A METRIC SURVEY OF CURVE COMPLEXES

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This is joint work with Howard Masur.

The *graph of curves* of a surface S has as its vertex set all isotopy classes of simple closed essential, nonperipheral curves in S. Two distinct vertices are connected by an edge if the classes in question have disjoint representatives. This graph $\mathcal{C}(S)$ (or rather, its clique complex) was introduced by Harvey [2] and has been used to study the mapping class group [7], Kleinian groups [9], and Heegaard splittings [4].

There is a veritable zoo of similar objects: $\mathcal{A}(S)$ the graph of arcs [1], $\mathrm{Sep}(S)$ the separating curve complex, the Hatcher-Thurston complex, the pants complex [3], and so on. Let $\mathcal{G}(S)$ be any of these. The vertices are all isotopy classes of multi-curves in S, and the edges of $\mathcal{G}(S)$ are the relation "small geometric intersection," typically disjointness. All edges are given length one. We wish to study the coarse properties of the resulting metric space. The model theorem in this direction is due to Masur and Minsky [6]:

Theorem 1. *The graph of curves is Gromov hyperbolic.*

Following [7] or [5], if X is an essential subsurface of S then any of the above graphs admits a "cut-and-paste" map to $\mathcal{C}(X)$ as follows: Pick α a vertex of $\mathcal{G}(S)$. Isotope α to intersect X tightly. Pick any component of $\alpha \cap X$. This gives an arc α' in X. Let α'' be any nonperipheral (in X) component of the boundary of a neighborhood of $\alpha' \cup \partial X$. Then α'' is a *subsurface projection* of the vertex α to X and we have a coarse map $\pi_X \colon \mathcal{G}(S) \to \mathcal{C}(X)$. If every vertex of $\mathcal{G}(S)$ meets X nontrivially then the subsurface projection is everywhere defined. In this case we call X a *hole* for $\mathcal{G}(S)$.

From Lemma 2.3 of [7] it is straight-forward to show:

Lemma 2. *For any $\mathcal{G}(S)$ there is a constant $K > 0$ so that subsurface projection to any hole is K-Lipschitz.*

It is then easy to deduce:

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This work is in the public domain.

Corollary 3. *Suppose $\mathcal{G}(S)$ admits an action by the mapping class group and X and Y are disjoint holes. Then $\mathcal{G}(S)$ is not Gromov hyperbolic.*

It is a pleasant exercise to classify the holes for any of the standard examples given above. In particular one finds examples where all holes intersect. It is natural to conjecture a converse to Corollary 3:

Conjecture 4. *Suppose that $\mathcal{G}(S)$ admits an action by the mapping class group and any pair of holes X and Y intersect. Then $\mathcal{G}(S)$ is Gromov hyperbolic.*

A crucial step in proving the conjecture for any fixed $\mathcal{G}(S)$ would be to verify the *distance estimate*:

Conjecture 5. *The sum $\sum' d_X(\alpha, \beta)$ is within uniform multiplicative and additive error of the distance between α and β in $\mathcal{G}(S)$.*

Here the summation ranges over all holes X for $\mathcal{G}(S)$. The quantity $d_X(\alpha, \beta)$ equals the distance between $\pi_X(\alpha)$ and $\pi_X(\beta)$ in $\mathcal{C}(X)$. The "prime" on the summation indicates that all summands less than certain size are omitted.

We have verified the distance estimate and hyperbolicity for the arc complex. The techniques required are essentially contained in the two papers [6] and [7].

It is much more difficult to obtain the two conjectures for the *graph of disks*, $\mathcal{D}(V_g)$, defined by McCullough [8]. This graph has as vertex set all proper isotopy classes of essential disks in a genus g handlebody V_g. As usual the edges come from disjointness. As work-in-progress we have classified the holes for $\mathcal{D}(V)$ using the techniques of Masur and Minsky, Jaco-Shalen-Johannson theory, and an analysis of which surfaces admit pseudo-Anosov maps.

Suppose now that ∂V is identified with S. Then there is a relationship between $\mathcal{D}(V)$ and $\mathcal{C}(S)$. The former is included in the latter by the natural boundary map. In fact a pair of handlebodies V and W, both glued to S, specifies a three-manifold with *Heegaard splitting surface* S. Hempel [4] then defines the *distance* of S to be $d_S(V, W)$: the minimal distance in $\mathcal{C}(S)$ between the subgraphs $\mathcal{D}(V)$ and $\mathcal{D}(W)$. As an application of the classification of holes we obtain:

Algorithm 6. *Fix a genus g. There is a constant K and an algorithm which, given a Heegaard diagram $(S, \mathbb{D}, \mathbb{E})$, computes the distance $d_S(V, W)$ up to an additive error of at most K.*

References

[1] John L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Ann. of Math. (2)*, 121(2):215–249, 1985.

[2] Willam J. Harvey. Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, pages 245–251, Princeton, N.J., 1981. Princeton Univ. Press.

[3] A. Hatcher and W. Thurston. A presentation for the mapping class group of a closed orientable surface. *Topology*, 19(3):221–237, 1980.

[4] John Hempel. 3-manifolds as viewed from the curve complex. *Topology*, 40(3):631–657, 2001. arXiv:math.GT/9712220.

[5] Nikolai V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.

[6] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999. arXiv:math.GT/9804098.

[7] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000. arXiv:math.GT/9807150.

[8] Darryl McCullough. Virtually geometrically finite mapping class groups of 3-manifolds. *J. Differential Geom.*, 33(1):1–65, 1991.

[9] Yair N. Minsky. The classification of Kleinian surface groups, I: Models and bounds. arXiv:math.GT/0302208.

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