TUNNEL NUMBER ONE KNOTS IN S^3 AND GENUS ONE SPLITTINGS

Abstract. We give a family of knots in the three-sphere which have increasingly complicated bridge positions with respect to the standard genus one splitting.

1. Introduction

A knot $K \subset S^3$ has a (g, b) *position* if K may be isotoped to have exactly b bridges with respect to the standard genus g Heegaard splitting of S^3. Of course, if K has a (g, b) position then K also has a $(g, b+1)$ position.

The purpose of this note is to prove:

Theorem 1.1. *There is a family of tunnel number one knots $K_n \subset S^3$ so that K_n has no $(1, n)$ position.*

We remark that a tunnel number t knot must always have a $(t+1, 0)$ position of a special kind: the knot K may be isotoped to be a *primitive* curve K' lying on the boundary of V_{t+1}, the standard genus $t+1$ handlebody. (That is, K' meets some essential disk $D \subset V$ in exactly one point.)

2. Sketch of the proof

Let $V = V_2$ be the standardly embedded genus two handlebody in S^3. Let W be the closure of $S^3 \setminus V$. Let $S = V \cap W$ be the standard genus two Heegaard splitting of S^3. Consider $\mathcal{C}(S)$ the curve complex of S. Curves which are primitive on V form a subcomplex.

This subcomplex is best thought of in its relation to the disk complex $\mathcal{D}(V)$, and lies in a radius one neighborhood of $\mathcal{D}(V)$ inside of $\mathcal{C}(S)$. We are interested in primitive curves (in V) which are far away from $\mathcal{D}(W)$ inside of $\mathcal{C}(S)$. We deduce the existance of such from:

Proposition 2.1. *There is a sequence of disks $D_n \subset V$ so that $d_S(D_n, \mathcal{D}(W)) \geq n$. Here $d_S(\cdot, \cdot)$ denotes distance in the curve complex $\mathcal{C}(S)$.*

This is proved by finding a pseudo-Anosov map $f\colon S \to S$ which extends over V but not over W and then applying a criterion of Kobayashi [1].

Date: January 11, 2006.

The desired primitive curves are distance at most two from the disks D_n.

Suppose now that K_n is a primitive curve on V, meeting D_n exactly once. Pushing K_n slightly into V and removing an open regular neighborhood of K_n we obtain a compression body/handlebody splitting of $E(K_n)$ the exterior of K_n.

Claim 2.2. If n is large enough then the given genus two compression body/handlebody splitting of $E(K_n)$ has Hempel distance at least $n/2$.

We now require a slight generalization of work of Scharlemann and Tomova:

Theorem 2.3. *If S is a compression body/handlebody splitting of $E(K)$ and T is any generalized Heegaard splittings of $E(K)$ which is not a stabilization of S then the quantity $-\chi(T) + 2$ is an upper bound for the Hempel distance of S.*

We are now in a postion to prove the theorem: let K_n be as above and suppose that T is the generalized Heegaard splitting of $E(K_n)$ coming from a $(1, b)$ position of K_n. So T is a $2b$-holed torus and $-\chi(T) + 2 = 2b + 2$. Combining the theorems above we find that $2b + 2 \geq n/2$ and so $b \geq (n - 4)/4$. Thus b goes to infinity with n and we are done.

3. The details

There are lots.

References

[1] Tsuyoshi Kobayashi. Heights of simple loops and pseudo-Anosov homeomorphisms. In *Braids (Santa Cruz, CA, 1986)*, pages 327–338. Amer. Math. Soc., Providence, RI, 1988.