.1). □

Note that any quasi-isometric embedding $\phi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ extends to a one-to-one continuous map from $\partial\mathcal{C}(S)$ to $\partial\mathcal{C}(\Sigma)$.

Theorem 5.2. *There is a function* $\mathsf{H}\colon \mathbb{N} \to \mathbb{N}$*, depending only on* q *and the topology of* S *and* Σ*, with the following property. Suppose* (k, ℓ) *is a pair of* c*–cobounded laminations and* $\phi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ *is a* q*–quasi-isometric embedding. Then* $\kappa = \phi(k)$ *and* $\lambda = \phi(\ell)$ *are* $\mathsf{H}(\mathsf{c})$*–cobounded*

Proof. For every strict subsurface $\Omega \subset \Sigma$ we must bound $d_\Omega(\kappa, \lambda)$ from above. Now, if $d_\Sigma(\partial\Omega, [\kappa, \lambda]) \geq 2$ then by the Bounded Geodesic Image Theorem (2.5) we find $d_\Omega(\kappa, \lambda) \leq \mathsf{c}_0 = \mathsf{c}_0(\Sigma)$ and we are done.

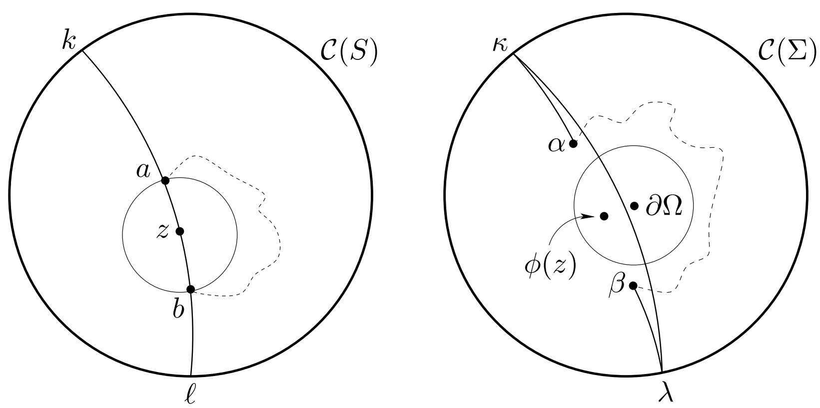

Figure 1. Points outside of an r–ball about z are sent by ϕ outside of an $(\mathsf{q} + \mathsf{M}_\Sigma + 2)$–ball about $\partial\Omega$.

Now suppose $d_\Sigma(\partial\Omega, [\kappa, \lambda]) \leq 1$. Note that $[\kappa, \lambda]$ lies in the M–neighborhood of $\phi([k, \ell])$, where $\mathsf{M} = \mathsf{M}_\Sigma$ is provided by Lemma 2.1. Choose a vertex $z \in [k, \ell]$ so that $d_\Sigma(\phi(z), \partial\Omega) \leq \mathsf{M} + 1$. Set $\mathsf{r} = \mathsf{q}(\mathsf{q} + 2\mathsf{M} + 3) + \mathsf{q}$. Thus

$$\begin{aligned} d_S(y, z) \geq \mathsf{r} &\implies d_\Sigma(\phi(y), \phi(z)) \geq \mathsf{q} + 2\mathsf{M} + 3 \\ &\implies d_\Sigma(\phi(y), \partial\Omega) \geq \mathsf{q} + \mathsf{M} + 2. \end{aligned}$$

Let a and b be the intersections of $[k, \ell]$ with $\partial\mathcal{B}(z, \mathsf{r})$, chosen so that $[k, a]$ and $[b, \ell]$ meet $\mathcal{B}(z, \mathsf{r})$ at the vertices a and b only. Connect a to b via a path P of length K, outside of $\mathcal{B}(z, \mathsf{r} - 1)$, as provided by Lemma 5.1.

Let $\alpha = \phi(a)$ and $\beta = \phi(b)$. Now, any consecutive vertices of P are mapped by ϕ to vertices of $\mathcal{C}(\Sigma)$ that are at distance at most $2\mathsf{q}$. Connecting these by geodesic segments gives a path Π from α to β.

Note that Π has length at most $2\mathsf{qK}$. Since every vertex of $\phi(P)$ is $(\mathsf{q}+\mathsf{M}+2)$–far from $\partial\Omega$ every vertex of Π is $(\mathsf{M}+2)$–far from $\partial\Omega$. So every vertex of Π cuts Ω. It follows that $d_\Omega(\alpha,\beta) \leq 4\mathsf{qK}$, by Lemma 2.4.

All that remains is to bound $d_\Omega(\kappa,\alpha)$ and $d_\Omega(\beta,\lambda)$. It suffices, by the Bounded Geodesic Image Theorem, to show that every vertex of $[\kappa,\alpha]$ cuts Ω. The same will hold for $[\beta,\lambda]$.

Every vertex of $[\kappa,\alpha]$ is M–close to a vertex of $\phi([k,a])$. But each of these is $(\mathsf{q}+\mathsf{M}+2)$–far from $\partial\Omega$. This completes the proof. $\square$

6. The induced map on markings

In this section, given a quasi-isometric embedding of one curve complex into another we construct a coarsely Lipschitz map between the associated marking complexes.

Let $\mathcal{M}(S)$ and $\mathcal{M}(\Sigma)$ be the marking complexes of S and Σ respectively. Let $p\colon \mathcal{M}(S) \to \mathcal{C}(S)$ and $\pi\colon \mathcal{M}(\Sigma) \to \mathcal{C}(\Sigma)$ be maps that send a marking to some curve in that marking.

Theorem 6.1. *Suppose that $\xi(S) \geq 2$ and $\phi\colon \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ is a q–quasi-isometric embedding. Then ϕ induces a coarse Lipschitz map $\Phi\colon \mathcal{M}(S) \to \mathcal{M}(\Sigma)$ so that the diagram*

$$\begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{\Phi} & \mathcal{M}(\Sigma) \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi} \\ \mathcal{C}(S) & \xrightarrow{\phi} & \mathcal{C}(\Sigma) \end{array}$$

commutes up to an additive error. Furthermore, if ϕ is a quasi-isometry then so is Φ.

Proof. For a marking m and laminations k and ℓ, we say the triple (m,k,ℓ) is c–admissible if

- $d_S\big(m,[k,\ell]\big) \leq 3$ and
- the pairs (k,m), (m,ℓ) and (k,ℓ) are c–cobounded.

For c large enough and for every marking m, Proposition 3.3 shows that there exists a c-admissible triple (m,k,ℓ).

Given a c-admissible triple (m,k,ℓ) we will now construct a triple (μ,κ,λ) for Σ. Let α be any curve in $\phi(m) \subset \mathcal{C}(\Sigma)$, $\kappa = \phi(k)$ and $\lambda = \phi(\ell)$. Note that

$$d_\Sigma\big(\alpha,[\kappa,\lambda]\big) \leq 4\mathsf{q} + \mathsf{M}_\Sigma, \tag{6.2}$$

by the stability of quasi-geodesics (Lemma 2.1). Also (κ,λ) is a $\mathsf{H}(\mathsf{c})$–cobounded pair, by Theorem 5.2. Let β be a closest point projection of

α to the geodesic $[\kappa, \lambda]$. By Lemma 2.9, the pair (β, κ) is $(\mathsf{H}(\mathsf{c}) + \mathsf{c}_1)$–cobounded. Using Lemma 3.1, there is a marking μ so that $\beta \in \operatorname{base}(\mu)$ and (μ, κ) are $(\mathsf{H}(\mathsf{c}) + \mathsf{c}_1 + \mathsf{c}_2)$–cobounded. Therefore, for $\mathsf{C} = 2\mathsf{H}(\mathsf{c}) + \mathsf{c}_1 + \mathsf{c}_2$ the triple (μ, κ, λ) is C–admissible. Define $\Phi(m)$ to be equal to μ.

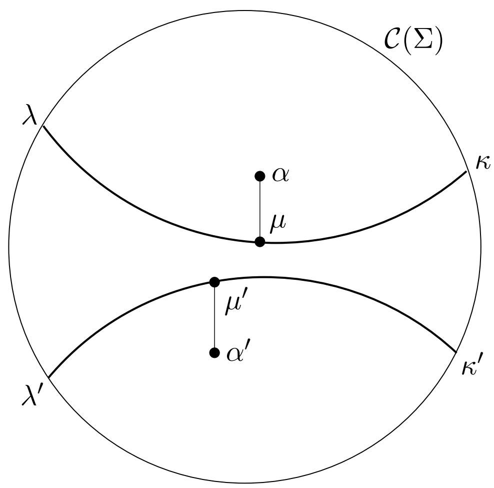

FIGURE 2. Markings μ and μ' are bounded apart.

We now prove Φ is coarsely well-defined and coarsely Lipschitz. Suppose that m and m' differ by at most one elementary move and the triples (m, k, ℓ) and (m', k', ℓ') are c-admissible. Let (μ, κ, λ) and $(\mu', \kappa', \lambda')$ be any corresponding C–admissible triples in Σ, as constructed above. (See Figure 2.) We must show that there is a uniform bound on the distance between μ and μ' in the marking graph. By Lemma 2.7, it suffices to prove:

Claim. For every subsurface $\Omega \subseteq \Sigma$, $d_\Omega(\mu, \mu') = O(1)$.

Now, Lemma 2.6 gives $d_S(m, m') \le 4$. Deduce

$$d_\Sigma(\phi(m), \phi(m')) \le 5\mathsf{q}.$$

Therefore,

$$\begin{aligned} d_\Sigma(\mu, \mu') &\le d_\Sigma\big(\mu, \phi(m)\big) + d_\Sigma\big(\phi(m), \phi(m')\big) + d_\Sigma\big(\phi(m'), \mu'\big) \\ &\le 2(7\mathsf{q} + \mathsf{M}_\Sigma + 2) + 5\mathsf{q}. \end{aligned}$$

On the other hand, for any strict subsurface $\Omega \subset \Sigma$, we have

$$d_\Omega(\mu, \mu') \le d_\Omega(\mu, \kappa) + d_\Omega(\kappa, \kappa') + d_\Omega(\kappa', \mu').$$

The first and third terms on the right are bounded by C. By Theorem 5.2, the second term is bounded by $\mathsf{H}(2\mathsf{c}+4)$. This is because, for every strict subsurface $Y \subset S$,

$$d_Y(k,k') \leq d_Y(k,m) + d_Y(m,m') + d_Y(m',k') \leq 2\mathsf{c}+4.$$

This proves the claim; thus Φ is coarsely well-defined and coarsely Lipschitz.

Now assume that ϕ is a quasi-isometry with inverse $f\colon \mathcal{C}(\Sigma) \to \mathcal{C}(S)$. Let Φ and F be the associated maps between marking complexes. We must show that $F \circ \Phi$ is close to the identity map on $\mathcal{C}(S)$. Fix $m \in \mathcal{M}(S)$ and define $\mu = \Phi(m)$, $m' = F(\mu)$. For any admissible triple (m,k,ℓ) we have, as above, an admissible triple (μ,κ,ℓ). Since $f(\kappa) = k$ and $f(\lambda) = \ell$ it follows that (m',k,ℓ) is also admissible for a somewhat larger constant.

As above, by Lemma 2.7 it suffices to show that $d_Z(m,m') = O(1)$ for every essential subsurface $Z \subset S$. Since $f \circ \phi$ is close to the identity, commutivity up to additive error implies that $d_S(m,m')$ is bounded. Since (m,k) and (m',k) are cobounded, the triangle inequality implies that $d_Z(m,m') = O(1)$ for strict subsurfaces $Z \subset S$. This completes the proof. □

7. Rigidity of the curve complex

Theorem 7.1. *Suppose that $\xi(S) \geq 2$. Then every quasi-isometry of $\mathcal{C}(S)$ is bounded distance from a simplicial automorphism of $\mathcal{C}(S)$.*

Proof. Let $f\colon \mathcal{C}(S) \to \mathcal{C}(S)$ be a q-quasi-isometry. By Theorem 6.1 there is a Q-quasi-isometry $F\colon \mathcal{M}(S) \to \mathcal{M}(S)$ associated to f. By Theorem 1.5 the action of F is uniformly close to the induced action of some homeomorphism $G\colon S \to S$. That is,

$$d_{\mathcal{M}}\big(F(m), G(m)\big) = O(1). \tag{7.2}$$

Let $g\colon \mathcal{C}(S) \to \mathcal{C}(S)$ be the simplicial automorphism induced by G. We need to show that f and g are equal in $\mathrm{QI}(\mathcal{C}(S))$. Fix a curve $a \in \mathcal{C}(S)$. We must show the distance $d_S(f(a), g(a))$ is bounded by a constant independent of the curve a. Choose a marking m containing a as a base curve. Note that $d_S(a, p(m)) \leq 2$, thus

$$d_S\big(f(a), f(p(m))\big) \leq 3\mathsf{q}.$$

By Theorem 6.1, for every marking $m \in \mathcal{M}(S)$,

$$d_S\big(f(p(m)), p(F(m))\big) = O(1).$$

From Equation 7.2 and Lemma 2.6 we have

$$d_S\big(p(F(m)), p(G(m))\big) = O(1).$$

Also, $g(a)$ is a base curve of $G(m)$, hence

$$d_S\big(p\big(G(m)\big), g(a)\big) \leq 2.$$

These four equations imply that

$$d_S\big(f(a), g(a)\big) = O(1).$$

This finishes the proof. □

Appendix A. Classifying mapping class groups

For any compact, connected, orientable surface S let $\mathcal{MCG}(S)$ be the extended mapping class group: the group of homeomorphisms of S, considered up to isotopy. We will use Ivanov's characterization of Dehn twists via *algebraic twist subgroups* [12], the action of $\mathcal{MCG}(S)$ on $\mathcal{PML}(S)$, and the concept of a *bracelet* [2] to give a detailed proof of:

Theorem A.1. *Suppose that S and Σ are compact, connected, orientable surfaces with $\mathcal{MCG}(S)$ isomorphic to $\mathcal{MCG}(\Sigma)$. Then either*

- *S and Σ are homeomorphic,*
- *$\{S, \Sigma\} = \{S_1, S_{1,1}\}$, or*
- *$\{S, \Sigma\} = \{\mathbb{S}, \mathbb{D}\}$.*

By the classification of surfaces, S is determined up to homeomorphism by the two numbers $\mathsf{g} = \text{genus}(S)$ and $\mathsf{b} = |\partial S|$. So to prove Theorem A.1 it suffices prove that $\xi(S) = 3\mathsf{g} - 3 + \mathsf{b}$, the complexity of S, and g, the genus, are *algebraic*: determined by the isomorphism type of $\mathcal{MCG}(S)$.

Remark A.2. Theorem A.1 is a well-known folk-theorem. A version of Theorem A.1, for pure mapping class groups, is implicitly contained in [12] and was known to N. Ivanov as early as the fall of 1983 [14]. Additionally, Ivanov and McCarthy [16] prove Theorem A.1 when $\mathsf{g} \geq 1$ (see also [15]).

There is also a "folk proof" of Theorem A.1 relying on the fact that the rank and virtual cohomological dimension are algebraic and give two independent linear equations in the unknowns g and b. However, the formula for the vcd changes when $\mathsf{g} = 0$ and when $\mathsf{b} = 0$. Thus there are two infinite families of pairs of surfaces which are not distinguished by these invariants.

These difficult pairs can be differentiated by carefully considering torsion elements in the associated mapping class groups. We prefer the somewhat lighter proof of Theorem A.1 given here.

The *rank* of a group is the size of a minimal generating set. The *algebraic rank* of a group G, $\mathrm{rank}(G)$, is the maximum of the ranks of free abelian subgroups $H < G$. Now, the algebraic rank of $\mathcal{MCG}(S)$ is equal to $\xi(S)$, when $\xi(S) \geq 1$ (Birman-Lubotzky-McCarthy [5]). When $\xi(S) \leq 0$ the algebraic rank is zero or one.

So when $\xi(S) \geq 1$ the complexity is algebraically determined. There are only finitely many surfaces having $\xi(S) < 1$; we now dispose of these and a few other special cases.

Low complexity. A Dehn twist along an essential, non-peripheral curve in an orientable surface has infinite order in the mapping class group. Thus, the only surfaces where the mapping class group has algebraic rank zero are the sphere, disk, annulus, and pants. We may compute these and other low complexity mapping class groups using the *Alexander method* [7].

For the sphere and the disk we find

$$\mathcal{MCG}(\mathbb{S}),\ \mathcal{MCG}(\mathbb{D}) \cong \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ is the group of order two generated by a reflection.

For the annulus and pants we find

$$\mathcal{MCG}(\mathbb{A}) \cong K_4 \quad \text{and} \quad \mathcal{MCG}(S_{0,3}) \cong \mathbb{Z}_2 \times \Sigma_3,$$

where K_4 is the Klein 4–group and Σ_3 is the symmetric group acting on the boundary of $S_{0,3}$. Here, in addition to reflections, there is the permutation action of $\mathcal{MCG}(S)$ on ∂S.

The surfaces with algebraic rank equal to one are the torus, once-holed torus, and four-holed sphere. For the torus and the once-holed torus we find

$$\mathcal{MCG}(\mathbb{T}),\ \mathcal{MCG}(S_{1,1}) \cong \mathrm{GL}(2, \mathbb{Z}).$$

The first isomorphism is classical [28, Section 6.4]. The second has a similar proof: the pair of curves meeting once is replaced by a pair of disjoint arcs cutting $S_{1,1}$ into a disk.

Now we compute the mapping class group of the four-holed sphere:

$$\mathcal{MCG}(S_{0,4}) \cong K_4 \rtimes \mathrm{PGL}(2, \mathbb{Z}).$$

The isomorphism arises from the surjective action of $\mathcal{MCG}(S_{0,4})$ on the Farey graph. If ϕ lies in the kernel then ϕ fixes each of the slopes $\{0, 1, \infty\}$ setwise. Examining the induced action of ϕ on these slopes and their intersections shows that ϕ is either the identity or one of the three "fake" hyperelliptic involutions. It follows that $\mathcal{MCG}(S_{0,4})$ has no center, and so distinguishes $S_{0,4}$ from S_1 and $S_{1,1}$.

All other compact, connected, orientable surfaces have algebraic rank equal to their complexity and greater than one (again, see [5]). Among these $S_{1,2}$ and S_2 are the only ones with mapping class group having nontrivial center [7]. As the complexities of $S_{1,2}$ and S_2 differ, their mapping class groups distinguish them from each other and from surfaces with equal complexity. This disposes of all surfaces of complexity at most three *except* for telling $S_{0,6}$ apart from $S_{1,3}$. We defer this delicate point to the end of the appendix.

Characterizing twists. Recall that if $a \subset S$ is a essential non-peripheral simple closed curve then T_a is the *Dehn twist* about a. We call a the *support* of the Dehn twist. If a is a separating curve and one component of $S \setminus a$ is a pair of pants then a is a *pants curve.* In this case there is a *half-twist*, $T_a^{1/2}$, about a. (When a cuts off a pants on both sides then the two possible half-twists differ by a fake hyperelliptic.)

We say that two elements $f, g \in \mathcal{MCG}(S)$ *braid* if f and g are conjugate and satisfy $fgf = gfg$. From [24, Lemma 4.3] and [16, Theorem 3.15] we have:

Lemma A.3. *Two powers of twists commute if and only if their supporting curves are disjoint. Two twists braid if and only if their supporting curves meet exactly once.* □

The proof generalizes to half-twists along pants curves:

Lemma A.4. *Two half-twists commute if and only if the underlying curves are disjoint. Two half-twists braid if and only if the underlying curves meet exactly twice.* □

Now, closely following Ivanov [12, Section 2], we essay an algebraic characterization of twists on nonseparating curves inside of $\mathcal{MCG}(S)$. Define a subgroup $H < \mathcal{MCG}(S)$ to be an *algebraic twist subgroup* if it has the following properties.

- $H = \langle g_1, \ldots, g_k \rangle$ is a free abelian group of rank $k = \xi(S)$,
- for all i, j the generators g_i, g_j are conjugate inside of $\mathcal{MCG}(S)$,
- for all i and n the center of the centralizer, $Z(C(g_i^n))$, is cyclic, and
- for all i, the generator g_i is not a proper power in $C(H)$.

We have:

Theorem A.5. *Suppose that $\xi(S) \geq 2$ and $\mathcal{MCG}(S)$ has trivial center. Suppose that $H = \langle g_1, \ldots, g_k \rangle$ is an algebraic twist group. Then the elements g_i are all either twists on nonseparating curves or are all half-twists on pants curves. Furthermore, the underlying curves for the g_i form a pants decomposition of S.*

Proof. By work of Birman-Lubotzky-McCarthy [5] (see also [24]) we know that there is a power m so that each element $f_i = g_i^m$ is either a power of Dehn twist or a pseudo-Anosov supported in a subsurface of complexity one. Suppose that f_i is pseudo-Anosov with support in $Y \subset S$; then the center of the centralizer $Z(C(f_i))$ contains the group generated by f_i and all twists along curves in ∂Y. However, this group is not cyclic, a contradiction.

Deduce instead that each f_i is the power of a Dehn twist. Let b_i be the support of f_i. Since the f_i commute with each other and are not equal it follows that the b_i are disjoint and are not isotopic. Thus the b_i form a pants decomposition. Since the g_i are conjugate the same holds for the f_i. Thus all the b_i have the same topological type.

If S has positive genus then it follows that all of the b_i are nonseparating. If S is planar then it follows that $S = S_{0,5}$ or $S_{0,6}$ and all of the b_i are pants curves.

Now fix attention on any $h \in C(H)$, the centralizer of H in $\mathcal{MCG}(S)$. We will show that h preserves each curve b_i. First recall that, for any index i and any $n \in \mathbb{Z}$, the element h commutes with f_i^n. Let $\ell \in \mathcal{PMF}(S)$ be any filling lamination. We have

$$f_i^n(\ell) \to b_i \quad \text{as} \quad n \to \infty,$$

so

$$h \circ f_i^n(\ell) \to h(b_i) \quad \text{and} \quad f_i^n \circ h(\ell) \to b_i.$$

It follows that $h(b_i) = b_i$ for all i.

Fix attention on any $b_\ell \in \{b_i\}$. Suppose first that the two sides of b_ℓ lie in a single pair of pants, P. If ∂P meets b_ℓ and no other pants curve then $S = S_{1,1}$, a contradiction. If ∂P meets only b_ℓ and b_k then b_ℓ is nonseparating and b_k is separating, a contradiction. We deduce that b_ℓ meets two pants, P and P'. Now, if h interchanges P and P' then S is in fact the union of P and P'; as $\xi(S) > 1$ it follows that $S = S_{1,2}$ or S_2. However, in both cases the mapping class group has non-trivial center, contrary to hypothesis.

We next consider the possibility that that h fixes P setwise. So $h|P$ is an element of $\mathcal{MCG}(P)$. If $h|P$ is orientation reversing then so is h; thus h conjugates f_ℓ to f_ℓ^{-1}, a contradiction. If $h|P$ permutes the components of $\partial P \setminus b_\ell$ then either b_ℓ cuts off a copy of $S_{0,3}$ or $S_{1,1}$ from S; in the latter case we have b_i's of differing types, a contradiction.

To summarize: h is orientation preserving, after an isotopy h preserves each of the b_i, and h preserves every component of $S \setminus \{b_i\}$. Furthermore, when restricted to any such component P, the element h is either isotopic to the identity or to a half twist. The latter occurs only when $P \cap \partial S = \delta_+ \cup \delta_-$, with $h(\delta_\pm) = \delta_\mp$.

So, if the b_i are nonseparating then h is isotopic to the identity map on $S \smallsetminus \{b_i\}$. It follow that h is a product of Dehn twists on the b_i. In particular this holds for each of the g_i and we deduce that T_{b_i}, the Dehn twist on b_i, is an element of $C(H)$. Now, since $Z(C(g_i)) = \mathbb{Z}$ we deduce that the support of g_i is a single curve. By the above g_i is a power of T_{b_i}; since g_i is primitive in $C(H)$ we find $g_i = T_{b_i}$, as desired.

The other possibility is that the b_i are all pants curves. It follows that $S = S_{0,5}$ or $S_{0,6}$. Here h is the identity on the unique pants component of $S \smallsetminus \{b_i\}$ meeting fewer than two components of ∂S. On the others, h is either the identity or of order two. So h is a product of half-twists on the $\{b_i\}$. As in the previous paragraph, this implies that $g_i = T_a^{1/2}$. $\square$

Bracelets. We now recall a pretty definition from [2]. Suppose g is a twist or a half-twist. A *bracelet* around g is a set of mapping classes $\{f_i\}$ so that

- every f_i braids with g (and so is conjugate to g),
- if $i \neq j$ then $f_i \neq f_j$ and $[f_i, f_j] = 1$, and
- no f_i is equal to g.

Note that bracelets are algebraically defined. The *bracelet number* of g is the maximal size of a bracelet around g.

Claim A.6. A half-twist on a pants curve has bracelet number at most two.

Proof. If two half-twists braid, then they intersect twice (Lemma A.4). Thus, the pants they cut off share a curve of ∂S. If two half-twists commute, the pants are disjoint (again, Lemma A.4). Therefore, there are at most two commuting half-twists braiding with a given half-twist. $\square$

On the other hand:

Claim A.7. Suppose that $S \neq \mathbb{T}$. Then a twist on a nonseparating curve has bracelet number $2\mathsf{g} - 2 + \mathsf{b}$.

Proof. Suppose that a is a non-separating curve with associated Dehn twist. Let $\{b_i\}$ be curves underlying the twists in the bracelet. Each b_i meets a exactly once (Lemma A.3). Also, each b_i is disjoint from the others and not parallel to any of the others (again, Lemma A.3). Thus the b_i cut S into a collection of surfaces $\{X_j\}$. For each j let $a_j = a \cap X_j$ be the remains of a in X_j. Note that $a_j \neq \emptyset$. Each component of ∂X_j either meets exactly one endpoint of exactly one arc of a_j or is a boundary component of S. Now:

- if X_j has genus,

- if $|\partial X_j| > 4$, or
- if $|\partial X_j| = 4$ and a_j is a single arc,

then there is a non-peripherial curve in X_j meeting a_j transversely in a single point. However, this contradicts the maximality of $\{b_i\}$. Thus every X_j is planar and has at most four boundary components. If X_j is an annulus then S is a torus, violating our assumption. If X_j is a pants then a_j is a single arc. If $X_j = S_{0,4}$ then a_j is a pair of arcs. An Euler characteristic computation finishes the proof. □

Proving the theorem. Suppose now that $\xi(S) \geq 4$. By Theorem A.5 any basis element of any algebraic twist group is a Dehn twist on a nonseparating curve. Here the bracelet number is $2\mathsf{g} - 2 + \mathsf{b}$. Thus $\xi(S)$ minus the bracelet number is $\mathsf{g} - 1$. This, together with the fact that $\xi(S)$ agrees with the algebraic rank, gives an algebraic characterization of g.

When $\xi(S) \leq 2$ the discussion of low complexity surfaces proves the theorem. The same is true when $\xi(S) = 3$ and $\mathcal{MCG}(S)$ has nontrivial center.

The only surfaces remaining are $S_{0,6}$ and $S_{1,3}$. In $\mathcal{MCG}(S_{0,6})$ every basis element of every algebraic twist group has bracelet number two. In $\mathcal{MCG}(S_{1,3})$ every basis element of every algebraic twist group has bracelet number three. So Theorem A.1 is proved. □

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