SECTION A: PROJECT SUMMARY

The study of 3-manifolds has recently become the subject of public attention. For example, in a recent issue of *Scientific American* Jeff Weeks describes how mathematicians and physicists are using techniques from 3-manifold topology to try to determine the shape of our universe. Knot theory, a branch of 3-manifold topology, has been in the news for several years now. It has become one of the many tools used by biochemists to study the actions of certain enzymes on strands of DNA.

A natural way to decompose a 3-manifold into two simple, identical pieces is called a *Heegaard splitting*. The goal of the present proposal is to classify low genus Heegaard splittings of 3-manifolds that are constructed via sufficiently complicated surface automorphisms. Geometrically, these manifolds should be thought of as having a region homeomorphic to $\{\text{surface}\} \times I$, where the I direction is long. Constructions of 3-manifolds by surface automorphisms arise naturally throughout 3-manifold topology. Examples include Heegaard splittings, surface bundles, Dehn fillings, and Haken hierarchies.

The proposed research will be disseminated to the mathematical community through publication and presentation at professional meetings. By keeping up with current research in topology the PIs will be better able to direct senior projects, a requirement for graduation at at least one of the PIs' institutions.

SECTION C: PROJECT DESCRIPTION

Heegaard splittings, introduced by Poul Heegaard in [Hee98], are a classical way to study the topology of 3-manifolds. A 3-manifold that is homeomorphic to the regular neighborhood of a (finite) connected graph in $\mathbb{R}^3$ is called a *handlebody*. Let M be a closed 3-manifold and $H \subset M$ an embedded surface. We say H is a *Heegaard splitting* of M if it separates M into two handlebodies (See Figure 1).

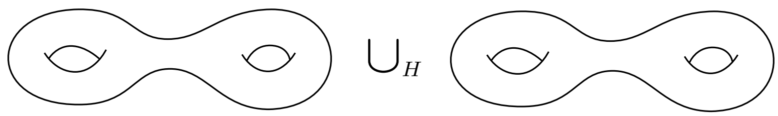

FIGURE 1

The utility of Heegaard splittings is due to their intimate relationship with other techniques used to study 3-manifolds:

(1) Given a Heegaard splitting of genus g one can write down a rank g presentation of $\pi_1(M)$. In most known examples a minimal genus Heegaard splitting gives a presentation of the fundamental group of minimal rank.
(2) For certain handle decompositions of M, the Heegaard splitting H is the boundary of the union of the zero and one-handles (and is simultaneously the boundary of the union of the two and three-handles).
(3) Heegaard splittings arise as level surfaces of height functions on M.
(4) Given a bumpy Riemannian metric on M, a "strongly irreducible" Heegaard splitting can be realized as an index 1 minimal surface [PR87].
(5) Any "strongly irreducible" Heegaard splitting can be isotoped into a normal form with respect to a fixed triangulation of M [Rub95].

Hence, we see connections between Heegaard theory and the algebra, topology, and geometry of 3-manifolds.

Given any Heegaard splitting of a 3-manifold one can always construct a higher genus splitting by adding a handle in a trivial way. Hence, if a manifold has a genus g Heegaard splitting then it has splittings of all genera higher than g. What is of most interest, then, are those Heegaard splittings of low genus. The goal of the present proposal is to classify low genus Heegaard splittings of 3-manifolds that are constructed via sufficiently complicated surface automorphisms. Geometrically, these manifolds should be thought of as having a region homeomorphic to $\{\text{surface}\} \times I$, where the I direction is long. Constructions of 3-manifolds by surface automorphisms arise naturally throughout 3-manifold topology. Examples include Heegaard splittings, surface bundles, Dehn fillings, and Haken hierarchies.

Authors who have analyzed low genus Heegaard splittings in the presence of a complicated surface automorphism include

- Lackenby [Lac02], Rubinstein [Rub], and the PIs [BS] for surface bundles,
- Moriah and Rubinstein [MR97] and Rieck and Sedgwick [RS01] for Dehn fillings
- Lackenby [Lac] (using results of Soma [Som02]) and Souto (in progress) for Haken manifolds containing an acylindrical surface.

With the exceptions of the work of Rieck and Sedgwick and the PIs, all of the above results were obtained with geometric techniques. The techniques described here are purely combinatorial, and therefore may yield more general results. Our results can often be phrased as follows: If some surface automorphism used to construct a 3-manifold is "sufficiently complicated" then the low genus Heegaard splittings are *standard*, *i.e.* they are constructed in some canonical way from Heegaard splittings of the "cut-open" manifold.

There are currently no results describing the low genus Heegaard splittings of a manifold, if the only information given is the existence of a "sufficiently complicated" Heegaard splitting. See Section 8.1 below.

1. The curve complex

Let H be a closed, connected, orientable surface. We say a curve embedded in H is *essential* if it does not bound a disk. We now define a 1-complex $\Gamma(H)$. For each isotopy class of essential curve in H there is a vertex of $\Gamma(H)$. Two such vertices are connected by an edge if there are representatives of the corresponding equivalence classes that are disjoint. The complex $\Gamma(H)$ is the one-skeleton of a complex which is commonly referred to as the *curve complex* of H.

The path metric on $\Gamma(H)$ gives a well-defined distance between any two vertices. If $\mathcal{V}$ and $\mathcal{W}$ are sets of vertices in $\Gamma(H)$ then we define the distance between $\mathcal{V}$ and $\mathcal{W}$ to be the smallest distance between a vertex of $\mathcal{V}$ and a vertex of $\mathcal{W}$. This definition allows us to create measures of complexity for various common structures in 3-manifolds. For example, the following definition is due to J. Hempel:

Definition 1.1. (Hempel [Hem01]) Let H denote a Heegaard splitting of a 3-manifold M. Then H separates M into handlebodies V and W. Let $\mathcal{V}$ and $\mathcal{W}$ denote the sets of vertices of $\Gamma(H)$ which correspond to loops bounding disks in V and W, respectively. The *distance* $d(H)$ of the Heegaard splitting H is defined to be the distance between $\mathcal{V}$ and $\mathcal{W}$.

One nice feature of this definition is that it allows us to quickly recall most of the standard terms in the theory of Heegaard splittings. A Heegaard splitting H

- is *reducible* if $d(H) = 0$, and *irreducible* otherwise,
- is *weakly reducible* if $d(H) \leq 1$, and *strongly irreducible* otherwise [CG87],
- has the *disjoint curve property* if $d(H) \leq 2$ [Tho99], and is *full* otherwise [Sch].

In an irreducible 3-manifold (*i.e.* one in which every embedded 2-sphere bounds a 3-ball) those Heegaard splittings that are reducible are precisely the ones that come from adding trivial handles to lower genus splittings (*i.e.* from *stabilizing*). Hence, what is of most interest are the irreducible Heegaard splittings. One goal of this proposal is to give a classification of irreducible Heegaard splittings of irreducible, orientable closed 3-manifolds, with respect to their distance.

The first step in such a classification was taken by S. Schleimer, who showed the following:

Theorem 7.1 (Schleimer). *For any closed orientable three-manifold M there is a constant $C(M)$ as follows: if $H \subset M$ is a Heegaard splitting with genus at least $C(M)$ then H has the disjoint curve property.*

This result will be discussed in more detail in Section 7.1.

2. Splittings of Haken 3-manifolds

After a great deal of work (see assorted papers of Bonahon, Otal, Moriah, Lustig, Schultens, Sedgwick, Cooper, Scharlemann) Heegaard splittings of closed orientable geometric manifolds are very well understood — except for the hyperbolic case. There, relatively little is known (but see Bachman, Cooper and White [BCW]).

By Thurston's geometrization theorem [Thu82] any closed, orientable, irreducible, atoroidal 3-manifold which contains an incompressible surface is hyperbolic. A compact, orientable, irreducible 3-manifold that contains a 2-sided incompressible surface is said to be *Haken*. A large part of our program is to classify the distances of all Heegaard splittings of such 3-manifolds. An important part of this program is a result of K. Hartshorn [Har02], who showed the following:

Theorem 2.1 (Hartshorn). *Let M be a closed, orientable, irreducible 3-manifold with Heegaard splitting H. Suppose M contains a 2-sided, incompressible surface F. Then the distance of H is bounded above by twice the genus of F.*

An important special case of Haken 3-manifolds are those that fiber over the circle S^1. Let M be such a 3-manifold. Then M can be obtained from the product $F \times I$ by identifying $F \times \{0\}$ with $F \times \{1\}$ via the map $\phi : F \to F$. We define the *translation distance* $d(\phi)$ of the map ϕ to be the minimum distance between any essential loop in F and its image under ϕ, as measured in the 1-complex $\Gamma(F)$. In [BS] the PI's jointly established the following result:

Theorem 7.3 (Bachman-Schleimer). *Suppose H is a Heegaard splitting of a surface bundle. If $-\chi(H)$ is less than the translation distance of the monodromy (i.e. if $g(H) \le \frac{1}{2}d(\phi) + 1$) then H is weakly reducible.*

This result will be discussed in more detail in Section 7.2.

Combining Theorems 7.1, 2.1 and 7.3 leads to the picture of the Heegaard splittings of a surface bundle depicted in Figure 2. It is the conjecture of the PIs that this picture is essentially valid for all Haken 3-manifolds. To establish this the PIs would need to find a suitable generalization of Theorem 7.3. This is precisely why the present proposal focuses only on the low genus Heegaard splittings. This is the topic of the next section.

3. Heegaard splittings of amalgamated 3-manifolds

Haken 3-manifolds naturally fall into two categories; those that contain a separating incompressible surface and those that do not. Our first goal is to find a replacement for Theorem 7.3 in the setting where M contains a separating incompressible surface. In this case M decomposes into two 3-manifolds, X and Y, each of whose boundary is a copy of F. To recover M these copies of F are glued via some automorphism ϕ.

Assuming X and Y are acylindrical there are canonical finite collections of curves $\Delta(X)$ and $\Delta(Y)$ in ∂X and ∂Y, respectively. Geometrically, one may think of these curves as being shortest in the unique hyperbolic metric making the boundary totally geodesic. We now define the *gluing distance* $d(\phi)$ of the map ϕ to be the distance between the sets $\Delta(X)$ and $\phi^{-1}(\Delta(Y))$ in the curve complex of ∂X.

The PIs are currently exploring the following conjecture.

Conjecture 1. *Suppose H is a Heegaard splitting of $X \cup_\phi Y$, where X and Y are acylindrical. If $-\chi(H)$ is less than the gluing distance (i.e. if $g(H) \le \frac{1}{2}d(\phi)+1$) then H is weakly reducible.*

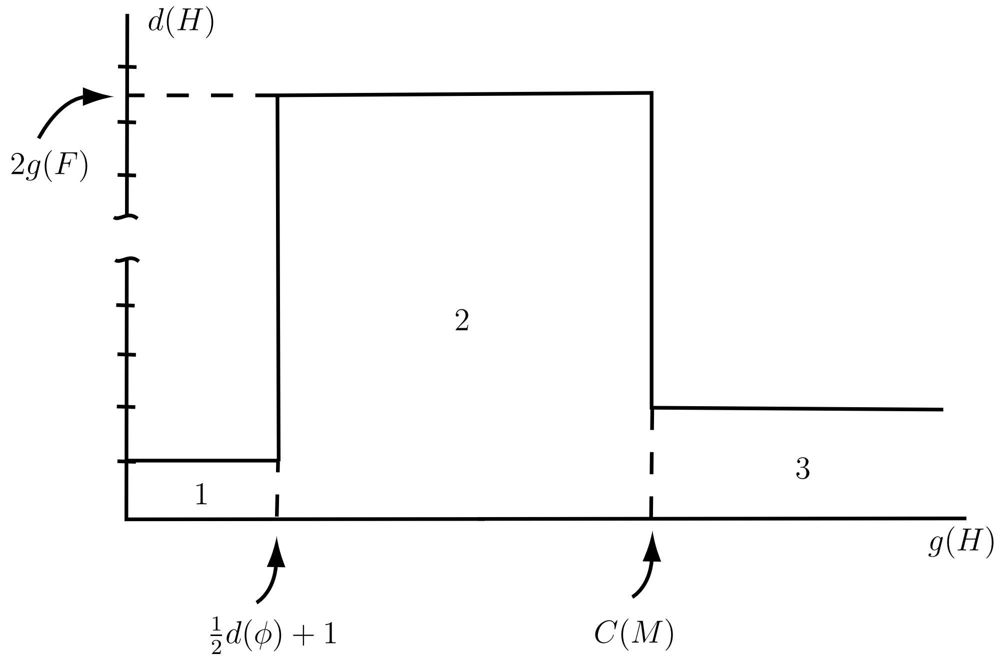

FIGURE 2. In region 1 all irreducible Heegaard splittings have distance exactly one. In region 3 any such splitting can have distance one or two.

Although this conjecture seems very similar to Theorem 7.3 any possible proof must be quite different (contrast Section 7.2 with Section 4). If true, Conjecture 1 would have the following consequence:

Conjecture 2. *Suppose H is a Heegaard splitting of $X \cup_\phi Y$. If $-\chi(H)$ is less than $d(\phi)$ then H is obtained by amalgamating splittings of X and Y.*

This conjecture is a strengthening of a recent result of Lackenby [Lac]:

Theorem 3.1 (Lackenby)**.** *If ϕ is a sufficiently high power of a pseudo-Anosov map, then the minimal genus Heegaard splitting of $X \cup_\phi Y$ is obtained by amalgamating splittings of X and Y. Thus the Heegaard genus of $X \cup_\phi Y$ is exactly*

$$g(X \cup_\phi Y) = g(X) + g(Y) - g(\partial X).$$

Here $g(X)$ denotes the minimal genus Heegaard splitting of X and $g(\partial X)$ denotes the genus of its boundary. Note that since X has boundary the definition of Heegaard splitting must be generalized. See [CG87].

Lackenby's proof relies on the hyperbolic geometry of $X \cup_\phi Y$ investigated by Soma [Som02]. The hyperbolic structure depends sensitively on the map ϕ. We expect our methods to yield the more general result because the gluing distance $d(\phi)$ is a much coarser invariant.

4. Proof sketch for Conjecture 1

Our approach to Conjecture 1 is based on a lemma from a forthcoming paper by both PIs and E. Sedgwick. This lemma is as follows:

Lemma 4.1. *Let M be a compact, irreducible, orientable 3-manifold whose boundary, if non-empty, is incompressible. Suppose $M = X \cup_F Y = V \cup_H W$, where F is incompressible, orientable, connected, closed, and non-boundary parallel and H is a Heegaard surface. Then either H is an amalgamation of splittings of X and Y or there are properly embedded surfaces $H_X \subset X$ and $H_Y \subset Y$ with boundaries on F such that at least one of the following holds:*

1. *The surfaces H_X and H_Y are incompressible, non-boundary parallel, and satisfy $\partial H_X = \partial H_Y$ and $\chi(H_X) + \chi(H_Y) \geq \chi(H)$.*
2. *After possibly exchanging X and Y we may assume H_X is incompressible and non-boundary parallel, H_Y is strongly irreducible, $\partial H_X = \partial H_Y$ and $\chi(H_X) + \chi(H_Y) \geq \chi(H)$.*
3. *The surfaces H_X and H_Y are incompressible, non-boundary parallel, and satisfy $\partial H_X \cap \partial H_Y = \emptyset$ and $\chi(H_X) + \chi(H_Y) - 1 \geq \chi(H)$.*

The proof of Lemma 4.1 follows from a careful sweepout argument. To make use of Lemma 4.1 to prove Conjecture 1 we first fix triangulations of X and Y (these triangulations do not have to agree on F).

Now, suppose H_X is an incompressible, non-boundary parallel surface properly embedded in X. Then ∂H_X defines a set of vertices in the 1-complex $\Gamma(\partial X)$. Let $\{H_X^i\}_{i=0}^n$ denote a sequence of properly embedded surfaces in X, where $H_X^0 = H_X$, H_X^i is obtained from H_X^{i-1} by a ∂-compression, and H_X^n is both incompressible and ∂-incompressible. Note that it follows that $n \leq -\chi(H_X)$.

For each i the curves ∂H_X^i are at a distance of at most one from the curves ∂H_X^{i-1} in $\Gamma(F)$. Hence, the curves $\{\partial H_X^i\}$ define a path of length at most $n \leq -\chi(H_X)$. The final surface H_X^n is incompressible and ∂-incompressible, and hence can be made normal with respect to the triangulation of X by a result of Haken [Hak61]. That is, we can make H_X^n intersect every tetrahedron in a collection of triangles and quadrilaterals, as in Figure 3.

All normal surfaces can be represented as a sum over a finite generating set, the so called *fundamental surfaces*. Furthermore, both boundary length (*i.e.* the number of intersections with the 1-skeleton) and Euler characteristic are additive with respect to this sum. Since, by a result of Jaco and Oertel [JO84], there can be no annular summands for H_X^n we can

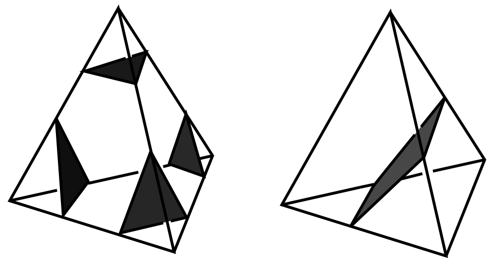

Figure 3

bound the length of ∂H_X^n in terms of its Euler characteristic. Once we have bounded length, it is not difficult to show that ∂H_X^n is a bounded distance from $\Delta(X)$ in $\Gamma(F)$.

Summing all this up, we can now show that there is a bound on the distance between the sets ∂H_X and $\Delta(X)$ as measured in $\Gamma(F)$, in terms of $\chi(H_X)$.

In Y things are a bit trickier. According to Lemma 4.1 we must now face the possibility that H_Y is strongly irreducible, *i.e.* that every compressing disk on one side must meet every compressing disk on the other. The relevant theorem that will replace the result of Haken's which we used to normalize H_X is the following:

Theorem 4.2 (Bachman [Bac01]). *If every compressing and boundary compressing disk on opposite sides of H_Y intersect then H_Y can be made almost normal.*

Here an almost normal surface is one which intersects every tetrahedron in a collection of triangles and quadrilaterals, except for exactly one piece. The exceptional piece can be either two normal disks connected by an unknotted tube (Figure 4 left), two disks connected by a "half tube" along ∂Y (Figure 4 middle), or an octagon (Figure 4 right).

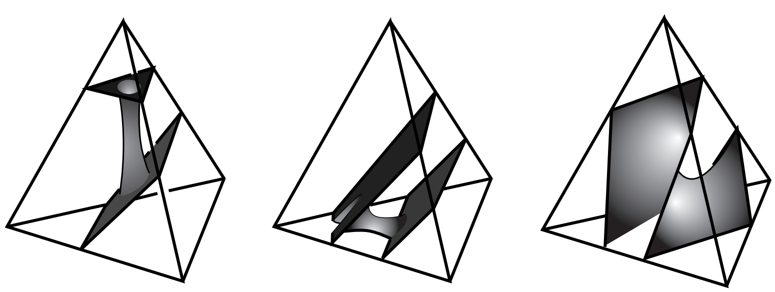

Figure 4

Although the surface H_Y given by Lemma 4.1 must have the property that compressing disks on opposite sides intersect, this may not be true of the ∂-compressing disks. Hence, we may have to do a sequence of *weak reductions* which consist of nothing more than pairs of simultaneous boundary compressions, on opposite sides, or a simultaneous compression and ∂-compression. Each such weak reduction represents a step of at most two for ∂H_Y in the 1-complex $\Gamma(F)$. But note that the the negative Euler characteristic of the resulting surface also decreases by a similar amount.

Once we have obtained an almost normal surface H'_Y by weak reducing H_Y we would be able to conclude that its boundary is a bounded distance from $\Delta(Y)$, if we knew there were no annular summands for H'_Y. Note that we cannot use the result of Jaco and Oertel mentioned previously, since H'_Y is not incompressible. Ignoring this for now, we once again obtain the result that there is a bound on the distance between the sets ∂H_Y and $\Delta(Y)$ as measured in $\Gamma(F)$, in terms of $\chi(H_Y)$.

Finally, Lemma 4.1 shows that ∂H_X is distance at most one away from ∂H_Y in $\Gamma(F)$, and establishes a relationship between $\chi(H_X)$, $\chi(H_Y)$ and $\chi(H)$. Putting all this together Conjecture 1 follows.

The missing step then is to gain control over the normal annuli in Y. To handle this we turn toward Jaco and Rubinstein's program for finding *1-efficient triangulations* of irreducible, atoroidal 3-manifolds. In this program they find triangulations in which there are serious restrictions on the toroidal summands of strongly irreducible Heegaard splittings. Both PIs have studied this program in detail, and hope to be able to mimic it to produce triangulations of X and Y in which the desired surfaces have restricted annular summands.

5. Non-separating essential surfaces

Haken 3-manifolds that do not contain separating incompressible surfaces necessarily contain non-separating ones. Let M be such a manifold, containing the non-separating surface F. Let M' denote the manifold obtained from M by removing a regular neighborhood of F. Then M' has two boundary components that are each copies of F. The manifold M can be recovered from M' by gluing via a homeomorphism ϕ.

As above, we make the simplifying assumption that M' is acylindrical. In this case there are canonical collections of curves Δ_1 and Δ_2 on each boundary component. We now define the distance $d(\phi)$ of ϕ to be the distance between Δ_1 and $\phi^{-1}(\Delta_2)$ in $\Gamma(F)$, the curve complex of F.

We conjecture that Figure 2 is still the correct picture. To prove this we need to show the following:

Conjecture 3. *Suppose H is a Heegaard splitting of $M = \cup_\phi M'$, where M' is acylindrical. If $-\chi(H)$ is less than the gluing distance (i.e. if $g(H) \leq \frac{1}{2}d(\phi)+1$) then H is weakly reducible.*

In this setting we believe we can prove that a suitable counterpart to Lemma 4.1 holds. Assuming this is the case much of the proof sketch from Section 4 goes through. In particular, once we have proven Conjecture 1, there should be no problem finding a suitably nice triangulation of M' (*i.e.* one with restricted normal annuli).

If Conjecture 3 is true, we believe the following holds:

Conjecture 4. *Suppose H is a Heegaard splitting of $M = \cup_\phi M'$. If $-\chi(H)$ is less than $d(\phi)$ then $g(H) \geq g(M') + 1$.*

Here $g(M')$ denotes the minimal genus Heegaard splitting of M' among all those which do not separate the boundary components.

6. Heegaard splittings of toroidal 3-manifolds

As mentioned in Section 2, of all *geometric* 3-manifolds the least is known about Heeegaard splittings of the hyperbolic ones. But what about Heegaard splittings of toroidal 3-manifolds that do not admit a geometry without further decomposition? It follows immediately from Theorem 2.1 that such splittings have distance at most two. But in fact we can say more. In work in progress the PIs, together with E. Sedgwick, consider the case where there is a separating essential torus. Our goal is to show the following:

Conjecture 5. *Let X and Y be manifolds with toroidal boundary. For "sufficiently complicated" gluings $\phi : \partial X \to \partial Y$ the manifold $X \cup_\phi Y$ admits only weakly reducible (i.e. distance one) Heegaard splittings.*

In this context "sufficiently complicated" means that there is no overlap between the set of boundary slopes in ∂X (a finite set) and the preimage, under ϕ, of the set of boundary slopes in ∂Y. The proof will follow quickly from Lemma 4.1 and Theorem 4.2, as follows.

Assume the surface F of Lemma 4.1 is a separating torus. Using Haken's work [Hak61] we can normalize the incompressible pieces H_X and/or H_Y. If H_Y is strongly irreducible then, using Bachman's result on "almost-normalizing" Heegaard splittings *with boundary* (see Theorem 4.2), we can find an almost normal form for H_Y. A paper of Jaco and Sedgwick [JS03] shows that a finite number of slopes on ∂X and ∂Y can be the boundaries of normal or almost normal surfaces. If we assume the map used to glue X to Y is sufficiently complicated then these slopes do not match up. We conclude that H must not have been strongly irreducible.

What remains is the non-separating case. Any 3-manifold with a non-separating essential torus can be obtained from a 3-manifold with two torus boundary components by gluing these components together by some automorphism ϕ. What we would like to show is the following:

Conjecture 6. *For "sufficiently complicated" such ϕ the resulting 3-manifold admits only weakly reducible Heegaard splittings.*

To establish this we expect to be able to use many of the same techniques as in the separating case above, although we will no longer be able to appeal to the Jaco-Sedgwick result [JS03].

7. Results from prior NSF support

In this section we discuss relevant work performed by Schleimer during his NSF postdoc at the University of Illinois at Chicago.

7.1. **Full Heegaard splittings.** Recall our notation: a Heegaard splitting H of a closed orientable three-manifold M is a surface dividing the manifold into a pair of handlebodies, V and W. A splitting has the *disjoint curve property* (as defined by Thompson [Tho99]) if there is a pair of properly embedded disks $D \subset V$, $E \subset W$ and an essential curve $\gamma \subset H$, with the boundaries of D and E disjoint from γ. That is to say, $H \setminus (\partial D \cup \partial E)$ has nontrivial topology. This definition naturally arises when studying the distance of a Heegaard splitting. As noted above, H has the disjoint curve property (DCP) if and only if its Hempel distance is two or less. Finally we say H is *full* if it does not admit such a triple (D, E, γ).

We have proved the following:

Theorem 7.1 (Schleimer)**.** *For any closed orientable three-manifold M there is a constant $C(M)$ as follows: if $H \subset M$ is a Heegaard splitting with genus at least $C(M)$ then H has the disjoint curve property.*

Combining this with work of Jaco and Rubinstein, and now-standard techniques of Casson and Gordon, gives:

Theorem 7.2. *Any closed orientable three-manifold M admits only finitely many full Heegaard splittings.*

Note that several false conjectures of the form "M has only finitely many splittings of type 'X' " have been made. See [Wal78] or [Ale67]. We believe that Theorem 7.2 gives a serious restriction on the Heegaard splittings of a generic three-manifold, as desired by [Wal78] or [Ale67].

Let us now sketch the proof of Theorem 7.1. For full details please consult [Sch]. First note that if a splitting is weakly reducible then it has the DCP. So we restrict attention to strongly irreducible splittings. Fix a triangulation of M. We now use the Rubinstein-Stocking theorem (see [Sto00]) to isotope a given strongly irreducible Heegaard splitting H to be almost normal. (Geometrically this may be thought of as fixing a metric on M and then realizing any strongly irreducible splitting as a surface with uniformly bounded sectional curvatures.)

Now cut the triangulation of M along H. When H has large enough genus we see the handlebody V (and similarily for W) divided into two regions: a "I-bundle region" where the pieces of the triangulation form an I-bundle over a surface and the complementary region, the "core region". (The rough geometric idea is that there is an ϵ, depending only on the metric, such that the I-bundle is the set of points in V which are within ϵ of two *separated* points of H. Here two points are separated if their distance, in H, is much bigger than 2ϵ.)

A section of the vertical boundary of the I-bundle for V is a *short* link in M; that is a link where no component is very long in the chosen triangulation (metric) on M. We remove a regular neighborhood of this link to get an irreducible, boundary-irreducible submanifold $N \subset M$. Choose a section F of the intersection of N with the I-bundle. This surface F is incompressible and boundary-incompressible in N. Thus, choosing a triangulation for N (which does not have many more tetrahedra than that of M), we may normalize F.

Now, since we may take H to have very large genus relative to the triangulation of N, the same holds of F. It follows from classical work of Haken [Hak61], together with a lemma of [JO84], that the surface F is *annular*; there is an essential annulus properly embedded in (N, F), which is not parallel into the boundary. This annulus, and a careful combinatorial argument, will show that H has the DCP. (For a much easier, but related theorem, see Thompson's proof [Tho99] that in toroidal manifolds *all* Heegaard splittings have the DCP.)

7.2. **Heegaard splittings of surface bundles.** Here we discuss joint work with David Bachman, which is the beginning of the program outlined in our proposal.

For the remainder of this section we study the Heegaard splittings of a *surface bundle* $M(\phi)$ which is obtained as follows: Fix a closed orientable surface F and a homeomorphism

$\phi : F \to F$. Taking $F \times I$, glue $(x, 1)$ to $(\phi(x), 0)$. We will refer to ϕ as the *monodromy* of the surface bundle $M(\phi)$.

As ϕ acts in a natrual fashion on the curve complex of F (see above) the translation distance of ϕ should effect the topology of $M(\phi)$. In fact we have:

Theorem 7.3 (Bachman-Schleimer)**.** *Suppose H is a Heegaard splitting of a surface bundle. If $-\chi(H)$ is less than the translation distance of the monodromy then H is weakly reducible.*

Following the lead of Casson and Gordon this theorem should be read as "if a Heegaard splitting of $M(\phi)$ has low genus then it can be obtained from an even lower genus Heegaard splitting or essential surface." In fact we prove:

Corollary 7.4 (Bachman-Schleimer)**.** *Suppose H is a Heegaard splitting of a surface bundle. If $-\chi(H)$ is less than the translation distance of the monodromy then H is a stablization of the standard splitting.*

Let us now sketch a proof of the contrapositive of Theorem 7.3. Fix H a strongly irreducible Heegaard splitting of $M(\phi)$. Now, there is a natural foliation of the manifold coming from the surface bundle structure. Label the leaves of the foliation $F(s)$ where s ranges from 0 to 2π and $F(0) = F(2\pi)$. Similarly there is a singular foliation of $M(\phi)$ coming from the given Heegaard splitting H – choose a height function on $M(\phi)$ with spines for V and W at heights 0 and 1 respectively, and every level set $H(t)$ isotopic to H.

Using Cerf theory (as in the work of Rubinstein-Scharlemann [RS96]) we analyze the intersection of $F(S)$ and $H(t)$. The strong irreducibility of H, together with combinatorial work, implies the existence of a level $H(t_0)$ which intersects all (except finitely many) of the fibres $F(s)$ in curves which are essential on both $H(t_0)$ and $F(s)$. From these we extract a sequence of curves which provide a path in the curve complex of the fibre. This path gives the desired bound.

8. Individual Projects

In this section we discuss projects that the PIs are working on separately, but nonetheless have relevance for their joint program, as outlined above.

8.1. **Bounding distance via Heegaard genus.** Saul Schleimer and Marc Lackenby are currently working on:

Conjecture 7. *If H and K are Heegaard splittings of M, and K is not a stabilization of H, then the distance of H is bounded above by twice the genus of K.*

The idea that the genus of K bounds the distance of H is prompted by the analogy between strongly irreducible splittings and incompressible surfaces, and by Theorem 2.1.

Our idea here is to mimic the proof technique of Stocking in [Sto00]. Instead of taking a triangulation we fix spines V_0 and W_1 for the splitting H. We isotope these until they are in thin position with respect to the sweepout given by K. We then look for a thick level, K_t, with the following property: for every level H_s all curves of $K_t \cap H_s$ are essential in H_s or are inessential in both surfaces. This should be thought of as making K *almost normal* with respect to the sweepout H_s.

Equipped with the above there are standard techniques (as in Lemma 4.2 of [BS]) which will give the desired result.

8.2. **Uniqueness of Heegaard splittings in Amalgamated 3-manifolds.** Bachman is currently exploring the following conjecture:

Conjecture 8. *Let X and Y be manifolds with torus boundary which have at most one Heegaard splitting of each genus. If one glues X to Y with a "sufficiently complicated" map then the resulting 3-manifold has at most one Heegaard splitting of each genus.*

In previous work Bachman has shown that a 3-manifold has non-isotopic Heegaard splittings of some genus if and only if it contains a *critical* surface [Bac02]. *Criticality* is a combinatorial condition on the compressing disks for a surface, analogous to the condition of strong irreducibility.

To establish Conjecture 8 the first step would be to prove a lemma analogous to Lemma 4.1. Such a lemma should follow from a 2-parameter Cerf-theory type argument, and establish the following:

Conjecture 9. *Assume H is a critical Heegaard splitting, F is essential, and F separates M into X and Y. Then there are non-∂-parallel incompressible, strongly irreducible, or critical surfaces $H_X \subset X$ and $H_Y \subset Y$ with disjoint, non-empty boundaries, such that $\chi(H_X) + \chi(H_Y) \geq \chi(H)$.*

Now, assume $X \cup_\phi Y$ has non-isotopic Heegaard splittings of some genus. Then it contains a critical surface by [Bac02]. If this surface was not isotopic into X or Y then it would follow from Conjecture 9 that there are incompressible, strongly irreducible, or critical surfaces *with boundary* in X and Y. In the first case we obtain a normal boundary slope by [Hak61]. In the second there is an almost normal boundary slope by [Bac01]. What Bachman must do, then, is to establish that there is a normal form for critical surfaces with boundary (as he has done for closed critical Heegaard surfaces [Bac] and strongly irreducible Heegaard surfaces with boundary [Bac01]).

Assuming this works out, and the map gluing X to Y is sufficiently complicated, then the conclusion would be that X or Y contains a critical surface. It then follows from [Bac02] that X or Y has non-isotopic Heegaard splittings of some genus, contradicting the hypotheses of Conjecture 8.

8.3. **Dehn Filling.** Bachman and E. Sedgwick have noted that finding a normal form for critical surfaces with non-empty boundary may yield other fruit as well. Suppose X is a manifold with torus boundary, and H and G are non-isotopic Heegaard splittings of X. If α is a slope on ∂X then let $X(\alpha)$ denote the manifold obtained by gluing a solid torus to X in such a way so that α bounds a disk. Now assume α is some slope on ∂X such that H and G are isotopic in $X(\alpha)$ (or, more generally, the stabilization bound between H and G is lower in $X(\alpha)$ than in X). Then there appears to be a critical surface with boundary α in X. We now need to show that this surface has an appropriate normal form in X, and that there are a finite number of slopes on ∂X that can bound such a surface. The proof of the latter assertion should be similar to the result of Jaco and Sedgwick, which says that there are a finite number of normal and almost normal boundary slopes [JS03]. We conclude that this phenomenon should only happen for a fairly restricted class of fillings. This is analogous to the result of Rieck and Sedgwick [RS01], which says that a strongly irreducible Heegaard splitting of X can become weakly reducible only after a restricted class of fillings.

Human Resources Impact Statement

David Bachman

Bachman has demonstrated a long history of educational activities at all levels. This started when Bachman was a second year graduate student and became an instructor for Uri Triesmann's *Emerging Scholars Program*. This program consisted of challenging workshops designed to improve the retention of underrepresented groups in the sciences.

Later in graduate school Bachman took on a program called *Saturday Morning Math Group*. This program offered high school students exposure to topics in advanced mathematics not usually experienced until graduate school. Four mornings each semester students were invited to come to the university, where they would listen to talks and do activities related to advanced mathematics. Bachman is hoping to start a similar program at Cal Poly.

As a postdoc at Portland State University Bachman organized the undergraduate mathematics club. This featured bi-weekly meetings in which speakers from both in and out of academia were solicited to talk about their relationship with mathematics. Bachman also administered the Putnam exam and ran practice sessions for interested students. Also while at Portland State Bachman was co-organizer for the Cascade Topology Conference, a semi-annual meeting of in the Pacific Northwest.

Finally, as a post-doc at the University of Illinois at Chicago Bachman organized the department's Geometry, Topology, and Dynamical Systems weekly seminar.

Saul Schleimer

Schleimer has, for the last three years, been the organizer of the UIC three-manifold seminar. Although the seminar is primarily focused on research-level talks on topics of current interest, it has also had many graduate student speakers who were both local and from further afield. Other activities directed at graduate education include assisting at two graduate summer schools and participation in several graduate level seminars during the school year.

In addition, he has twice served as a referee for the journal *Geometriae Dedicata*, which proved to be very instructive.

SECTION D: REFERENCES CITED

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[Thu82] W. P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

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SECTION E: BIOGRAPHICAL SKETCH

a. Professional Preparation.

State University of New York at Binghamton	Mathematics	B.S. 1993
The University of Texas at Austin	Mathematics	Ph.D. 1999
Portland State University	Mathematics	1999-2000
The University of Illinois at Chicago	Mathematics	2000-2002

b. Appointments.

Cal Poly State University, San Luis Obispo	Asst. Prof.	2002-
The University of Illinois at Chicago	Research Asst. Prof.	2000-2002
Portland State University	Asst. Prof.	1999-2000

c. Publications.

Links to all publications and preprints are available at:

http://www.calpoly.edu/∼dbachman

(1) Publications most closely related to the proposed project.
   (a) Heegaard Splittings with Boundary and Almost Normal Surfaces. *Topology and its applications*, 116 (2001) 153-184.
   (b) Distance and bridge position, (with S. Schleimer) to appear in *Pacific Journal of Mathematics.*
   (c) Surface Bundles versus Heegaard Splittings, (with S. Schleimer) submitted for publication (February 2003).
   (d) Critical Heegaard Surfaces. *Transactions of the American Mathematical Society* 354 (2002), 4015-4042.
   (e) 2-Normal Surfaces. Submitted for publication, July 2002.
(2) Other significant publications.
   (a) Large embedded balls and Heegaard genus in negative curvature, (with D. Cooper and M. White) to appear in *Algebraic & Geometric Topology.*
   (b) Thin position for tangles, (with S. Schleimer) *Journal of Knot Theory and its Ramifications* Vol.12, No.1 (2003) 117-122.
   (c) A note on Kneser-Haken finiteness. To appear in *Proceedings of the American Mathematical Society.*

(d) Normalizing Heegaard-Scharlemann-Thompson Splittings. Submitted for publication, February 2002.

d. Synergistic Activities.

Curriculum Developer Ongoing

"A Geometric Approach to Differential Forms"
(see http://www.calpoly.edu/∼dbachman)

Conference Organizer May 2000

Cascade Topology Conference, Portland State University

Organizer 1999-2000

Undergraduate Math Club, Portland State University

Coordinator 1995-1996

Saturday Morning Math Group, UT Austin

Instructor 1994-1995

Emerging Scholars Program, UT Austin

e. Collaborators & Other Affiliations.

(1) **Collaborators.**
Saul Schleimer, The University of Illinois at Chicago
William Jaco, Oklahoma State University
Mathew White, Cal Poly State University at San Luis Obispo
Daryl Cooper, University of California, Santa Barbara

(2) **Graduate and Postdoctoral Advisors.**
Cameron McA. Gordon, The University of Texas at Austin
Steven Bleiler, Portland State University
Peter Shalen, The University of Illinois at Chicago

(3) **Thesis Advisor and Postgraduate-Scholar Sponsor.**
None

SECTION F: BUDGET JUSTIFICATION

A. Salaries and Wages

(1) 4 units of release time per year are requested to give the PI enough time to further develop his career goals. The normal teaching load at Cal Poly is 12 units (3 classes that meet 4 hours per week) per quarter. In support of an NSF grant, the PI's department will adjust the PI's teaching load for the Winter and Spring quarters to enable the faculty to perform support for the project. The requested release time would be used in the Fall quarter.

(2) The usual two-ninths salary is requested for research activities during the summer.

E. Travel

(1) **Domestic.** Four domestic trips per year are estimated for conference attendance and visits to colleagues/coauthors. Expenses per trip are estimated as follows:

$ 500	travel
$ 350	hotel (5 nights @ $ 70 per night)
$ 270	per diem (6 days @ $ 45 per day)
$1120	total per trip
$4480	total per year

(2) **Foreign.** Two foreign trips per year are estimated for international conference attendance. Expenses per trip are estimated as follows:

$1500	travel
$ 700	hotel (7 nights @ $ 100 per night)
$ 520	per diem (8 days @ $ 65 per day)
$2720	total per trip
$5440	total per year

G. Other Direct Costs

(1) **Materials and Supplies.** $3500 in the first year for the purchase of a laptop projector for conference presentations. There is currently no such projector owned by the PI's department. $500 per year for general supplies.

(2) **Publication Costs.** $500 per year to cover costs of reproduction and distribution of research papers.

(3) **Consultant Services.** Money to bring colleagues to visit the PI at Cal Poly is requested. Should such a visitor also give a colloquium talk the department will pay an

honorarium. Four such visits per year are estimated. Expenses per visit are estimated as follows:

\$ 500	travel
\$ 280	hotel (4 nights @ \$ 70 per night)
\$ 225	per diem (5 days @ \$ 45 per day)
\$1005	total per trip
\$4020	total per year