Degeneration and Regeneration of Geometric Structures on Three-Manifolds

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Abstract

This thesis studies the degeneration of geometric structures on manifolds. We begin with a foliation with transverse geometric structure on a manifold M; this can be regarded as a kind of degenerate geometric structure on M. We investigate when the foliation can be "approximated" by non-degenerate geometric structures on M. This can be made precise in terms of developing maps describing geometric structures and foliations on M.

To approximate a foliation by non-degenerate geometric structures in this way, we carry out two steps :

(1) Find a (suitable) family of representations starting with the holonomy representation for the foliation.

(2) Show that there are geometric structures with these representations as holonomy representations.

We introduce a 1-form giving a "tangent vector" to a family of developing maps and obtain "regeneration conditions" on the form guaranteeing that step (2) is possible. For a 1-dimensional foliation, these conditions can be also be expressed in terms of cohomology, making them easier to apply.

To carry out step (1), we study the topology of certain spaces of representations using cohomological techniques. In particular, we consider the representations of certain 3-manifold groups into the group of hyperbolic isometries $PSL_2\mathbb{C}$.

Applying the previous results, we investigate when certain foliations of 3-manifolds with non-hyperbolic geometric structures (modelled on the geometries : $\mathbf{E}^3$, $\mathbf{H}^2\times\mathbf{R}$, $P\tilde{SL}_2\mathbf{R}$ and Solv) can arise as limits of degenerating hyperbolic structures. In general, the initial foliation with transverse geometric structure must satisfy restrictive topological and geometric conditions for such deformation to be possible.

We also give a new proof of a formula of Schläfli describing the derivative of volume for a smooth family of polyhedra in a space of constant curvature. Using this result, we describe the variation of volume as hyperbolic structures are deformed.

Finally, we illustrate how our results can sometimes be used to find the exact boundary of hyperbolic Dehn surgery space. We discuss the space of hyperbolic structures on the figure eight knot complement in detail.

Acknowledgements

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I dedicate this work to my parents, in appreciation for the support and encouragement they have always given me.

Contents

Introduction

In this thesis, we study the deformation of geometric structures on manifolds. In general, the structures we consider will be incomplete with special kinds of singularities. We will be particularly interested in understanding the ways that these structures can degenerate.

These ideas have been used to construct geometric structures on 3-manifolds. Every closed 3-manifold M contains a link Σ such that $M-\Sigma$ has a complete hyperbolic structure (see e.g. Myers [My]). One could try to deform this hyperbolic structure to obtain incomplete hyperbolic structures on M with cone type singularities along Σ. If the cone angle could be increased from 0 to 2π then we would obtain a hyperbolic structure on M.

This idea was first used by Thurston [Th1] to prove that if M is a hyperbolic manifold with boundary consisting of tori, then "almost all" manifolds obtained by Dehn surgery on M also have complete hyperbolic structures. (Here "almost all" means that finitely many surgeries are excluded for each boundary component of M.)

In general, these hyperbolic cone manifold structures will degenerate in some way, before the cone angles reach 2π. By analyzing the kinds of degeneration that can occur when cone angles are $\leqslant\pi$, Thurston [Th3] was able to use this approach to show the existence of geometric structures on orbifolds with singular locus of dimension at least one.

Some examples

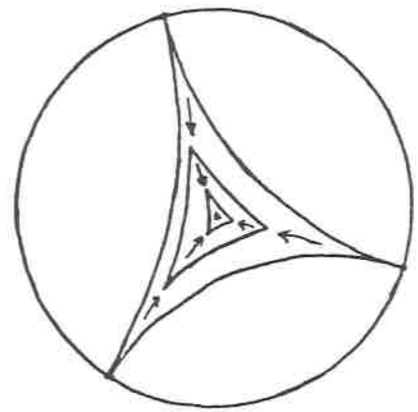
We first give some 2-dimensional examples to illustrate the kind of behaviour we will consider.

Example 1 : Cone manifold structures on the three-punctured sphere.

In the hyperbolic plane, there is a unique ideal triangle with all vertices on the circle at infinity, and three angles equal to zero. By doubling such a triangle, we obtain the unique

complete hyperbolic structure on the 3-punctured 2-sphere.

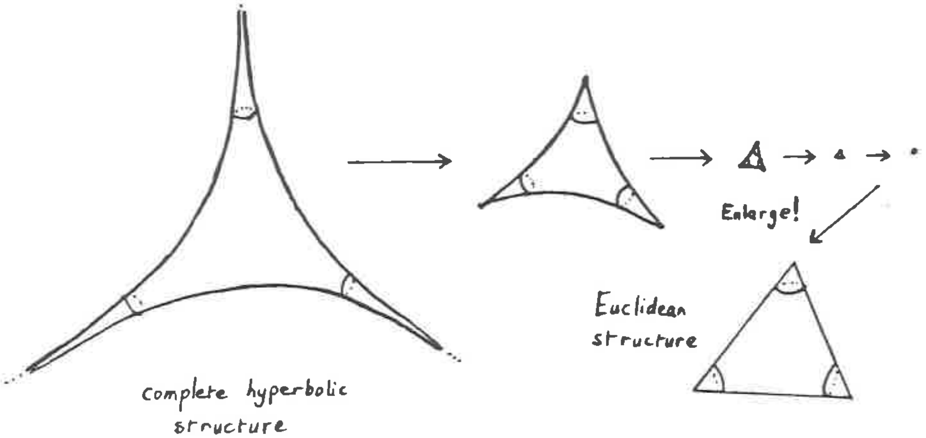
Now we can deform the ideal triangle by moving the vertices inwards from infinity, keeping the three sides of equal length. This gives a family of (finite) hyperbolic triangles with positive angles. If we continue moving the vertices of our triangle radially inwards, the triangles become smaller and smaller, and finally shrink to a point. Then shapes of the triangles become closer to Euclidean triangles, and the angle sum approaches π as the triangles shrink. By rescaling so that the diameter of the triangles stays constant, there is geometric convergence to a Euclidean limit.



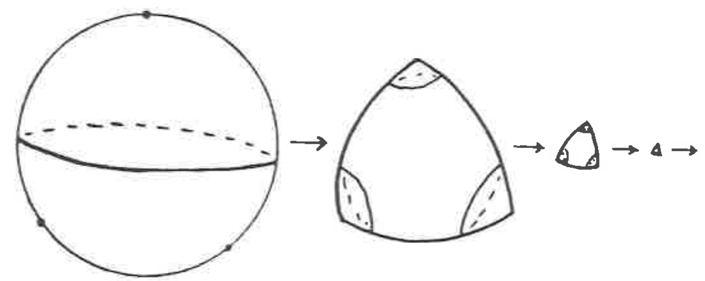
Triangles in H^2.

By doubling such triangles, we obtain an *incomplete* hyperbolic structures on the 3-punctured sphere. The metric completion gives a hyperbolic cone manifold structure on the 2-sphere with three cone points corresponding to the vertices of the triangles. Near each cone point, the metric is modelled on a cone with angle equal to twice the angle of the triangle at the corresponding vertex.

By doubling a family of triangles shrinking to a point, we obtain a family of hyperbolic cone manifold structures on S^2 with 3 cone points, which begin with the complete hyperbolic structure (with cone angles zero) and shrink to a point as the sum of cone angles increases to 2π. By rescaling the metrics so that the diameter remains constant, we obtain a limiting Euclidean cone manifold structure with sum of the cone angles equal to 2π.



By using spherical triangles, instead of hyperbolic triangles we can continue to increase the angles. This gives a family of spherical cone manifold structures on S^2 with 3 cone points, starting with a non-singular structure (when cone angles are 2π) and shrinking to a point as the sum of the cone angles decreases to 2π.

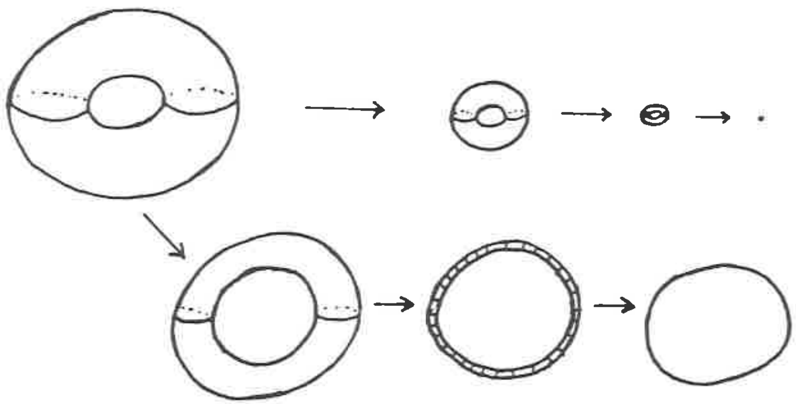


In this way we obtain a continuous family of cone manifold structures with metrics changing from hyperbolic to Euclidean to spherical as the cone angles increase. For instance, using equilateral triangles we obtain three equal cone angles θ varying between 0 and 2π: hyperbolic for $0<\theta<2\pi/3$, Euclidean for $\theta=2\pi/3$ and spherical for $2\pi/3<\theta<2\pi$.

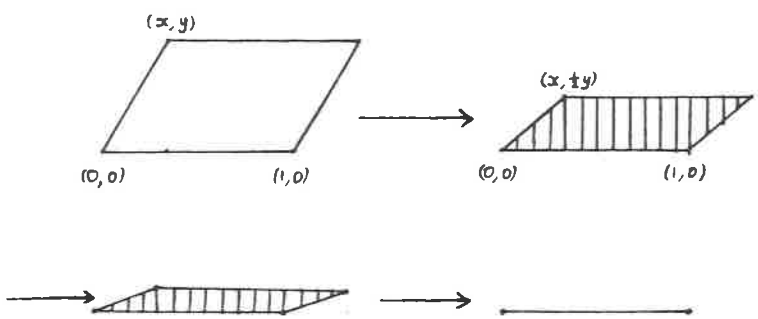
Example 2 : Euclidean structures on the torus.

Any Euclidean structure on the torus can be obtained from a parallelogram by identifying opposite edges by translation. There are several ways these structures can degenerate as the Euclidean structure is deformed.

The simplest deformation is rescaling of the Euclidean metric by factors approaching zero. Then the tori shrink to a point in the limit. It is also easy to construct a sequence of Euclidean tori shrinking down to a circle as limit. In this case, there is foliation arising from the limiting process, consisting of closed geodesics with lengths approaching zero.



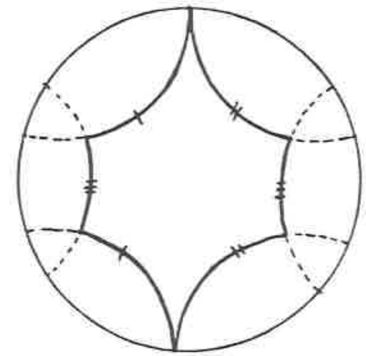
More generally consider a family of Euclidean tori T_t, $t>0$ obtained from the parallelogram in $\mathbb{R}^2$ with sides given by vectors $(1,0)$, (x,ty), with $y>0$.



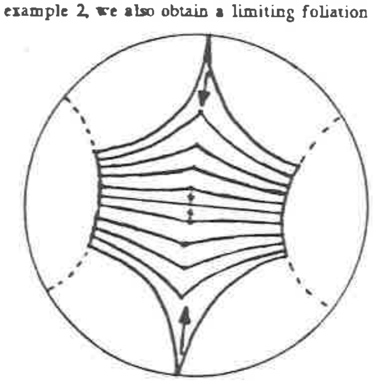
Again a limiting foliation arises, and the limit of the tori T_t is obtained by shrinking each leaf of the foliation to a point.

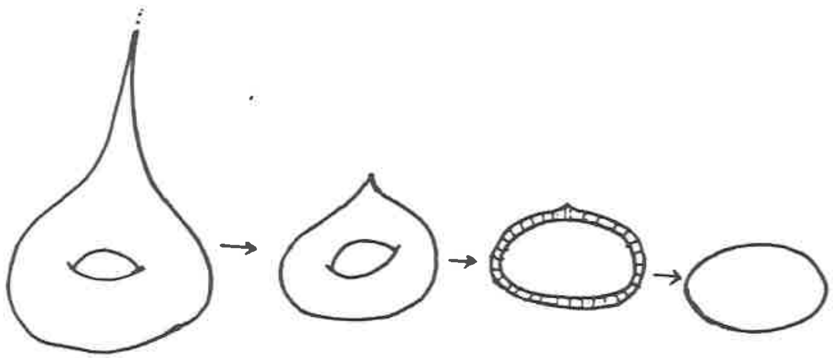
Example 3 : Limits of hyperbolic structures on a punctured torus

We begin with a complete hyperbolic structure on a once-punctured torus obtained by gluing together the sides of a hexagon in $\mathbb{H}^2$ with angles $0, \pi/2, \pi/2, 0, \pi/2, \pi/2$ as in the following figure.



We fix two edges and move the two ideal vertices in from infinity. By gluing up these hyperbolic polyhedra, we obtain a family of hyperbolic structures on the torus with one cone point, with cone angle varying between 0 and 2π. As the angle approaches 2π, the tori degenerate to a circle. As in example 2, we also obtain a limiting foliation by circles whose length goes to zero.





This example can also be easily modified to obtain a family of hyperbolic cone manifold structures on the torus shrinking to a point in the limit.

Contents of thesis

In this thesis, these examples will be generalized as follows. We begin with a foliation with transverse geometric structure on a manifold M; this can be regarded as a kind of degenerate geometric structure on M. We investigate when the foliation can be "approximated" by non-degenerate geometric structures on M. This can be made precise in terms of developing maps describing geometric structures and foliations on M.

Let (G, X) be a geometry, where G is a group acting transitively and analytically on X. A (G, X)-structure or a foliation with a transverse (G, X)-structure on a manifold M can be described by a developing map $D:\tilde{M} \to X$, with associated holonomy representation $\rho:\pi_1(M) \to G$. The developing map is a local diffeomorphism for a geometric structure on M, and a submersion for a foliation with transverse geometric structure. Then the main problem we consider is the following.

Given a foliation defined by a submersion $D:\tilde{M} \to X_0$ and holonomy representation $\rho:\pi_1(M) \to G_0$, can we find a smooth family of developing maps $D_t:\tilde{M} \to X$, and holonomy representations $\rho_t:\pi_1(M) \to G$, $t \geqslant 0$, such that

(1) D_t is a local diffeomorphism for $t > 0$,

(2) $D_0 = D$?

Then D_t describe a family of (G, X)-structures on M degenerating to the given foliation with transverse (G_0, X_0)-structure.

To approximate a foliation by non-degenerate geometric structures in this way, we need to carry out two steps :

(1) Find a family of representations starting with the holonomy representation for the foliation. (2) Show that there are geometric structures with these representations as holonomy representations.

We begin by considering the second question. In chapter one, we introduce a 1-form ω with values in the Lie algebra of G giving a "tangent vector" to a family of developing maps. The cohomology class of ω is determined by the Zariski tangent vector to the corresponding family of holonomy representations.

In chapter two, we find conditions on ω which guarantee that a foliation can be approximated by non-degenerate geometric structures. In the case of a 1-dimensional foliation, we show that these "regeneration conditions" can be expressed in terms of cohomology, making them much easier to apply.

In chapter three, we study the topology of certain spaces of representations using cohomological techniques. In particular, we consider the representations of certain 3-manifolds with boundary into the group of hyperbolic isometries $PSL_2\mathbb{C}$.

In chapter four, we apply the previous results to three-manifolds. For example, we investigate when certain foliations of manifolds with non-hyperbolic geometric structures (modelled on the geometries : E^3, $H^2 \times R$, $PS\tilde{L}_2R$ and Solv) can arise as limits of degenerating hyperbolic structures. In general, the initial foliation with transverse geometric structure must satisfy restrictive topological and geometric conditions for such deformation to be possible.

In chapter five, we study the variation of volume as geometric structures are deformed. First we give a new proof of a result of Schläfli giving the derivative of volume for a smoothly varying family of polyhedra in a space of constant curvature. Then we apply this result to obtain a formula for the variation of volume for hyperbolic and spherical manifolds with cone type or Dehn surgery type singularities. In particular, the derivative of volume only depends on the geometry along the singular locus.

In chapter six, we show how the previous results can sometimes be used to find the "local" boundary of hyperbolic Dehn surgery space. We discuss the case of Dehn surgery on the figure eight knot in detail. Finally, we consider briefly some of the open problems arising from this work.

There is also an appendix describing some of the standard results on group cohomology and de Rham cohomology with coefficients in a flat vector bundle.

It is worth noting that chapters 5 and 6 are largely independent of the other chapters. The references to previous sections consist mainly of definitions and some examples.

Conventions : All manifolds and maps are assumed to be smooth unless otherwise indicated.

CHAPTER 1

Deformations of geometric structures on manifolds

1. Geometric structures on manifolds

In this chapter we begin to study the deformation and degeneration of geometric structures on manifolds. We will consider *locally homogeneous* geometric structures on a manifold M, locally modelled on a homogeneous space $X = G/H$, where G is a (finite dimensional) Lie group and H is a closed subgroup. Then G acts transitively and analytically on X, and the pair (G, X) defines a geometry in the sense of Thurston ([Th1, chap.3], [Th2]).

Definition 1.1. A (G, X) geometric structure on a manifold M is given by a covering of M by open sets U_i and diffeomorphisms $\phi_i : U_i \to X$ to open subsets of X, giving coordinate charts on M, such that all the transition maps are restrictions of elements in G.

Given such a geometric structure there is a *developing map* D: $\tilde{M} \to X$, where $\tilde{M}$ is the universal covering of M, defined as follows. Begin with an embedding $\phi_1 : U_1 \subset M \to X$ giving a coordinate chart on M. If $\phi_2 : U_2 \to X$ is another coordinate chart with $U_1 \cap U_2 \neq \emptyset$, there is a unique $g \in G$ such that $g \circ \phi_2 = \phi_1$ on $U_1 \cap U_2$. So ϕ_1 extends to a map $\phi : U_1 \cup U_2 \to X$, with $\phi = \phi_1$ on U_1 and $\phi = g \circ \phi_2$ on U_2. In this way, we can extend ϕ_1 by analytic continuation along paths in M. Since the result of the analytic continuation only depends on the homotopy class of the path involved, we obtain a well defined map $D : \tilde{M} \to X$. Then D is a local diffeomorphism satisfying the equivariance condition

$$D(\gamma m) = \rho(\gamma) D(m) \qquad (*)$$

for $m \in \tilde{M}$, $\gamma \in \pi_1(M)$, where $\rho : \pi_1(M) \to G$ is a homomorphism called the *holonomy representation* for the geometric structure. Here γm denotes the image of m under the covering transformation on $\tilde{M}$ corresponding to $\gamma \in \pi_1(M)$. (See Thurston [Th1, chap.3] for more details.) Note that D and ρ are not uniquely defined; changing the original coordinate chart ϕ_1 by an element $g \in G$ gives a new developing map $g \circ D$ with corresponding holonomy representation $g \circ \rho \circ g^{-1}$.

Any collection of coordinate charts as in definition 1.1, can be enlarged to a maximal such collection; call this a maximal (G, X) atlas on M.

Definition 1.2. Two (G, X)-structures on a fixed manifold M are *isomorphic* if the maximal (G, X) atlases defining the structures are equal. This is equivalent to requiring that developing maps for the structures differ by composition with an element of G.

Two (G, X)-structures on M_1 and M_2 are *isomorphic* (written $\cong$) if there is a diffeomorphism M_1 to M_2 taking a maximal atlas for M_1 to a maximal atlas for M_2. Equivalently, their developing maps differ by composition with a diffeomorphism $\tilde{M}_1 \to \tilde{M}_2$ covering a diffeomorphism $M_1 \to M_2$ and an element of G.

Definition 1.3. A (G, X)-structure on a manifold is *complete* if the developing map $D: \tilde{M} \to X$ is a diffeomorphism. In the case that G a group of isometries of a Riemannian metric on a manifold X without boundary, this agrees with the other usual definitions, for example :

(1) every geodesic in M can be extended indefinitely

or

(2) M is a complete metric space : every Cauchy sequence converges.

In the case where G acts on X with compact point stabilizers, every (G, X) structure on a closed manifold is complete. (See Thurston [Th1, chap.3].)

We will be interested in finding geometric structures on non-compact manifolds. Without any completeness condition, such structures may be badly behaved near infinity. For our purposes, it will be much too restrictive to require structures to be complete. For example, we will be especially interested in deforming finite volume, hyperbolic structures on 3-manifolds; however, Mostow's rigidity theorem shows that there is a *unique* complete structure. We will add extra conditions in the next section to control the singularities allowed.

We now outline another approach to describing a geometric structure on a manifold. Given a representation $\rho: \pi_1(M) = \Gamma \to G$ there is an associated foliated X-bundle $X(\rho)$ over M defined as follows. Let $\tilde{M}$ denote the universal cover of M. Then there is an action of Γ on $\tilde{M} \times X$ given by $\gamma: (m, x) \mapsto (\gamma m, \rho(\gamma)x)$, $m \in \tilde{M}$, $x \in X$, where Γ acts on $\tilde{M}$ by covering transformations. Then $X(\rho)$ is the quotient space $(\tilde{M} \times X)/\Gamma$ and the projection $\tilde{M} \times X \to \tilde{M}$ induces the bundle projection $X(\rho) \to M$. Moreover, there is a natural foliation of $X(\rho)$ by the images of $\tilde{M} \times x$, $x \in X$ with a transverse (G, X) structure (see definition 2.1) coming from the fibres X of the bundle.

A geometric (G, X) structure on M with holonomy ρ is given by $X(\rho)$ together with a section of this bundle transverse to the foliation. Sections of the bundle always exist if the fibre X is contractible (e.g. for $X = \mathbf{H}^n$ or $\mathbf{E}^n$), however finding sections transverse to the foliation is not always possible. Given a transverse section, one obtains local coordinate charts on M as in definition 1.1 by projecting to a fibre X along the leaves of the foliation. Conversely, given coordinate charts $\phi_i: U_i \to X$ covering M we can construct a developing map $D: \tilde{M} \to X$ as above. Then the map $\tilde{s}: \tilde{M} \to \tilde{M} \times X, m \mapsto (m, D(m))$ gives a section transverse to the foliation. (A map $D: \tilde{M} \to X$ satisfying the above equivariance condition (*) gives a map transverse to the foliation if and only if D is an immersion (i.e. local diffeomorphism).)

In the case where $\rho(\Gamma)$ is a discrete subgroup of G, the map $X \to X/\rho(\Gamma)$ is a covering (or orbifold covering if $\rho(\Gamma)$ has torsion elements) we can identify $\tilde{M}$ with X and $\rho(\Gamma)$ with the group of covering transformations. Then $X(\rho) \cong X \times X/(x, y) \sim (\gamma x, \gamma y)$, $\gamma \in \rho(\Gamma)$ and the diagonal map $X \to X \times X$, $x \mapsto (x, x)$ gives a section transverse to the foliation.

When M is non-compact, we wish to obtain geometric structures with certain restricted kinds of singularities so we will impose extra conditions on the sections "at infinity".

2. Allowed singularities In this section, we introduce two types of singularities of geometric structures that will be allowed in the following work.

Definition 1.4. Cone type singularities

We define these for geometric structures modelled on constant curvature geometries as follows. A geometric structure on a manifold M with *cone type singularities* is given by a triangulation of M by geometric simplices with totally geodesic faces glued together by isometries. We then say that M is a (hyperbolic, spherical or Euclidean) *cone manifold*. (One could give a definition in a much more general setting, but this won't be needed for our purposes.)

Such a cone manifold has a smooth metric of constant curvature K on the complement of the codimension-2 skeleton, and an orthogonal cross section to each codimension-2 face is locally a 2-dimensional cone of curvature K. The angle of this cone is the *cone angle* along this face. A neighbourhood of the face is obtained by gluing together simplices around the face so that the sum of dihedral angles is equal to the cone angle. The *singular locus* Σ, consisting of the points where the metric is not smooth, is exactly the union of codimension-2 faces where the cone angle is not 2π. We will also say that "$M-\Sigma$ has a geometric structure with cone type singularities".

Definition 1.5. Dehn surgery type singularities.

First we introduce a model for these singularities. Let (G, X) be a Riemannian geometry, L a totally geodesic codimension two submanifold of X, and N an ϵ-neighbourhood of L in X. Let H be the subgroup of G leaving L invariant. Then H preserves $N-L$ so lifts to a group $\tilde{H}$ of isometries of the universal cover $(\widetilde{N-L})$ of $N-L$.

Let M be a non-compact manifold with a (G, X)-structure. Then there is a *Dehn surgery singularity* at an end of M if there is a neighbourhood E of the end such that

(1) the image of the developing map $D: \tilde{E} \to X$ is contained in $N-L$ as above,

and

(2) D lifts to a diffeomorphism $\tilde{D}: \tilde{E} \to (\widetilde{N-L})$ which is the developing map for a $(\tilde{H}, (\widetilde{N-L}))$-structure on E.

Let $\tilde{\rho}: \pi_1(E) \to \tilde{H}$ be the holonomy representation for this structure. Then $\Gamma = \tilde{\rho}(\pi_1(E))$ is a discrete subgroup of $\tilde{H}$, and $(\widetilde{N-L}) \to (\widetilde{N-L})/\Gamma$ is a covering. It follows that the geometric structure on the end of M is modelled on $(\widetilde{N-L})/\Gamma$.

Throughout this thesis we will also assume that $\partial(\widetilde{N-L})/\Gamma$ is *compact*. However, there are situations where it might be useful to remove this restriction.

We will be primarily interested in the 2- and 3-dimensional cases.

In the 2-dimensional case, L is a point and H is the group $SO(2) \cong S^1$ of rotations about L. Then $\tilde{H} \cong \mathbf{R}$ can be parametrized so that $\theta \in \mathbf{R}$ corresponds to the lift to $(\widetilde{N-L})$ of a rotation of N about L by angle θ. (Note that θ can be defined as an element of $\mathbf{R}$ rather than $\mathbf{R}\ mod 2\pi$, so that $\theta = 0$ for the identity element of $\tilde{H}$.) A discrete subgroup Γ of $\tilde{H} \cong \mathbf{R}$ is cyclic, and if $\theta \in \mathbf{R}$ is a generator of Γ then the quotient $(\widetilde{N-L})/\Gamma$ is a cone with angle θ. So Dehn surgery singularities are exactly cone type singularities in dimension two. Similarly, in any dimension, cone type singularities along a totally geodesic codimension-2 submanifold of M form an important special case of Dehn surgery singularities.

For the case of a 3-dimensional geometry (G, X), L is a geodesic in X, and $N \tilde{-} L$ is diffeomorphic to $(\widetilde{D^2-\text{point}}) \times \mathbf{R} \cong \mathbf{R}^3$. Isometries in $\tilde{H}$ can be parametrized by elements of $\mathbf{R}^2$ so that (l, θ) corresponds to the lift of an element of H acting as translation by l along L and rotation by θ around L. The action of $\tilde{H}$ on $(\widetilde{N-L})$ is equivalent to the action of a subgroup of $\mathbf{R}^2$ on $\mathbf{R}^3$ by translations. Then Γ is discrete subgroup of $\mathbf{R}^2$ and $E \cong T^2 \times [0, \infty)$ is topologically a solid torus with its core circle removed.

Following Thurston [Th1, chap.4] we parametrize the singularity at E as follows. For each element x in the homology group $H_1(E; \mathbf{Z}) \cong \pi_1(M) \cong \mathbf{Z}^2$, the holonomy of x is an isometry of G preserving an axis L in X and lifts to an element $h(x) \in \tilde{H} \subset \mathbb{R}^2$. This gives a homomorphism $h : H_1(T;\mathbb{Z}) \rightarrow \mathbb{R}^2$ with image the discrete group $\Gamma \cong \mathbb{Z}^2$. Now h extends to an isomorphism $h : H_1(T;\mathbb{R}) \rightarrow \mathbb{R}^2$, and we define the *(generalized) Dehn surgery coefficient* at the end E to be the unique class $c \in H_1(T;\mathbb{R})$ satisfying $h(c) = (0, 2\pi)$.

In the hyperbolic case, there is a complex structure on $G = PSL_2(\mathbb{C})$ and it is useful to think of h as a complex valued function : $h(x) = l(x) + i\theta(x)$. Then the Dehn surgery coefficient c satisfies $h(c) = 2\pi i$. (Compare [Th1, chap.4]).

Fixing a basis a, b for the homology group $H_1(T;\mathbb{Z}) \cong \mathbb{R}^2$, the Dehn surgery coefficient is (p, q) where p, q are the unique real numbers such that

$$p\,h(a) + q\,h(b) = (0, 2\pi)$$

Remarks : (1) If a hyperbolic structure on a 3-manifold is complete at an end, there is a local model analogous to that given in the above definition. Take N to be a horoball (a neighbourhood of point L at infinity); and $H \cong \hat{H} \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \cong \mathbb{R}^2$ the group of hyperbolic isometries preserving $N' = N - L = \widetilde{N - L}$. (This arises as a limit of the spaces $(\widetilde{N - L})$ where geodesics L are chosen with both their endpoints converging to a single point at infinity.) Then the local model for a complete end is N/Γ where $\Gamma \cong \mathbb{Z}^2$ is a discrete subgroup of $H \cong \mathbb{R}^2$.

(2) The name "(generalized) Dehn surgery type singularity" comes from the following observations. (We explain this further in remark (3) below.) When p and q are relatively prime integers, an incomplete geometric structure on M with an (p, q) Dehn surgery singularity can be completed to give a non-singular, complete geometric structure on the manifold $M_{(p,q)}$ obtained by topological (p, q) Dehn filling on M. Thus, $M_{(p,q)}$ denotes the manifold obtained by gluing a solid torus V to ∂M so that a meridian curve of V is identified with a simple closed curve homologous to $pa + qb$ on ∂M.

When p, q are integers with greatest common divisor $gcd(p, q) = d > 1$ there is a non-singular structure on branched covers of $M_{(\frac{p}{d}, \frac{q}{d})}$ with branching index d over C. More

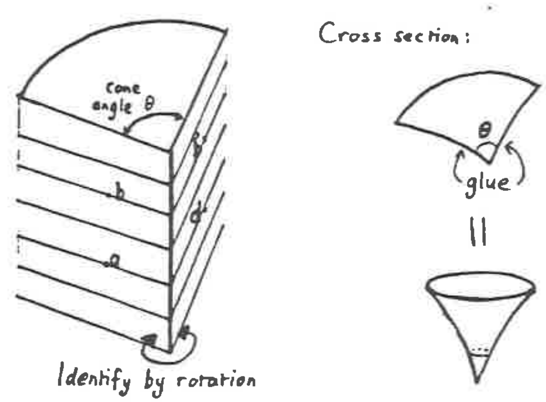
generally, if $(p,q) = r(s,t)$ where s,t are relatively prime integers and r is rational then taking the metric completion gives structures on $M_{(p,q)}$ having cone-type singularities with cone angle $2\pi/r$ along the core circle $\tilde{C}$ of the added solid torus V.

When the ratio of p to q is irrational, the metric completion of M is the compactification of M obtained by adding a single point corresponding to the end E.

Examples of Dehn surgery type singularities

(a) Cone type singularities:

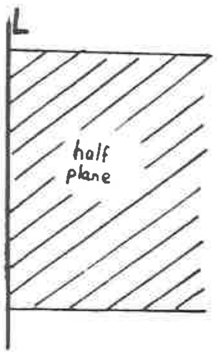
Such a singularity with cone angle θ is obtained (locally) by gluing together the sides of a wedge of X of angle θ by a rotation.

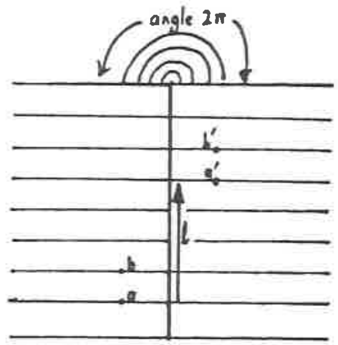


(b) Shear type singularities:

Take $\mathbb{R}^3$ cut open along a half plane, bounded by a straight line L. Glue the sides together after a translation of length l parallel to L, then quotient out by a translation through distance 1 parallel to L. Then the quotient is topologically an open solid torus $V = S^1 \times \mathbb{R}^2$ having a Dehn surgery type singularity along its core circle C. Choosing a basis a, b for $H_1(V - C;\mathbb{Z})$ where a is a meridian and b a longitude for V, we have $h(a) = (l, 2\pi)$, $h(b) = (1, 0)$ and the

Dehn surgery coefficient is $(1, -l)$.





Identify by translation (and 2π rotation)

(c) The most general local form of singularity is a combination of types (a) and (b): Glue together the sides of a wedge by a translation plus rotation, then quotient by a $\mathbf{Z}$ action preserving the axis.

(d) The most common and interesting examples of geometric structures on 3-manifolds with Dehn surgery type singularities are those obtained in "hyperbolic Dehn surgery". Let M be a 3-manifold with c cusps, having a complete hyperbolic structure M_0. Then Thurston has shown that there is a c-dimensional complex manifold of representations $\pi_1(M) \to \mathrm{isom}(\mathbf{H}^3) = PSL_2(\mathbf{C})$ near the holonomy representation for M_0 corresponding to incomplete hyperbolic structures on M near M_0 with Dehn surgery type singularities. Using these observations and remark 2 above, Thurston proves that almost all manifolds obtained by topological Dehn surgery on M have (non-singular) hyperbolic structures. (See [Th1, chap.5], [C-S].)

(e) A singular manifold modelled on $\mathbf{R}^3$ with $\mathbf{R}^3$ acting by translations.

Let A be a flat annulus in a 3-torus T^3, with boundary consisting of two parallel geodesics.

Cut T^3 open along A and reglue by a translation in the direction of the boundary geodesics. Then the holonomy around each component of ∂A is a translation preserving the geodesic. This gives an $(\mathbf{R}^3, \mathbf{R}^3)$-structure on T^3 with Dehn surgery type singularities along ∂A.

Remarks:

(3) Let M be the interior of a 3-manifold $\overline{M}$ with boundary consisting of tori T_i. A geometric structure on M with generalized Dehn surgery type singularities can be described by a developing map $D: \tilde{M} \to X$ which extends smoothly to a map $\hat{D}: \tilde{\overline{M}} \to X$ restricting to a submersion from $\tilde{T}_i$ to a geodesic in X on each lift $\tilde{T}_i$ of T_i.

On each boundary torus T_i, there is an induced foliation F whose leaves are the preimages of points in the geodesic $l = \hat{D}(\tilde{T}_i)$ with a (signed) invariant transverse measure induced from length along l. It follows that the foliation is isotopic to a linear foliation of T_i. Choosing an orientation on l, we can also define a (signed) measure along each leaf of F, given by the total angle of rotation around the axis l. (More precisely, the length of a curve α in T_i is the limit of the total rotational angle for a sequence of curves in M approaching α.) Then the generalized Dehn surgery coordinate at T_i is the homology class in $H_1(T_i; \mathbf{R})$ of a "segment of a leaf" of F having total angle 2π. (One should approximate multiples of the segment by simple closed curves to give a precise definition of this homology class.)

The metric completion of M can be described as follows. (Compare [Th1, chap.4].) Extend the metric on M over the boundary components T_i using the *degenerate* metric given by the transverse measure on F. Thus, the distance between two points of T_i is the infimum of total measures of curves on T_i joining the points. Then the metric completion $\hat{M}$ of M is obtained from $M \cup \partial M$ by identifying points at distance zero apart.

If the foliation F on T_i consists of simple closed curves then the completion (near T_i) is obtained by adding a single circle C whose length is the measure of a "shortest" simple closed curve transverse to F. Topologically, $\hat{M}$ is obtained from M by Dehn filling : $\hat{M} = M \cup V$ where V is a solid torus glued to T_i so that each leaf of F bounds a disc in V. Metrically, $\hat{M}$

has cone type singularities along C, with cone angle equal to the total angular measure of each leaf of F. So $\hat{M}$ has a smooth metric exactly when the Dehn surgery coefficient is an indivisible element of $H_1(T_i;\mathbb{Z})\subset H_1(T_i;\mathbb{R})$.

Otherwise, F is a foliation of T_i by lines of irrational slope and the metric completion $\hat{M}$ is the one-point compactification of M.

(4) There exist Dehn surgery type singularities with all possible Dehn surgery coefficients for the 3-dimensional geometries containing geodesics with 2-dimensional stabilizers. All geodesics have this property in the isotropic geometries : $(\mathrm{isom}(\mathbb{H}^3), \mathbb{H}^3)$, $(\mathrm{isom}(S^3), S^3)$ and $(\mathrm{isom}(\mathbb{E}^3, \mathbb{E}^3))$. For the fibred geometries with point stabilizer 1-dimensional, only the fibres have 2-dimensional stabilizers. For the other geometries, geodesics have stabilizers of dimension 1.

For geodesics with 1-dimensional stabilizers, there is still a 1-dimensional set of generalized Dehn surgery coefficients which make sense. The identity component of the group H of isometries preserving an axis consists of a one-parameter group of screw motions (translation plus rotation) along the axis. $\tilde{H}\subset\mathbb{R}\times\pi\mathbb{Z}$ is generated by lifts of screw motions and lifts of "rotations" by angles $n\pi$, $n\in\mathbb{Z}$.

3. Topology on geometric structures

Let M_t, $0\leq t\leq 1$, be a 1-parameter family of geometric structures on M.

Definition 1.6. We say that the geometric structures M_t *vary continuously* (in the C^k topology) if we can choose developing maps $D_t:\tilde{M}\to X$ for M_t which vary continuously in the weak (or compact-open) C^k topology on the space of C^∞ maps $\tilde{M}\to X$. We say that M_t is a *smooth family* of geometric structures if the structures vary continuously in the C^∞ topology. Note that the corresponding holonomy representations $\rho_t:\pi_1(M)\to G$ also vary smoothly : $\rho_t(\gamma)$ is a smooth path in G for each $\gamma\in\pi_1(M)$.

Of basic importance in our approach to the deformation of geometric structures on manifolds is the following observation. Given the holonomy representation $\rho:\pi_1(M)\to G$ for a (G,X)-structure on M, all nearby representations $\pi_1(M)\to G$ near ρ also correspond to geometric structures on M. (Compare [We1], [Th1, chap.5], [L].)

We will give a proof of a more general result in section 5 below. Intuitively, the result follows from the fact that transversality is an open condition : Think of a geometric structure as a section of an X-bundle transverse to a foliation as in section 1. Since the bundle and foliation vary smoothly with the representation, it is easy to believe that representations giving geometric structures are open.

In the next section, we introduce some machinery to make this argument rigorous.

4. Infinitesimal Deformations

In this section, we study infinitesimal deformations of representations and geometric structures on manifolds. First we review some standard cohomology theory for studying the deformation of representations. Then we show how to define a differential form describing an infinitesimal deformation of developing maps. We refer the reader to the appendix of this thesis for notation, definitions and references for basic properties of the group cohomology and de Rham cohomology used below.

We will be interested in deforming certain kinds of degenerate (G,X)-structures on a manifold M. Such a structure will be given by a smooth map $D:\tilde{M}\to X$ satisfying an equivariance condition $D(\gamma m)=\rho(\gamma)D(m)$, where ρ is a "holonomy" representation $\pi_1(M)\to G$. It will be convenient to use the term "developing map" for any such map D (not necessarily an immersion). Such a map D gives a non-singular (G,X)-structure on M if and only if D is an immersion (i.e. local diffeomorphism). Then D is a developing map for the (G,X)-structure on M, as defined in section 1.

Let $D_t:\tilde{M}\to X$ be a smooth family of developing maps describing a deformation of (possibly degenerate) (G,X)-structures on a manifold M. Then there are several ways to obtain a tangent vector to the deformation, giving an element in a certain cohomology group $H^1(M)$.

4.1. Tangents to representations

First, we obtain a tangent vector to the family of holonomy representations $\rho_t : \Gamma = \pi_1(M) \to G$, with $\rho_0 = \rho$. See [R], [We3] for further details.

For each $\gamma \in \Gamma$, the derivative $\frac{d}{dt}(\rho_t(\gamma))|_{t=0} \in TG_{\rho(\gamma)}$ is a tangent vector to G at $\rho(\gamma)$. We identify tangent space at g in G with the Lie algebra $\mathfrak{g} = TG_1$ of G, by right translation : $v \in T_gG \mapsto vg^{-1} \in \mathfrak{g}$, writing vh for the derivative of right translation by $h \in G$ applied to $v \in TG$. With this identification, we obtain a map $\dot\rho : \Gamma \to \mathfrak{g}$, with

$$\dot\rho(\gamma) = \frac{d}{dt}(\rho_t(\gamma))|_{t=0} \cdot \rho_0(\gamma)^{-1} .$$

By differentiating the relation

$$\rho_t(\gamma_1\gamma_2) = \rho_t(\gamma_1)\rho_t(\gamma_2)$$

we see that $\dot\rho$ satisfies the cocycle condition :

$$\dot\rho(\gamma_1\gamma_2) = \dot\rho(\gamma_1) + \mathrm{Ad}\rho_0(\gamma_1) \cdot \dot\rho(\gamma_2) .$$

It follows that $\dot\rho$ represents a 1-cocycle in $Z^1(\Gamma; Ad\rho)$ for the group Γ with coefficients in the Γ-module $\mathfrak{g}$, where Γ acts by the representation $\Gamma \xrightarrow{\rho_0} G \xrightarrow{Ad} \mathrm{Aut}(\mathfrak{g})$. Here, $Ad : G \to \mathfrak{g}$ is the adjoint representation of G on its Lie algebra $\mathfrak{g}$. (See appendix for details.)

Now consider a trivial deformation of ρ: given by conjugation $\rho_t = g_t \cdot \rho \cdot g_t^{-1}$, where g_t is a smooth path in G starting at the identity. Then differentiation shows that

$$\dot\rho(\gamma) = \dot g - \mathrm{Ad}\rho(\gamma)\dot g$$

where $\dot g \in \mathfrak{g}$ is the tangent vector to g_t at $t = 0$. So $\dot\rho = d\dot g$ is a coboundary in $B^1(\Gamma; Ad\rho)$.

Thus, given a curve of (conjugacy classes of) representations $\rho_t : \Gamma \to G$ we obtain a cohomology class in $H^1(\Gamma; Ad\rho)$, the group cohomology of Γ with coefficients in the Lie algebra $\mathfrak{g}$

of G, regarded as a Γ-module by the representation $\Gamma \xrightarrow{\rho} G \xrightarrow{Ad} \mathrm{Aut}(\mathfrak{g})$. We will also write $H^1(\Gamma; \mathfrak{g}_{Ad\rho})$ for this cohomology group.

Let M be a manifold which is a $K(\Gamma, 1)$. Then the group cohomology $H^i(\Gamma; Ad\rho)$ is naturally isomorphic to the (simplicial, singular, Čech or de Rham) cohomology $H^i(M; E(\rho))$ of M with coefficients in the flat vector bundle $E(\rho)$ constructed as follows. Let $\rho : \Gamma = \pi_1(M) \to G$ be the holonomy map for M and let $\tilde M$ be the universal cover of M. Then Γ acts on $\tilde M \times \mathfrak{g}$ by $\gamma : (m, x) \mapsto (\gamma m, \gamma x)$ where Γ acts on $\tilde M$ by covering transformations and on $\mathfrak{g}$ by the $Ad\rho$ representation. Then the quotient $\tilde M \times \mathfrak{g} / \Gamma$ is a vector bundle $E(\rho)$ over M with projection induced from the natural projection $\tilde M \times \mathfrak{g}$ onto $\tilde M$. Moreover, $E(\rho)$ has a canonical flat connection such that the images of $x \times \mathfrak{g} \subset \tilde M \times \mathfrak{g}$ give horizontal sections of $E(\rho)$.

Remark : If $\pi_1(M) = \Gamma$ but M is not necessarily a $K(\Gamma, 1)$, $H^i(M; E(\rho)) \cong H^i(\Gamma; Ad\rho)$ for $i = 0, 1$ and there is an injection $H^2(\Gamma; Ad\rho) \to H^2(M; E(\rho))$.

Using the de Rham theorem, we can obtain a closed 1-form ω on M with values in the flat vector bundle $E(\rho)$, representing the cohomology class $\dot\rho$ tangent to the representations ρ_t. The explicit map from de Rham cohomology to group cohomology is given by integration along paths in M. To make this precise, it is convenient to work in the universal cover $\tilde M$ of M. The form ω lifts to a $\mathfrak{g}$-valued form $\tilde\omega$ on $\tilde M$ satisfying the equivariance condition

$$\tilde\omega(\gamma v) = Ad\rho(\gamma)\tilde\omega(v)$$

for $v \in T\tilde M$, $\gamma \in \Gamma$. (More formally, $\tilde\omega$ is a form with values in the product bundle $\tilde M \times \mathfrak{g}$ over $\tilde M$; however we will always think of $\tilde\omega$ as a form with values in $\mathfrak{g}$, using the canonical (and flat) projection $\tilde M \times \mathfrak{g} \to \mathfrak{g}$.)

Fixing $x \in \tilde M$, we obtain a cocycle $z : \Gamma \to \mathfrak{g}$ corresponding to ω defined by

$$z(\gamma) = \int_x^{\gamma x} \tilde\omega$$

where the integral is taken over any path in $\tilde{M}$ joining x to γx. It is easy to check that different choices of x only change z by a coboundary. Note also that if $f:\tilde{M}\to \mathfrak{g}$ is an integral of $\tilde{\omega}$:

$$f(p)=\int_x^p \tilde{\omega}$$

then

$$f(\gamma p)=\int_x^{\gamma x}\tilde{\omega}+\int_{\gamma x}^{\gamma p}\tilde{\omega}=z(\gamma)+Ad\rho(\gamma)\cdot f(p)$$

This observation is often useful for determining the cohomology class of a form.

We will see below, how to *directly* obtain a 1-form ω in $\Omega^1(M;E(\rho))$ describing a deformation of geometric structures on M. It turns out that the form ω gives more information than the group cocycle $\dot{\rho}$, describing an infinitesimal deformation of *developing maps* rather than just a deformation of the holonomy representations.

4.2. Forms tangent to a deformation

Let $D_t:\tilde{M}\to X$, $t\in J=[0,1)$ be any smooth family of "developing maps" (not necessarily local diffeomorphisms), with holonomy representations $\rho_t:\pi_1(M)\to G$. Then we define differential forms tangent to the deformation as follows. We choose smooth maps $g_t:\tilde{M}\to G$ such that

$$D_t(m)=g_t(m)\cdot D_0(m),\quad g_0(m)=1 \tag{*}$$

and

$$g_t(\gamma m)=\rho_t(\gamma)g_t(m)\rho_0(\gamma)^{-1}, \tag{**}$$

for all $m\in\tilde{M}$, $t\in J$, $\gamma\in\pi_1(M)$.

Remark : If g_t were *uniquely* determined by (*), then (**) would follow immediately from

the equivariance property of D. In general, g_t is not uniquely determined by these conditions. (For instance, we will be interested in the case where the image of D_0 is a single point!)

To see that such g_t exist in general, consider the bundles EG and EX over $M\times J$ ($J=[0,1)$) with fibres G and X respectively, defined as follows. Each $\gamma\in\pi_1(M)$ acts on $\tilde{M}\times J\times G$ by $\gamma:(m,t,g)\mapsto(\gamma m,t,\rho_t(\gamma)g\rho_0(\gamma)^{-1})$ and on $\tilde{M}\times J\times X$ by $\gamma:(m,t,x)\mapsto(\gamma m,t,\rho_t(\gamma)x)$. We define EG and EX as the quotients of these group actions

$$EG=\tilde{M}\times J\times G/\pi_1(M),\quad\text{and}\quad EX=\tilde{M}\times J\times X/\pi_1(M)$$

with bundle projection induced by projection onto $\tilde{M}\times J$. Given D_0 there is a natural fibre preserving map $\pi:EG\to EX$ induced by $(m,t,g)\mapsto(m,t,g\cdot D_0(m))$. Further, π is a submersion, since the map $G\to X, g\in G\mapsto g x$ is a submersion for any fixed $x\in X$. The map $(m,t)\to(m,t,D_t(m))$ gives a section $s:M\times J\to EX$. Using the homotopy lifting property for $\pi:EG\to EX$ we can lift s to a section $\tilde{s}:M\times J\to EG$ such that $\pi_*\tilde{s}=s$ and $\tilde{s}=1$ on $M\times 0$. Then $\tilde{s}$ lifts to a map $M\times J\to M\times J\times G$, $(m,t)\mapsto(m,t,g_t(m))$ such that g_t satisfies the conditions (*) and (**).

Let ∂g denote the derivative $\dfrac{\partial g_t}{\partial t}:\tilde{M}\to\mathfrak{g}=TG_1$. By differentiating (**), we see that ∂g satisfies the equivariance condition

$$\partial g(\gamma m)=\frac{\partial}{\partial t}\Big|_0\rho_t(\gamma)+\rho_0(\gamma)\partial g(m)\rho_0(\gamma)^{-1}$$

or

$$\partial g(\gamma m)=\dot{\rho}(\gamma)+Ad\rho(\gamma)\cdot\partial g(m) \tag{***}$$

where $\dot{\rho}:\Gamma\to\mathfrak{g}$, is the cocycle tangent to the family of representations ρ_t at $t=0$.

We define $\dot{g}$ to be the $\mathfrak{g}$ valued 1-form on $\tilde{M}$ given by $\dot{g}=d(\dfrac{\partial g_t}{\partial t}):T\tilde{M}\to\mathfrak{g}$. Then $\dot{g}$ is a closed form, and differentiating (***) shows that

$$\dot{g}(\gamma v)=Ad\rho(\gamma)\cdot\dot{g}(v)$$

for $v \in T\tilde{M}$. Hence, $\dot{g}$ descends to a 1-form ω on M with values in $E(\rho)$; we write $\omega \in \Omega^1(M; E(\rho))$. Moreover, using (***) we see that the cohomology class in $H^1(M; E(\rho))$ represented by ω corresponds to the tangent vector $\dot{\rho} \in H^1(\pi_1(M); Ad\rho)$ to the path of representations ρ_t, as defined in 4.1. In particular, this shows that the cohomology class of ω is independent of the choices made above.

Remark : Suppose we vary the developing maps by a smooth path h_t in G with $h_0 = 1$, replacing D_t by $h_t \cdot D_t$. Then we can replace $g_t(m)$ by $h_t g_t(m)$, $\frac{\partial g_t}{\partial t}(m)$ by $\frac{\partial g_t}{\partial t}(m) - \frac{\partial h_t}{\partial t}$; hence the forms $\dot{g}$ and ω are unchanged.

5. Integrating infinitesimal deformations

We now show that there are always developing maps *(not necessarily immersions)* corresponding to 1-parameter families of representations.

Proposition 1.7. *Let $D: \tilde{M} \to X$ be a developing map with holonomy $\rho: \Gamma = \pi_1(M) \to G$. Let ρ_t be a smooth 1-parameter family of representations $\rho_t: \Gamma \to G$, with Zariski tangent vector at $t=0$ representing a class $c \in H^1(\Gamma; Ad\rho) \cong H^1(M; E(\rho))$. Then there is a smooth 1-parameter family of developing maps $D_t: \tilde{M} \to G$ with holonomy representations ρ_t such that $D_0 = D$. Moreover, if ω is any $E(\rho)$-valued form on M representing the cohomology class c, we can choose D_t such that the derivative of D_t at $t=0$ is represented by the form ω.*

Proof. For each $t \geq 0$, we have a cohomology class $c_t = \frac{\partial \rho_t}{\partial t} \in H^1(\Gamma; Ad\rho_t)$. These depend smoothly on t in the following sense : c_t is represented by a map $\pi_1(M) \to g$ which varies smoothly with t. We claim that there are closed 1-forms $\omega_t \in \Omega^1(M; E(\rho_t))$ such that $\omega_0 = \omega$, ω_t represents the cohomology class c_t and ω_t depends smoothly on t in the following sense. Each ω_t lifts to a g-valued form $\tilde{\omega}_t$ on $\tilde{M}$ satisfying the equivariance condition $\tilde{\omega}_t(\gamma v) = Ad\rho_t(\gamma)\tilde{\omega}_t(v)$, for $\gamma \in \pi_1(M)$, $v \in T\tilde{M}$. We require that $\tilde{\omega}_t$ be chosen to vary smoothly with t.

Essentially this can be seen by examining an *explicit* form of the de Rham isomorphism, (see e.g. [B-T]). However, the fact that the coefficient bundle is varying introduces a slight complication, so we give another proof.

A smooth family of cohomology classes $c_t \in H^i(\Gamma; Ad\rho_t)$, $t \in J$ can also be thought of as a single class in $H^i(\Gamma; C^\infty(J, g)_{Ad})$, where the coefficient module consists of all smooth maps $f: J \to g$ with Γ-action given by :

$$(\gamma \cdot f)(t) = Ad\rho_t(\gamma) \cdot f(t) \qquad (*)$$

for $\gamma \in \Gamma$, $t \in J$. The corresponding de Rham cohomology group is $H^i(M; E)$ where E is the flat vector bundle with fibre $C^\infty(J, g)$ defined as follows. Each $\gamma \in \Gamma = \pi_1(M)$ acts on $\tilde{M}$ by covering transformations and on $C^\infty(J, g)$ by (*). Then E is the quotient of $\tilde{M} \times C^\infty(J, g)$ by the diagonal action of Γ, $(m, f) \mapsto (\gamma m, \gamma f)$.

A cohomology class in $H^i(\Gamma; C^\infty(J, g))$ is given by a smooth family of i-cocycles $z_t: \Gamma \to g$, while a cohomology class in $H^i(M; E)$ is given by a smooth family of closed forms $\eta_t \in \Omega^i(M; Ad\rho_t)$. So it follows immediately from the de Rham isomorphism theorem that we can find a smooth family of forms $\eta_t \in \Omega(M; E(\rho_t))$ representing the classes c_t. Now, $\eta_0 \neq \omega$ in general, but $\omega - \eta_0 = d\alpha_0$ for some $\alpha_0 \in \Omega^0(M; E(\rho_0))$. It is easy to extend α_0 to a smooth family $\alpha_t \in \Omega^0(M; E(\rho_t))$ of sections of $E(\rho_t)$. Then we can take $\omega_t = \eta_t + d\alpha_t$.

Given forms $\tilde{\omega}_t$ as above we obtain the required deformation of the developing maps by integration. Fix a basepoint x_0 in $\tilde{M}$ and define $\alpha_t: \tilde{M} \to g$ by $\alpha_t(m) = \int_{x_0}^{m} \omega_t$. Then we obtain $g_t: \tilde{M} \to G$ by solving the first order differential equation : $\frac{\partial g_t(m)}{\partial t} = \alpha_t(g_t(m)) \cdot g_t(m)$, with initial condition $g_t(m) = 1$, for $t = 0$. Finally, $D_t(m) = g_t(m) \cdot D_0(m)$. □

Remark : If D is a local diffeomorphism, then D_t is also a local diffeomorphism for all t near 0, so we obtain a family of non-degenerate geometric structures on M with holonomies ρ_t.

Other proofs of this result can be found in [Th1, chap.5] and [L] (based on the argument of [We1]). The condition on the derivative is not included in the other proofs.

In the next chapters we will apply this result in the case where D is not a local diffeomorphism.

The following addendum shows that generalized Dehn surgery type singularities are open.

Proposition 1.8. *In the above situation, assume that ρ corresponds to a geometric structure with generalized Dehn surgery type singularities. Then there are geometric structures with generalized Dehn surgery type singularities corresponding to all representations near ρ.*

Proof. This is similar to an argument of Thurston in [Th1, chap.5]. Let E be an end of M having a Dehn surgery type singularity. Then there is a geodesic L (preserved by $\rho_0(\pi_1(E))$) and neighbourhood N of L in X, such that $D = D_0 : \tilde{E} \to (N-L) \subset X$ lifts to $\tilde{D} : \tilde{E} \to (\widetilde{N-L})$. Now truncate E to obtain a compact deformation retract $K \subset E$, and choose a compact fundamental domain $F \subset \tilde{E}$ projecting to F. Then the distance from $D_0(F)$ to L is positive. So the distance from $D_t(F)$ to the axis L_t preserved by $\rho_t(\pi_1(E))$ is also positive, for all t near 0. Hence $D_t : \tilde{K} \to (N_t - L_t)$ lifts to $\tilde{D}_t : \tilde{K} \to (\widetilde{N_t - L_t})$. Then D_t extends over $\tilde{E}$ "linearly" to complete the proof. □

CHAPTER 2

Regeneration of geometric structures

6. Deforming foliations to geometric structures

In this chapter, we consider the situation where a family of geometric structures on a manifold degenerates to a lower dimensional geometric limit. The limiting process often gives rise to a foliation of the manifold, with a transverse geometric structure. The recent work of Cheeger and Gromov ([C-G],[P]) on "collapsing of Riemannian manifolds" studies similar phenomena from a different point of view.

In the introduction, we have given some 2-dimensional examples of the kind of behaviour we have in mind. Many 3-dimensional examples arise as hyperbolic structures with Dehn surgery type singularities degenerate. Some explicit examples of such degeneration for the figure eight knot complement are given in [Th1, chap.4]. The kinds of possible degeneration occurring for hyperbolic cone manifolds with cone angles $\leqslant\pi$ are analyzed in detail by Thurston [Th3] using the idea of geometric limits. (See also [Gr].)

Our approach is rather different. We begin with a degenerate geometric structure on a manifold, and try to "approximate" by non-degenerate structures on M. We will call this "regeneration" of the degenerate structure. The basic question we consider is the following.

Given a foliation with transverse geometric structure on a manifold M, when can it be deformed to obtain "nearby" non-degenerate geometric structures on M?

This can be made precise as follows. Let M be an n-dimensional manifold, possibly with boundary, and let (G, X) be an n-dimensional geometry. Let X_0 be a submanifold of X, invariant under a subgroup G_0 of G such that G_0 acts transitively on X_0. Then (G_0, X_0) is also a geometry.

Definition 2.1. A *foliation F on M with transverse (G_0, X_0)-structure* is given by a covering of M by open sets U_i and submersions $\phi: U_i \to X_0$ onto open subsets of X_0, such that all transition maps are restrictions of elements of G_0. The leaves of the foliation in a chart U_i are given by the preimages of points in X under ϕ_i.

As in section 1, there is a *developing map* $D: \tilde{M} \to X_0$, which is a submersion satisfying an equivariance condition

$$D(\gamma m) = \rho(\gamma) D(m),$$

for all $\gamma \in \Gamma = \pi_1(M)$, $m \in \tilde{M}$, where $\rho: \Gamma \to G_0$ is a homomorphism, called the *holonomy representation* for the structure. The components of preimages of points under D give the leaves of a foliation $\tilde{F}$ of $\tilde{M}$, which is invariant under the group $\pi_1(M)$ of covering transformations and covers the foliation F on M.

Such a foliation F is *complete* if the developing map D is a fibration.

Basic problem :

Throughout this chapter we assume that we are given such a foliation F (not necessarily complete) with developing map $D: \tilde{M} \to X_0 \subset X$ and holonomy $\rho: \pi_1(M) \to G_0 \subset G$. (Note : We could take X_0 a single point or $X = X_0$; then F would be a foliation of codimension zero or dimension zero!)

Our aim is to find conditions so that there is a smooth 1-parameter family of developing maps $D_t: \tilde{M} \to X$, $t \geqslant 0$, with holonomy representations $\rho_t: \Gamma \to G$ satisfying the conditions:

(1) $D_0 = D$

(2) D_t is a local diffeomorphism for $t > 0$.

Then each D_t, $t > 0$ will define a non-degenerate (G, X)-structure on M; and these structures degenerate to the foliation F as $t \to 0$. We will also obtain conditions enabling us to restrict the kind the singularities involved.

Remarks : In chapters 4 and 6, we will apply the results obtained here to construct many examples of such degeneration. The examples will be mostly for the case where M is a 3-manifold and the geometry is a constant curvature geometry : $(G, X) =$ $(\mathrm{isom}(\mathbb{H}^3), \mathbb{H}^3)$, $(\mathrm{isom}(S^3), S^3)$ or $(\mathrm{isom}(\mathbb{E}^3), \mathbb{E}^3)$. However, most of the theory developed applies in any dimension.

7. Conditions for regeneration of foliations

Let $D: \tilde{M} \to X_0 \subset X$ be a submersion defining foliations $\tilde{F}$ on $\tilde{M}$ and F on M. If $D_t: \tilde{M} \to X$ is a smooth family of developing maps with $D_0 = D$, then we have defined an associated form $\omega \in \Omega^1(M; E(\rho))$ representing the infinitesimal deformation of D. We now obtain conditions on ω ensuring that D_t are local diffeomorphisms for all $t > 0$ sufficiently small, whenever D_t has ω as a tangent vector. For simplicity, we always assume that the tangent bundle TF to the foliation is *orientable*.

Example :

The following simple example illustrates the crucial idea underlying this section. Suppose that $A_t: \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth family of linear maps with $A_0(x, y) = (x, 0)$. Then a simple "non-degeneracy" condition on the *derivative* $\partial A = \dfrac{\partial A}{\partial t}$ will ensure that the maps A_t are *isomorphisms* for all $t > 0$ sufficiently close to zero.

Write $A_t = \begin{pmatrix} a_t & c_t \\ b_t & d_t \end{pmatrix}$ and $\dfrac{\partial A_t}{\partial t} = \begin{pmatrix} \partial a & \partial c \\ \partial b & \partial d \end{pmatrix}$. Then $A_t = \begin{pmatrix} 1+t\partial a & t\partial c \\ t\partial b & t\partial d \end{pmatrix} + O(t^2)$. So $\det A_t = td + O(t^2)$ and A_t is invertible for $t > 0$ near 0 if $d \neq 0$. (*)

Here is a more canonical formulation of this condition. The preimages of points under A_0 give a foliation F of $\mathbb{R}^2$ by vertical lines $x \times \mathbb{R}$. Then the derivative ∂A applied to a vertical tangent vector $v = (0, y)$, $y \neq 0$, is $\partial A(v) = (c, d)y$. So the condition (*) is that the vertical component of $\partial A(v)$ should be non-zero on vectors $v \neq 0$ tangent to the foliation F.

General case :

In general, we must ensure that the "Jacobian" of D_t is non-zero for all $t>0$. More precisely, let v denote the volume form on X. Then we require that $D_t^* v$ (the induced volume form on $\tilde{M}$) is non-degenerate on $\tilde{M}$. Choose a smooth frame field $e_1, \ldots, e_n$ on $\tilde{M}$ such that $e_1, \ldots, e_k$ gives a basis for the tangent space $T\tilde{F}$ to the foliation $\tilde{F}$ at each point. Let

$$v_t(m) = D_t^* v(e_1, \ldots, e_n) = v(dD_t(e_1), \ldots, dD_t(e_n))$$

where e_i are evaluated at the point $m \in \tilde{M}$.

Then we must have $v_t \neq 0$ for all $t>0$. Now if $k = \dim F$ then v_t vanishes to order $k-1$ at $t=0$, and we require that :

$$\frac{\partial^i v_t}{\partial t^i} = 0 \quad \text{for } i<k \text{, and} \quad \frac{\partial^i v_t}{\partial t^i} \neq 0 \quad \text{for } i=k \tag{*}$$

at all points of $\tilde{M}$. Clearly, this is sufficient to guarantee $v_t \neq 0$ for all positive t near 0. (This is the simplest possible case; in general the kth derivative could also be zero. See the remark below for some further discussion.)

We now evaluate these derivatives at a point $m \in \tilde{M}$. First we adjust $D_t : \tilde{M} \to X$ and $g_t : \tilde{M} \to G$ by a smooth path h_t in G so that $D_t(m) = D(m)$ and $g_t(m) = 1$ for all t. This leaves v_t unchanged since v is invariant under the isometry group G, and leaves ω unchanged by a remark from section 4.2. Then

$$\frac{\partial v_t}{\partial t}\Big|_0 = \sum_i v(dD_0(e_1), \ldots, \dot{D}(e_i), \ldots, dD_0(e_n)) \tag{**}$$

where $\dot{D} = \frac{\partial}{\partial t}\Big|_0 (dD_t) : T\tilde{M}_m \to TX_{D(m)}$. (Note that $dD_t : T\tilde{M}_m \to TX_{D(m)}$ for all t, so the t derivative can also be regarded as a map into $TX_{D(m)}$ by the usual identification.)

For $i = 1, \ldots, k$ $dD_0(e_i) = 0$ since e_i is tangent to $\tilde{F}$. So (**) vanishes unless $k=1$, in which case there is a single non-vanishing term :

$$\frac{\partial v_t}{\partial t} = v(\dot{D}(e_1), dD_0(e_2), \ldots, dD_0(e_n))$$

In general, the first non-vanishing term is :

$$\frac{\partial^k v_t}{\partial t^k} = v(\dot{D}(e_1), \ldots, \dot{D}(e_k), dD_0(e_{k+1}), \ldots, dD_0(e_n))$$

and the regeneration condition (*) becomes

$$v(\dot{D}(e_1), \ldots, \dot{D}(e_k), dD_0(e_{k+1}), \ldots, dD_0(e_n)) \neq 0$$

We can rewrite this condition in terms of the form ω describing the infinitesimal deformation. The lift $\dot{g}$ of ω to $\tilde{M}$ is a g-valued form. We can think of an element of the Lie algebra g of G as a Killing vector field on X as follows. Let $\phi : G \times X \to X$ be the action of G on X. Then $v \in g$ corresponds to the vector field $\phi_*(v)$ on X given by $\phi_*(v)(x) = \frac{d}{dt}\Big|_0 \phi(\exp(tv), x)$ for $x \in X$. We write $v(x)$ for $\phi_*(v)(x)$.

Claim : $\dot{D}(v) = \dot{g}(v)(D(m))$ for all $v \in T\tilde{M}_m$.

To see this, first note that since $g_t(m) = 1$ for all t, $\dot{g} = d(\frac{\partial g_t}{\partial t}) = \frac{\partial}{\partial t}(dg_t) : T\tilde{M}_{m_0} \to g$. Since $D_t(m) = \phi(g_t, D_0(m))$, it follows that

$$\dot{D}(v) = \frac{\partial}{\partial t} d\phi(dg_t(v), 0) = d\phi(\frac{\partial}{\partial t}(dg_t(v)), 0) = \dot{g}(v)(D_0(m))$$

using $dg_0(v) = 0$.

So the regeneration condition is that $\dot{g}(e_1) \wedge \ldots \wedge \dot{g}(e_k) \wedge dD(e_{k+1}) \wedge \ldots \wedge dD(e_n)$ should be a non-zero volume form on X at every point $m \in \tilde{M}$.

It will be convenient to abbreviate this somewhat, so we write $\omega(F)$ for $\wedge^k \omega$ evaluated on a oriented k-vector tangent to F, and $[TX_0]$ for a oriented $n-k$-vector tangent to X_0. For example, letting $\bar{e}_i \in TM$ denote the image of $e_i \in T\tilde{M}$ under projection to M, we can take $\omega(F) = \omega(\bar{e}_1) \wedge \ldots \wedge \omega(\bar{e}_k)$ at $m \in M$ and $[TX_0] = dD(e_{k+1}) \wedge \ldots \wedge dD(e_n)$ at $D(m) \in X_0$. Then we can rewrite the regeneration condition as

$$\omega(F)\wedge[TX_0]\neq 0$$

Summarizing this discussion we obtain :

Theorem 2.2. *Let $D:\tilde{M}\to X_0$ be the developing map and $\rho:\pi_1(M)\to G_0$ the holonomy for a foliation F with transverse (G_0, X_0) geometric structure. Let $\rho_t:\pi_1(M)\to G$ be a smooth family of representations such that $\rho_0=\rho$. Assume that the tangent vector $\dot{\rho}\in H^1(M; Ad\rho)$ to ρ_t can be represented by a $E(\rho)$-valued 1-form ω satisfying*

$$\omega(F)\wedge[TX_0]\neq 0$$

Then there is a smooth family of developing maps maps $D_t:\tilde{M}\to X$, $t\geq 0$ such that $D_0=D$, D_t is a local diffeomorphism for $t>0$, and ω is a tangent vector to the family D_t.

In particular, each D_t with $t>0$ defines a non-degenerate geometric structure on M.

□

Remark 2.3. We needn't assume the kth deriv is non-zero in (*). The same argument can be applied much more generally : If have an analytic family of developing maps then for each $i=1,\ldots k$, $dD_t(e_i)=c_it^{n_i}+O(t^{n_i+1})$ for some integer $n_i>0$ and $c_i\in TX$. Then v_t is also analytic function of t satisfying $v_t=ct^n+O(t^{n+1})$, where $n=\sum_{i=1}^{k} n_i$ and $c\in\mathbb{R}$. Again we obtain a regeneration condition by requiring that the nth derivative of v_t is non-zero.

Remark 2.4. Here is a slightly different formulation of the non-degeneracy condition on ω. The form ω describing an infinitesimal deformation gives a map $\bar{\omega}:T\tilde{M}\to TX\to\nu(X_0)$, where $TX\to\nu(X_0)$ is orthogonal projection onto the normal bundle to X_0 in X. This map is defined by $\bar{\omega}(v)=n(\omega(v)(D(m)))$, for $v\in TM_m$, $m\in\tilde{M}$, where nx is the component of x orthogonal to X_0.

Then the "non-degeneracy condition" on ω holds if and only if $\bar{\omega}$ takes tangent spaces $T\tilde{F}$ to the leaves of $\tilde{F}$ to fibres of the normal bundle $\nu(X_0)$ by an isomorphism.

Remark 2.5 Limiting geometric structures.

A 1-form $\omega\in\Omega^1(M;E)$ lifts to a 1-form $\tilde{\omega}$ on $\tilde{M}$ with values in $\mathfrak{g}$. Choosing a base point $m_0\in\tilde{M}$, $\tilde{\omega}:T\tilde{M}\to\mathfrak{g}$ can be integrated to give a map $\tilde{f}:\tilde{M}\to\mathfrak{g}$, where $\tilde{f}(m)=\int_{m_0}^{m}\tilde{\omega}$. We then obtain a map $f:\tilde{M}\to\nu(X_0)$, where $f(m)$ is the component of the vector field $\tilde{f}(m)$ in the normal bundle to X_0 at the point $D(m)$.

Then $f(\gamma m)=Ad\rho(\gamma)f(m)+nz(\gamma)$ for all $\gamma\in\pi_1(M)$, $m\in\tilde{M}$. Hence, there is a "developing map" $\bar{D}:\tilde{M}\to X_0\times\nu(X_0)$ where γ acts on $\nu(X_0)$ by

$$(x, t)\mapsto(\rho(\gamma)x,\, n.z(\gamma)+Ad\rho(\gamma)t).$$

Here $t\in\nu(X_0)$ is a vector normal to X_0 at the point $x\in X_0$. Then $\bar{D}$ is an immersion if and only if ω satisfies the non-degeneracy condition given in the remark above.

Often, $\bar{D}$ can be used to construct a limiting metric and geometric structure on M with $\bar{D}$ as the developing map. We illustrate this for the simplest case, when X_0 is a point. Then $\nu=\nu(X_0)=TX_{x_0}$ is a vector space with Euclidean metric given by the Riemannian metric on X. Since $Ad\rho(\gamma)$ acts on ν by rotations it follows that the action of each $\gamma\in\pi_1(M)$ given by (*) is a Euclidean isometry. So we obtain a limiting Euclidean geometric structure on M in this case. The limiting metric thus constructed is uniquely defined up to rescaling.

There are many other situations where a limiting metric arises in a natural way from the degeneration of geometric structures. We will discuss this further in [Ho].

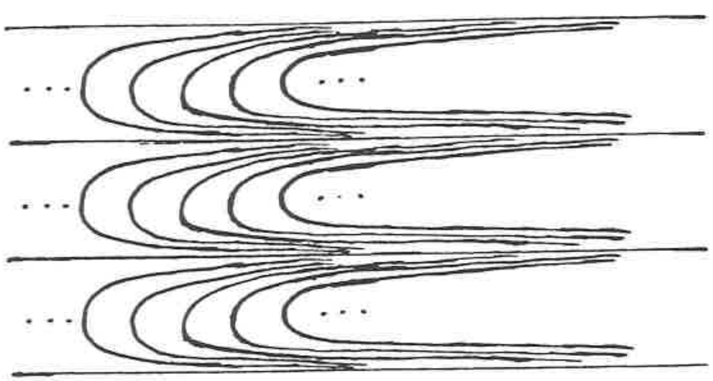
8. Dehn surgery type singularities for foliations

We introduce a notion of "Dehn surgery type singularities" for certain 1-dimensional foliations of manifolds.

Topological Dehn surgery singularities

First we describe a (topological) model foliation. Let $V \supset V_1 \supset V_2$ be Euclidean spaces with $\dim V = n$, $\dim V_1 = n-1$ and $\dim V_2 = n-2$. Let $p: V \to V_1$ be the orthogonal projection. Then the preimages of points under p give the leaves of a foliation F of $V \cong \mathbb{R}^n$ by straight lines. Now F lifts to a foliation $\tilde{F}$ on the universal cover $(\widetilde{V-V_2})$ of $V-V_2$, defined by the submersion $\tilde{p}:(\widetilde{V-V_2}) \to V-V_2 \to V_1$.

In the 2-dimensional case, the foliation of $(\widetilde{V-V_2}) \cong \mathbb{R}^2$ looks like:



Explicitly, we can take $V = \mathbb{C}$, $V_1 = \mathbb{R}$, $V_2 = 0$, $p: z \mapsto \mathrm{Re}(z)$. Taking $\exp: \mathbb{C} \to \mathbb{C}-0$ as the universal covering, we see the leaves of F are given by $\{z \in \mathbb{C}: \mathrm{Re}(e^z) = c\}$ for $c \in \mathbb{R}$. For $c = 0$, the leaves are straight lines $\mathbb{R}+n\pi i, n \in \mathbb{Z}$. In between each pair of straight lines, we have a Reeb type foliation.

The 3-dimensional case is obtained by taking the product of the 2-dimensional case with $\mathbb{R}$.

Let H be the group of isometries of V preserving V_1 and V_0. Then H lifts to a group $\tilde{H}$ of isometries of $(\widetilde{V-V_2})$ preserving the foliation $\tilde{F}$. If Γ is a discrete subgroup of H, then $\tilde{F}$ descends to a foliation on $(\widetilde{V-V_2})/\Gamma$.

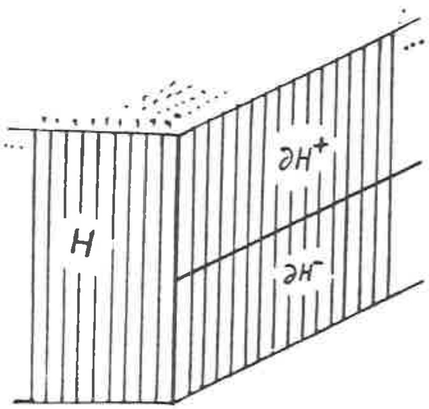
Definition 2.6. A foliation with *(topological) Dehn surgery type singularities* is modelled on such a foliation.

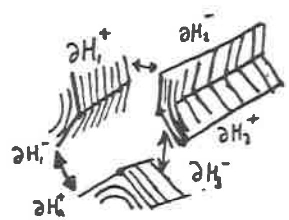
In the case $n = 2$, $\tilde{H} \cong \mathbb{Z}$ where $n \in \mathbb{Z}$ corresponds to the lift of a rotation by $n\pi$ around the point V_0. With the notation introduced above, n corresponds to the map $\mathbb{C} \to \mathbb{C}$, $x+yi \mapsto x+yi+n\pi i$. If $\Gamma \cong \mathbb{Z}$ is the subgroup of $\pi\mathbb{Z} \cong \tilde{H}$ generated by $n\pi$, then the induced foliation on $\mathbb{C}/\Gamma$ has an n-pronged singularity:

$n=1$ $n=2$ $n=3$

If $n = 3$, then V_2 is a geodesic in V. Elements of $\tilde{H}$ can be parametrized by elements of $\mathbb{R}\times\pi\mathbb{Z}$ so that $(l, n\pi)$ corresponds to the lift of a rotation by $n\pi$ about the geodesic plus translation by distance l along V_2. In this case, $\Gamma \cong \mathbb{Z}^2 \subset H$ so the quotient $(\widetilde{V-V_2})/\Gamma \cong \mathbb{R}^3/\mathbb{Z}^2$ is topologically $T^2\times\mathbb{R}$.

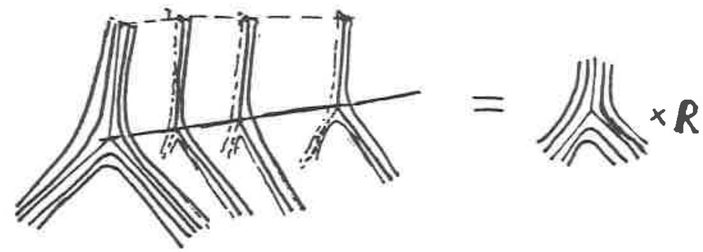
The local structure of the foliation can be seen as follows. Let $H \cong \mathbb{R}^3_+$ be a half space foliated by parallel lines, restricting to a foliation of $\partial H \cong \mathbb{R}^2$. Let $L \subset \partial H$ be a line orthogonal to the foliation of ∂H dividing ∂H into 2 components ∂H^+ and ∂H^-.





Then a foliation with Dehn surgery singularities of type (2) is obtained by gluing together copies $H_1, H_2, \ldots H_n$ of H so that ∂H_i^+ is glued to ∂H_{i+1}^- (i taken mod n) by a linear map respecting the foliations. The resulting foliation of $\mathbf{R}^3-$line depends on the choice of gluing maps; however the foliation is determined by n and the translational holonomy $l \in \mathbf{R}$ giving the total shearing along the line of singularities. (Note that if $n = 2$ then the foliation is topologically non-singular; although the transverse structure will be singular unless $l = 0$.)

The simplest case is when $l = 0$. Then we have a two dimensional foliation with an n-prong singularity at a point crossed with $\mathbf{R}$.



The general case is obtained by cutting this example open along a half plane bounded by the singular axis, and regluing the sides after a translation by l along the axis.

Geometric Dehn surgery singularities

For foliations with transverse geometric structure, we construct model singularities as follows. Let $p: Y \to X_0$ be a fibration, defining a complete 1-dimensional foliation F with transverse (G_0, X_0)-structure. Let $L \subset Y$ be a codimension two submanifold of Y, which is transverse to the fibres, so that the projection $p: L \to X_0$ is an immersion onto a totally geodesic codimension one submanifold of X_0.

Let N be a tubular neighbourhood of L in Y. Then the foliation $F|N-L$ lifts to a foliation $\tilde{F}$ of the universal cover $\widetilde{N-L}$. Let $\tilde{H}$ be the group of diffeomorphisms of $\widetilde{N-L}$ which preserve the foliation F and project to isometries of X_0 preserving $p(L)$.

Definition 2.7. A *Dehn surgery type singularity* for a foliation with transverse (G_0, X_0)-structure is modelled on $\widetilde{N-L}/\Gamma$ with foliation induced by $\tilde{F}$, where Γ is a discrete subgroup of $\tilde{H}$.

Topologically, these singularities are of the form discussed above. The singularities can be parametrized as follows.

In the 3-dimensional case, each element of Γ can be thought of as translation by distance d along L (measured in X_0) composed with a rotation by angle $\theta \in \pi\mathbf{Z}$. The holonomy of each element in $\pi_1(\partial)$ is the pair $(d, \theta) \in \mathbf{R}\times\pi\mathbf{Z}$.

These singularities can also be parametrized by a *generalized Dehn surgery coefficient* $c \in H_1(T; \mathbf{R})$ as in chapter 1 (section 2). The holonomy $\pi_1(T) \cong H_1(M; \mathbf{Z}) \to \mathbf{R}\times\pi\mathbf{Z} \subset \mathbf{R}^2$ extends to a linear isomorphism $h: H_1(T; \mathbf{R}) \to \mathbf{R}^2$. Then $c \in H_1(T; \mathbf{R})$ is the unique class satisfying $h(c) = (0, 2\pi)$.

In this situation, only a restricted set of Dehn surgery coefficients are possible. If a_1, a_2 is a base for $H_1(T; \mathbf{Z})$ then $h(a_j) = (l_j, n_j\pi)$, where $n_j \in \mathbf{Z}$ and $l_j \in \mathbf{R}$. Then the Dehn surgery coefficient $c = (c_1, c_2)$ satisfies $c_1(l_1, n_1\pi) + c_2(l_2, n_2\pi) = (0, 2\pi)$. Looking at the second coordinates shows that c satisfies a linear equation with integer coefficients of the form $c_1 n_1 + c_2 n_2 = 2$, with $n_1, n_2 \in \mathbf{Z}$.

Examples 2.8.

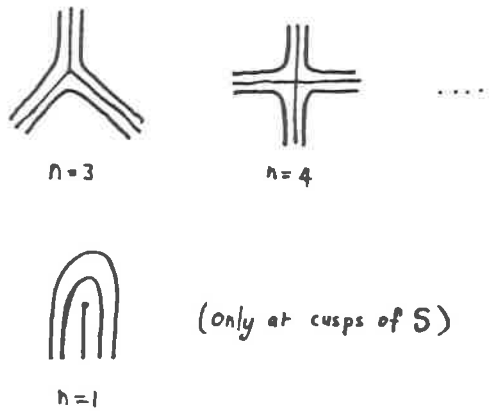
(a) Let M be a closed manifold with a 1-dimensional foliation F and Γ a codimension-2 submanifold of M such that each component of Γ is transverse to F. Then F gives foliation on $M-\Sigma$ with (topological) Dehn surgery type singularities. If $\hat{M}$ is any covering of $M-\Sigma$ then F lifts to a foliation of $\hat{M}$ with Dehn surgery singularities.

(b) Assume that M is a 3-dimensional Seifert fibre space with base orbifold B, and let $p: M \to B$ be the projection. Then the fibres of p give a foliation F of M by circles. Now B has a metric of constant curvature (except when B is a 2-sphere with one or two cone points). Choosing such a metric defines a transverse (G_0, X_0)-structure for F, modelled on either $\mathbf{H}^2$, $\mathbf{E}^2$ or S^2. In fact, M has a (G, X)-structure modelled on one of the six Seifert fibred geometries. (See [Th1, chap.5 (revised version)].)

Let Σ be a simple closed curve in M projecting to a geodesic in the base B. If N is a small tubular neighbourhood of a component L of Σ in M, then the composition $p: N-L \subset M \to B$ lifts to a map $\tilde{p}: \widetilde{N-L} \to \tilde{B}$. Moreover, the group $\pi_1(N-L) \cong \mathbf{Z}^2$ of covering transformations project to isometries of B preserving $p(L)$. Hence, F restricts to a (G_0, X_0)-foliation on $M-\Sigma$ with Dehn surgery singularities. Here, the holonomy $\pi_1(N-L) \to \tilde{H} \subset \mathbf{R}\times\pi\mathbf{Z}$ is given by is $h(a) = (0, 2\pi)$, $h(b) = (l, 2\pi)$ where l is the length of $p(L)$ and a, b are meridian and longitude on ∂N.

(c) Let $\phi: T \to T$ be an Anosov diffeomorphism of the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. Then ϕ is covered by a linear map with matrix $A \in GL(2, \mathbf{Z})$ having two distinct real eigenvalues $\lambda, 1/\lambda$. There are two linear foliations F_1, F_2 of T^2 invariant under ϕ by lines in the directions of the two eigenvectors of A. Each of these foliations extends to a 1-dimensional foliation of the mapping torus of ϕ : the bundle over S^1 obtained from $T^2 \times [0, 1]$ by identifying $(s, 0)$ with $(\phi(s), 1)$ for $s \in T^2$. Further, there are transverse measures μ_1, μ_2 satisfying $\phi_* \mu_1 = \lambda\mu_1$, $\phi_* \mu_2 = \lambda\mu_2$ From this it follows easily that there is a transverse hyperbolic structure. (In fact, an affine structure.)

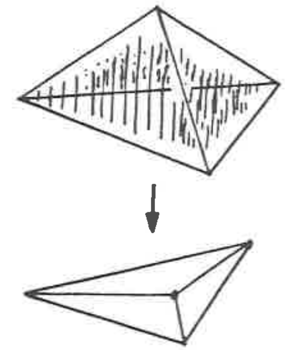
Similarly, let $\phi: S \to S$ be a pseudo-Anosov diffeomorphism of a surface S. Then there are two transverse foliations F_1, F_2 on S invariant under ϕ, with isolated singularities of the form :



Again each of these foliations gives rise to a 1-dimensional foliation of the mapping torus of ϕ, with transverse hyperbolic structure. These foliations have Dehn surgery type singularities, modelled on the above two dimensional singularities crossed with S^1, with angle $n\pi$ corresponding to an n-pronged singularity on S.

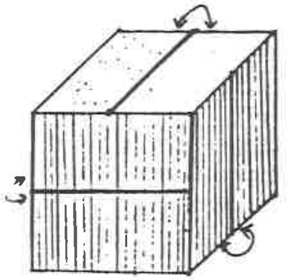
In each case, we obtain a holonomy representation (giving the signed length of curves with respect to the transverse measure), $\pi_1(M-\Sigma) \to PSL_2\mathbf{R}$ and developing map $D: \widetilde{M-\Sigma} \to \mathbf{H}^2$.

(d) The transversely hyperbolic foliations discussed in [Th1, chap.4] also have singularities of Dehn surgery type. These arise as follows. Suppose that we have a hyperbolic structure with Dehn surgery on a 3-manifold given by gluing together ideal hyperbolic simplices. It often happens that the hyperbolic structure A family of such hyperbolic structures M_t may degenerate in the following way. All simplices are positively oriented for $t>0$, but flatten out simultaneously as $t \to 0$. In this case, there is a natural projection in each topological simplex to a flattened simplex lying in $\mathbf{H}^2$.



This gives rise to a foliation with transverse hyperbolic structure and limiting developing map $\tilde{M} \to \mathbf{H}^2$. These foliations always have Dehn surgery type singularities, and the Dehn surgery coefficient for the foliation is the limit of the Dehn surgery coefficients for the M_t.

(e) Every 3-manifold has 3-orthogonal foliations with transverse Euclidean structure having Dehn surgery type singularities along a link. Further, the singularities are all of the form $(l, \theta) = (0, \pi)$ or $(0, 4\pi)$. This is an immediate consequence of the recent result of [H-L-M-W] showing that the Borromean rings are universal : more precisely, every 3-manifold is a branched covering of the 3-sphere branched over the Borromean rings, with branching indices 1, 2 and 4. (Compare the following figure.)



Fold as shown to obtain Borromean rings in S^3

The following result is the main justification for introducing the idea of Dehn surgery singularities for foliations. Suppose that a foliation can be approximated by non-degenerate geometric structures. Then the next result shows that if the foliation has Dehn surgery singularities then the geometric structures can be chosen to also have Dehn surgery singularities.

Theorem 2.9. *Let $D_t : \tilde{M} \to X$, $t>0$ be a smooth family of developing maps such that D_t is an immersion for $t>0$. Assume $D_0 : \tilde{M} \to X_0$ is a submersion defining a foliation on M with Dehn surgery type singularities. Then there are developing maps $D_{t'} : \tilde{M} \to X$, $t \geqslant 0$, such that $D_{0'} = D_0$, $D_{t'} = D_t$ except in an (arbitrarily small) neighbourhood of the ends of M, and $D_{t'}$ is the holonomy for a geometric structure on M with Dehn surgery type singularities.*

Proof. Let E be an end of M where the foliation has a Dehn surgery type singularity. We have seen in section 7 (remark 2) that if ω satisfies the regeneration condition then there is an induced immersion $\overline{D} : \tilde{M} \to \nu(X_0)$, where $\nu(X_0)$ denotes the normal bundle to X_0 in X. The developing map $D : \tilde{M} \to X_0$ is just the composition of $\overline{D}$ with the bundle projection $\nu(X_0) \to X_0$. We can write $\dot{D}(m) = (D(m), n(m))$ and $\overline{D} = (D, \bar{n})$ where $n \in TX_{D(m)}$ and $\bar{n} \perp X_0$.

Then

$$D_t(m) = \exp_{D(m)}(tn(m)) + O(t^2)$$

$$= \exp_{D_t(m)}(t\bar{n}(m)) + O(t^2) \qquad (*)$$

Now, assume that F has a Dehn surgery type singularity at the end E. Then there is $\overline{L} \subset \nu(X_0)$ such that $\overline{D} : \tilde{E} \to \nu(X_0) - \overline{L}$. Now truncate E to obtain a compact $K \subset E$. Let $F \subset \tilde{E}$ be a compact fundamental domain for the action of $\pi_1(E)$ on $\tilde{K}$. Then the distance from $\overline{D}(F)$ to $\overline{L}$ is positive. Let L be the image of $\overline{L}$ in X_0. This is the axis preserved by $\rho_0(\pi_1(E))$. It follows from equation $(*)$ above that there is $c>0$ such that the distance from $D_t(F)$ to the axis L_t preserved by $\rho_t(\pi_1(E))$ is given by $ct + (t^2)$, so is positive for all $t>0$ near 0. Hence $D_t : \tilde{K} \to (N_t - L_t)$ lifts to $\tilde{D}_t : \tilde{K} \to (\widetilde{N_t - L_t})$. Moreover, $\overline{D}$ is a diffeomorphism onto its image; hence so is D_t. Since $\pi_1(E)$ acts on $\nu(X_0)$ as covering transformations, so does $\tilde{\rho}_t(\pi_1)$. Then extend over $\tilde{E}$ "linearly" to complete the proof. □

9. Cohomology conditions for regeneration

In this section, we obtain more useful necessary conditions for regeneration of foliations. Let M be a compact *orientable* manifold (possibly with boundary) and let F be a 1-dimensional foliation with transverse (G_0, X_0)-structure on the interior of M. Assume that the foliation has Dehn surgery type singularities. Then we want to find (G, X)-structures on M with Dehn surgery type singularities, degenerating to the foliation F. For simplicity we also assume that X_0 is a codimension-1 submanifold of X with *trivial* normal bundle and that the tangent bundle TF to the foliation F is *oriented*.

Let $D : \tilde{M} \to X_0 \subset X$ be the developing map for F with holonomy $\rho : \pi_1(M) \to G_0 \subset G$, and write E for the flat vector bundle $E(\rho)$ defined in section 4.1. Then there is a bundle map $n : E \to M \times \mathbb{R}$ to the trivial $\mathbb{R}$ bundle on M induced by the map taking $(m, v) \in \tilde{M} \times g$ to the component $v(D(m))^{\perp}$ of $v(D(m))$ in the direction of the positive unit normal to X_0 in X at the point $D(m)\in X_0$. Here $v\in g$ is regarded as a Killing vector field on X. n can also be regarded as a section of the dual flat vector bundle E^*.

Then n induces a map $n\cdot : \Omega^k(M;E)\to\Omega^k(M;\mathbf{R})$. Further, there is a dual bundle map $n^* : M\times\mathbf{R}\to E^*$, inducing a map $n^* : \Omega^k(M;\mathbf{R})\to\Omega^k(M;E^*)$. In general n won't be a flat map so there will not be induced maps on cohomology; however n is parallel along each leaf of the foliation F.

In particular, $n\cdot\omega$ is a $\mathbf{R}$-valued form on M and the regeneration condition from theorem 2.2 (or 2.4) can be written in the form

$$n\cdot\omega(v)>0 \qquad (*)$$

for all $v\in TM$ which are positively oriented tangent vectors to F. We will abbreviate this, writing : $n\cdot\omega>0$ on TF.

We assume first that all the leaves of F are closed as subsets of $int M$; then each leaf extends to compact, properly embedded submanifold of M. (This follows from the local structure of Dehn surgery type singularities.) For example, let N be a closed manifold with Seifert fibration and Σ a link in N transverse to the fibres. Then the foliation of N restricts to a foliation F of $M = N-U(\Sigma)$ of this form. The leaves of F consist of circles and open intervals whose closure in $\overline{M}$ is an interval with endpoints in ∂M.

Now an obvious necessary condition for (*) is that the integral $\int_L n\cdot\omega$ of $n\cdot\omega$ over each (oriented) leaf L of F should be positive.

It will be convenient to write this in terms of a form dual to L. First we recall the definition of the Poincaré dual η_L to a submanifold of a compact n-manifold M. Let $i:\partial M\subset M$ be the inclusion and let $\Omega^k(M,\partial M;\mathbf{R})$ denote the space of $\mathbf{R}$-valued k-forms ω on M such that $i^*\omega = 0$. (We call these "forms vanishing on ∂M".)

For every oriented k-dimensional submanifold $j:L\subset M$ properly embedded in M, integration over L defines a linear functional on $\Omega^k(M,\partial M;\mathbf{R})$:

$$L[\omega] = \int_L j^*\omega.$$

This induces a linear functional on the relative cohomology $H^k(M,\partial M;\mathbf{R})$, since

$$L[d\omega] = \int_L d(j^*\omega) = \int_{\partial L} j^*\omega = 0$$

for $\omega\in\Omega^{k-1}(M,\partial M;\mathbf{R})$.

Using Poincaré duality $H^k(M,\partial M)^*\cong H^{n-k}(M)$, we obtain a closed form $\eta_L\in\Omega^{n-k}(M;\mathbf{R})$ Poincaré dual to L, characterized by :

$$\int_L j^*\omega = \int_M \omega\wedge\eta_L$$

for all $\omega\in\Omega^k(M,\partial M;\mathbf{R})$.

In particular, for each leaf $L\subset M$ as above, we obtain a closed form η_L representing a class in $H^{n-k}(M;\mathbf{R})$. This is also the dual of a homology class $[L]\in H_k(M,\partial M;\mathbf{R})$; we will refer to $[L]$ and η_L as the "foliation cycle" carried by L. (This terminology makes more sense, if we think of L as a k-cycle on $int M$ with *closed* support representing an element in the homology group $H_{k,closed}(int M;\mathbf{R})$ dual to $H^{n-k}(int M;\mathbf{R})$. See [B-T] for some further discussion of homology with closed supports.)

Applying n^* gives a form $n^*\eta_L\in\Omega^{n-k}(M;E^*)$ which is closed since n is parallel on each leaf of F. This gives a linear functional n^*L defined by

$$n^*L[\omega] = \int_M \omega\wedge n^*\eta_L \qquad (*)$$

defined on certain E-valued k-forms. It is clear that this definition makes sense for E-valued forms vanishing on ∂M; and since $n\cdot\omega$ is closed there is an induced map on cohomology $H^k(M;E)\to\mathbf{R}$. We need to extend the definition to a larger class.

Assume that $D:\tilde{M}\to X_0\subset X$ corresponds to a geometric structure or foliation with Dehn surgery type singularities. Then each end of M has a neighbourhood diffeomorphic to a quotient $\tilde{N}-L/\Gamma$, where N is a neighbourhood of a codimension-2 submanifold $L\subset X$ where Γ is a discrete subgroup of $\tilde{H}$, the lifts of isometries of G preserving L. We fix such a neighbourhood U of each end of M, and give $\tilde{U} \cong \widetilde{N-L}$ a Euclidean metric such that $\tilde{H}$ acts by Euclidean isometries. (The Euclidean metric on the normal bundle to L in X is invariant under H; use the exponential map to obtain a suitable metric on $N-L$ and $\widetilde{N-L}$.) This gives a flat connection (actually a product structure) on the tangent bundle to U and we will say that a form on M with values in flat vector bundle E is *parallel* on U if its covariant derivative is zero. (Equivalently, using the connections on U and E, a form ω locally gives maps $\omega_u : TM_0 \to E_0$ depending smoothly on $u \in U$. ω is parallel iff the map ω_u is locally constant.)

Let $\Omega_p^1(M;E)$ denote the space of E-valued forms ω on M such that ω is constant on a neighbourhood of each end of $int M$. Let $H_p^1(M;E)$ denote the cohomology of the complex $\{\Omega_p^*(M;E), d\}$.

Proposition 2.10. *If ρ corresponds to a geometric structure or foliation with Dehn surgery type singularities then the inclusion $\Omega_p^*(M;E(\rho)) \subset \Omega^*(M;E(\rho))$ induces an isomorphism on cohomology: $H_p^1(M;E(\rho)) \cong H^1(M;E(\rho))$.*

Proof. We use the following elementary lemma, leaving the proof as an exercise for the reader. (It also follows from theorem 3.17, using the fact that $E \cong \nu(L)$ is a flat orthogonal vector bundle over ∂M.)

Lemma 2.11. *Let U be a neighbourhood of an end of M as above. Then every class in $H^*(U;E)$ can be represented by a unique parallel form.*

□

(1) We first show that every cohomology class $c \in H^1(M;E)$ can be represented by a form ω such that

(*) ω is constant on a neighbourhood of each component ∂ of ∂M, with values in the infinitesimal centralizer of the axis in X_0 preserved by $\rho(\pi_1(\partial))$.

(Hence, $n \lrcorner \omega$ vanishes on ∂M.)

By the lemma, the restriction $i^*c \in H^1(U;E)$ can be represented by such a unique constant form $\alpha \in \Omega^1(U;E)$ satisfying condition (*). Let β be any form in $\Omega^1(M;E)$ representing c. Then $\alpha - \beta$ is a coboundary on U, say $\alpha - \beta = d\sigma$ where σ is a section of E over U. Extend σ to a section $\tilde{\sigma}$ of E and take $\omega = \beta + d\tilde{\sigma}$.

(2) Moreover, if ω as above is a coboundary $\omega = d\alpha$ with $\alpha \in \Omega^0(M;E)$ then ω is a coboundary on U; so $\omega = 0$ on U. It follows that α is constant on U; hence $\alpha \in \Omega_p^0(M;E)$. □

It follows that $n \lrcorner \omega \to 0$ at each end of M and we have

Proposition 2.12. *For each leaf L of F, there is a continuous linear functional n^*L on $\Omega_p^1(M;E)$ defined by:*

$$n^*L[\omega] = \int_L n \lrcorner \omega = \int_M \omega \wedge n^* \eta_L$$

This induces a linear functional on cohomology : $n^*L : H^1(M;E) \cong H_p^1(M;E) \to \mathbb{R}$ depending only on the cohomology class $[n^*L] \in H^1(M;E^*)$.

□

Remark 2.13 Regeneration Conditions.

The condition that $\int_L n \lrcorner \omega > 0$ is the same as the condition that $n.L[\omega] > 0$. The previous proposition implies that this can be written in terms of group cohomology, since $H^1(M;E) \cong H^1(\pi_1(M);Ad\rho)$. Explicitly this can be done as follows.

(1) If L is a closed leaf, with homotopy class $[L]$ in $\pi_1(M)$ then the condition becomes : there is a cocycle $z \in Z^1(\pi_1(M), Ad\rho)$ representing the cohomology class of ω such that

REGEN II $\int n \lrcorner \omega = n \cdot z([L]) > 0$.

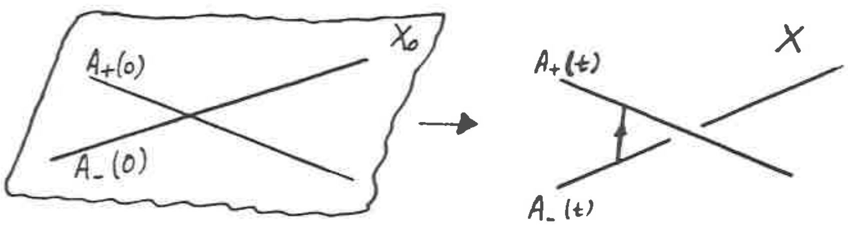
(More precisely, we mean that $z([L]) \in g$ regarded as a Killing vector field points in the positive normal direction to X_0 at all points of $D(\tilde{M})$.) This condition is independent of choice of base point used to obtain the homotopy class $[L]\in\pi_1(M)$: Putting $l=[L]$, we have $\rho(l)=1$ hence

$$z(ala^{-1}) = z(a)+Ad\rho(a)z(l)-Ad\rho(ala^{-1})z(a) = Ad\rho(a)z(l)$$

However, the vector fields corresponding to $Ad\rho(a)z(l)$ differ by an isometry $\rho(a)$ preserving X_0. Hence, $nz(l)>0$ if and only if $nz(ala^{-1})>0$.)

(2) Let L be a oriented leaf meeting ∂M. Lift L to a path $\tilde{L}$ in $\tilde{M}$ joining two components ∂_-, ∂_+ of $\partial\tilde{M}$. Let $\Gamma_i\subset\pi_1(M)$, $i=+,-$ be the group of covering transformations preserving ∂_i; then $D_t(\partial_i)=A_i(t)$ is the axis preserved by $\rho_t(\Gamma_i)$. The axes $A_-(t)$, $A_+(t)$ lie in X_0 and intersect for $t=0$. The condition that $\int_L n\omega>0$ is exactly the condition :

(REGEN2) the axes move apart infinitesimally for $t>0$, with $A_+(t)$ lying above $A_-(t)$.



(This condition can also be written in terms of group cohomology but we won't use this form.)

In fact these cohomology conditions $(REGEN1)$ and $(REGEN2)$ are enough to ensure that ω is cohomologous to a form satisfying the positivity condition $n\omega>0$ on TF. In the

next section we will give a proof of this by using techniques of Sullivan from [Su]. We will also obtain a cohomological set of regeneration condition when the leaves of the foliation are not necessarily closed.

10. Sufficiency of cohomology conditions

Now we prove that the homological conditions for the deformation of foliations to geometric structures are actually sufficient. This section is an extension of ideas introduced by Sullivan for finding real-valued forms positive on the leaves of a foliation. The reader should consult [Su] for more details.

First we introduce more notation. Let $\Omega_{\mathbb{R}}=\Omega^k(M;\mathbb{R})$ denote the $\mathbb{R}$-valued k-forms on M. Let $\Omega_{\mathbb{R}}^*$ denote the dual space of currents : continuous linear functionals on $\Omega_{\mathbb{R}}$. Let $\Omega=\Omega_f^k(M;E)$ denote the space of E-valued k-forms ω on M, parallel on a fixed compact neighbourhood of ∂M. Let Ω^* denote the dual space of continuous linear functionals on Ω.

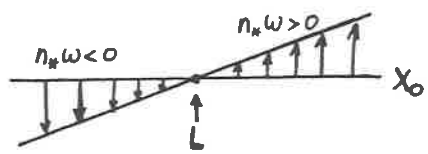
Let F be a foliation on $intM$ and assume that ρ is flat along every leaf of F. Let $CF\subset\Omega_{\mathbb{R}}^*$ denote the convex cone of foliation currents : the closure of finite convex combinations of Dirac currents, given by evaluating forms on the tangent k-vector to F at any point of $intM$. Let $C\subset\Omega^*$ denote the image $n^*(CF)$ under the map induced by n. We wish to find forms $\omega\in\Omega$ such that $\omega(C)>0$.

We now use the approach of Sullivan, with C as "structure cone". The idea of the argument is the same as that in [Su]; however there is a slight technical complication since our structure cone may be defined on a non-compact manifold.

Lemma 2.14. *There are forms $\omega\in\Omega$ positive on C.*

Proof. These are easily constructed locally : this is clear except perhaps on a neighbourhood U of an end of M. To construct a suitable form on U note the developing map $D:\tilde{U}\to X_0$ has image contained in one of the two components of X_0-L. Choose ω to be a constant form whose value on TF is an infinitesimal rotation with axis L, with direction of rotation chosen so that $n.\omega>0$ on the component of X_0-L containing *image*(D). (Note : $n.\omega<0$ on the other side of L.)

Cross section:



Finally, piece together these local forms using a partition of unity. □

The following result is the crucial point needed to apply Sullivan's ideas in our situation. Some control on the behaviour of forms at infinity is needed to obtain compactness here; this is the reason for the use of forms parallel near the ends of M.

Proposition 2.15. *The cone C is a compact convex cone in Ω^*.*

Proof. Let $\omega\in\Omega$ be a form such that $n.\omega$ is positive on C. Let $\bar{C}$ denote the set of $c\in C$ such that $c(\omega)=\int_c \omega = 1$. We will show that $\bar{C}$ is compact.

Now $\bar{C}$ is a closed subset of C and is compact if and only if the set of values $\eta(\bar{C})$ is bounded for any fixed form $\eta\in\Omega$ (by the uniform boundedness principle, cf.[Su]). Let U be a neighbourhood of the ends of M such that η and ω are constant forms on U, with $M-U$ compact. Then there is a constant $K>0$ such that :

$$|n.\eta(v)|\leqslant K\,|n.\omega(v)|$$

for all vectors v tangent to the foliation F. (Such an inequality holds on U since $n.\eta$ is just a multiple of $n.\omega$ on U. There is a similar inequality on $M-U$ by compactness.) Hence, for a finite sum of Dirac currents $c\in C$ we have $c(\eta)\leqslant K c(\omega)$. It follows that $c(\eta)\leqslant K$ for all $c\in\bar{C}$. □

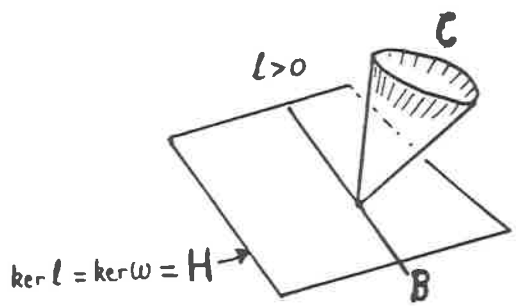
Let B and Z denote the subspaces of boundaries and cycles in Ω^*. Thus
$B=\{\, l\in\Omega^* : l(\omega)=0$, for all closed forms $\omega\in\Omega\,\}$ and
$Z=\{\, l\in\Omega^* : l(\omega)=0$, for all exact forms $\omega\in\Omega\,\}$.

Theorem 2.16. *Assume that*

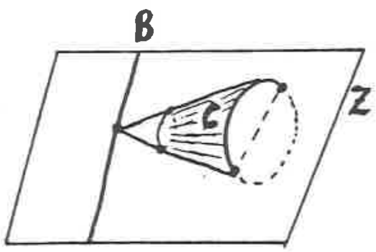
$$C\cap B=\{0\}. \qquad (**)$$

Then a cohomology class $u\in H^1_{\rho}(M;E)$ can be represented by a form $\omega\in\Omega(M;E)$ such that $n.\omega(v)>0$ for all $v\in TF$ tangent to the foliation F if and only if $n.u(f)>0$ for all foliation cycles $f\in H^{c-1}(M;\mathbb{R})$.

Proof. (The argument is essentially the same as in [Su].) First note that Z and B are closed subspaces of Ω^*. Since C is a compact convex cone and $B\cap C=\{0\}$ it follows from the Hahn-Banach theorem that there is a hyperplane $H\supset B$ which supports C (i.e. $H\cap B=\{0\}$). Equivalently, there is a continuous linear functional $l:\Omega^*\to\mathbb{R}$ such that $l\geqslant 0$ on C, with $\ker l=H$. By [de R], forms in $\Omega(M;E)$ and currents in $\Omega(M;E)^*$ are duals. Since $\Omega\subset\Omega(M;E)$ it follows that $\Omega^{**}\subset\Omega(M;E)^{**}=\Omega(M;E)$. Hence l is defined by a form $\omega\in\Omega(M;E)$ with kernel H, such that $\omega(C)>0$: $l(\eta)=\int_M \omega\wedge\eta$ for all $\eta\in\Omega$. Since ω vanishes on B we have $\int_M \omega\wedge d\eta=0$, for all $\eta\in\Omega(M;E)$ parallel near infinity. This implies easily that $d\omega=0$ on $\operatorname{int}M$; hence $d\omega=0$ on M. So there are closed forms positive on M.



The transverse geometric structure on F gives a transverse invariant measure μ on F, which defines a foliation cycle in $f \in \Omega^{\cdot}_{B}$. (f is given by integrating forms over leaves of F with a weighting given by the transverse measure μ.) Then $n^{\cdot} f \in \Omega^{\cdot}$ is also a cycle; hence $C \cap Z \neq \{0\}$. It follows that C, Z, B are arranged as follows.



By a slight extension of the previous argument, we then find that a cohomology class $u \in H^1_F(M; E)$ can be represented by a "positive" form $\omega \in \Omega(M; E)$ such that $\omega(C) > 0$ if and only if u is positive when evaluated on the homology classes of cycles in C. $\square$

To apply this result in our situation, we need to understand the foliation cycles for F. We will restrict our attention to the case where each leaf of F is a closed subset of $\operatorname{int} M$. Locally (in a coordinate chart) any foliation current is given by integration over leaves with respect to some measure on a transversal to the leaves. If all leaves are closed sets, then any extreme point of the cone of foliation currents is given by integrating a Dirac measure: concentrated at a single leaf. It follows that the extreme points of the cone of foliation cycles are just the foliation cycles $[L]$ supported by leaves. Hence, the condition $n.u(f) > 0$ for all foliation cycles f is equivalent to the conditions of the previous section : $n^{\cdot} L[\omega] = \int_L \omega > 0$ for all leaves. Further the non-degeneracy condition (**) becomes : there is no leaf L such that $\int_L \omega = 0$ for all closed forms $\omega \in \Omega^1(M; E)$. But this is impossible if $u([L]) > 0$ where $u \in H^1(M; E)$ is the cohomology class of ω. So we have proved :

Theorem 2.17. *Let F be as above. Assume that every leaf of F is a closed subset of $\operatorname{int} M$. Then a cohomology class $u \in H^1(M; E)$ can be represented by a form $\omega \in \Omega^1(M; E)$ satisfying the regeneration condition : $n.\omega > 0$ on TF if and only if $u([L]) > 0$ for all leaves L of F.*

$\square$

Combining this with theorem 2.2, we obtain the following criterion for regeneration of geometric structures.

Theorem 2.18. *Let $D: \tilde{M} \to X$ be a submersion, and $\rho: \pi_1(M) \to G$ a holonomy map such that $D(\gamma m) = \rho(\gamma) D(m)$ for $\gamma \in \pi_1(M)$, $m \in \tilde{M}$. Let $c \in H^1(M; E_\rho)$ represent the tangent to a smooth family of representations $\rho_t: \pi_1(M) \to G$. Assume that c is positive when evaluated* *over every foliation cycle f for F (i.e. $c(f)>0$ for all f). Then there is a smooth family of developing maps $D_t:\tilde{M}\to G$ such that $D_0=D$. D_t is a local diffeomorphism for $t>0$, and the derivative to D_t at $t=0$ represented by the form ω.*

In particular, there is a family of (G,X)-structures on M converging to the (possibly degenerate) structure given by $D=D_0$.

□

Further, from theorem 2.9 we have the following

Addendum. *If F has Dehn surgery type singularities, then we can ensure that the D_t, $t>0$ define (G,X)-structures with Dehn surgery type singularities.*

Remark 2.19 Special Case.

If D defines a 1-dimensional foliation on $int M$ with all leaves closed (as subsets of $int M$), then the regeneration condition $c(f)>0$ in the theorem reduces to:

there is a cocycle $z\in Z^1(\pi_1(M); Ad\rho)$ representing the cohomology class c such that

(REGEN1) $n z([L])>0$ whenever L is a leaf diffeomorphic to S^1

and

(REGEN2) the axes preserved by $\rho(\partial_1)$ and $\rho(\partial_2)$ move apart infinitesimally for $t>0$, when L is a leaf joining boundary components ∂_1 and ∂_2 of ∂M.

In other words, the lengths of all leaves become positive under the infinitesimal deformation.

CHAPTER 3

Cohomology Theory of Deformations

Let $R(\Gamma,G)$ denote the space of all representations $\Gamma\to G$. Let *gen* be a set of generators and *rel* a set of relations for Γ. Then $R(\Gamma,G)$ is an analytic subset of G^{gen} consisting of maps $gen\to G$ satisfying the relations *rel*. If G is an algebraic group and Γ is finitely generated then $R(\Gamma,G)$ is an algebraic variety. The group G acts on $R(\Gamma,G)$ by conjugation and the quotient $R(\Gamma,G)/G$ is of particular interest because it parametrizes locally the space of (G,X)-structures on M. (Compare propositions 1.7, 1.8 and [L].)

In chapter 1, section 4.1, we noted the following

Lemma 3.1. *$Z^1(\Gamma,g_{Ad\rho})$ is the Zariski tangent space to $R(\Gamma,G)$ at a representation ρ. $B^1(\Gamma,g_{Ad\rho})$ is the Zariski tangent space to the orbit $G.\rho$.* □

Hence, in some sense, the dimension of $H^1(\Gamma,G)$ gives the dimension of $R(\Gamma,G)/G$ at ρ. However, the spaces involved are singular in general so more work is needed to obtain a good understanding of the local structure of $R(\Gamma,G)$ and $R(\Gamma,G)/G$.

As general references for this chapter, we suggest the books [Br], [R] and the papers [We3], [G1], [G2], [J-M]. The appendix to this thesis also contains a summary of notation and properties of group cohomology used below.

1.1. Obstructions to deforming representations

We have seen that every smooth 1-parameter family of representations $\Gamma\to G$ gives a Zariski tangent vector in $Z^1(\Gamma,G)$. However, the *tangent cone* to $R(\Gamma,G)$ consisting of Zariski tangent vectors which extend to 1-parameter families of representations $\Gamma\to G$ is often a proper subset of the Zariski tangent space. If the tangent cone is equal to the Zariski tangent space, then $R(\Gamma, G)$ is locally a manifold with dimension given by $Z^1(\Gamma, g)$. (See Whitney [Wh2] for more information on tangent cones.)

In fact there is a sequence of obstructions in $H^2(\Gamma; Ad\rho)$ to the extension of a Zariski tangent vector in $Z^1(\Gamma; Ad\rho)$ to a 1-parameter family of representations. We now indicate how this comes about and explicitly give the first obstructions.

Let $\rho_t : \Gamma \to G$ be a smooth 1-parameter family of representations with $\rho_0 = \rho$. Put $\rho_t(g) = \exp(f_t(g)) \cdot \rho(g)$ for $g \in \Gamma$, where $f_t : \Gamma \to g$ has Taylor series expansion $f_t(g) = \sum_{i=1}^{\infty} t^i a_i(g)$.

For ρ_t to be a representation we have

$$\rho_t(gh) = \rho_t(g) \cdot \rho_t(h)$$

for all $g, h \in \Gamma$. Equivalently, $\rho_t(gh)\rho(gh)^{-1} = \rho_t(g)\rho(g)^{-1} \cdot \rho(g)(\rho_t(h)\rho(h)^{-1})\rho(g)^{-1}$, or

$$\exp(\sum_{i=1}^{\infty} t^i a_i(gh)) = \exp(\sum_{i=1}^{\infty} t^i a_i(g)) \cdot \exp(\sum_{i=1}^{\infty} t^i Ad\rho(g) \cdot a_i(h)) \qquad (*)$$

Using the Campbell-Baker-Hausdorff formula

$$\exp(a) \cdot \exp(b) = \exp(a+b+\frac{1}{2}[a, b]+\frac{1}{12}[[a, b], b]-\frac{1}{12}[a, b], a]+\ldots)$$

and expanding (the logarithms of) both sides of $(*)$ in Taylor series, we obtain a sequence of equations for the coefficients $a_i(g)$. In particular, the terms of degree ≤ 2 give:

$$a_1(gh) - a_1(g) - Ad\rho(g) \cdot a_1(h) = 0 \qquad (1)$$

$$a_2(gh) - a_2(g) - Ad\rho(g) \cdot a_2(h) = \frac{1}{2}[a_1(g), Ad\rho(g) \cdot a_1(h)] \qquad (2)$$

These equations can be interpreted in terms of group cohomology of Γ with coefficients in the Γ-module $Ad\rho = g_{Ad\rho}$. Equation (1) says that $a_1 : \Gamma \to g$ is a cocycle in $Z^1(\Gamma; g)$. Equation (2) says that $da_2 = [a_1, a_1]$ where $[\ ,\ ] : C^1(\Gamma; g) \times C^1(\Gamma; g) \to C^2(\Gamma; g)$ denotes the cup product using coefficient pairing $[\ ,\ ] : g \times g \to g$. (See appendix.) In particular, $[a_1, a_1] = 0$ in $H^2(\Gamma; g)$.

12. Conditions for a representation space to be a manifold

In this section, we give some cohomological conditions for a space of representations $R(\Gamma, G)$ to be manifold. The main ideas here go back to Weil [We3].

First, we note the following easy consequence of the implicit function theorem. (For a proof see [We3] or [R].)

Lemma 3.2. *Let A, B, C be smooth manifolds and $\alpha : A \to B$, $\beta : B \to C$ be smooth maps. Suppose there is a point $c \in C$ such that $\beta \circ \alpha(A) = \{c\}$. Let $a \in A$ be a point such that the sequence of derivatives*

$$TA_a \xrightarrow{d\alpha} TB_{\alpha(a)} \xrightarrow{d\beta} TC_c$$

is exact. Then $\alpha(A) = \beta^{-1}(c)$ is a manifold in a neighbourhood of $\alpha(b)$. □

As a first application, we outline Weil's proof of the following result.

Theorem 3.3. *Let $\rho : \Gamma \to G$ be a representation such that $H^1(\Gamma, Ad\rho) = 0$. Then ρ is locally rigid: giving an isolated point in $R(\Gamma, G)$.*

Proof. Let gen, rel be sets of generators and relators for Γ. Define maps $G \xrightarrow{\alpha} G^{gen} \xrightarrow{\beta} G^{rel}$ where $\alpha(g)(\gamma) = g\rho(\gamma)g^{-1}$, for all $\gamma \in gen$ and β is the evaluation map given by substituting a point of G^{gen} into the relations of rel. Then it is clear that $\beta \circ \alpha(G) = * = (1, 1, \ldots, 1) \in G^{rel}$, $\beta^{-1}(*) = R(\Gamma, G)$ and $\alpha(G) = G \cdot \rho$ is the orbit of ρ in $R(\Gamma, G)$.

We identify the tangent spaces of G, G^{gen}, G^{rel} with g, g^{gen}, g^{rel} (using right translation back to the identity $1 \in G$). Then the derivatives of α, β are just the coboundary maps d^0, d^1 in a complex $g \xrightarrow{d^0} g^{gen} \xrightarrow{d^1} g^{rel}$ whose cohomology gives $H^*(\Gamma, Ad\rho)$. (To see this consider a CW-complex X with $\pi_1(X) = \Gamma$ made up of one 0-cell, one 1-cell for each generator in gen and one 2-cell for each relation in rel. Then the cellular homology $H^1(X; Ad\rho) \cong H^1(\Gamma; Ad\rho)$ is exactly the homology of the above complex.)

If $H^1(\Gamma, Ad\rho) = 0$ then the above sequence is exact, and it follows from the lemma that $R(\Gamma, G) = G\cdot\rho$ near ρ, as desired. □

The following result is well known. We give a proof analogous to the above proof of Weil's theorem.

Theorem 3.4. *Let $\rho : \Gamma \to G$ be a representation such that $H^2(\Gamma, Ad\rho) = 0$. Then $R(\Gamma, G)$ is a manifold near ρ, whose dimension is the dimension of $Z^1(\Gamma, Ad\rho)$.*

Proof. Let X be a CW-complex with a single vertex such that $\pi_1(X) = \Gamma$ and $\pi_j(X) = 0$ for $j = 2$. Then $H^i(\Gamma, Ad\rho)$ is isomorphic to the cellular cohomology groups $H^i(X; g_{Ad\rho}) = H^i(X; Ad\rho)$ for $i = 0, 1, 2$. (See [Br].)

Let X_i be the set of (oriented) i-cells in X. Let $C^i = C^i(X; G)$ be the set G^{X_i} of maps assigning an element $c(\sigma)$ of G to each i-cell $\sigma \in X_i$, such that $c(\bar{\sigma}) = c(\sigma)^{-1}$ where $\bar{\sigma}$ denotes the cell σ with orientation reversed. It will be convenient to parametrize the cells as simplices, and we let $\sigma_0, \ldots, \sigma_i$ denote the vertices of i-simplex σ and label faces by giving an ordered list of vertices as subscripts.

The representation $\rho : \pi_1(X) \to G$ can be realized as an element $\rho \in C^1(X; G)$ satisfying the cocycle condition $\rho(\sigma_{01})\rho(\sigma_{12})\rho(\sigma_{20}) = e$ for every 2-simplex σ of X. (For instance, if ρ is the holonomy representation for a geometric structure on X then we can take $\rho(edge)$ to be the holonomy along the edge obtained from the developing map.)

We define maps

$$C^1(X; G) \xrightarrow{A} C^1(X; G) \times C^2(X; G) \xrightarrow{B} C^2(X; G) \times C^2(X; G)$$

as follows.

B onto 2nd factor is $D^1 f\, 1^{-1}$

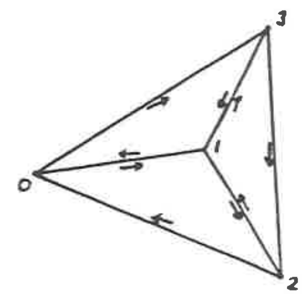
$$A(f) = (f, D^1 f), \text{ where } D^1 f(\sigma) = f(\sigma_{01}) f(\sigma_{12}) f(\sigma_{20})$$

for $f \in C^1$, $\sigma \in X_2$ and

$$B(f, g) = D_f^2 g, \text{ where } D_f^2 g(\sigma) = f(\sigma_{01}) g(\sigma_{123}) f(\sigma_{01})^{-1} g(\sigma_{031})^{-1} g(\sigma_{023})^{-1} g(\sigma_{012})^{-1}$$

for $f \in C^1$, $g \in C^2$, $\sigma \in X_3$.

Then $B \cdot A = (1)$ where (1) denotes the map whose value on all i-simplices is the identity element $1 \in G$. (Compare the following figure.)



(terms cancel in pairs)

Also, $A^{-1}(C^1 \times (1))$ is exactly $R(\Gamma, G)$.

We identify the tangent space to G at a point with the Lie algebra g by right translation to the tangent space at the identity. Then if $\rho \in R(\Gamma, G)$, the derivatives of A at ρ and B at $A(\rho) = (\rho, (1))$ are given by:

$$dA = 1 \oplus d^1 : C^1(X; g) \to C^1(X; g) \oplus C^2(X; g)$$

$$dB = 0 + d^2 : C^1(X; g) \oplus C^2(X; g) \to C^3(X; g)$$

where the maps d^i are exactly the boundary maps in the complex $C^i(X; g_{Ad\rho})$ giving the (cellular) cohomology $H^i(\Gamma; Ad\rho) = H^i(X; Ad\rho)$.

Now since $H^2(X; Ad\rho) = 0$, $im\, d^1 = ker\, d^2$ and it follows that the sequence of derivatives

$$C^1(X; g) \xrightarrow{dA} C^1(X; g) \oplus C^2(X; g) \xrightarrow{dB} C^3(X; g) \oplus C^4$$

is exact. Hence by lemma 3.4, A is a submersion onto a neighbourhood of $(\rho, (1))$ in $B^{-1}(1) \subset C^1 X; G) \times C^2(X; G)$. In particular, $R(\Gamma, G) = A^{-1}(C^1(X; G) \times (1))$ is a manifold of the same dimension as $kerd^1 = Z^1(\Gamma, Ad\rho)$. □

We will also need a relative version of this result.

Proposition 3.5. *Let X be a simplicial complex and Y a subcomplex. Assume X is connected and let $Y_1, \ldots, Y_n$ be the components of Y. Let $\Gamma = \pi_1(X)$ and $\Gamma_i = \pi_1(Y_i)$ for $i = 1, \ldots, n$ (for any choice of base points). Let ρ be a representation $\rho : \Gamma \to G$ restricting to representations $\rho_i : \Gamma_i \to G$. Assume that $R(\Gamma_i, G)$ is a manifold near ρ_i for each i. Then if image $(H^2(X, A) \to H^2(X)) = 0$, $R(\Gamma, G)$ is a manifold near ρ.*

There is also an equivalent version in terms of the representations of fundamental groupoids, i.e. G-valued cocycles $Z^1(X;G)$ and $Z^1(Y;G)$.

Proposition 3.6. *Let X be a simplicial complex and Y a subcomplex. Let ρ be cocycle representing an element of $H^1(X;G)$. Assume that $Z^1(Y;G)$ is a manifold near the restriction $\rho|Y$. Assume that $im(H^2(X,Y;g) \to H^2(X;g)) = 0$. Then $Z^1(X;G)$ is a manifold near ρ.*

Proof. Define $C^i(X;G)$, $C^i(Y;G)$ and maps A, B, D^i as above. Put $Z^1(X;G) = A^{-1}(C^1(X;G)\times(1))$ (corresponding to representations in $R(\Gamma, G)$) and define $Z^1(Y;G)$ similarly. Let $r : C^i(X;G) \to C^i(Y;G)$ be the restriction map and $C^i(X, Y; G) = r^{-1}(1)$ the preimage of the trivial cocycle having value 1 on each i-simplex of Y.

Since r is a submersion, $S = r^{-1}(Z^1(Y;G))$ is a manifold containing $Z^1(X;G)$. Further, $D^1(S) \subset C^2(X, Y; G)$ and there are maps:

$$S \xrightarrow{a} C^1(X;G)\times C^2(X,Y;G) \xrightarrow{b} C^2(X;G)$$

where a is the restriction of $A = 1 \oplus D^1 : C^1(X;G) \to C^1(X;G)\times C^2(X;G)$ to S, and b is B restricted to $C^2(X,Y;G)$. If $\rho \in kerD^1$ (giving a representation in $R(\Gamma, G)$), then we have seen above that $da = j \oplus d^1$ and $db = 0 + d^2$ where $j : TS \subset C^1(X;g)$ is induced by inclusion and d^i are the coboundary maps for computing the cohomology with local coefficients in g. Using this observation, the condition that $im(H^2(X,Y;g) \to H^2(X;g)) = 0$ implies that $kerdb \subset imda$. By the construction of the maps a, b we have $b \circ a(s) = (1)$ for all $s \in S$; hence $imda \subset kerdb$. By lemma 3.2, it follows that a is a submersion onto a manifold $a(S)$; hence $Z^1(X;G) = a^{-1}(C^1(X;G)\times(1))$ is a manifold. □

13. Local properties of representation spaces

We now study some local properties of the representation space $R = R(\Gamma, G)$. First we recall a fundamental result on the structure of real analytic varieties. A proof for the case of real algebraic varieties can be found in Whitney [Wh1]. As Whitney observes in the introduction to [Wh1], the same proof applies in the analytic case.

Theorem 3.7. *Let V be an analytic subset of $\mathbb{R}^n$. Let M be the set of points in V where the dimension of the Zariski tangent space is minimal. Then M is a smooth manifold with tangent space equal to the Zariski tangent space at each point, and $V - M$ is a proper analytic subset of V.*

□

Remark : This fails, in general, if V is given by the zero set of a family of smooth, non-analytic functions. (For example, take a C^∞ function on $\mathbb{R}$ vanishing on a closed interval.)

Let ρ_0 be a representation in R. For each ρ in R, write H^i_ρ for the cohomology group $H^i(\Gamma, g_{Ad\rho})$. Then the following proposition gives some stability properties of these groups and their relation with the local properties of the space of representations.

Proposition 3.8.

(1) $\dim H^i_\rho \leq \dim H^i_{\rho_0}$, *for all representations* $\rho \in R$ *near* ρ_0.

(2) If $\dim H^i_\rho$ *is constant in a neighbourhood of* ρ_0 *in* R *then the coboundary maps* d^{i-1} *and* d^i *have constant rank in this neighbourhood of* ρ_0.

(3) If d^1 *has constant rank at all points of* R *near* ρ_0, *then* $R(\Gamma, G)$ *is a manifold near* ρ_0 *of dimension equal to the dimension of kernel* $d^1 = Z^1$.

(4) If $R(\Gamma, G)$ *is a manifold near* ρ_0 *and* H^0 *has constant rank near* ρ_0, *then there are neighbourhoods* U *of* ρ_0 *in* $R(\Gamma, G)$ *and* V *of* 1 *in* G *such that the "local" quotient* U/V *is a manifold of dimension equal to the dimension of* H^1.

Proof. Parts (1) and (2) follow, from the fact that the rank (and nullity) of linear maps are lower (upper) semicontinuous functions of the map. The coboundary maps d^i giving Lie group cohomology with coefficients in $Ad\rho$ depend continuously on the representation ρ. But, the dimension of image d^i can not increase in a limit, while the dimension of kernel d^i can not decrease in a limit. Hence, the dimension of $H^i_\rho = kerd^i / imd^{i-1}$ cannot decrease in a limit, as $\rho \to \rho_0$. Moreover, $\dim H^i$ strictly increases unless the rank of d^i and d^{i-1} both remain constant in a neighbourhood of ρ_0.

Part (3) follows from the observations that d^1 can be identified with the derivative of D^1 and $(D^1)^{-1}(1) = R(\Gamma, G)$, using the previous theorem.

The hypothesis of part (4) implies that the G-orbits on $R(\Gamma, G)$ near ρ are manifolds of (constant) dimension equal to the rank of d^0. Hence U/V is a manifold of dimension $\dim Z^1 - \dim imd^0 = \dim H^1$. □

The following is a refinement of part (4), in the case where G is an algebraic group.

Proposition 3.9. *Assume that* G *is an algebraic group,* $R(\Gamma, G)$ *is a manifold near* ρ *and* $\dim H^0$ *is constant near* ρ. *Then there is a neighbourhood* U *of* ρ *in* $R(\Gamma, G)$ *such that the quotient* U/G *is a manifold of dimension equal to the dimension of* H^1.

Proof. Consider the action of the group G by conjugation on the algebraic variety $R = R(\Gamma, G)$. Since this is an algebraic group action it follows that each orbit is a locally (Zariski-) closed subset of R. (This a standard fact for algebraic group actions. The map $G \to L$ given by $g \mapsto gx$ is a morphism of algebraic varieties. So the image Gx contains a dense open subset U of its Zariski closure $\overline{Gx}$. Hence $GU = Gx$ is an open subset of $\overline{Gx}$.)

It follows that if two nearby representations in R are conjugate, then they are conjugate by elements of G near 1. So the result follows from part (4) of the previous proposition. □

Remark 3.10.

In general the global quotient $R(\Gamma, G)/G$ will be badly behaved, e.g. non-Hausdorff. However, we will be primarily concerned with the local structure of $R(\Gamma, G)$ so the preceding results will suffice for our purposes. For information on global quotient spaces $R(\Gamma, G)/G$ and the associated character variety, see [G1], [G2], [C-S] and [J-M].

Proposition 3.8 also gives another proof of the following results from the previous section.

Corollary 3.11. *(1) If* $H^1 = 0$ *at* ρ *then* $R(\Gamma, \rho)$ *is a manifold near* ρ *equal to the orbit of* ρ *under* G. *The image of* ρ *in* $R(\Gamma, G)/G$ *is an isolated point.*

(2) If $H^2 = 0$ *at* ρ *then* $R(\Gamma, G)$ *is a manifold near* ρ *of dimension equal to* $\dim Z^1$.

□

3.4. Some properties of cohomology

3.4.1. Poincare duality

One important property of the cohomology groups $H^*(M; E)$ with coefficients in a flat vector bundle (or local coefficient system) E is Poincaré duality. Given a pairing $A \otimes B \to C$ of flat vector bundles (or local coefficient systems) there is an associated cup product

$$\cup : H^k(M;A)\otimes H^l(M;B)\to H^{k+l}(M;C).$$

Let M be an orientable closed manifold of dimension n. When a pairing $A\otimes B\to \mathbf{R}$ is non-degenerate, we have Poincaré duality: the cup product gives a non-degenerate bilinear form

$$H^k(M;A)\otimes H^{n-k}(M;B)\to H^n(M;\mathbf{R})\cong\mathbf{R}.$$

When M is an orientable n-manifold with boundary ∂M there is a relative version : the cup product gives a non-degenerate bilinear form

$$H^k(M;A)\otimes H^{n-k}(M,\partial M;B)\to H^n(M,\partial M;\mathbf{R})\cong\mathbf{R}.$$

Proofs of these results can be found, for example, in [Sw, chap. XI]; alternatively one can give proofs via de Rham theory by extending the usual arguments for the case of real coefficients (e.g. [B-T]).

We will be interested in the case where $A=B$ are vector bundles of infinitesimal isometries with fibres isomorphic to the Lie algebra g of the isometry group G. Then one such pairing comes from the Killing form $\langle\ ,\ \rangle : g\times g\to \mathbf{R}$ on the Lie algebra g. Another natural pairing comes from the Lie bracket $[\ ,\ ] : g\times g\to g$ on the fibres. The first pairing gives Poincaré duality when the Killing form is non-degenerate i.e. when the Lie algebra g is semisimple. This holds, for example, when G is the isometry group of a sphere or hyperbolic space.

More generally, we obtain a non-degenerate pairing $g\times g\to \mathbf{R}$, invariant under the adjoint action, whenever G is reductive, i.e. $Ad(G)\subset GL(g)$ is semisimple. We can then use the trace pairing $(A,B)\mapsto tr(AB^t)$ on $GL(g)$.

14.2. Kronecker pairing

For any flat vector bundle E, let E^* denote the dual vector bundle. Then there is an isomorphism

$$H^i(Y;E)^*\cong H_i(Y;E^*)$$

induced by the Kronecker pairing (evaluation of cohomology on homology)

$$H^1(Y;E^*)\times H_1(Y;E)\to \mathbf{R}.$$

14.3. Euler characteristic

It is also useful to recall that the Euler characteristic $\chi(M)$ of M is given by :

$$(\dim E)\cdot\chi(M)=\sum_{i=0}^{n}(-1)^i\dim H^i(M;E).$$

15. Some cohomology calculations

In this section we compute cohomology groups $H^i(\Gamma;Ad\rho)$ for certain representations $\rho:\Gamma\to PSL_2(\mathbf{C})$, where Γ is the fundamental group of a surface or 3-manifold. All dimensions given will be *complex* dimensions.

Case (a) Let S be a surface of genus g. We compute the cohomology groups $H^i=H^i(S;Ad\rho)$ for some representations $\rho:\pi_1(S)\to PSL_2(\mathbf{C})$.

Suppose that ρ is "irreducible". (We really mean that $\rho(\pi_1(S))$ fixes no point on the sphere at infinity for $\mathbf{H}^3$ or ρ lifts to an irreducible representation $\pi_1\to SL_2(\mathbf{C})$.) Then $H^0=0$ since the centralizer of $\rho(\pi_1(S))$ is trivial, so $H^2=0$ by Poincaré duality. Thus, $\dim_{\mathbf{C}}H^1=-\dim_{\mathbf{C}}(sl_2\mathbf{C})\cdot\chi(S)=-3\chi(S)=6g-6$.

For a torus, $g=1$, representations are usually reducible. In the generic case, one generator of $\pi_1(S)=\mathbf{Z}^2$ is a hyperbolic transformation of $\mathbf{H}^3$ and its axis must be preserved by the other generator. Then there is a one (complex) dimensional infinitesimal centralizer (corresponding to the infinitesimal rotations and translations along the axis which commute with the $\rho(\mathbf{Z}^2)$ action). In this case, we obtain $H^0=H^2=\mathbf{C}$ and $H^1=\mathbf{C}^2$. The same result holds when the image of ρ consists of parabolic elements. (In this case, there is a 1-dimensional family of infinitesimal parabolic elements commuting with the $\rho(\mathbf{Z}^2)$ action.)

Remark 3.12. In both cases, the dimension of H^1 is constant for representations near ρ. It follows from proposition 3.8 that the space of representations is a manifold of complex dimension 4 near ρ. Further, $\dim H^0 = 1$ is constant near ρ so the "local quotient space" of representations near ρ up to conjugacy is a manifold of complex dimension 2. It is an easy exercise to see that this space can be parametrized locally by traces or "complex angles" of images of two generators.

There are also isolated irreducible representations of $\mathbf{Z}^2$. For example, there is an irreducible representation $\mathbf{Z}^2 \to PSL_2(\mathbf{C})$ whose image is the subgroup $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \subset PSL_2(\mathbf{C})$ generated by 180° rotations about 3 orthogonal axes in $\mathbf{H}^3$. Then $H^0 = H^1 = H^2 = 0$.

Case (b) Now we consider the case of a compact 3-manifold M, with boundary ∂M containing no spheres. Let $\rho : \pi_1(M) \to PSL_2(\mathbf{C})$ be a representation such that

(*)

(i) the centralizer of $\rho(\Gamma)$ in $PSL_2(\mathbf{C})$ is trivial

(ii) for each torus boundary component $T \subset \partial M$, $\rho(\pi_1(T)) \neq 1$ preserves either a geodesic in $\mathbf{H}^3$ or a point at infinity. (i.e. $\rho(\pi_1(T))$ is not a subgroup of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.)

(iii) for each non-torus component S of ∂M, the restriction of ρ to $\pi_1(S)$ has trivial centralizer.

(Condition (*) holds, for example, whenever ρ is geometric : corresponding to a hyperbolic structure with cone or Dehn surgery type singularities at torus boundary components and complete at other boundary components.)

We consider two special cases.

(1) ∂M is a union of n tori

We have an cohomology exact sequence

$$\ldots \longrightarrow H^i(M, \partial M) \longrightarrow H^i(M) \longrightarrow H^i(\partial M) \longrightarrow H^{i+1}(M, \partial M) \longrightarrow \ldots \quad (**)$$

Now $H^0(M) = H^0(M, \partial M) = 0$ since ρ has trivial centralizer, so by Poincaré duality, the third cohomology groups also vanish. By the previous example, we know that $H^1(\partial M) = \mathbf{C}^{2n}$ and $H^0(\partial M) = H^2(\partial M) = \mathbf{C}^n$. In the exact sequence

$$H^1(M) \xrightarrow{\alpha} H^1(\partial M) \xrightarrow{\beta} H^2(M, \partial M)$$

the maps α and β are dual maps (with respect to Poincaré duality) so $\dim im\alpha = \dim im\beta = \frac{1}{2}\dim H^1(\partial M) = n$. Let x denote the dimension of $ker\alpha = ker(H^1(M) \to H^1(\partial M)) = im(H^1(M, \partial M) \to H^1(M))$. Then using the exact sequence (**) it follows that $\dim H^i(M) = H^i(M, \partial M) = n + x$, for $i = 1, 2$.

Remark 3.13.

A slight refinement of this argument will prove useful. Let ∂_i denote the ith boundary component of M. Then we can write $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = \beta_1 + \ldots + \beta_n$, where

$$H^1(M) \xrightarrow{\alpha_i} H^1(\partial_i) \xrightarrow{\beta_i} H^2(M, \partial M).$$

Then α_i and β_i are dual maps and $\dim im\alpha = \dim im\beta = \frac{1}{2}\dim H^1(\partial_i)$, for each i.

(2) Each component S_k of ∂M has genus > 1, and the restriction of ρ to each $\pi_1(S_k)$ is irreducible. (This is the generic case, and holds for instance when ρ is the holonomy representative of a complete hyperbolic structure on $\operatorname{int} M$.)

Then a similar argument shows that for $i = 1, 2$, $\dim H^i(M) = H^i(M, \partial M) = \frac{-3}{2}\chi(\partial M) + x$, where $x = \dim ker(H^1(M) \to H^1(\partial M)) = \dim im(H^1(M, \partial M) \to H^1(M))$.

(3) Let M be a compact 3-manifold with boundary containing no spheres, and let $\rho : \pi_1(M) \to PSL_2(\mathbf{C})$ be an representation satisfying the conditions (*). Then by combining cases (1) and (2), the cohomology with $Ad\rho$ coefficients is given by $\dim H^i(M) = \dim H^i(M, \partial M) = \frac{-3}{2}\chi(\partial M) + t + x$ for $i = 1, 2$, where t is the number of tori

components of M, and $x = \dim ker(H^1(M) \to H^1(\partial M)) = \dim im(H^1(M, \partial M) \to H^1(M))$.

Remark : Similar cohomology calculations were noted by Thurston in a Princeton University course in Spring 1982.

From these calculations we obtain the following result, originally proved by a different method in [Th1] (see also [C-S]).

Theorem 3.14. *Let M, Γ be as above and let R be an irreducible component of $R(\Gamma, PSL_2(\mathbf{C}))$ which contains a representation, satisfying the conditions (*). Then R has complex dimension $\geq N+3 = \frac{-3}{2}\chi(M)+t+3$.*

If $\rho \in R$ is a representation such that $\dim Z^1(\Gamma, Ad\rho) = N+3$ or R has dimension $N+3$ at ρ, then R is a complex manifold of dimension $N+3$ near ρ. Further, the conjugacy classes of representations near ρ form a complex manifold of dimension N. If ∂M consists of tori T_i, then the representations near ρ are parametrized up to conjugacy by traces or complex lengths of (suitably chosen) elements $a_i \in \pi_1(T_i)$.

Proof. There is a Zariski dense set of representations $\rho \in R$ such that ρ satisfies (*) and near ρ, R coincides with $R(\Gamma, PSL_2(\mathbf{C}))$ and is a smooth manifold of dimension equal to $\dim R$. Further, at such points $\dim Z^1(\Gamma, Ad\rho) = \dim R = \dim H^1(\Gamma, Ad\rho)+3$. But we have seen that for every such ρ, $\dim H^1(\Gamma; Ad\rho) \geq N$, and $\dim B^1(\Gamma; Ad\rho) = 3$, so the first claim follows. The next part follows from prop 3.6 or from the general fact that the Zariski tangent space to a complex variety R has dimension $\geq \dim R$ at every point and R is smooth where equality holds (cf. prop. 3.7).

If equality holds then $H^1(M) \to H^1(\partial M)$ is an injection. ($x = 0$ in the previous notation.) Let $R(\partial M)$ denote the product $R(\partial_1)\times \ldots \times R(\partial_n)$, where $\partial_1, \ldots, \partial_n$ are the components of ∂M. Then it follows that restriction induces an immersion of the local quotient spaces $R(M)/PSL_2\mathbf{C} \to R(\partial M)/PSL_2\mathbf{C}$. (By 3.9, these local quotient spaces are manifolds of

dimension $\dim H^1(M)$ and $H^1(\partial M)$.)

If ∂M consists of tori, then each $R(\partial_i)/PSL_2\mathbf{C}$ is parametrized by traces of two generators. (Remark 3.12) The final claim follows from the fact that $im(H^1(M) \to H^1(\partial_i))$ has dimension $\frac{1}{2}\dim H^1(\partial_i) = 1$ for all i (Remark 3.13). □

This will also prove useful for dealing with examples, in the next chapter.

The arguments used above extend immediately to compute cohomology for representations of surface and 3-manifold groups into reductive groups.

Remark 3.15.

The condition that $0 = im(H^1(M, \partial M) \to H^1(M))$, (i.e. $x = 0$), can be regarded as a kind of rigidity: roughly, there are no infinitesimal deformations of M fixed on ∂M. This rigidity is also reflected in the fact that the representations and geometric structures are determined (locally) by their restrictions to the boundary of M (theorem 3.14).

For example, let M be a 3-manifold with boundary consisting of t tori. Then it follows easily from the Mostow rigidity theorem (or local rigidity theorems of [We2], [GR]) that x vanishes at the holonomy representation $\rho \in R(\pi_1(M), PSL_2\mathbf{C})$ for a complete hyperbolic structure on M or a hyperbolic orbifold structure on M (i.e. with cone angles of the form $2\pi/n$, $n \in \mathbf{Z}$ along ∂M.) In each of these cases, $R(\pi_1(M), PSL_2\mathbf{C})/PSL_2\mathbf{C}$ is locally a manifold of dimension t. Thurston's hyperbolic Dehn surgery theorem follows from this result and the fact that all representations near ρ correspond to hyperbolic structures with generalized Dehn surgery type singularities (theorem 1.8).

In the next section we will show $x = 0$ for representations coming from certain Euclidean structures. We will use this in chapter 4, to deform Euclidean structures to "nearby" hyperbolic structures and spherical structures.

16. Cohomology of Euclidean orbifolds

Suppose that Q^n is a Euclidean orbifold, i.e. Q is a quotient $\mathbf{E}^n/\Pi$ where Π is a discrete subgroup of $\tilde{G} = \mathrm{isom}(\mathbf{E}^n)$. Let $\Sigma \subset Q$ be the singular locus of Q consisting of the image of the union $\tilde{\Sigma} \subset \mathbf{E}^n$ of fixed point sets of non-trivial elements of Π. Let $M = Q - \mathrm{int}N(\Sigma)$ where $N(\Sigma)$ is a closed tubular neighbourhood of Σ in Q. (In general Q is not a topological manifold, and we use tubular neighbourhood to mean the image in Q of a Π-invariant tubular neighbourhood of $\tilde{\Sigma}$ in $\mathbf{E}^n$.) Put $\Gamma = \pi_1(M) = \pi_1(Q-\Sigma)$, and let $\tilde{\rho} : \Gamma \to \tilde{G}$ be the holonomy representation for the Euclidean structure on Q. (So $im\, \rho = \Pi \subset \tilde{G}$.)

Now $\tilde{G}$ is a split extension $\mathbb{R}^n \to \tilde{G} \xrightarrow{\pi} SO(n)$ where π takes a Euclidean isometry to its rotational part, and $\mathbb{R}^n$ consists of translations. Let $\rho = \pi \circ \tilde{\rho} : \Gamma \to SO(n)$, and identify $SO(n)$ with the subgroups of $\mathrm{isom}(\mathbf{H}^n)$, $\mathrm{isom}(S^n)$ fixing a point. Then we will study the problem of deforming ρ in $R(\Gamma, \mathrm{isom}(\mathbf{E}^n))$, $R(\Gamma, \mathrm{isom}(\mathbf{H}^n))$ and in $R(\Gamma, \mathrm{isom}(S^n))$.

First we study the cohomology of M with coefficients in the flat vector bundles $E_E(\rho)$, $E_H(\rho)$ and $E_S(\rho)$ of infinitesimal isometries of euclidean, hyperbolic and spherical geometries. The following observation will be important.

Proposition 3.16. *The flat vector bundles $E_E(\rho)$, $E_H(\rho)$, and $E_S(\rho)$ are isomorphic. Moreover, in the 3-dimensional case, each flat bundle is isomorphic to the direct sum of two copies of the euclidean tangent bundle of Q.*

Proof. The first claim follows from the fact that the adjoint action of $SO(n)$ on infinitesimal translations and rotations about a point are the same in all cases. The second claim follows from the fact that if $n=3$ the adjoint actions on infinitesimal translations and rotations are isomorphic. Explicitly, if $A \in SO(3)$ and t is an infinitesimal rotation then the adjoint action of A is just rotation by $A : t \mapsto At$. If we represent an infinitesimal rotation by a vector r whose direction gives the axis of rotation and length gives the magnitude of rotation, then A acts by : $r \mapsto Ar$. (This can also been seen from the Lie algebra structures. The ad actions of $so(3)$ are clearly isomorphic for the three Lie algebras, and each splits as a direct sum.)

Finally, the action on infinitesimal rotations gives rise to the Euclidean tangent bundle since if $g_{ij} \in \mathrm{isom}(\mathbf{E}^3)$ are the transition maps for the Euclidean structure then the corresponding transition maps on the fibres of the two bundles are equal : the projection $\pi(g_{ij})$ and the derivative dg_{ij} both give the rotational part of g_{ij}. □

Thus, in the 3-dimensional case, it suffices to consider the cohomology of M with coefficients in the tangent bundle $T(M)$. More generally, the cohomology of an n-dimensional Euclidean orbifold M with coefficients in a flat, orthogonal vector bundle V is given by the following theorem.

Theorem 3.17. *Let M and V be as above. Then every cohomology class in the image of the natural map $(H^1(M, \partial M; V) \to H^1(M; V))$ can be represented by a parallel 1-form on M.*

Proof. (Due to Thurston.) Let $\omega \in \Omega(M, \partial M; V)$ be a closed V-valued 1-form on M vanishing on ∂M. Using the flat connection on V we can represent ω locally as a F-valued form where F is a (fixed) fibre of the bundle V. So for each point x in an open subset of M, ω gives a linear map $\omega(x) : T_x M \to F$ varying smoothly with x. We extend ω over all of Q by putting $\omega = 0$ on the neighbourhood $N(\Sigma)$ of Σ. Fixing a base point $x_0 \in M$ we obtain a map $f : \tilde{M} \subset X \to F$ by integrating ω along paths in M based at x_0. Since ω is closed, the integral $\int_c \omega(x)dx$ along a path c only depends on the homotopy class of c in M, and we can identify the collection of homotopy classes of paths in M based at x_0 with the universal cover of M in the usual way. Then f satisfies the equivariance condition :

$$f(\gamma x) = \gamma \cdot f(x) + c(\gamma), \quad \text{for all } \gamma \in \Gamma,\ x \in X$$

where γ acts on X by covering transformations for M and on F by the holonomy representation $\phi : \Gamma \to SO(F) \subset GL(F)$ for the flat vector bundle V and $c : \Gamma \to F$ is a cocycle representing the cohomology class in $H^1(\Gamma, F)$ corresponding to the class of ω in $H^1(M, V)$. Thus, the derivative df is a F-valued 1-form on X satisfying

$$df(\gamma x) = \gamma df(x), \quad \text{for all } \gamma \in \Gamma,\ x \in X.$$

For each $r>0$, we define a new map $f_r : \mathbf{E}^n \to F$ by averaging f over balls of radius r:

$$f_r(x) = \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} f(y)dy$$

$$= \frac{1}{\mathrm{vol}(B_r(0))} \int_{B_r(0)} f(x-y)dy$$

where $B_r(x)$ is the Euclidean ball of radius r centred at x and the integration is with respect to Lebesgue measure. Then f_r is also equivariant and its derivative df_r gives a V-valued 1-form ω_r on M.

We claim that by taking the limit as $r \to \infty$ of df_r we obtain a *parallel* V-valued form ω_∞ on M which is equivariantly homotopic, hence cohomologous, to ω. (However, the maps f_r will not converge in general.) To see this, note that

$$df_r(x) = \frac{1}{\mathrm{vol}(B_r(0))} \int_{B_r(0)} df(x-y)dy$$

$$= \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} df(y)dy$$

so $\|df_r\| < \|df\| = \sup_M |\omega| < \infty$, where $\|\phi\|$ denotes the supremum over $\mathbf{E}^n$ of the Euclidean norm $|\phi|$ of ϕ. Moreover,

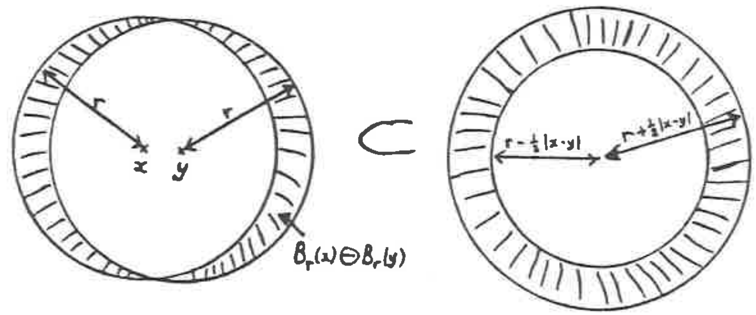
$$|df_r(x) - df_r(y)| \leqslant \frac{1}{\mathrm{vol}}(B_r)\Big| \int_{B_r(x)\ominus B_r(y)} df \Big|$$

$$\leqslant \|df\| \frac{\mathrm{vol}(B_r(x)\ominus B_r(y))}{\mathrm{vol}(B_r)}$$

$$\leqslant \|df\| \cdot const \cdot |x-y| \frac{\mathrm{vol}(\partial B_r)}{\mathrm{vol}(B_r)}$$

$$\leqslant \|df\| \cdot const \cdot \frac{|x-y|}{r}$$

where $\ominus$ denotes the symmetric difference.



Hence $\lim_{r\to\infty} \frac{|df_r(x) - df_r(y)|}{|x-y|} = 0$. So the df_r form a bounded equicontinuous family, and by the Ascoli-Arzela theorem there is a convergent subsequence with limit $\tilde{\omega}_\infty$ satisfying $d\tilde{\omega}_\infty = 0$ which descends to a parallel form on M as desired. □

Essentially the same argument gives a proof of the following theorem of Bieberbach.

Theorem 3.18. *Two compact Euclidean orbifolds are affinely equivalent if and only if their fundamental groups (in the orbifold sense) are isomorphic.*

Proof. Given an isomorphism of fundamental groups there exists an equivariant map between their universal covers:

$$f : \mathbf{E}^n \to \mathbf{E}^n$$

since both orbifolds are $K(\pi, 1)$'s. Then by applying the previous averaging argument and adjusting the averaged maps by translations so the images of a basepoint converge we obtain an equivariant homotopy from f to an affine equivariant map. □

Remark 3.19.

The averaging argument also applies for certain other geometries. In the case of a spherical structure, we get the stronger result that every closed form in $\Omega^1(M, \partial M; V)$ is cohomologous to zero in M, since f becomes constant after averaging over the compact n-sphere. So we obtain

Theorem 3.20. *Let Q be a spherical orbifold and V a flat orthogonal vector bundle over $M = Q - \mathrm{int}N(\Sigma)$. Then image$(H^1(M, \partial M; V) \to H^1(M; V)) = 0$.*

□

Remark 3.21.

In the above argument we need an invariant metric such that $\mathrm{vol}(\partial B_r)/\mathrm{vol}(B_r) \to 0$ as $r \to \infty$. This condition can also be satisfied for nilpotent groups. However, the proof doesn't apply in the hyperbolic case, and in fact there can be non-trivial harmonic forms on compact hyperbolic manifolds. But the result with coefficients Adp certainly follows for hyperbolic *orbifolds* by the rigidity theorem of [We2] ([GR]) (or Mostow's rigidity theorem). The recent results of Rivin on rigidity of convex polyhedra in $\mathbf{H}^3$ may lead to a proof for hyperbolic cone manifolds.

If the result is true for hyperbolic cone manifolds with cone angles $\leqslant \pi$, then it would probably follow from the techniques of [Th3] that hyperbolic cone manifolds (of fixed combinatorial type) with cone angles $\leqslant \pi$ form a connected space.

Remark 3.22.

The above argument has been generalized by Thurston to the case of Euclidean or spherical cone manifolds, *with cone angles less than or equal to π*. The proof involves averaging of 1-forms over ϵ-balls not meeting the singular locus Σ, then using harmonic extensions over an ϵ-neighbourhood of Σ. In such an averaging step, the norm of the covariant derivative of a 1-form is reduced by a constant factor $c < 1$ so by iterating this procedure a parallel 1-form is obtained. In the spherical case, every 1-form becomes trivial.

This gives the following results, which are used in the proof of the orbifold theorem [Th3].

Theorem 3.23 (Thurston). *Let Q be a Euclidean or spherical cone manifold with cone angles $\leqslant \pi$ along the singular locus Σ. Let V be a flat orthogonal vector bundle over $M = Q - \mathrm{int}N(\Sigma)$, where N is a tubular neighbourhood of Σ in Q. Then every cohomology class in the image of the natural map $(H^1(M, \partial M; V) \to H^1(M; V))$ can be represented by a parallel 1-form on M.*

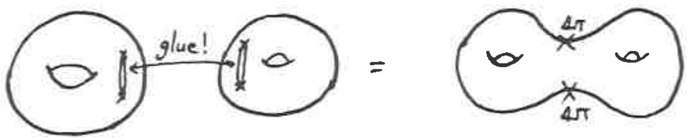
If Q is a spherical cone manifold as above, then image $(H^1(M, \partial M; V) \to H^1(M; V))$ is zero.

□

Remark 3.24.

When cone angles greater than π are allowed these results are no longer true in general. The proof outlined above fails because of the behaviour of the harmonic extensions (they no longer satisfy the maximum principle in general). The following are concrete examples of the failure.

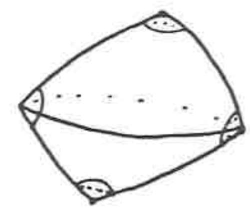
(a) Consider a Euclidean cone manifold structure S on a surface of genus 2, obtained by taking the 2-fold cover of a Euclidean torus branched over 2 points. More explicitly, take two copies of the torus cut open along a geodesic arc and glue together as indicated below.



Then S has 2 cone points with cone angle 4π. Now $H^1(S;\mathbb{R})\cong\mathbb{R}^4$ (using trivial coefficients!), but parallel 1-forms correspond to linear maps : $TS\to\mathbb{R}$ so make up only a 2-dimensional vector space.

(b) Consider spherical cone manifold structures on the two sphere with 4 cone points. By gluing together pairs of triangles along an edge we obtain a 5-dimensional family of quadrilaterals, parametrized by edge lengths of the triangles. But the angles form the 4-dimensional manifold, consisting of angles with sum greater than 2π.

By doubling these quadrilaterals we obtain a 5-dimensional manifold of spherical metrics on S^2 with 4 cone points. So these cone manifold structures can be deformed, keeping the cone angles fixed. Note that the sum of the cone angles is greater than 4π, so some cone angle is greater than π in these examples.



17. Cohomology of link complements

In this section, we compute some cohomology groups for complements of links in 3-manifolds, giving Zariski tangent spaces for certain spaces of representations. This will be used in chapter 4, to deform foliations associated with some non-hyperbolic geometric structures on 3-manifolds to nearby hyperbolic structures.

Let M be an orientable 3-manifold and Σ a link in M, with components $\Sigma_1,\ldots,\Sigma_n$. Let $\bar{\rho}:\pi_1(M)\to G=PSL_2(\mathbb{C})$ be a representation and let $\rho:\pi_1(M-\Sigma)\to G$ be obtained by composing $\bar{\rho}$ with the surjection $\pi_1(M-\Sigma)\to\pi_1(M)$, induced by inclusion. We will study the cohomology $H^*(M-\Sigma;E(\rho))$. Here and in the following calculation, $E(\rho)$ denotes the flat $\mathfrak{g}$ vector bundle associated with ρ on M or restricted to a subset of M.

It follows, in a standard way, from Poincaré duality for manifolds with boundary that there is also Lefschetz duality isomorphism : $H^i(M-\Sigma;E(\rho))\cong H_{3-i}(M,\Sigma;E(\rho))$. Using the exact Mayer-Vietoris sequence (with $E(\rho)$ coefficients):

$$0\longrightarrow H_2(M)\longrightarrow H_2(M,\Sigma)\longrightarrow H_1(\Sigma)\xrightarrow{i_*} H_1(M)$$

$$\longrightarrow H_1(M,\Sigma) \longrightarrow H_0(\Sigma) \xrightarrow{i_0} H_0(M)$$

we obtain

Lemma 3.25. *With $M \supset \Sigma$ as above,*

$$H^1(M-\Sigma;E(\rho)) \cong H_2(M,\Sigma;E(\rho)) \cong H_2(M;E(\rho)) \oplus ker(i_1)$$

$$H^2(M-\Sigma;E(\rho)) \cong H_1(M,\Sigma;E(\rho)) \cong coker(i_1) \oplus ker(i_0)$$

□

The simplest case is when $H^1(M)=0$. Then $H^1(M-\Sigma)\cong H^1(\Sigma)$. If $G=PSL_2\mathbb{C}$, then $H^1(M-\Sigma)\cong\mathbb{C}^n$ has dimension equal to the number of components of Σ. Let N be the compact manifold obtained by removing an open tubular neighbourhood of Σ from M. Then in the exact sequence

$$H^1(N,\partial N) \longrightarrow H^1(N) \longrightarrow H^1(\partial N)$$

the second map is an injection, so the first map is the zero map. By duality, $H^2(N,\partial N) \to H^2(N)$ is also the zero map. By theorem 3.6, we obtain :

Theorem 3.26. *If $H^1(M;E(\rho))=0$ then $R(\pi_1(M-\Sigma),G)$ is a manifold of dimension equal to $nd+z$ near ρ, where z is the dimension of the infinitesimal centralizer of ρ, and $d=\dim H^0(S^1;E(\rho))$.*

□

Now, $H_0(M;E(\rho))\cong g/<x-\gamma x,\gamma\in\pi_1(M)>$ (see appendix) and $H_0(\Sigma_i;E(\rho))\cong g/<x-\gamma x,\gamma\in\pi_1(\Sigma_i)>$. It follows that the inclusion $\Sigma_i\subset M$ induces a surjection $H_0(\Sigma_i)\to H_0(M)$. Hence, $i_0:H_0(\Sigma)=\bigoplus H_0(\Sigma_i)\to H_0(M)$ is surjective. Moreover, $H^0(M;E(\rho))\cong g^{\rho(\pi_1(M))}$ and $H^0(\Sigma_i;E(\rho))\cong g^{\rho(\pi_1(\Sigma_i))}$, where g^K denotes the infinitesimal centralizer of K in G. (See appendix.)

If G is a semi-simple Lie group, the Killing form gives a non-degenerate bilinear pairing on g invariant under the adjoint action. This gives a natural identification $g^*\cong g$ inducing an isomorphism of vector bundles $E(\rho)^*\cong E(\rho)$. Combining this with the Kronecker pairing isomorphism (section 14.2) shows that $H_i(Y;E(\rho))$ is canonically dual to $H^i(Y;E(\rho))$. In particular, dimension of H_0 is the same as $\dim H^0$ in the calculation above.

To study the map $i_1:H_1(\Sigma;E(\rho))\to H_1(M;E(\rho))$ we look at the dual map

$$i^*:H^1(M;E(\rho)^*)\to H^1(\Sigma;E(\rho)^*)=\bigoplus H^1(\Sigma_i;E(\rho)^*).$$

Here, $H^1(X;E(\rho)^*)$ represents the cotangent space to $R(\pi_1(X),G)/G$ and the map i^* is induced by restriction of a representation of $\pi_1(M)$ to the subgroups $\pi_1(\Sigma_i)$.

Now we consider the case $G=PSL_2(\mathbb{C})$. Then irreducible representations $\phi:\Gamma\to G$ are determined up to conjugacy by the set of traces $tr(\phi(\gamma)),\gamma\in\pi$. We can also parametrize these representations by the complex lengths $l(A)$ given by $trace(A)=2\cosh(l(A)/2)$, for $A\in SL_2(\mathbb{C})$. These well are defined modulo $2\pi i$. This gives an embedding $R(\Gamma,PSL_2(\mathbb{C}))/PSL_2(\mathbb{C})\to(\mathbb{C}/2\pi i\mathbb{Z})^\Gamma$, $\phi\mapsto(l(\phi(\gamma)))_{\gamma\in\Gamma}$. The differentials $(dl(\gamma))_{\gamma\in\Gamma}$ then span the (Zariski) cotangent space to $R(\Gamma,PSL_2(\mathbb{C}))/PSL_2(\mathbb{C})$, which we identify with $H^1(\Gamma;g_{Ad}\rho^*)$. With these identifications, the map i^* becomes the restriction $(dl(\gamma))_{\gamma\in\Gamma}\mapsto(dl(\Sigma_1),\ldots,dl(\Sigma_n))$.

Thus, we are reduced to the geometric problem of understanding the variation of complex lengths of the curves Σ_i as the representation $\pi_1(M)\to G$ is deformed.

In particular, if the lengths of $\Sigma_1,\ldots,\Sigma_n$ can be varied independently, then i^* is surjective, and i_1 is injective. Then $H^1(M-\Sigma;E(\rho))\cong H^1(M;E(\rho))$. This is the generic, uninteresting case: every representation of $\pi_1(M-\Gamma)$ near ρ factors through a representation of $\pi_1(M)$.

Otherwise, there is some relation between the differentials $dl(\Sigma_i)$: $\sum_i a_i dl(\Sigma_i)=0$ at ρ. This imposes a non-trivial geometric condition on ρ, which we will study in chapter 4, in some special cases.

Summarizing this discussion we have

Proposition 3.27. *Let $\Sigma \subset M$ be a link with components Σ_i, $i = 1, \dots, n$. Let $\rho : \pi_1(M-\Sigma) \to PSL_2C$ be a representation factoring $\pi_1(M)$ as above. Then $H^1(M-\Sigma; E(\rho)) \neq H^1(M; E(\rho))$ if and only if there is a relation between the differentials of the complex lengths of Σ_i at ρ.*

□

CHAPTER 4

Examples

18. Introduction

Suppose we have a family of hyperbolic structures on a 3-manifold M with developing maps $D_t : \tilde{M} \to X = \mathbf{H}^3$ converging to a submersion $D_0 : \tilde{M} \to X_0 \subset \mathbf{H}^3$. We can classify the possibilities according to the submanifold X_0.

The limiting holonomy is a homomorphism $\rho_0 : \pi_1(M) \to G_0 \subset G = PSL_2(\mathbf{C})$ where G_0 is the subgroup of $PSL_2(\mathbf{C})$ preserving X_0. If G_0 is non-trivial, then it is easy to see that X_0 is either a totally geodesic subspace of $\mathbf{H}^3$ or a horosphere. The possibilities are the following:

(1) X_0 = point, $G_0 = SO(3)$.

(2) X_0 = geodesic, $G_0 = \mathbf{C}^*$.

(3) $X_0 = \mathbf{H}^2$, $G_0 = PSL_2\mathbf{R}$.

(4) X_0 = horosphere, G_0 = affine group.

We discuss examples of degeneration in each of the categories (1)-(3). In our examples we begin with a closed 3-manifold or orbifold M having a non-hyperbolic geometric structure and a link Σ in M. We find hyperbolic structures on $M-\Sigma$ with Dehn surgery type singularities degenerating to a foliation.

For (1), we begin with a Euclidean orbifold; for (2) a solv manifold; for (3) a Seifert fibre space with hyperbolic orbifold as base, having an $SL_2\tilde{\mathbf{R}}$ or $\mathbf{H}^2 \times \mathbf{R}$ geometric structure.

Remarks:

For a non-degenerate hyperbolic structure the Zariski closure of $\rho(\Gamma)$ is $PSL_2(\mathbf{C})$. (Otherwise, one of the cases (1)-(4) applies and the volume of the representation ρ would be zero;

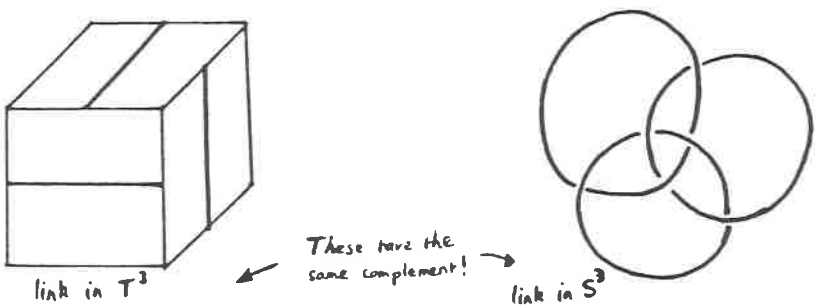
compare chapter 5.) In particular, this implies that the centralizer of ρ is trivial, so proposition 3.14 applies.

In each case, a non-hyperbolic geometric structure on M arises as a kind of limit of the hyperbolic structures on $M-\Sigma$: it can be reconstructed from the way in which the hyperbolic structures degenerate. In [Ho] we introduce a concept of deformation of geometries to study such changes.

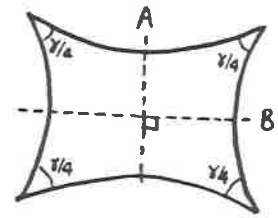
19. Geometric structures on the Borromean rings complement

We first consider an example where degenerating hyperbolic structures can be constructed explicitly by gluing together hyperbolic polyhedra.

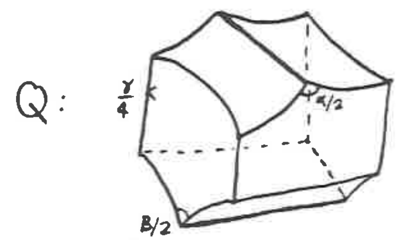
Let $M=S^1\times S^1\times S^1$ be a 3-torus and let Σ consist of three simple closed curves in orthogonal directions, say $S^1\times p\times q$, $q\times S^1\times p$, $p\times q\times S^1$ where p, q are distinct points in S^1. (Then $M-\Sigma$ is diffeomorphic to the Borromean rings complement.)



We begin with a hyperbolic structure on the once punctured torus F with a cone angle γ at the puncture. Assume that F is "rectangular": with the two horizontal curves of Σ projecting to geodesics in F meeting at an angle of 90°. (We will see below (theorem 4.4) that the hyperbolic foliation determined by projection $M\to F$ can only be deformed if this condition holds.)

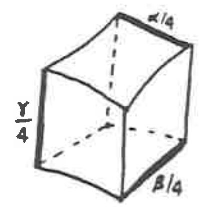


For such a punctured torus, there are generators a, b for $\pi_1(F)$ whose holonomies preserve two orthogonal geodesics A, B in H^2. Now bend H^2 up along A and down along B to give a polyhedron Q as shown below, with angle $\frac{1}{2}\alpha$ along A and $\frac{1}{2}\beta$ along B.



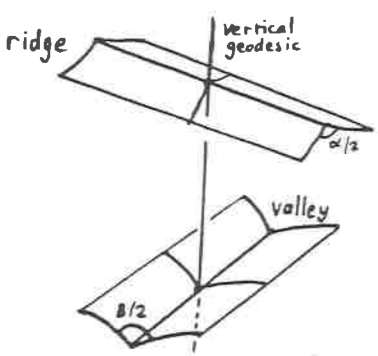
Such a polyhedron can be constructed whenever α, β are less than 2π. For example, the polyhedron can be obtained from four copies of a cube C having dihedral angles as shown below; such a hyperbolic cube exists by a theorem of Andreev [An].

C :



(other dihedral angles = $\frac{\pi}{2}$)

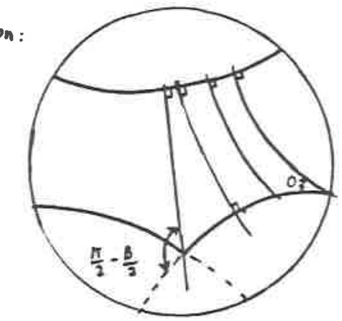
Here is a direct construction of Q, which also gives some idea of the variation in the shape of Q as the angles α, β, γ are varied. Begin with a "vertical" geodesic in $\mathbf{H}^3$ and two pairs of planes: one pair meeting in a ridge line with angle $\alpha/2$, the other meeting along a valley line with angle $\beta/2$. Further, the geodesics defining the ridge and the valley should be orthogonal to the vertical geodesic and to each other (after vertical translation!).



There is a one parameter family of planes orthogonal to the ridge meeting one side of the valley; with angles of intersection varying monotonely from 0 to $\pi/2-\beta/2$. Hence, there is a

unique such plane for which the angle is $\pi/2$.

cross section:



Similarly, we find three other vertical planes; by symmetry these meet at the same angle γ. Finally, it is easy to see that the angle γ varies from $\pi/2$ to 0 as the vertical distance from the valley to ridge varies from 0 to ∞.

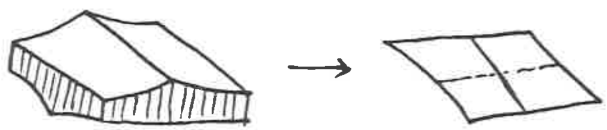
The polyhedron Q can be glued up to give a hyperbolic structure on $F\times I$ bent along a on top and bent along b on the bottom, with a vertical axis with cone angle γ. Let $P(\alpha, \beta; \gamma)$ denote this hyperbolic structure on $F\times I$.

By doubling $P(\alpha, \beta; \gamma)$ we obtain a hyperbolic cone manifold structure on M with cone angles α, β, γ along the components of Σ.

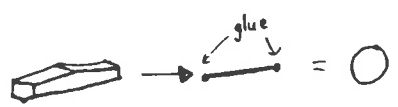
Proposition 4.1. *There is a 3-dimensional manifold of hyperbolic cone manifold structures on T^3 parametrized by cone angles $(\alpha, \beta, \gamma)\in[0, 2\pi)^3$.*

Further, these structures degenerate as any angle approaches 2π.

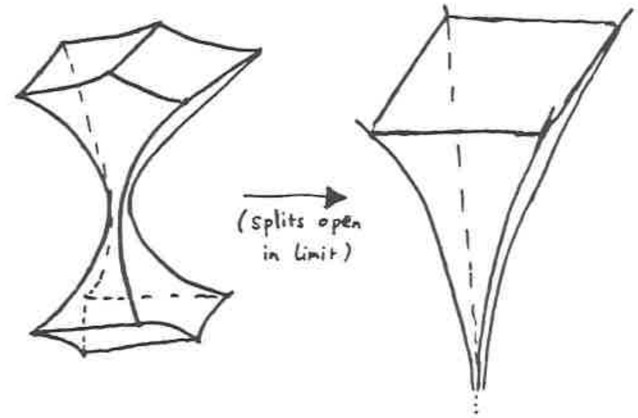
(a) As α, $\beta \to 2\pi$, with $\gamma<2\pi$ fixed, the polyhedra flatten out to give limiting transverse hyperbolic foliations. (In fact, we get $\mathbf{H}^2\times\mathbf{R}$ or $SL_2\mathbf{R}$ geometry structures.)



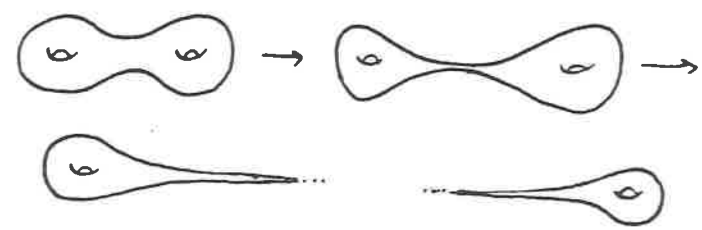
(b) As $\alpha, \beta, \gamma \to 2\pi$, the geometric limit can be either a point, circle or a line, depending on the exact mode of convergence. In each case there is a limiting Euclidean structure.



(c) If $\gamma \to 2\pi$ with $\alpha, \beta < 2\pi$ fixed, then the hyperbolic structures have diameter going to infinity, while two tori (with cone angle γ) become smaller and smaller, looking more and more like Euclidean tori as $\gamma \to 2\pi$. The limiting polyhedra give a hyperbolic structure on the manifold obtained from T^3–two horizontal curves by splitting open along two incompressible tori.

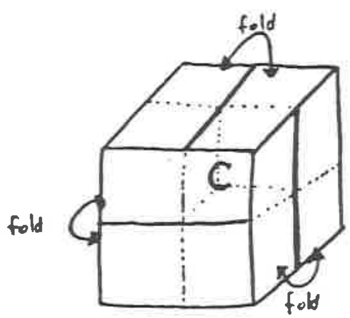


(The process is a three-dimensional analogue of the process of creating a cusp in a hyperbolic surface by pinching a curve to a point, in going to the boundary of Teichmüller space.)



Remarks:

(1) Similarly, one can obtain cone manifold structures on S^3 with the Borromean rings as the singular locus, with arbitrary cone angles $0<\alpha, \beta, \gamma<\pi$. This can be done by taking 8 copies of the cube C, and gluing them up as follows.



The kinds of degeneration occurring are analogous to those described above. In the case corresponding to (c), there is splitting along an incompressible, Euclidean 4-punctured sphere (with cone angles π).

(2) We can also start with different hyperbolic structures on two copies of $F\times[0,1]$. Then we can reglue after twisting along the bending curves a and b by arbitrary distances $t_a, t_b \in \mathbf{R}$ to obtain $\mathbf{H}^3$ and $\mathbf{H}^2\times\mathbf{R}$ structures with Dehn surgery type singularities about the three meridians for Σ. (With holonomies $(\alpha, t_a), (\beta, t_b), (\gamma, t_c) \in \mathbf{R}^2$, where $t_a, t_b \neq 0$.) However, the hyperbolic structures on the pieces $F\times[0,1]$ must be carefully chosen so that they fit together, to give a hyperbolic structure on M.

When shearing type singularities are allowed it is possible to obtain :

(e) geometric limit = point or circle, with limiting Nil geometry structure

(f) geometric limit = circle, with limiting solv geometry structure.

It may be possible to obtain a complete description of the possible limiting geometric structures arising from degenerating hyperbolic structures on the Borromean rings using this approach. (At least for the component of hyperbolic structures containing the complete hyperbolic structure.) But we will not discuss the most general case here.

20. Deforming Euclidean orbifolds In this section we apply the theory we have developed to the problem of deforming Euclidean orbifold structures to hyperbolic and spherical structures. The results are summarized in the following theorem.

Theorem 4.2. *Let Q be a 3-dimensional Euclidean orbifold with singular locus Σ and $M = Q - \mathrm{int}N(\Sigma)$. Let $\rho:\Gamma = \pi_1(M) \to SO(3)$ be the rotational part of the holonomy of Q.*

Assume that Q does not fibre over a circle with a Euclidean orbifold as fibre. Then the representation spaces $R(\Gamma, \mathrm{isom}(\mathbf{E}^3))$, $R(\Gamma, \mathrm{isom}(\mathbf{H}^3))$ and $R(\Gamma, \mathrm{isom}(S^3))$ are manifolds near ρ of real dimension $k+6 = \frac{1}{2}\dim H^1(\partial M; E(\rho))+6$. Moreover, each representation near ρ is determined by its restrictions to the fundamental groups of the boundary components of M.

If Q is 3-dimensional, there are k-(real) dimensional manifolds of hyperbolic and spherical structures on the underlying space of Q with cone like singularities along Σ, and $2k$ dimensional families of Euclidean, spherical and hyperbolic structures with Dehn surgery type singularities along Σ whenever Σ is a link. Moreover, the hyperbolic and spherical structures are determined locally by the cone angles or generalized Dehn surgery coordinates, while the Euclidean structures are determined up to rescaling by these coordinates and form a codimension one subspace of the space of Dehn surgery coordinates.

Proof. By theorem 3.17 (and 3.16), each class in $im(H^1(M, \partial M) \to H^1(M))$ (with Adp coefficients) is represented by a pair of parallel T_*M-valued 1-forms on M (which extend over Q). Such a form is completely determined by a linear map $f: T_mM \to T_mM$. There is always a trivial 1-dimensional family of such T_*M-valued forms corresponding to multiples of the identity map. (Geometrically these correspond to changing the Euclidean structure on M by rescaling the metric.) If ω is a non-trivial form its kernel $ker\,\omega$ or the orthogonal complement of $ker\omega$ gives a 2-plane field M tangent to a codimension-1 foliation of M. Then, we can use the following topological result.

Proposition 4.3 (Tischler[Ti]). *If a compact manifold M admits a codimension-1 foliation defined by a closed 1-form, then M fibres over the circle S^1.*

Proof. (Outline) Approximate the 1-form by a (nowhere vanishing) form ω' representing a rational cohomology class (i.e. in $H^1(M;\mathbf{Q})\subset H^1(M;\mathbf{R})$). Fix a base-point * in M and integrate ω' along paths starting at *. Integrating along different paths with the same endpoint changes the result by a "period" of ω': $\int_\gamma \omega'$ where γ is a closed curve in M. But the last integral is a rational number depending only on the homology class in $H_1(M;\mathbf{Z})$. So there is an integer multiple ω'' of ω' all of whose periods are integers. Then integrating ω'' gives a well-defined map $M\to\mathbf{R}/\mathbf{Z}=S^1$ which defines the desired fibration. □

Moreover, in our situation the same argument can be carried out equivariantly on the universal cover of Q, giving a fibration of Q such that the induced metric on each fibre is Euclidean.

If Q does not fibre over S^1 then we have $\mathrm{im}(H^1(M,\partial M;E(\rho))\to H^1(M;E(\rho))=0$, $\mathrm{im}(H^2(M,\partial M;E(\rho))\to H^2(M;E(\rho))=0$ and $\dim H^1(M)=2k$. So by theorem 3.14, we conclude that $R(\Gamma,G)$ is a manifold of dimension $2k+6$ near ρ, and representations near ρ are parametrized locally by complex angles or generalized Dehn surgery coordinates. (Here $G=\mathrm{isom}(\mathbf{E}^3)$), $\mathrm{isom}(\mathbf{H}^3)$) or $\mathrm{isom}(S^3)$).)

Now, the representations near ρ have the property that their "translational" parts are multiples of the Euclidean translational distances to first order (by 3.16). It follows from chapter 2, section 7, that there are hyperbolic and spherical geometric structures near the Euclidean structure corresponding to all representations sufficiently close to ρ. (This can also be seen directly using the geometric convergence of the enlarged hyperbolic and spherical manifolds to the Euclidean structure.) Moreover, if Σ is a link then all the nearby structures have generalized Dehn surgery type singularities (by 1.9). The k-dimensional subset of $R(\Gamma,G)/G$ consisting of representations such that each meridian is mapped to a pure rotation corresponds to cone-manifold structures near the original Euclidean structure. (This also holds if Σ is a graph, using an argument similar to that in 1.9. Compare [Th3].) □

Remark : This proof is essentially due to Thurston and extends to the case where Q is a Euclidean cone manifold with cone angles $\leqslant\pi$. This stronger result is used in the proof of Thurston's orbifold theorem [Th3].

21. Deforming Seifert fibre spaces

Let M be a Seifert fibre space over a 2-dimensional orbifold F, and let $p:M\to F$ be the projection. Then the fibres of p give a foliation F of M by circles. Choosing a geometric structure F_0 on F defines a transverse geometric structure for the foliation. Let $\Sigma\subset M$ be a link and let (F,F_0) denote the foliation with transverse geometric structure restricted to $M-\Sigma$. We want to investigate when (F,F_0) can be deformed to geometric structures on $M-\Sigma$.

Assume now that F admits a hyperbolic structure F_0. Then we begin with a holonomy representation $\rho:\pi_1(F)\to PSL_2(\mathbf{R})$ for the hyperbolic structure F_0. Here, $\pi_1(F)$ denotes the fundamental group of F in the orbifold sense: ρ gives an isomorphism onto a discrete subgroup of $PSL_2\mathbf{R}$ and $\mathbf{H}^2/\rho(\pi_1(F))$ is a hyperbolic orbifold, with cone points corresponding to the exceptional fibres of M. Assume that M and F are orientable. Then $\pi_1(M)$ is a extension :

$$1\longrightarrow \mathbf{Z}\longrightarrow \pi_1(M)\longrightarrow \pi_1(F)\longrightarrow 1$$

where $\mathbf{Z}$ is the central normal subgroup generated by a non-singular fibre in M. Every representation $\phi:\pi_1(F)\to G=PSL_2(\mathbf{C})$ near ρ has trivial centralizer, so the only extension of ϕ to a representation $\pi_1(M)\to G$ is the trivial one : with $\phi(\mathbf{Z})=1$. Hence, $R(\pi_1(M),G)=R(\pi_1(F),G)$ near ρ, and $H^1(M;g)=H^1(F;g)\cong\mathbf{C}^N$, where N is the dimension of the Teichmüller space of F.

Let Σ project to closed curves $C_1,\ldots,C_n$ on F. Now it follows from proposition 3.27 that $H^1(M-\Sigma;g)\neq H^1(M;g)$ if and only if there is a relation between the differentials restricted to $R(M,PSL_2(\mathbf{R}))$:

$$0=\sum_i a_i dl(C_i) \qquad (*)$$

where $a_i \in \mathbb{R}$. If $(*)$ holds, then the sum $\sum_i a_i l(C_i)$ defines a function on the Teichmüller space of F having a critical point at the hyperbolic structure corresponding to ρ.

Suppose that the curves C_i such that $a_i \neq 0$ *fill up* the orbifold F, so each component of $F-\bigcup_{a_i\neq 0} C_i$ is a disc (containing at most one cone point). If each a_i has the same sign, say $a_i>0$, then the work of Kerckhoff [K1] shows that ρ is the unique hyperbolic structure on F such that $\sum_i a_i l(C_i)$ is minimized. For such ρ, we have $H^1(M-\Sigma\,;g)\neq H^1(M\,;g)$

Further, if the curves C_i such that $a_i\neq 0$ don't fill up the surface F then any hyperbolic structure on F can be deformed so that the length of each C_i increases. So $(*)$ has no solutions in this case.

Say that a representation $\pi_1(M-\Sigma)\to G$ is *trivial* if it factors through $\pi_1(M)$. Then summarizing this discussion, we obtain the result:

Theorem 4.4. *A hyperbolic foliation (F, F_0) as above can be deformed to hyperbolic structures on M with Dehn surgery type singularities along Σ only if*

(1) the projection $\bigcup C_i$ of Σ to F fills up F.

(2) $\sum_i a_i l(C_i)$ has a critical point, at the hyperbolic structure F_0, for some $a_i \in \mathbb{R}$. If all $a_i>0$ then there is a unique such hyperbolic structure and the function is minimized at this point.

In fact, the holonomy representation $\pi_1(M-\Sigma)\to\pi_1(M)\to\pi_1(F)\xrightarrow{\rho} PSL_2\mathbb{R}$ can not be approximated by non-trivial representations $\pi_1(M-\Sigma)\to PSL_2\mathbb{C}$ unless conditions (1) and (2) hold.

□

Remarks:

(1) If Σ has at most two components, then critical points of $\sum_i a_i l(C_i)$ only occur when all the a_i have the same sign, say $a_i>0$. This is clear in the case of one component. For the case of two components, it follows from a result of Kerckhoff [K2] that if two earthquake paths are tangent at a point then their defining measured laminations coincide. For if $c_1, c_2>0$ then $c_1dl(C_1)=c_2dl(C_2)$ are tangent vectors to earthquakes along c_1C_1 and c_2C_2. Hence by Kerckhoff's result $C_1=C_2$ and $c_1=c_2$. Further, in the case of two components the set of all critical points of $\sum_i a_i l(C_i)$ as $a_i>0$ vary form an properly embedded copy of $\mathbb{R}$ in T (called a "line of minima" by Kerckhoff).

When Σ has more than two components, then there are critical points of $\sum_i a_i l(C_i)$ with not all A_i of the same sign. Examples can be easily be constructed in the case of a punctured torus. (Compare example 4.9.)

(2) The condition that geodesics must fill up F also follows from the fact that $M-\Sigma$ must be atoroidal when M has a hyperbolic structure with cone type singularities along Σ. (This follows from the fact that $M-\Sigma$ admits a complete metric of negative curvature; obtained by modifying the hyperbolic metric near Σ.)

(3) It would be useful to have a direct geometric argument showing that a linear combination of lengths of C_i must be critical.

We now begin to consider the converse question : are these sufficient for approximation by hyperbolic structures?.

Lemma 4.5. *Given a link $\Sigma\subset M$ and a representation ρ satisfying (1), (2) there is a family of representations $\rho_t:\pi_1(M-\Sigma)\to PSL_2\mathbb{C}$ with $\rho_0=\rho$ and ρ_t non-trivial for $t>0$: $\rho_t(m)\neq 1$ for m any meridian of Σ.*

Proof. The space R of representations factoring through $\pi_1(M)$ locally form a complex manifold of dimension equal to $\dim H^1(M; Ad\rho)$. Since $\dim H^1(M-\Sigma; Ad\rho) > \dim H^1(M; Ad\rho)$, there are representations of $\pi_1(M-\Sigma)$ which don't factor through $\pi_1(M)$. (The Zariski tangent space agrees with the usual tangent space for a complex algebraic manifold.)

Hence there is a curve ρ_t, $t \geq 0$, of such representations and meeting R in a single point (e.g. by Milnor's curve selection lemma [Mi2]). By construction, these cannot be trivial on any meridian for $t \neq 0$ □

Remark 4.6.

Given a family of representations as above, there is a corresponding family of (degenerating) hyperbolic structures provided the conditions $REGEN1$ and $REGEN2$ of section 9 are satisfied. (In fact it is not hard to see that it suffices to check $REGEN2$ in our situation.)

Often, when studying Dehn surgery spaces, the condition on axes can be immediately verified as follows. Suppose that we have a family of hyperbolic structures on a manifold given by gluing together positively oriented ideal hyperbolic simplices. If all the simplices simultaneously flatten out, with cross ratios becoming real, then there is a limiting hyperbolic foliation obtained as in example 2.8(d) (or [Th1, chap4]). So we have a family of hyperbolic structures degenerating to a hyperbolic foliation. If the limiting foliation occurs at a manifold point on the boundary of hyperbolic Dehn surgery space, then the foliation is non-singular. Then a result of Thurston, [Th1, chap.4] show that the Dehn surgered manifold is either a Seifert fibre space or a torus bundle over S^1 with Anosov monodromy. Further the foliation has the form described in example 2.8(b) or 2.8(c) of chapter 2.

In practice this situation occurs quite frequently among the Dehn surgery spaces for complements of knots in S^3, at least when the number of crossings is small. (See Weeks [We] for detailed information on the Dehn surgery spaces for knots with up to 9 crossings.) The case of the figure eight knot complement is considered in detail in chapter 6, and in [Th1, chap.4].

In general the behaviour of the axes depends delicately on the geometry of the situation. In the next sections, we will consider some special cases where the the movement of axes seems easier to understand. We hope to return to the general case in the future.

In section 22, we construct families of hyperbolic foliations with Dehn surgery type singularities on the products $F \times S^1$, where F is a hyperbolic surface. In section 23, we give some examples of hyperbolic structures with Dehn surgery singularities on $F \times S^1$ degenerating to foliations.

22. Hyperbolic foliations on Surface × circle

In this section we construct some hyperbolic foliations and $\mathbf{H}^2 \times \mathbf{R}$ structures with Dehn surgery type singularities on products of hyperbolic surfaces with S^1.

Let F be a closed hyperbolic surface, $M = F \times S^1$ and Σ a link in M consisting of horizontal simple closed curves $\Sigma_i = C_i \times h_i$, at "heights" satisfying $0 = h_0 < h_1 < \ldots < h_{n-1} < 1 = h_n$ in $\mathbf{R}$ mod $\mathbf{Z} = S^1$. We assume that each $C_i \subset F$ is essential : not contractible and not boundary parallel.

We try to construct $\mathbf{H}^2 \times \mathbf{R}$ structures on $M-\Sigma$ (with Dehn surgery singularities) as follows. Divide M up into slabs $S_i = F \times [h_i, h_{i+1}]$; then S_i meets Σ in two curves : a copy of C_i on top and a copy of C_{i-1} on bottom. Begin with a hyperbolic structure F_0 on F; this gives a $\mathbf{H}^2 \times \mathbf{R}$ structure on S_0. Now form a new hyperbolic structure F_1 on F by beginning with F_0 and twisting (to the left) by a distance d_1 along the geodesic $\tilde{C}_1$ homotopic to C_1 before regluing. (This change is a Fenchel-Nielsen deformation or earthquake along $\tilde{C}_1$.) Now the hyperbolic structures on $F_0 - \tilde{C}_1$ and $F_1 - \tilde{C}_1$ are isometric; we use such an isometry to glue the slabs S_0, S_1 together along $F - \tilde{C}_1$. This gives a $\mathbf{H}^2 \times \mathbf{R}$ structure on $S_0 \cup S_1$ with a shear type singularity along Σ_1. The holonomy around a meridian for Σ_1 is a (hyperbolic) translation by distance d_1 in the direction of $\tilde{C}_1$.

By repeating this construction we obtain a $\mathbf{H}^2\times\mathbf{R}$ structure on $F\times[0,1]$. On the boundary we have two hyperbolic structures F_0 and F_n. We obtain a $\mathbf{H}^2\times\mathbf{R}$ structure on $F\times S^1$ if and only if these structures agree. The change of structures going from F_0 to F_n is a composition of earthquakes $E_{C_i}(d_i)$, $i=1,\ldots,n$, where $E_C(d)$ denotes a left earthquake by distance $d\in\mathbf{R}$ along a simple closed geodesic (homotopic to) C. We call the composition $h=E_{C_n}(d_n)\circ E_{C_{n-1}}(d_{n-1})\circ\ldots\circ E_{C_1}(d_1)$ the holonomy for the $\mathbf{H}^2\times\mathbf{R}$ structure on $F\times I$. This holonomy should be regarded as a map on the Teichmüller space T of hyperbolic structures on F.

It is a classical result that T is an open ball diffeomorphic to $\mathbf{R}^N$, where $N=-3\chi(F)$. Thurston (see [Th4, [FLP]) has shown that T can be compactified to obtain a closed ball $\overline{T}=T\cup PM$ by adding a sphere PM consisting of "projective measured foliations" on F. Further, each $E_{C_i}(d_i)$ gives a smooth (actually analytic) map $T\to T$ which extends to a continuous map $\overline{T}\to\overline{T}$. Hence, the holonomy gives a continuous map $h:\overline{T}\to\overline{T}$. By the Brouwer fixed point theorem this map has a fixed point $x\in\overline{T}$.

If $x\in T$, this defines a hyperbolic structure F_0 on F which is isometric to the structure on F_n given by the construction above. Hence we obtain a $\mathbf{H}^2\times\mathbf{R}$ structure on M with Dehn surgery type singularities along Σ, such that the holonomy of a meridian of Σ_i is a translation by distance d_i.

If $x\in PM$ then there is a foliation of F " preserved by h" with transverse measure "multiplied by a constant under h". In particular, this gives a topological foliation of $F\times S^1$ with Dehn surgery type singularities along Σ and some vertical curves corresponding to singular points of the foliation of F.

In general, it is not clear whether h will have a single fixed point in T; however this can be seen easily in the case where F is a punctured torus. (Then $h\in SL_2\mathbf{R}$ acts on $T\cong\mathbf{H}^2$ by hyperbolic isometries. See example 4.9 below.) However, we can show that the fixed points of h vary smoothly with h using the implicit function theorem.

We begin with a non-singular $\mathbf{H}^2\times\mathbf{R}$ structure on $F\times S^1$, given by a hyperbolic structure F_0 on F crossed with S^1. We investigate when this can be deformed to obtain a family of $\mathbf{H}^2\times\mathbf{R}$ structures on $F\times S^1-\Sigma$ with prescribed shear type singularities with a translation by c_it as the holonomy of a meridian for Σ_i.

The "gluing map" h_t for such a structure is a composition of earthquakes $E_{C_i}(c_it)$, $i=1,\ldots,n$. Now Teichmüller space T can be parametrized by lengths of geodesics on F. This gives an embedding $l:T\to\mathbf{R}_+^\Gamma$, taking each hyperbolic structure to the set of lengths $(l(\gamma))_{\gamma\in\Gamma}$, where $l(\gamma)$ is the length of the closed geodesic in the homotopy class $\gamma\in\Gamma=\pi_1(M)$. We will now identify T with its image in $\mathbf{R}_+^\Gamma$. We want to find a smooth path $(l_t(\gamma))\in T$ satisfying the "gluing equation" $h_t(l_t(\gamma))=l_t(\gamma)$, for all $\gamma\in\Gamma$, $t\geqslant 0$, with $l_0(\gamma)=l(\gamma)$.

A first necessary condition is that

$$\frac{d}{dt}h_t(l_t(\gamma))=\frac{d}{dt}(E_{C_n}(c_nt)\circ\ldots\circ E_{C_1}(c_1t)(l(\gamma)))$$

for all $\gamma\in\Gamma$. Using the chain rule the derivative at $t=0$ is given by :

$$0=\sum_{i=1}^{n}c_i\frac{d}{dt}E_{C_i}(t)(l(\gamma)). \qquad (*)$$

The work of Kerckhoff [K1] or Wolpert [Wo], shows that the first variation of geodesic lengths under an earthquake along C is given by :

$$\frac{d}{dt}E_C(t)(l(\gamma))=\sum_{p\in C\cap\gamma}\cos(\theta_p(\gamma\to C))$$

where $\theta_p(\gamma\to C)$ denotes the positive (counterclockwise) angle from the geodesic representing γ to the geodesic representing C at an intersection point p. In particular, this shows that

$$\frac{d}{dt}E_{C_i}(t)(l(\gamma))=-\frac{d}{dt}E_\gamma(t)(l(C_i)),$$

so $(*)$ can be rewritten in the form :

$$0=\sum_{i=1}^{n}c_i\frac{d}{dt}E_\gamma(t)(l(C_i)) \qquad (**)$$

for all $\gamma \in \Gamma$. This is exactly the condition that the function $L = \sum_{i=1}^{n} c_i l(C_i)$ has a critical point at the initial hyperbolic structure F_0 (since the tangent vectors to earthquakes along simple closed curves in F span the tangent space to Teichmüller space.)

Now assume that all the c_i have the *same sign*, say $c_i > 0$. (Deformations are sometimes possible when this fails, see example 4.9 below.) Then Kerckhoff [K1] has shown that L has a unique critical point F_0 in T, giving a global minimum of L. Moreover, the same proof (or Wolpert's second derivative formulae [Wo]) shows that L has a *non-degenerate* critical point at F_0 (i.e. the second derivative of L has an invertible matrix).

Using this fact we can apply the implicit function theorem to conclude that the equations $h_t(l_t(\gamma)) = l_t(\gamma)$ have a unique solution $l_t(\gamma)$, $\gamma \in \Gamma$, depending smoothly on t, for all t near 0.

We can write $h_t = \exp(tX_t)$, where X_t is a smooth family of vector fields on T. Then for $t \neq 0$, $x_t \in T$ is a fixed point of h_t if and only if x_t is a zero of the vector field X_t. Now a zero of $X_0 = \frac{\partial}{\partial t} h_t |_{t=0}$ is exactly a critical point of L. By the work of Kerckhoff [K1], there is a unique such point $x \in T$ whenever C_i fill up F and all $c_i > 0$. Further, x is a non-degenerate critical point of L (with positive definite Hessian) by the work of Kerckhoff [K1] or Wolpert [Wo]. Hence, x is a non-degenerate zero of X_0 and it follows by the implicit function theorem that there is a smooth family x_t of zeroes of X_t for $t \in \mathbb{R}$ near 0, with $x_0 = x$. Then x_t is also a fixed point of h_t so we obtain the following result.

Theorem 4.7. *Assume $c_i > 0$. Then an $\mathbb{H}^2 \times \mathbb{R}$ structure $F_0 \times S^1$ can be deformed to obtain a family of $\mathbb{H}^2 \times \mathbb{R}$ structures on $F \times S^1 - \Sigma$, with $c_i t$-shear singularities along Σ, (for t near 0) if and only if F_0 is the unique hyperbolic structure on F such that $L = \sum_{i=1}^{n} c_i l(C_i)$ is minimal.*

□

Remark 4.8. Let $\rho_t : M - \Sigma \to PSL_2\mathbb{R}$ be the holonomy representations for the associated family of hyperbolic foliations. Then if f is the homotopy class of a fibre $point \times S^1$ then $\rho_t(f)$ is a positive rotation or each $t > 0$. This can be seen as follows. Let γ be any horizontal curve in the slab S_0. Then $\rho_t(f\gamma f^{-1}) = \rho_t(m_1 \ldots m_k \gamma)$ where m_i are certain meridians corresponding to points of intersection of γ with $\cup C_i$ in F. In particular, $k \neq 0$, and since each $\rho_t(m_i)$ is a translation to the left, it follows that the endpoints of the axis of $\rho_t(f\gamma f^{-1})$ lie to the left of the endpoints for $\rho_t(\gamma)$. Since this holds for every γ, it follows that $\rho_t(f)$ moves all points on the circle at infinity to the left; so $\rho_t(f)$ is a positive rotation.

Remarks:

A similar construction works for a punctured surface. In this case it is also useful to allow hyperbolic structures with cone points instead of cusps. In general the foliations will have additional singularities along vertical circles corresponding to the cusps.

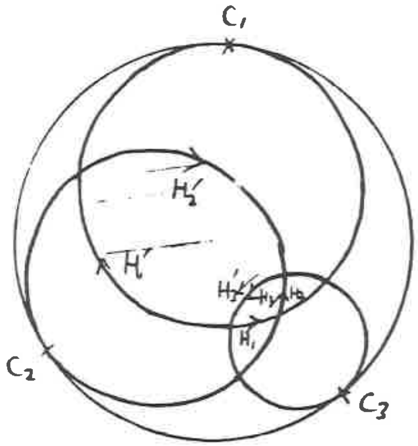
There is an analogous construction giving Euclidean structures with Dehn surgery type singularities in the case where F is a torus.

Example 4.9.

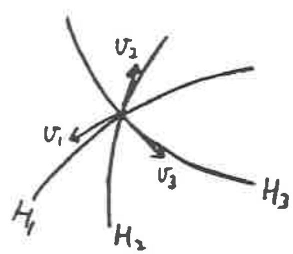
When F is a once punctured torus, the space of hyperbolic foliations on $F \times S^1$ with shear singularities along Σ can be understood globally. (Further, exactly the same approach applies in the case where F is a torus, to describe the Euclidean structures on $T^2 \times S^1$ with shear type singularities along Σ.)

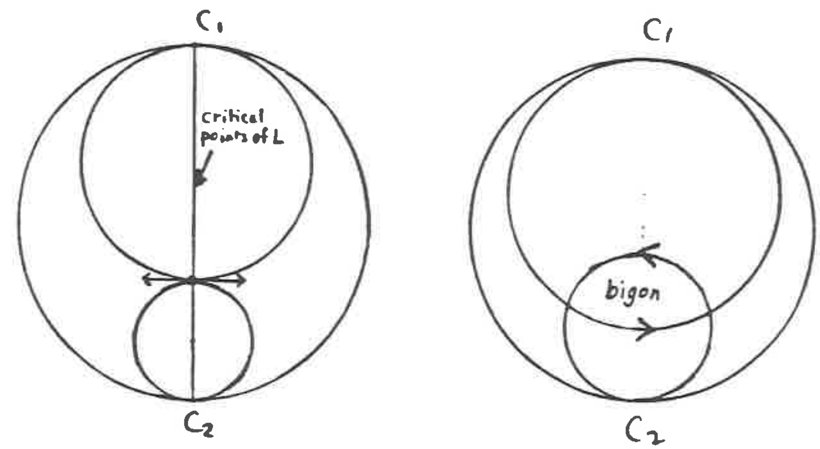
In this case the Teichmüller space of F can be identified with $\mathbb{H}^2$ so that a twist deformation about a simple closed curve C in F corresponds to a parabolic isometry of $\mathbb{H}^2$. The fixed point in the circle at infinity is a "rational" point determined by the slope of the curve C. It will be convenient to label this point of S^1 by C.

Let $h : T = \mathbb{H}^2 \to \mathbb{H}^2$ be the holonomy for the $\mathbb{H}^2 \times \mathbb{R}$ structure. $h = E_{C_n}(d_n) \circ E_{C_{n-1}}(d_{n-1}) \circ \cdots \circ E_{C_1}(d_1)$. Then h is a hyperbolic isometry so there is at most one solution in $\mathbf{H}^2$ to the gluing equation $h(x) = x$. Further, the solutions as h varies correspond exactly to closed cycles made up of arcs of horocycles $H_1, \ldots, H_n$ about the points of S^1 corresponding to the curves $C_1, \ldots, C_n$.

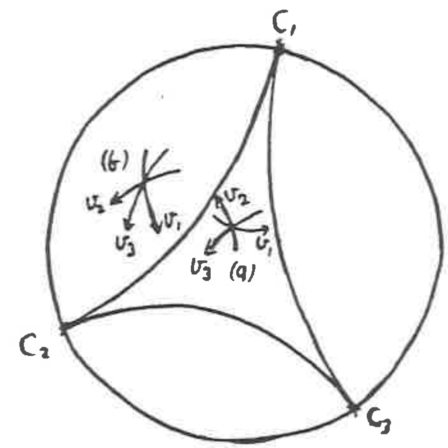


Moreover, critical points of $L = \sum_{i=1}^{n} c_i l(C_i)$ correspond to intersections of horocycles $H_1, \ldots, H_n$ such that some linear combination of tangent vectors v_i to H_i vanishes. (This follows since $0 = \sum_{i=1}^{n} c_i \frac{d}{dt} E_{C_i}(t)(l(\gamma)) = -\sum_{i=1}^{n} c_i \frac{d}{dt} E_\gamma(t)(l(C_i))$, as discussed above.) In particular, critical points of $a_1 l(C_1) + a_2 l(C_2)$ are exactly the points on the geodesic joining the points corresponding to C_1, C_2 on the boundary of $\mathbf{H}^2$. Solutions to the gluing equations are given by bigons as shown below.

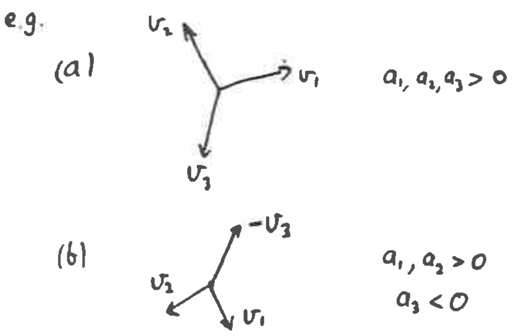




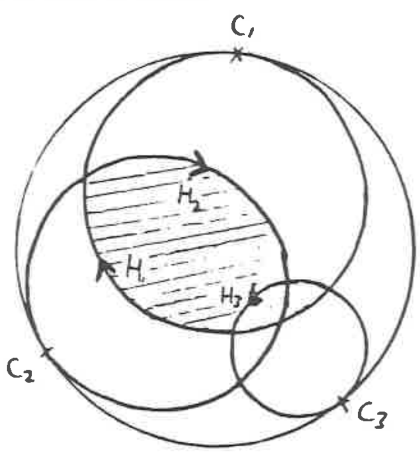
When Σ has three components then every point of $\mathbf{H}^2$ lies on exactly one horocycle through the points C_1, C_2, C_3. The tangent vectors v_i to the horocycles are linearly dependent so satisfy $\sum_i v_i = 0$. Orienting v_i consistently, we see that the signs of a_i are all the same only if the point x lies in the triangle in $\mathbf{H}^2$ with vertices $C_1, C_2, C_3 \in S^1$.



e.g.



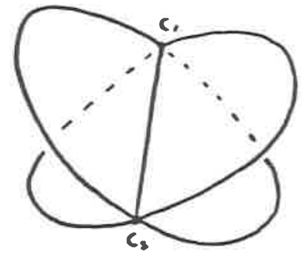
Similarly, one can see that there are global solutions to the gluing equations with coefficients not all the same sign. We illustrate this below.



A solution with $a_1 < 0$, $a_2 < 0$, $a_3 > 0$.

It is not hard to give a global description of the space of solutions to the gluing equations, at least when the number of components of Σ is small. For example, for two components the

space consists of two copies of $\mathbf{H}^2$ meeting along the geodesic joining the points C_1, C_2. (One component consists of non-singular structures, the other one corresponds to hyperbolic foliations along Σ.)



It is fun to find the space of hyperbolic foliations for the case of 3 components. We leave this as an exercise for the reader.

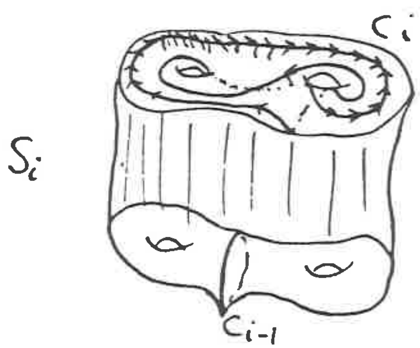
Remarks : The hyperbolic foliations in the punctured torus case will generally have additional vertical singularities along $\partial F \times S^1$. Further hyperbolic foliations can be obtained using hyperbolic structures on the torus with a single cone point, instead of a cusp.

It is also instructive to study the kinds of degeneration that occur approaching the boundary of the space of hyperbolic foliations.

This construction can be used to "explain" some of the planes of hyperbolic foliations observed on the boundary of hyperbolic Dehn spaces. A foliation with a shear singularity along C with translation distance equal to the length of C gives a non-singular hyperbolic foliation on a manifold obtained by Dehn surgery along C. (The new manifold is a bundle over S^1 with monodromy changed by a Dehn twist along C.) This gives a way to pass continuously from one surface bundle to another through hyperbolic foliations. We will discuss this further in the future.

23. Hyperbolic structures on surface × circle

We can try to construct hyperbolic ($\mathbf{H}^3$) structures on $F \times S^1$ with Dehn surgery singularities along Σ in an analogous way. The basic idea is to obtain hyperbolic structures on the slabs S_i so that the top of the slab is a copy of F_i *bent up along* C_i and the bottom of the slab is a copy of F_i bent down along C_{i-1}.



If the underlying hyperbolic structure on the top of S_{i-1} is the same as that on the bottom of S_i (ignoring bending) for each i, then we can glue adjacent slabs together by isometries so that the holonomy round each meridian of Σ is a pure rotation. More generally, we can allow the structures on the top of S_{i-1} and bottom of S_i to differ by shearing as in our previous construction of foliations; then the slabs are glued together by rotations plus translations. (In the case where F is a punctured torus, this can sometimes be done directly as in section 19 above.)

In the general case we will prove a local result, making use of our previous results on hyperbolic foliations. Because of the complex structure on $R = R(\pi_1 M-\Sigma), PSL_2\mathbb{C})$ the tangent cone is a complex cone. (see [Wh2].) Hence if v is a tangent vector in the cone to the set of real points $R_{\mathbb{R}} = R(\pi_1(M-\Sigma), PSL_2\mathbb{R})$ then cv is in the tangent cone to R. Now we can check the positivity condition $REGEN1$ as follows.

Since the holonomies for fibres are rotations by $r>0$ by remark 4.7, multiplying by $c = x+iy \in \mathbb{C}$ gives holonomy with "complex length" $(x+iy)ir = -yr+ixr$. So if $y = imag(c) < 0$ then the holonomy of the fibre has a positive translational part. So condition $REGEN1$ is verified.

The condition $REGEN2$ on the movement of axes is not so easy to check, and it will not generally be satisfied even if $REGEN1$ holds. (When it fails we will generally obtain developing maps for hyperbolic structures with fold type singularities, at best.) To understand the movement of axes it will usually be necessary to obtain more detailed information on the families of representations constructed in section 22. In particular, we need to know the tangent vector $\dot{x}$ to the curve $x_t \in T$ satisfying the "gluing equation" $h(x_t) = x_t$. We will not consider the general case at the moment.

In the special case, where Σ consists of two components there is a clear geometric picture enabling us to construct the infinitesimal deformations of representations and geometric structures quite explicitly. This gives:

Theorem 4.10. *Assume that $\Sigma \subset F \times S^1$ consists of two horizontal curves projecting to curves C_1, C_2 filling up F. Let F_0 be a hyperbolic structure such that $L = \sum_{i=1}^{2} a_i \ell(C_i)$ is minimal. Then there are hyperbolic structures with Dehn surgery singularities along Σ degenerating to the hyperbolic foliation defined by F_0. For these structures the meridians of Σ have holonomies $2\pi i + c_i(t)$, $t \in \mathbb{C}$ small, with $\dot{c}_i(0) = c_i$ and $\mathrm{Im}(c_i(t)) < 0$ for all i. There are hyperbolic foliations when all $\mathrm{Im}(c_i(t)) = 0$.*

(The condition that $\mathrm{Im}(c_i(t))<0$ says that the rotational part of the holonomy of the ith meridian is less than 2π.)

Proof. The idea is as follows. Begin with the initial hyperbolic structure on F and corresponding representation ρ of $\pi_1(F)$. Now we can deform F and the representation ρ by Thurston's bending construction. (See [J-M] for a detailed discussion.) Bending F along C_1 by (complex) angle $\frac{1}{a_1}t$ gives a new representation $\rho(t)$ of $\pi_1(M)$ corresponding to a quasi-Fuchsian manifold whose convex core has F bent by the angle $\frac{1}{2}a_1t$ on the top. Similarly, we can bend F by (complex) angle $-\frac{1}{2}a_2t$, giving a representation $\rho'(t)$ and a quasi-Fuchsian manifold with F bent by angle $\frac{1}{2}a_2t$ on the bottom of its convex core.

If these representations $\rho(t)$, $\rho'(t)$ were actually actually the *same*, then we would have a hyperbolic structure on $F\times[0,1]$ bent up by angle $\frac{1}{2}a_1t$ on top and bent down by angle $\frac{1}{2}a_2t$ on the bottom. Doubling this would give a hyperbolic structure on $F\times S^1$ with the desired holonomies a_1, a_2 around the components of Σ. This was done explicitly for the case where F is a once punctured torus in section 19.

In general $\rho(t)$, $\rho'(t)$ will be different. However the condition $a_1dl(C_1)+a_2dl(C_2)=0$, implies that *infinitesimally* the deformations $\rho\to\rho(t)$ and $\rho\to\rho'(t)$ are the same (up to conjugacy). (This follows since equations (*) and (**) of section 22 are equivalent.) It follows that the above picture is correct infinitesimally.

To make this rigorous, we can explicitly construct a Zariski tangent vector $z\in Z^1(\pi_1(M-\Sigma), sl_2C)$. On the slab S_1 we take $z=z_1=\dot\rho(0)$; on slab S_2 we take $z=z_2=-\dot\rho'(0)$. Now z_1 and z_2 agree on $\pi_1(F-C_1)$ (nothing changes on the complement of the bending locus). Further, z_1 and z_2 agree up to a coboundary on $\pi_1(F-C_2)$ since $\dot\rho(0)$ and $\dot\rho'(0)$ agree up to a coboundary. It follows that z_1, z_2 extend to a cocycle for $\Gamma=\pi_1(M-\Sigma)$. (Here

we use Van Kampen's theorem and the description of $Z^1(\Gamma, g)$ in terms of generators and relations: see [We3] or the proof of theorem 3.4.) We leave it to the reader to check that the holonomies of the meridians have the form $c_i(t)$ with $\dot c_i(0)=a_i$.

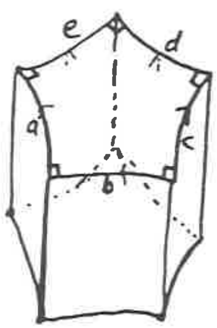
We claim that z is a multiple cv, $c\in\mathbf{C}$ of the real tangent vector obtained in section 22. This follows from the fact that the vector v is uniquely determined by the a_i. Hence z is a tangent to a family of representations $\Gamma\to PSL_2(\mathbf{C})$. Further, the infinitesimal movement of axes is completely determined by the infinitesimal deformation z. If a_1 has negative imaginary part (say), F is bent upward along C_1; in particular the axis corresponding to C_1 in the slab S_1 moves above the axis for C_2. Similarly for each pair of adjacent axes.

Hence the axes move apart in the desired directions and REGENT is verified. □

We conclude this section with some examples illustrating other possible singular loci for hyperbolic cone manifold structures on $F\times S^1$ degenerating to foliations.

Examples 4.11. Begin with a hyperbolic structure F on a surface and a collection C of closed geodesics on F such that $F-C$ is a union of of right angled pentagons. We will construct some families of hyperbolic cone manifolds degenerating to F, with singular locus projecting to C.

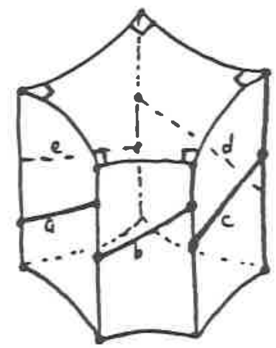
(a) Begin with a pentagonal prism P. By Andreev's theorem [An] there are hyperbolic structures P with dihedral angles a, b, c, d, e along the top edges and $\pi/2$ along the other edges as shown below, whenever the angles a, b, c, d, e are all less than $\pi/2$.



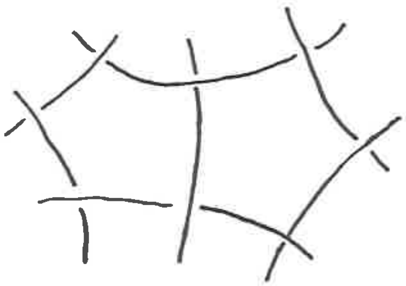
(Dihedral angles $\frac{\pi}{2}$ except where indicated.)

Doubling such polyhedra along the top and bottom faces gives a hyperbolic structure on structures on $pentagon \times S^1$. By reflecting across the vertical faces, we obtain a hyperbolic structure on $F \times S^1$ with cone type singularities along a graph $\tilde{C}$ projecting homeomorphically to C in F. As the cone angles $4a, 4b, 4c, 4d, 4e$ approach 2π, these hyperbolic structures flatten out, giving F as the geometric limit.

(b) Begin with a pentagonal prism and add five extra edges, sloping upwards on the vertical faces as shown below, to give a polyhedra Q having the same combinatorial type as a dodecahedron.



By Rivin [Ri] or a direct construction, there are hyperbolic structures on Q with arbitrary dihedral angles $a,b,c,d,e < \pi$ along the five added edges and other angles $\pi/2$. Again, these polyhedra can be reflected around to give hyperbolic structures on $F \times S^1$ degenerating to F as the cone angles approach 2π. This time, the singular locus is a link consisting of curves projecting to C, arranged in a weaving pattern. (If the singular locus "slopes up" around the boundary of a $pentagon \times S^1$, then it "slopes down" around each adjacent $pentagon \times S^1$.) Locally the singular locus has the form shown below.



In particular, this example (suggested by Thurston) shows that the singular locus need not be isotopic to a union of geodesics in any Seifert fibred geometry.

An interesting open problem is to characterize the possible singular loci for hyperbolic structures on Seifert fibred spaces degenerating to foliations.

24. Deforming Solv geometry structures

Let M be a torus bundle over the circle with Anosov monodromy ϕ and let Σ be the "zero section" of the bundle, so $M-\Sigma$ is the corresponding bundle over S^1 with once punctured torus as fibre. We begin with a representation $\rho:\pi_1(M-\Sigma)\to C'\subset PSL_2C$ where C' is a subgroup of PSL_2C preserving a geodesic $L\subset \mathbf{H}^3$, and a developing map $D:M\tilde{-}\Sigma\to L\subset\mathbf{H}^3$ mapping the lift of each fibre of $M-\Sigma\to S^1$ to a point. We wish to deform D to obtain a family of developing maps for hyperbolic structures on $M-\Sigma$ degenerating to a circle as geometric limit. We begin by studying the space of representations $R(\pi_1(M-\Sigma), PSL_2C)$ near ρ.

(1) First we consider the cohomology $H^1(\pi_1(M-\Sigma); Ad\rho)$ using the approach of section 17. Now $\Gamma=\pi_1(M)$ is an extension :

$$1\longrightarrow \pi_1(T^2)\longrightarrow \pi_1(M)\longrightarrow \pi_1(S^1)\longrightarrow 1.$$

Choosing generators g_1, g_2 for $\pi_1(T^2)\cong\mathbf{Z}^2$ and a generator t for $\pi_1(S^1)\cong\mathbf{Z}$, $\Gamma=\pi_1(M)$ has a presentation

$$\Gamma = \langle g_1, g_2, t \mid [g_1, g_2]=1,\ tg_it^{-1}=\phi(g_i),\ i=1,2\rangle.$$

There is an obvious family of representations of $\pi_1(M)$ factoring through $\pi_1(S^1)=\mathbf{Z}$. It follows that $H^1(M)\to H^1(\Sigma)$ is surjective and section 17 shows that $H^1(M)\to H^1(M-\Sigma)$ is an isomorphism. So it suffices to consider $H^1(M; Ad\rho)=H^1(\Gamma; Ad\rho)$.

Let $c:\Gamma\to g=sl_2(\mathbf{C})$ be a cocycle in $Z^1(\Gamma, Ad\rho)$. Then c is completely determined by its values on the generators of Γ : $c(g_i)=\begin{bmatrix} z_i & x_i\\ y_i & -z_i\end{bmatrix}$, $i=1,2$ and $c(t)$. Such an assignment gives a cocycle if and only if c satisfies the relations $c(tg_1t^{-1})=c(g_1^{\alpha}g_2^{\beta})$, and $c(tg_2t^{-1})=c(g_1^{\gamma}g_2^{\delta})$ where ϕ has matrix $\Phi=\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix}$. Using the cocycle condition, these reduce to :

$$\rho(t)c(g_1)\rho(t)^{-1}=\alpha c(g_1)+\beta c(g_2) \tag{1}$$

$$\rho(t)c(g_2)\rho(t)^{-1}=\gamma c(g_1)+\delta(g_2)$$

This shows that $c(t)$ can be an arbitrary element of $sl_2(\mathbf{C})$, since $c(t)$ doesn't occur in the equations (1). So the (complex) dimension of Z^1 is $3+\dim\{x_i, y_i, z_i$ satifying (1)$\}$.

Now assume that $\rho(t)$ is diagonalizable. By conjugating ρ if necessary we can assume that $\rho(t)=\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}$. Then (1) reduces to the three matrix equations :

$$\Phi x=\lambda^2 x,\ \Phi y=\lambda^{-2}y,\ \Phi z=z \tag{2}$$

where $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $y=\begin{bmatrix}y_1\\ y_2\end{bmatrix}$, $z=\begin{bmatrix}z_1\\ z_2\end{bmatrix}$. Thus, x, y, z are eigenvectors of Φ with eigenvalues $\lambda^2, \lambda^{-2}, 1$ if they are not zero.

Thus, if λ^2 is an eigenvalue of Φ, then $z=0$ and x,y lie in 1-(complex)-dimensional eigenspaces of Φ so $\dim_{\mathbf{C}} Z^1(\Gamma, Ad\rho) = 3+1+1 = 5$. Moreover the centralizer of $\rho(\Gamma)$ is the 1-dimensional group of diagonal matrices, so $\dim H^0(\Gamma, Ad\rho) = 1$ and $\dim H^1 = \dim ker d^1 - \dim im d^0 = 5-2 = 3$.

If λ^2 is not an eigenvalue of Φ then $x=y=z=0$ and $\dim_{\mathbf{C}} Z^1(\Gamma, Ad\rho) = 3$.

If $\rho(t) = 1$, the cohomology is just the usual cohomology of M with trivial (untwisted) coefficients in $\mathbf{C}^3$. Thus, $\dim H^0 = 3$, $\dim H^1 = 9$, $\dim H^2 = 9$, $\dim H^3 = 3$.

(2) Next we find forms representing the cohomology classes in $H^1(M) \cong H^1(M-\Sigma)$.

We describe forms on the universal cover $\tilde{M}$ of M, which we identify with $\mathbf{R}^3$. Explicitly, let the covering transformations corresponding to $a=[g_1]$, $b=[g_2]$, $c=[t] \in \pi_1(M)$ be given by :

$$a:(x,y,z) \mapsto (x+1, y, z)$$

$$b:(x,y,z) \mapsto (x, y+1, z)$$

$$c:(x,y,z) \mapsto (\Phi x, \Phi y, z+1)$$

Let ω be a $\mathbf{g} = sl_2\mathbf{C}$-valued form on $\mathbf{R}^3$ with "constant coefficients":

$$\omega = p \otimes dx + q \otimes dy + r \otimes dz$$

where $p, q, r \in \mathbf{g}$. Then ω descends a form on M if and only if ω satisfies the equivariance condition :

$$\omega(\gamma_* v) = Ad\rho(\gamma)\omega(v) \tag{3}$$

for all $v \in T\tilde{M}$, $\gamma \in \pi_1(M)$. Now the cocycle in $z \in Z^1(M, Ad\rho)$ corresponding to ω is given by integration over cycles (see section 4.1), so $p = z(g_1)$, $q = z(g_2)$, $r = z(t)$. It is easy to check that the equivariance condition (3) is exactly the condition that $z(g_1), z(g_2), z(t)$ gives a cocycle in $Z^1(M, Ad\rho)$. Hence, every element of $H^1(M, Ad\rho)$ can be represented by a form with constant coefficients. The coboundaries form a 2-(complex)-dimensional space corresponding

to cocycles z such that $z(\gamma) = x - Ad\rho(\gamma)x$ for $x \in \mathbf{g}$; explicitly : $z(a) = z(b) = 0$, $z(t) = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$.

Remark 4.12. Choosing coordinates e_1, e_2 on $\mathbf{R}^2 \times 0$ in the eigendirections of Φ, ω can be written in the more transparent form:

$$\omega = \begin{bmatrix} 0 & 0 \\ u_1 & 0 \end{bmatrix} \otimes de_1 + \begin{bmatrix} 0 & u_2 \\ 0 & 0 \end{bmatrix} \otimes de_2 + r \otimes dz$$

(3) We now compute the cup products $[u,u] \in H^2(M-\Sigma, Ad\rho)$ which give the first obstruction to extending the Zariski tangent vector u to a path in $R(\pi_1(M-\Sigma), PSL_2(\mathbf{C}))$. Using the notation introduced above, we can represent u by a form

$$\omega = p \otimes dx + q \otimes dy + r \otimes dz$$

where $p = \begin{bmatrix} 0 & x_1 \\ y_1 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 0 & x_2 \\ y_2 & 0 \end{bmatrix}$, $r = \begin{bmatrix} z & x \\ y & -z \end{bmatrix}$. In de Rham cohomology, the map $[\ ,\ ]: H^1 \times H^1 \to H^2$ is given by exterior product with coefficient pairing $[\ ,\ ]: \mathbf{g} \times \mathbf{g} \to \mathbf{g}$. Hence, $[u,u]$ is represented by the 2-form :

$$[\omega, \omega] = 2([p,q] \otimes dx \wedge dy + [p,r] \otimes dx \wedge dz + [q,r] \otimes dy \wedge dz)$$

Using the above notation, we have

Lemma 4.13. *A 2-form with constant coefficients $a \otimes dx \wedge dy + b \otimes dx \wedge dz + c \otimes dy \wedge dz$ is equivariant if and only if $a = Ad\rho(t)a$, $b = q$, $c = r$ for some q, r as above. This is a coboundary if and only if $b = c = 0$.*

Proof. We leave the first part as an exercise for the reader. The form $a \otimes dx \wedge dy$ is a coboundary since the volume form on a non-compact manifold (e.g. a punctured torus) is exact. To check other classes are non-trivial, take a wedge product with a (suitably chosen) 1-form in $H^1(M, \partial M)$ using Killing form as pairing, and integrate over M. $\square$

Thus, we find

Proposition 4.14. *Let $u \in H^1(M-\Sigma; Ad\rho)$ be represented by a form $\omega = p \otimes dx + q \otimes dy + r \otimes dz$ as above. Then $[u, u] = 0$ in H^2 if and only if $[p, r] = [q, r] = 0$, i.e. $p = q = 0$ or $r = 0$.*

□

Remark 4.15. In the notation of the previous remark

$$\omega = \begin{vmatrix} 0 & 0 \\ u_1 & 0 \end{vmatrix} \otimes de_1 + \begin{vmatrix} 0 & u_2 \\ 0 & 0 \end{vmatrix} \otimes de_2 + r \otimes dz$$

represents $u \in H^1$ such that $[u, u] = 0 \in H^2$ iff $u_1 = u_2 = 0$ or $r = 0$.

Remark 4.16.

We can describe the representations $\pi_1(M) \to PSL_2(\mathbf{C})$ near ρ quite explicitly as follows. Since $\rho(t)$ is a non-parabolic element of $PSL_2(\mathbf{C})$, $\rho'(t)$ is also non-parabolic for all ρ' near ρ. Since $\rho(\Gamma)$ is a solvable subgroup of $PSL_2(\mathbf{C})$ it follows easily that ρ' is a reducible representation.

By conjugation, we can assume that

$$\rho(t) = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{vmatrix}$$

as above. Then for $x \in \mathbf{Z}^2$

$$\rho(x) = \begin{vmatrix} 1 & p_1(x) \\ 0 & 1 \end{vmatrix} \quad \text{or} \quad \rho(x) = \begin{vmatrix} 1 & 0 \\ p_2(x) & 1 \end{vmatrix}$$

for some homomorphisms $p_1, p_2 : \mathbf{Z}^2 \to \mathbf{C}$. Since $txt^{-1} = \phi(x)$ for all $x \in \mathbf{Z}^2$ it follows that

$$p_1 \circ \phi = t^2 p_1 \text{ and } p_2 \circ \phi = t^{-2} p_2 \tag{4}$$

Extend p_i to linear maps $p_i : \mathbf{R}^2 \to \mathbf{C}$ and let e_i denote the eigenvector of ϕ with eigenvalue μ_i, for $i = 1, 2$. Then (*) shows that

$$\mu_i p_1(e_i) = t^2 p_1(e_i)$$

and

$$\mu_i p_2(e_i) = t^{-2} p_2(e_i).$$

Hence, $p_1 = 0$ unless t^2 is an eigenvalue of ϕ and $p_2 = 0$ unless t^{-2} is an eigenvalue of ϕ, say $t^2 = \mu_1$ and $t^{-2} = \mu_2$. Then p_i is projection onto the eigenspace $\mathbf{C}e_i$: $p_i(e_j) = c_i \delta_{ij}$, for some $c_i \in \mathbf{C}$. Moreover, different choices of $c_i \neq 0$ give conjugate representations, so we can assume $c_i = 1$. Let $\bar{\eta} : \pi_1(M) \to PSl_2\mathbf{R}$ denote this representation and let η be the induced representation of $\pi_1(M-\Sigma)$.

More geometrically, there are Anosov foliations of the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ by curves parallel to the eigenvectors of ϕ, which are invariant under T^2. The natural (linear) transverse measures m_1, m_2 to these foliations are multiplied by the eigenvalues μ_1, μ_2 under the action of ϕ. Hence, there is an induced 1-dimensional foliation on M. Further, the foliations have transverse hyperbolic structures with holonomy given by ϕ as above with $c_i \in \mathbf{R}$. Then the maps $p_i : \pi_1(T^2) \to \mathbf{C}$ give the length of a curve with respect to the transverse measures m_i, with suitable signs.

It is technically easier to understand the representation space $R(\pi_1(M-\Sigma), PSL_2\mathbf{C})$ near η than ρ. (Roughly, we are resolving the singularity at ρ by "blowing up" a point.) First note that the preceding remarks show that every representation in $R(\pi_1(M), G)$ near η is conjugate to η. Hence $H^1(M; Ad\eta) = 0$. It follows from lemma 3.25 that $H^1(M-\Sigma) \cong H^1(\Sigma) \cong \mathbf{C}^n$, where n is the number of components of Σ. Since η has trivial centralizer, $\dim Z^1(M-\Sigma) = \dim H^1 + 3 = n + 3$.

Using theorem 3.14, we obtain

Proposition 4.17. *$R = R(\pi_1(M-\Sigma), G)$ is a complex manifold of dimension $n+3$ near η. Representations near η are parametrized up to conjugacy by the traces of meridians of the components of Σ.*

□

In particular, there are irreducible representations near η. We now use this to show that there are irreducible representations near ρ.

Let η_t, $t \geq 0$ be a path of representations in $R(\pi_1(M-\Sigma), PSL_2\mathbb{C})$ with $\eta_0 = \eta$ and η_t irreducible for $t>0$. Write $\eta_t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}$ where $a_t, b_t, c_t, d_t : \pi_1(M-\Sigma) \to \mathbb{C}$. Then $c_t = O(t^{2k})$ for some $k>0$. (In fact it is not hard to see that we can assume $k = 1$, since the inclusion of coefficient modules $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset sl_2\mathbb{C}$ induces an isomorphism on cohomology groups $H^1(M-\Sigma)$.) Let $\Delta_t = \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix}$ and $\rho_t = \Delta_t \eta_t \Delta_t^{-1}$. Then ρ_t is a continuous path of irreducible representations with $\rho_0 = \rho$. It follows that there is a component R of $R(\pi_1(M-\Sigma), PSL_2\mathbb{C})$ such that $\rho \in R$ and R contains irreducible representations. From 3.14, it follows that R has dimension $\geq 1+3 = 4$; however the tangent cone to R lies in the 4 dimensional subvariety of the Zariski tangent space $Z^1(\pi_1(M-\Sigma), Ad\rho)$ determined by the equations $[u, u] = 0$. Hence R is a manifold of dimension 4 near ρ, with tangent space represented by forms

$$\omega = \begin{pmatrix} 0 & 0 \\ u_1 & 0 \end{pmatrix} \otimes de_1 + \begin{pmatrix} 0 & u_2 \\ 0 & 0 \end{pmatrix} \otimes de_2.$$

It is now easy to apply our previous results to obtain

Theorem 4.18. *Let M be a torus bundle over the circle with Anosov monodromy ϕ and $\Sigma \subset M$ the zero section. Then D, ρ as above can be deformed to a family of hyperbolic structures on $M-\Sigma$ if and only if $\rho(t)$ is a translation by $\log(\lambda)$, where $\lambda > 1$ is the larger eigenvalue of the matrix representing ϕ.*

In fact, there is a 1-(complex)-dimensional manifold of conjugacy classes of representations near ρ, parametrized by the trace of a meridian for Σ. There is a submanifold of (real) dimension one corresponding to hyperbolic foliations on $M-\Sigma$ with Dehn surgery type singularities; all the other representations near ρ correspond to hyperbolic structures on $M-\Sigma$ with Dehn surgery type singularities.

Proof. All that remains is to verify the regeneration condition from theorem 2.2. But this can be done directly here, using the constant coefficient forms ω obtained above. Now $\begin{pmatrix} 0 & 0 \\ u_1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & u_2 \\ 0 & 0 \end{pmatrix}$ give parabolic elements of $sl_2\mathbb{C}$ representing infinitesimal translations along two horospheres tangent at a point. It follows easily that the regeneration condition holds if and only if $u_1, u_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. □

Remark 4.19.

As discussed above, there is hyperbolic foliation on M with holonomy $\bar{\eta} : \Gamma \to PSL_2\mathbb{R}$, and corresponding developing map $\bar{D} : \tilde{M} \to \mathbf{H}^2 \subset \mathbf{H}^3$. Then there are corresponding maps $\eta : \pi_1(M-\Sigma) \to PSL_2\mathbb{C}$ and $D : \widetilde{M-\Sigma} \to \mathbf{H}^3$ obtained via the inclusion $M-\Sigma \subset M$. These maps can also be deformed to obtain hyperbolic structures on $M-\Sigma$ with Dehn surgery singularities. This is an easy consequence of the work in this section; we leave the details to the reader.

CHAPTER 5

Variation of volume under deformations

25. The Schläfli Differential Formula

In this chapter we study the variation of volume under deformations of geometric structures on manifolds of constant curvature. The basic ingredient used is the following theorem, originally proved in the spherical case by Schläfli in 1858.

Theorem 5.1. *Let X_t be a smooth one-parameter family of polyhedra in a simply-connected n-dimensional space of constant curvature K. Then the derivative of the volume of X_t satisfies the equation :*

$$(n-1)\,K\,d\mathrm{Vol}(X_t) = \sum_F V_{n-2}(F)\,d\theta_F \qquad (1)$$

where the sum is over all codimension-2 faces of X_t, V_{n-2} denotes the $(n-2)$-dimensional volume, and θ_F denotes the dihedral angle at F.

In the 2-dimensional case, the theorem is equivalent to the Gauss-Bonnet theorem for polygons. The formula becomes

$$K\,dA(X_t) = \sum d\theta_v \qquad (2)$$

where θ_v is the angle at vertex v and A denotes area. Integrating this gives the Gauss formula

$$KA = \sum \theta_v + constant,$$

and the constant of integration can be determined by considering the case where the polygon flattens out to a straight line segment.

We now give a new, non-computational proof of Theorem 5.1 using the classical 2-dimensional version (2).

Proof. First we outline the argument, then fill in some details at the end. It suffices to consider the case where the polyhedra are simplices. Choose an arbitrary totally geodesic, 2-dimensional plane P and consider the intersection $P \cap X_t$. Applying the 2-dimensional case (2) gives

$$K\,dA(P\cap X_t) = \sum d\theta_v \qquad (3)$$

summed over all vertices v of $P \cap X_t$. Now we just integrate (1) over all planes P to prove the theorem!

For any simply connected space of constant curvature K, there is a measure on the set of (totally geodesic) 2-planes which is invariant under the group of all isometries and this is unique up to scaling. We integrate (3) with respect to this measure, using the normalization of [Santaló].

Integrating the cross sectional areas $A(P\cap X_t)$ over all planes gives $c\,\mathrm{Vol}(X_t)$, where $c = \frac{V_{n-1}V_{n-2}}{V_1V_0}$ and V_k is the volume of the unit k-sphere. In the Euclidean case, this follows immediately from Fubini's theorem - integrate first over a family of parallel planes then over all directions. See [Santaló] for a proof in the general case. So integrating the left side of (3) gives

$$\int LHS(3) = Kc\,d\mathrm{Vol}(X_t) \qquad (4)$$

Now, consider the right hand side of (3). The vertices of cross sections occur exactly at intersections of P with codimension-2 faces F of the polyhedra X_t. The angle occurring at each point v of $F \cap P$ depends on the orientation of the plane P, so first we average over all planes through a point v. To find this average angle, consider the intersections of P and X_t with a small $(n-1)$-dimensional sphere centred at the point v. This sphere meets X_t in a lune

L whose angle is exactly the dihedral angle along face F, and meets P in a segment of a great circle whose length is proportional to the angle of $P\cap X_t$ at v. Hence (after rescaling so the sphere has radius one) the average angle at v is the average length of intersections of great circles with L, which is proportional to the $(n-1)$-volume of L so proportional to the dihedral angle θ_F. In fact, by considering the case when L is the whole sphere we see that the average angle at a vertex on F is the dihedral angle θ_F. Moreover, the measure of 2-planes meeting a codimension-2 face F is $\frac{c}{n-1}V_{n-2}(F)$ where c is as in (4). It follows immediately that

$$\int R.H.S.(3) = \frac{c}{n-1}\sum_F V_{n-2}(F)\,d\theta_F \tag{5}$$

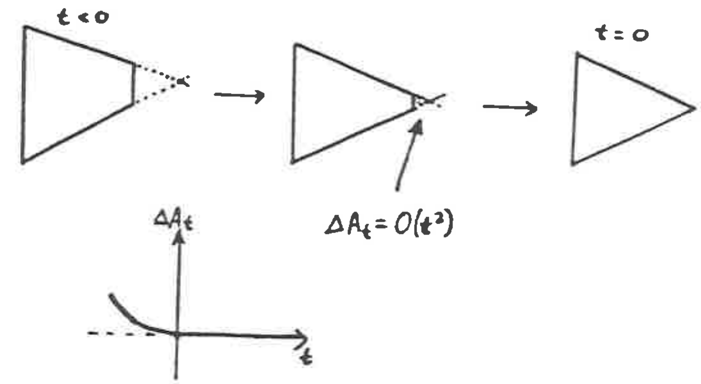
completing the outline of the proof.

In the argument outlined we have implicitly interchanged the order of differentiation and integration, writing

$$\frac{\partial}{\partial t}\mathrm{Vol}(X_t) = \frac{\partial}{\partial t}\int A(X_t\cap P)\,dP = \int \frac{\partial}{\partial t}A(X_t\cap P)\,dP$$

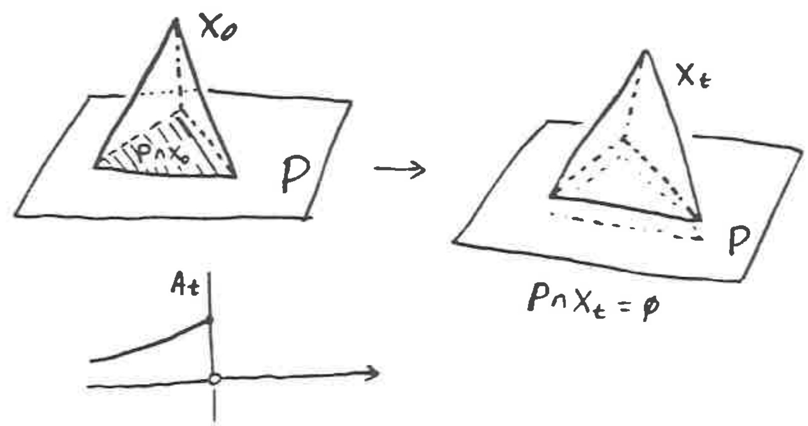
where the integrations are over the set Π of all 2-planes P. This needs justification, since the integrand $A_t(P) = A(P\cap X_t)$ can be non-differentiable (even discontinuous!) for some P. We now study the behaviour of $A_t(P)$ in more detail.

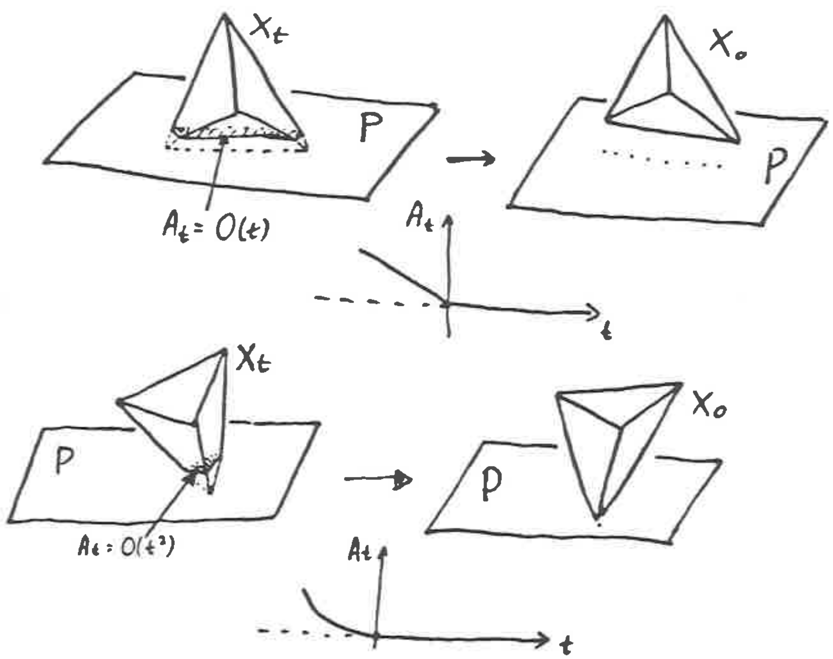
It is easy to see that A_t is a differentiable function of t at $t=0$ unless P is a *support plane* for $X = X_0$ (i.e. $P\cap X\neq\varnothing$ and there are planes P' arbitrarily close to P such that $P'\cap X = \varnothing$.) For P not a support plane, $P\cap X_t$ is the convex hull of a set of points (not necessarily all distinct) varying smoothly with t. The area of this set is actually a C^∞ function of t at $t = 0$ if the points stay distinct as $t\to 0$. If two points coalesce as $t\to 0$ we have the situation:



and the area is still a twice differentiable function of t (but not C^2).

If P is a support plane, then A_t can be discontinuous at $t = 0$ if P meets 3 vertices of X, and Lipschitz but not differentiable at $t = 0$ if P meets 2 vertices of X. (A_t is C^1 if P meets a single vertex of X.) The following figure illustrates these situations in the 3-dimensional case.



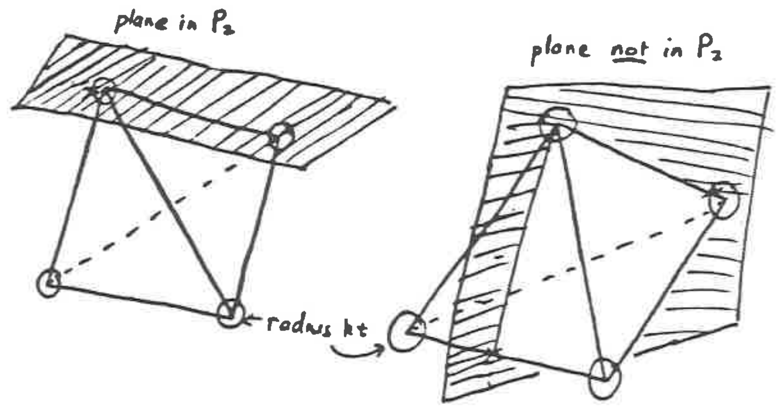


We now estimate the contribution made to $\int A_t(P)dP$ by planes close to support planes of X. For this we give an explicit form for the invariant measure described above. One way to describe the measure on r-planes in an n-dimensional space of constant curvature K is as follows. Choose a basepoint *. Then a plane P is determined by the $n-r$-plane L_{n-r} perpendicular to P at *, and a point $x = P \cap L_{n-r}$ in L_{n-r}. Let ρ be the length of a perpendicular from * to P, let $d\sigma_{n-r}$ denote the measure on $n-r$ planes through * and $d\mu_{n-r}$ the measure on L_{n-r}. Then the invariant measure on r-planes is given by :

$$dL_r = \cos^r(K^{1/2}\rho)\, d\sigma_{n-r} \wedge d\mu_{n-r} \qquad (*)$$

(See [Santaló, p.306]).

Now we make some estimates. There is a constant $k>0$ such that each vertex of X_t is within distance kt of the corresponding vertex of X_0, for all t sufficiently small. Let $B_i(kt)$ denote the ball of radius kt about the ith vertex of X_0. Let $P_j(t)$ denote the planes *close to support planes for* X passing through exactly j of the balls $B_i(kt)$, $j = 1, 2, 3$.



Now, the planes of $P_j(t)$ pass close to a face f of X_t containing j vertices. Choosing a basepoint on f and using (*) we see that $meas(P_j(t)) \leqslant const(kt)^j$. (These planes have angular measure $O(t^{j-1})$ when $n \geqslant 3$, by a dimension counting argument, and distances to base point $O(t)$.) Moreover, $A(P \cap X_t) \leqslant const(kt)^{3-j}$ so the contribution to $\int A(P \cap X_t)dP$ by all such planes is of order $O(t^3)$.

Let Π_t be the planes in Π which are not in any $P_j(t)$, $j = 1, 2, 3$ and which do not pass through a vertex of X_0. Then the above estimate shows that

$$Vol(X_t) = \int_{\Pi_t} A_t(P)dP + O(t^3) ;$$

hence

$$\frac{\partial}{\partial t}Vol(X_t) = \lim_{t\to 0}\int_{\Pi_t}\frac{\partial}{\partial t}A_t(P)dP$$

Now our previous arguments show that

$$\int_{\Pi_t}\frac{\partial}{\partial t}A_t(P)dP = \sum_F V_{n-2}(F)\,\frac{\partial}{\partial t}\theta_F + E$$

where the term E gives the error introduced by integrating $d\theta_F$ over Π_t instead of Π. So $|E| \leq const\, kt$ and letting $t \to 0$ completes our proof of the Schläfli formula. □

Remark : The theorem also applies to the case of hyperbolic polyhedra with some ideal vertices. When $n \neq 3$, no change in the statement or proof of the theorem is needed. In the 3-dimensional case, some edge lengths $V_{n-2}(P)$ become infinite. However, the theorem remains valid if we remove small horoball neighbourhoods of the ideal vertices before measuring edge lengths. (The right hand side of (1) is easily seen to be independent of the choice of horoballs, using the fact that the sum of dihedral angles at an ideal vertex is constant. To prove the result for ideal polyhedra, approximate the polyhedra truncated along horospheres by polyhedra truncated along totally geodesic planes. As the horospheres used are moved out to infinity, the difference in volumes of the two approximations approaches zero, and the result follows by using the theorem for the truncated polyhedra with totally geodesic faces.)

Application : Using the Schläfli formula for ideal simplices we can give a simple proof of the following.

Proposition 5.2. *The regular ideal simplex is the unique simplex of maximal volume in* $\mathbf{H}^3$.

Proof. Since any simplex is contained in an ideal simplex with all its vertices at infinity, it suffices to consider ideal simplices. We will show that for an ideal simplex of maximal volume, horospheres about the 4 vertices can be chosen so that every pair of horospheres meets

tangentially. It then follows from elementary geometry that the simplex is regular.

Notice that the Schläfli formula is valid using any set of horospheres (if two overlap, the corresponding edge length is negative). We can always choose these horospheres so that four of the six edge lengths are zero. Then the Schläfli formula reduces to :

$$-2dV = 0\cdot d\theta_1 + l_2 d\theta_2 + l_3 d\theta_3$$

where l_i are the non-zero edge lengths, θ_i the corresponding angles. For a simplex of maximal volume, $dV = 0$ for all deformations of the simplex. But the angles θ_i can be varied arbitrarily subject to the constraint $\sum_{i=1}^{3} d\theta_i = 0$. Choosing $(d\theta_1, d\theta_2, d\theta_3) = (-l_2-l_3, l_2, l_3)$ shows that we must have $l_2 = l_3 = 0$, i.e. all the horospheres meeting tangentially. □

Remark : Thurston has observed that by using recent results of Rivin [Ri] characterizing the possible dihedral angles of convex polyhedra in $\mathbf{H}^3$, this argument can be extended to other polyhedra. For example, the convex polyhedron of maximal volume in a given combinatorial type is obtained by gluing together ideal triangles in the symmetric way : so that horospheres from adjacent vertices meet tangentially.

§6. Applications to cone manifolds

In this section, we apply the Schläfli formula to study the volume of manifolds having metrics of constant curvature with cone-like singularities. The following theorem shows that the variation in volume for a family of cone manifold structures is completely determined by the changes in geometry *along the singular locus*.

Theorem 5.3. *Let C_t be a smooth family of (curvature K) cone manifold structures on a manifold with fixed topological type of singular locus. Then the derivative of volume of C_t satisfies*

$$(n-1)\,K\,dVol(C_t) = \sum_{\Sigma} V_{n-2}(\Sigma)\,d\theta_{\Sigma}$$

where the sum is over all components Σ of the singular locus of C, and θ_Σ is the cone angle along Σ.

Proof. Divide C_t into geometric simplices, varying smoothly with t. (This can be done as follows. Choose a (sufficiently) fine geodesic triangulation of C_0, with the one skeleton containing Σ. This lifts to a triangulation of $\tilde{C}_0$ mapped isometrically by the developing map for C_0. Form a geodesic triangulation for C_t using the images of the original vertices under the developing map for C_t.) Applying the Schläfli formula to each simplex and adding shows that the variation of volume is given by

$$(n-1)\,K\,dVol(C_t) = \sum_F V_{n-2}(F)\,d\theta_F$$

summed over all codimension 2 faces of the triangulation of C_t, where θ_F denotes the cone angle along F. However, at any non-singular face F the cone angle is 2π for all t so $d\theta_F=0$. So the right hand side reduces to a sum over faces F in the singular locus Σ. □

As a corollary we see that the volume of cone manifolds satisfy the following remarkable monotonicity properties:

Corollary 5.4. *If $K>0$, then the volume increase strictly monotonically as any cone angle is increased. If $K<0$, then the volume decreases monotonically as the cone angles increase.* □

(In fact this result holds, suitably interpreted, even if the combinatorial type of the singular locus is allowed to change.)

27. Variation of volume in hyperbolic Dehn surgery space

Let M be a complete hyperbolic manifold of finite volume with n cusps. Then M is homeomorphic to the interior of a compact manifold with boundary consisting of n tori T_j.

By Mostow's rigidity theorem, M has a unique complete hyperbolic structure but if we consider incomplete hyperbolic structures on M we obtain a deformation space which is a complex manifold of dimension n near the complete structure called *hyperbolic Dehn surgery space*. This space $H(M)$ consists of hyperbolic structures on M with generalized Dehn surgery type singularities (see section 2). We now apply the Schläfli formula to describe the variation of volume in hyperbolic Dehn surgery space.

This space can be parametrized locally as in section 2. For each cusp $T_j\times[0,\infty)$ choose a basis α_j, β_j for $\pi_1(T_j)$. Except at the complete structure, the holonomies $hol(\alpha_j)$, $hol(\beta_j)\in$ isom$(\mathbf{H}^3)$ preserve a common axis in $\mathbf{H}^3$, so act as translations plus rotations about this axis. Let $l_{\alpha_j}, l_{\beta_j}$ and $\theta_{\alpha_j}, \theta_{\beta_j}$ denote the translation distances and rotation angles for $hol(\alpha_j)$, $hol(\beta_j)$. Then $u_j = l_{\alpha_j} + i\theta_{\alpha_j}$ (or $v_j = l_{\beta_j} + i\theta_{\beta_j}$) give a complex analytic coordinate system on hyperbolic Dehn surgery space. (Alternatively, we could define u_j (respectively v_j) as the logarithm of the ratio of the eigenvalues of $hol(\alpha_j)$ (respectively $hol(\beta_j)$) considered as elements of $PGL(2,\mathbf{C}) \cong$ isom$(\mathbf{H}^3)$.)

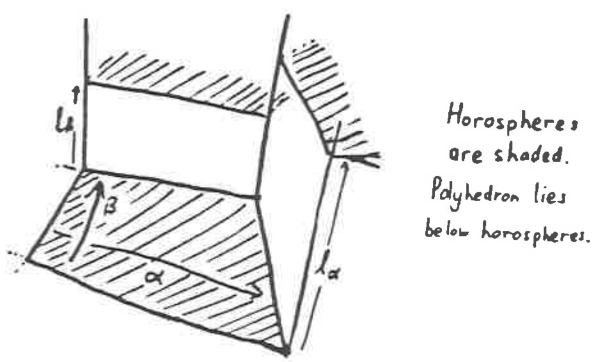
Theorem 5.5. *The derivative of volume in hyperbolic Dehn surgery space is given by:*

$$dVol = -\frac{1}{2}\sum_j (l_{\beta_j}\,d\theta_{\alpha_j} - l_{\alpha_j}\,d\theta_{\beta_j}) \tag{*}$$

Here the orientations are chosen so that near the jth cusp, α_j, β_j, n_j for a positively oriented basis for TM, where n_j is normal to tori T_j and points towards the cusp.

Proof. First we choose a topological triangulation of M by ideal tetrahedra. Thurston [Th5] has shown that this is possible for any complete finite volume hyperbolic manifold with cusps. Moreover, any (possibly incomplete) hyperbolic structure near the complete structure on M corresponds to a triangulation by geometric ideal tetrahedra in $\mathbf{H}^3$ (see [Th1, chap.4]). (The idea of the proof is to consider the volume of a fundamental region made up of ideal tetrahedra. We study the variation of volume of these tetrahedra, by using the version of Schläfli's theorem given in the remark following the proof of theorem 5.1. To do this we first truncate the tetrahedron along horospheres.)

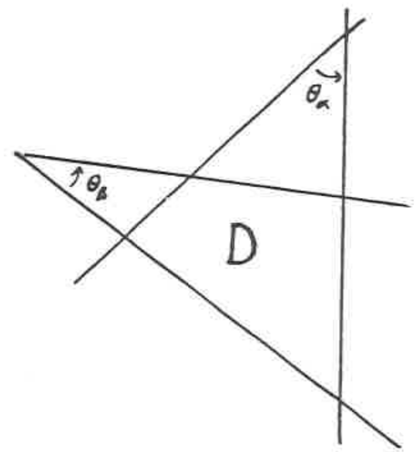
At each cusp of M we obtain an induced triangulation of the torus T_j by intersecting with ideal tetrahedra. For the complete structure on M we can obtain a Euclidean structure on T_j by piecing together Euclidean triangles formed by intersecting the ideal tetrahedra at a cusp with suitably chosen horospheres. In the general (incomplete) case, we can choose the horospheres to match up over a "fundamental domain" D for each torus T_j, but the horospheres will move towards or away from the cusp as we move from one side of the fundamental domain to the other, by a distance determined by the holonomy for the hyperbolic structure. Suppose that the generators α_j and β_j for $\pi_1(T_j)$ correspond to face pairing transformations gluing together "sides" of D to give T_j. Then the horospheres move a distance l_{α_j} (respectively l_{β_j}) towards the cusp as we move across D in the "direction" of α_j (respectively β_j).



Now consider how the polyhedron formed by truncating along these horospheres varies as we deform the hyperbolic structure on M. The only changes in lengths and dihedral angles

occur

over the boundary of D. To see how the angles change consider the following figure, showing a projection of the polyhedron as viewed from the cusp.



It follows that

$$\theta_{\beta_j} = n\pi + \sum \text{dihedral angles}$$

summed over the sides of D paired by α_j, and

$$-\theta_{\alpha_j} = m\pi + \sum \text{dihedral angles}$$

summed over sides of D paired by β_j, for some integers n and m. It follows that the derivative of volume of the truncated polyhedron is given by equation (*), and since this is independent of the choice of truncating horospheres this also gives the derivative of volume for hyperbolic structures on M. □

It is enlightening to make use of the complex structure on hyperbolic Dehn surgery space and rewrite this formula in terms of the complex lengths u_j, v_j of α, β. For notational convenience, we assume there is a single cusp and put $u = l_\alpha + i\theta_\alpha$, $v = l_\beta + i\theta_\beta$. We will compare the right hand side of (*) with $udv - vdu$. We have

$$\begin{aligned} \mathrm{Im}(udv - vdu) &= (\theta_\alpha dl_\beta - \theta_\beta dl_\alpha) + (l_\alpha d\theta_\beta - l_\beta d\theta_\alpha) \\ &= 2(\theta_\alpha dl_\beta - \theta_\beta dl_\alpha) + (l_\alpha d\theta_\beta - l_\beta d\theta_\alpha) - (\theta_\alpha dl_\beta - \theta_\beta dl_\alpha) \\ &= 4dV - d(\mathrm{Im}(u\bar{v})) \end{aligned}$$

Since u, v are holomorphic functions, it follows that $V(u) - V(0) - \frac{1}{4}\mathrm{Im}(u\bar{v})$ is the imaginary part of the holomorphic function $\frac{1}{4}\int (udv - vdu)$.

At a manifold point, we can choose α to be a meridian, and β a longitude for the solid torus added in Dehn surgery. Then $u = 2\pi i$ and $u\bar{v} = 2\pi(\theta_\beta + il_\beta)$; so $\mathrm{Im}(u\bar{v})$, $\mathrm{Re}(u\bar{v})$ give the length and "torsion" (i.e. rotational part of holonomy) for the core circle of the added solid torus. Writing $L(u) = \frac{1}{2\pi}\mathrm{Im}\, u\bar{v}$ for this analytic extension of this "core geodesic length", we have a result of Neumann-Zagier [N-Z]:

Proposition 5.6. *Near the complete hyperbolic structure,* $V(u) - V(0) - \frac{\pi}{2}L(u)$ *is the imaginary part of the holomorphic function* $\frac{1}{4}\int (udv - vdu)$. □

Remarks : (a) Since the right hand side of the formula (*) only depends on the representation not on the choice of developing maps, it follows that the volume V extends by analytic continuation so that the formula (*) holds throughout the component of $R(\pi_1(M); PSL_2(\mathbb{C}))$ containing the holonomy of the complete structure. For each such representation, there is a map of an ideal triangulation of M into M determined by a solution to the hyperbolic gluing equations. However, some simplices will be mapped in by orientation reversing maps in general, so this will not yield a non-singular hyperbolic structure on M.

We can still define the *volume* of such a representation to be the sum of signed volumes of the ideal simplices. By analytic continuation, it follows that this volume agrees with $V(0) + \frac{\pi}{2}L(u) + \frac{1}{4}\,\mathrm{Im}\int (vdu - udv)$. In fact, this volume is independent of choice of triangulation so is a well-defined geometric invariant for a manifold with Dehn surgery singularities. Indeed, the volume only depends on the holonomy representation together with Dehn surgery coordinates. This can be seen by interpreting the volume in terms of sections of the associated X-bundle with controlled (Dehn surgery type) behaviour at infinity, and applying Stokes' theorem.

(Actually , there is no need to use analytic continuation here; the same proof applies in this situation, using maps of an ideal triangulation into M, rather than embeddings.)

(b) Using analytic continuation, it follows that an analogous formula applies to spherical and Euclidean structures with Dehn surgery type singularities, obtained by deforming hyperbolic structures. To see this, consider the space $GS(M)$ of constant curvature metrics on M with Dehn surgery type singularities. (We allow the curvature K to take any real value.) This space has a natural real analytic structure and volume is an analytic function of the metric and the Dehn surgery coordinates on this space. Since the formula

$$2K\,dVol = \sum_j (l_{\beta_j} d\theta_{\alpha_j} - l_{\alpha_j} d\theta_{\beta_j})$$

holds for hyperbolic structures near the complete structure, it follows that the formula holds throughout the component of $GS(M)$ containing the complete hyperbolic structure on M.

In particular, Euclidean structures satisfy the relation

$$0 = \sum_j (l_{\beta_j} d\theta_{\alpha_j} - l_{\alpha_j} d\theta_{\beta_j})$$

so the Dehn surgery coordinates lie in a codimension one submanifold E of $H_1(\partial M; \mathbb{R})$. From chapter 4, it follows that near a Euclidean cone manifold with cone angles $\leqslant \pi$, the set of Dehn surgery coordinates for Euclidean structures in $GS(M)$ actually coincides with all of E.

Moreover, E also agrees locally with the zero set of volume $V = 0$.

(c) It is natural to ask : what is significance of the real part of the function $\int udv - vdu$? In fact, this essentially gives the Chern-Simons invariant $CS(u)$ of the hyperbolic structure on M, when the parameter u corresponds to a manifold. Precisely, the following relationship holds.

Theorem 5.7 (Yoshida).

$$\frac{1}{2\pi}\int udv - vdu = [\frac{2}{\pi}(Vol(u) - Vol(0)) + \sum_i l_i] + i[4\pi(CS(u) - CS(0) + \sum_i \theta_i] \bmod 2\pi\mathbb{Z}$$

where l_i, θ_i give the length and torsion of the geodesic added to complete the hyperbolic structure on $M(u)$.

This was conjectured by Neumann-Zagier [N-Z] and proved by Yoshida in [Y].

It would be interesting to give a different proof of this result using integral geometry. It would also be nice to have a geometric interpretation of the analytic function $CS + torsion$ (or η) for manifolds with Dehn surgery singularities.

28. Applications of the Schläfli formula

We give some applications of the Schläfli formula to hyperbolic Dehn surgery. In particular, we study the level sets of volume and show that in certain cases, the Schläfli formula can be used to find the (local) boundary of hyperbolic Dehn surgery space. Throughout this section M will denote a hyperbolic manifold with boundary consisting of t tori.

Proposition 5.8. *The level sets of volume are star-shaped with respect to to the point at infinity (i.e. the complete structure), using Dehn surgery coordinates. In fact, volume is strictly decreasing along every ray from infinity towards the origin in $H_1(\partial M;\mathbb{R}) \cong \mathbb{R}^2$.* □

Next, note that the level sets of volume satisfy the equation

$$\sum_i l_{\alpha_i} d\theta_{\beta_i} - l_{\beta_i} d\theta_{\alpha_i} = 0 \tag{1}$$

This can be interpreted in terms of Dehn surgery coordinates as follows. Recall that, using the basis α_i, β_i for $H_1(T_i;\mathbb{R})$ the Dehn surgery coefficients are given by (μ_i, λ_i) satisfying the equations

$$\mu_i l_{\alpha_i} + \lambda_i l_{\beta_i} = 0 \tag{2i}$$

$$\mu_i \theta_{\alpha_i} + \lambda_i \theta_{\beta_i} = 2\pi \tag{3i}$$

$i = 1, \ldots, n$.

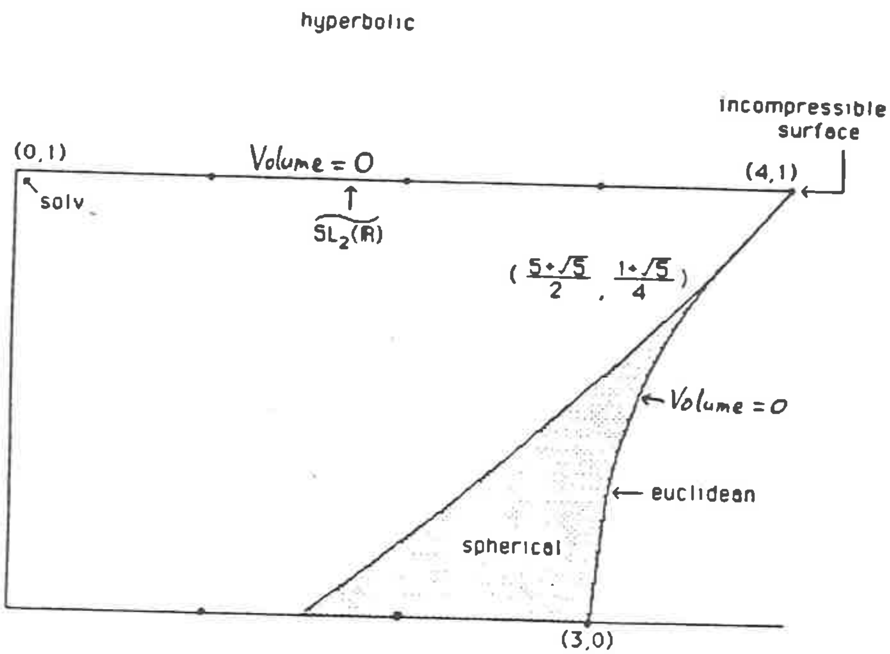
For a vector $(d\theta_{\alpha_i}, d\theta_{\beta_i})$ in the line tangent to (3i) (keeping other $\theta_\alpha, \theta_\beta$ fixed), $\mu d\theta_\alpha + \lambda d\theta_\beta = 0$. Hence $(d\theta_{\alpha_i}, d\theta_{\beta i})$ is proportional to $(l_{\alpha_i}, l_{\beta_i})$ because of (2i). It follows that $l_{\alpha_i} d\theta_{\beta_i} = l_{\alpha_i} d\theta_{\beta_i}$. Hence, the line defined by (2i) is tangent to the level set of volume at the point with Dehn surgery coordinates (μ_i, λ_i). Hence, the real n-dimensional subspace defined by all the equations (2i), $i = 1, \ldots, n$ is tangent In fact the level set can be obtained as the envelope of these subspaces.

Proposition 5.9. *Each level set of volume is an envelope of n-dimensional planes defined by:*

$$\mu_i \theta_{\alpha_i} + \lambda_i \theta_{\beta_i} = 2\pi$$

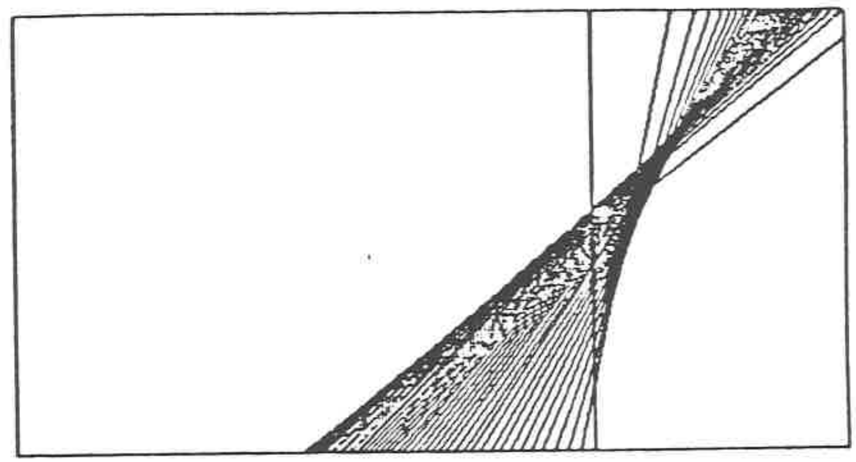
$, i = 1, \ldots, n$. □

Here is an example, showing the level set volume $= 0$ for hyperbolic Dehn surgery on the figure eight knot complement. The second figure exhibits this set as an envelope of straight lines corresponding to representations with zero volume. (See chapter 6, for an exact description of these representations and of the level set.) I thank Rob Johnson for his assistance in preparing these figures.



Dehn Surgeries on the Figure Eight Knot

(Kinds of geometric structure at integer points are also shown — see chapter 6.)



Envelope of lines corresponding to orthogonal representations gives the curved part of the level set volume = 0.

The level set *volume* = 0 is of special interest. Among simpler knot complements (at least), the representions where volume is zero often correspond to real representations $\Gamma \to PSL_2\mathbb{R}$ or orthogonal representations $\Gamma \to SO(3)$. In these cases, the level set volume = 0 (in Dehn surgery coordinates) can be described quite explicitly as follows.

(1) For real representations : $hol(\alpha)=(l_\alpha,\theta_\alpha)=(l_\alpha,n\pi)$ and $hol(\beta)=(l_\beta,m\pi)$ for some integers $n,m\in\mathbf{Z}$. It follows that the level set volume is locally given by the straight line : $n\mu+m\lambda=2$.

(2) For orthogonal representations : $hol(\alpha)=(0,\theta_\alpha)$, $hol(\beta)=(0,\theta_\beta)$. Then the level set is given locally as the envelope of the lines $\mu\theta_\alpha+\lambda\theta_\beta=2\pi$.

(Examples of both these phenomena are evident in the figure eight knot example.)

Schläfli's formula can also be used to study the variation of volume near various "interesting" points in the representation space. For this, we need some information on the relation between lengths and angles.

(1) Behaviour of volume near the complete hyperbolic structure

Each of u and v gives holomorphic coordinates on R in a neighbourhood of the complete representation, with $u=v=0$ corresponding to ρ_0. (See 3.14, 3.15 and [Th1, chap.5].) It follows that $v/u=c+O(u)$ near ρ_0, for some $\sigma\in\mathbf{C}$, and $\mathrm{Im}\sigma>0$ by our choice of orientations. Thurston shows in [Th1, chap.5] (see also [N-Z]) that σ defines the shape of the Euclidean tori at the cusp $T^2\times[0,\infty)$ in the complete hyperbolic structure. The holonomies $\rho_0(\alpha),\rho_0(\beta)$ are parabolic elements. By conjugation we can assume that these elements fix the point at infinity and that $\rho_0(\alpha)=\begin{vmatrix}1&1\\0&1\end{vmatrix}$; then σ is defined by $\rho_0(\beta)=\begin{vmatrix}1&\sigma\\0&1\end{vmatrix}$.

It then follows from proposition 5.5 that

$$V(u)-V(0)=-\frac{\pi}{2}L(u)+O(u^3)$$

In fact the error term is of order $O(u^4)$ since all functions involved are even in u.

It is also enlightening to rewrite the result in terms of Dehn surgery coordinates $(\mu,\lambda)=c\in H_1(T^2;\mathbf{R})$. Since $\mu u+\lambda v=2\pi i$, and $v=\sigma u+O(u^2)$ it follows that $u=\dfrac{2\pi i}{\mu+\sigma\lambda}+\_$, and

$$u\bar{v}=\mathrm{Im}(\bar{\sigma})|u|^2+\_=\frac{-\mathrm{Im}\sigma\, 4\pi^2}{\mu^2+|\sigma|^2\lambda^2}+\_$$

where the terms omitted are of order $\dfrac{1}{\mu^2+\lambda^2}$. If we give T^2 the Euclidean metric coming from the complete hyperbolic structure normalized so $l(\alpha)=1$, then the $\mathrm{Im}\sigma$ is the area $A(T^2)$ of T^2, while $\mu^2+|\sigma|\lambda^2=l(c)^2$, where $l(c)$ is the length of the geodesic in T^2 representing the homology class $c\in H_1(M;\mathbf{Z})$. Then we have :

Proposition 5.10. $V(c)-V(0)=-\pi^2\dfrac{A(T^2)}{l(c)^2}+O(\dfrac{1}{l(c)^4})$, *for c near ∞ in $H_1(T^2;\mathbf{R})$.* □

In particular, the level sets of volume near the complete structure are close to circles when the natural Euclidean metric is used on $H_1(T^2;\mathbf{R})\cong\tilde{T}^2$.

(This gives new proofs of results of Neumann and Zagier in [N-Z].)

We can also obtain results as hyperbolic structures degenerate to foliations. We now describe the dependence of volume on cone angles for degenerations of the kinds discussed in the last chapter. For simplicity, we restrict to the case of hyperbolic cone manifolds degenerating. The behaviour of the "complex volume" for all hyperbolic or spherical manifolds with Dehn surgery type singularities near a limiting structure can be also described using the same ideas.

(2) Euclidean limits

In this case, the hyperbolic manifolds shrink uniformly to a point as $t\to 0$. We can parametrize as in chapter 4, so that at time t all lengths are of order t, then $l(t)=l_0t+O(t^2)$ and $V(t)=V_0t^3+O(t^4)$, where l_0,V_0 denote the length of singular locus and volume in the limiting Euclidean structure. Using $\dfrac{dV}{dt}=-\dfrac{1}{2}l(t)\dfrac{d\theta(t)}{dt}$ gives $\Delta\theta=\theta(t)-\theta(0)=\dfrac{-3V_0}{l_0}t^2+O(t^3)$, hence

$$V(t) = -\left[\frac{l_0^3}{27V_0}\right]^{1/2}(\Delta\theta)^{3/2} + \ldots$$

(Note that this is independent of the choice of Euclidean metric, since $\frac{l_0^3}{V_0}$ is invariant under rescaling.)

(3) $PSL_2\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$ limits

In this case, we have hyperbolic manifolds with distances shrinking linearly in the fibre direction, converging to a 2-dimensional hyperbolic orbifold as limit. We parametrize as in chapter 4, so that at time t the fibres have length of order t. Then $V(t) = at + O(t^2)$, and since the singular locus is transverse to the foliation its length satisfies $l(t) = l_0 + O(t)$, where l_0 is the limiting hyperbolic length of the singular locus. It follows that

$$V(t) = -\frac{1}{2}l_0\Delta\theta + O(t^2)$$

(4) Solv geometry limits

Here the underlying manifold fibres over a circle with torus as fibre. We have a family of hyperbolic manifolds with metrics shrinking in the fibre direction, with a circle as the geometric limit. We parametrize so lengths along the fibres are of order t at time t. Then $V(t) = at^2 + O(t^3)$, and $l(t) = l_0 + O(t)$ since the singular locus is transverse to the fibres, where l_0 is the limiting hyperbolic length of the singular locus. It follows that

$$V(t) = -\frac{1}{2}l_0\Delta\theta + \ldots$$

(5) Nil geometry limits

Orbifolds (and manifolds) with Nil geometry structure can also occur as limits of hyperbolic manifolds. We did not consider this case in chapter 4, however it is discussed in [Th3] and can occur as follows. As in the spherical case we have hyperbolic manifolds shrinking to a point, but after rescaling so that diameter remains constant there is a 2-dimensional Euclidean orbifold as geometric limit. The underlying manifold is a Seifert fibre space with the Euclidean orbifold as base, and the singular locus is transverse to the fibres. We can parametrize so that at time lengths in the fibre direction are of order t^2, while lengths transverse to the fibres are of order t. Then $V(t) = V_0t^4 + O(t^5)$ and $l(t) = l_0t + O(t^2)$, where V_0, l_0 are the volume and length of geodesic in the limiting Nil geometry structure. (These are taken with respect to a specific choice of metric on Nil arising naturally from the degeneration of hyperbolic metrics, as explained in the remark below.) Hence, $\Delta\theta = \theta(t) - \theta(0) = -\left[\frac{8V_0}{3l_0}\right]t^3$ and

$$V(t) = \left[\frac{3l_0\Delta\theta}{8V_0^{1/4}}\right]^{4/3} + \ldots$$

Remarks : At first sight, the term $\frac{l_0}{V_0^{1/4}}$ may seem incorrect, for dimensional reasons. This term is not independent of the choice of a metric on *Nil* with full 4-dimensional isometry group G. However, there is a particular equivalence class of G-invariant metrics determined by the degeneration of hyperbolic structures. To see this, first note that the space of G-invariant metrics on *Nil* is 2-dimensional : the metric can only be altered by rescaling the Euclidean metric on the base and rescaling the metric on the fibres. (No other change gives a metric with 4-dimensional isometry group.)

We obtain a particular family of G-invariant metrics from the degeneration of hyperbolic structures M_t, as follows. At time t, any fibres F of M_t has length of the form $c_1t^2+O(t^3)$ while the length of any "horizontal" geodesic $C\subset M_t$ has length of the form $c_2t+O(t^2)$, where $c_1, c_2\in\mathbb{R}$. We can specify the limiting G-invariant metric on M by requiring that F has length c_1 and C has length c_2. Here, c_1, c_2 are not uniquely determined. If the parameter t is changed they are replaced by c_1s^2, c_2s for some $s>0$. So the metric in the horizontal direction is rescaled by s while the metric in the vertical direction is rescaled by s^2. Under such a change of metric, $l_0 \to sl_0$ and $V_0 \to s^4V_0$, so the ratio $\frac{l_0}{V_0^{1/4}}$ is unchanged.

CHAPTER 6

Finding the boundary of hyperbolic Dehn surgery space

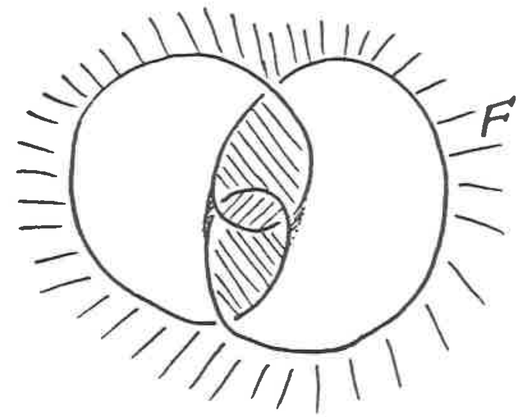
29. An Example : The figure eight knot complement

In this section, we study the Dehn surgery space for the complement of the figure eight knot K in S^3. In particular, we show how the results from previous sections can be useful for finding the exact local boundary of the hyperbolic region.

Throughout this section, M will denote the complement of an open tubular neighbourhood of the figure eight knot in S^3. We will study the space $H(M)$ of hyperbolic structures on M with Dehn surgery type singularities in some detail.

29.1. Representations

It will be convenient to use the fact that M fibres over S^1, with a once punctured torus F as fibre. (F is a minimal genus Seifert surface for the knot as shown below.)



The monodromy for the bundle can be represented by the matrix

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

To obtain a presentation for $\pi_1(M)$, we can choose generators a, b for the free group $\pi_1(F)$ such that the isomorphism on $\pi_1(F)$ induced by the monodromy is :

$$\phi : a \mapsto ab, \qquad b \mapsto bab$$

(This choice is convenient since the commutator $[a, b]$ is fixed by the monodromy; for any choice of generators $[a, b]$ is fixed up to conjugacy, since the monodromy preserves ∂F.) Then

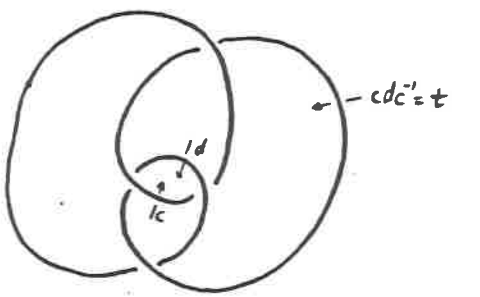
$$\pi_1(M) = \langle a, b, t \mid tat^{-1} = ab, tbt^{-1} = bab \rangle \qquad (1)$$

where t represents a meridian for the knot and projects to a generator for S^1.

There is also a presentation of $\pi_1(M)$ with two generators and one relation

$$\pi_1(M) = \langle c, d \mid cdc^{-1} = dcd^{-1}cdc^{-1}d^{-1} \rangle \qquad (2)$$

obtained from the Wirtinger presentation for the knot group with c, d as shown below :



These sets of generators are related by:

$$t = cdc^{-1}, \qquad a = cd^{-1}, \qquad b = dcd^{-1}c^{-1}$$

and

$$c = b^{-1}tb, \qquad d = a^{-1}ta.$$

We want to study the representation space $R(\pi_1(M), PSL_2(\mathbf{C}))$ however it will be more convenient to work with representations into $SL_2(\mathbf{C})$ rather than $PSL_2(\mathbf{C})$. In fact, any representation $\rho : \pi_1(M) \to PSL_2(\mathbf{C})$ lifts to a pair of representations $\pm\tilde{\rho} : \pi_1(M) \to SL_2(\mathbf{C})$. (This can be seen directly from the second form of presentation for $\pi_1(M)$: Choose arbitrary lifts of $\rho(c)$, $\rho(d)$ to $SL_2(\mathbf{C})$. If the relation of (2) holds this gives a representation into $SL_2(\mathbf{C})$; otherwise we obtain a representation after changing the sign of the lift of $\rho(c)$.)

We first describe the representations $\pi_1(F) \to SL_2(\mathbf{C})$ which extend to representations ρ of $\pi_1(M)$. Let $A = \rho(a)$, $B = \rho(b)$. Then the restriction $\rho | \pi_1(F)$ is determined by the pair (A, B). If $\rho | \pi_1(F)$ is irreducible then (A, B) is completely determined up to conjugacy by the traces trA, trB and $trAB$. Further, $\rho | \pi_1(F)$ is irreducible if and only if $tr[A, B] \neq 2$. (See

e.g. [C-S].) Moreover, an irreducible representation in $SL_2(\mathbf{C})$ has centralizer ± 1; so if $A, A', B, B' \in SL_2(\mathbf{C})$ with $trA = trA'$, $trB = trB'$, $trAB = trA'B'$ and $tr[A, B] \neq 2$ then there is a $C \in SL_2(\mathbf{C})$ such that $CAC^{-1} = A'$ and $CBC^{-1} = B'$ and C is unique up to sign.

It follows that the irreducible representations $\rho : \pi_1(M) \to SL_2(\mathbf{C}))$ can be parametrized by triples $(tr\rho(a), tr\rho(b), tr\rho(ab)) \in \mathbf{C}^3$ satisfying the equations

$$tr\rho(a) = tr\rho(\phi(a)) \tag{3}$$

$$tr\rho(b) = tr\rho(\phi(b))$$

$$tr\rho(ab) = tr\rho(\phi(ab))$$

For any word w in A, B, trw can be expressed as a polynomial in trA, trB and $trAB$ using the well known relations

$$tr(XY) + tr(XY^{-1}) = tr(X)tr(Y) \qquad \text{and} \qquad tr(XY) = tr(YX)$$

for all $X, Y \in SL_2(\mathbf{C})$. In particular, using

$$tr(BAB) + tr(BAB^{-1}) = tr(BA)tr(B)$$

and

$$tr(AB \cdot BAB) + tr((AB)^{-1}BAB) = tr(AB)tr(BAB)$$

gives

$$tr(\rho(b)) = tr(BAB) = tr(B)tr(AB) - tr(A)$$

$$tr(\rho(ab)) = tr(AB)tr(BAB) - tr(B)$$

Similarly,

$$tr([A, B]) = tr(ABA^{-1}B^{-1}) = (trA)^2 + (trB)^2 + (trAB)^2 - trA\,trB\,trAB - 2$$

Combining this with the previous remarks gives the following

Lemma 6.1. *The irreducible representations $\rho : \pi_1(M) \to SL_2(\mathbf{C})$ can be parametrized by the points $(\alpha, \beta, \gamma) \in \mathbf{C}^3$, $\alpha = tr\rho(a)$, $\beta = tr\rho(b)$, $\gamma = tr\rho(ab)$ satisfying the equations*

$$\alpha = \gamma \qquad \beta = \beta\gamma - \alpha \qquad \gamma = \gamma(\beta\gamma - \alpha) - \beta. \tag{4}$$

ρ restricts to an irreducible representation of $\pi_1(F)$ if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 2 \neq 2. \qquad \square$$

In fact, the third equation is a consequence of the first two, and the solutions (α, β, γ) are given by :

$$\gamma = \alpha \qquad\qquad \alpha + \beta = \alpha\beta \tag{5}$$

Remarks : (a) The traces α, β, γ parametrize the character variety for representations $\pi_1(M) \to SL_2(\mathbf{C})$.

(b) The parametrization used here is convenient for studying hyperbolic structures on a punctured torus bundle. The method used here can be used to give another proof of the results from chapter 4 in this case. The important observation is that one of the equations in (3) can *always* be eliminated. This follows from the fact that $[a, b]$ is fixed up to conjugacy by the monodromy; so

$$tr\rho(a)^2 + tr\rho(b)^2 + tr\rho(ab)^2 - tr\rho(a)tr\rho(b)tr\rho(ab)$$

$$= tr\rho(\phi a)^2 + tr\rho(\phi b)^2 + tr\rho(\phi ab)^2 - tr\rho(\phi a)tr\rho(\phi b)tr\rho(\phi ab)$$

From this it follows that any one of the equations in (3) is a consequence of the other two. Hence, the character variety is of complex dimension ≥ 1.

29.2. Connection with ideal triangulations

This description can be related to Thurston's in [Th1, chap.4] as follows. The commutator $l = [a, b] \in \pi_1(M)$ is the homotopy class of a (standard) longitude for the figure eight knot.

Let $v = \log H'(l)$ be the logarithm of the derivative of the holonomy of l, in Thurston's notation. Then $\rho(l) \in PSL_2(\mathbf{C})$ is represented by a matrix in $SL_2(\mathbf{C})$ conjugate to $\pm\begin{bmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{bmatrix}$; so $tr\rho([a, b]) = \pm 2\cosh(v/2)$. The complete hyperbolic structure on M corresponds to $v = 0$; since the corresponding holonomy representation is irreducible on $\pi_1(F)$ we must have : $tr\rho([a,b]) = -2\cosh(v/2)$.

Thurston constructs hyperbolic structures on M by gluing together two ideal simplices. These structures are parametrized by cross ratios z, w describing the shapes of the ideal simplices, subject to the equation

$$z(1-z)w(1-w) = 1 \tag{6}$$

A solution to these equations gives a "simplicial developing map" $D : \tilde{M} \to \mathbf{H}^3$ defining a hyperbolic structure on M if z, w have positive imaginary parts (i.e. the simplices are both positively oriented). D maps each simplex in a (topological) ideal triangulation of M to a geodesic ideal simplex in $\mathbf{H}^3$.

Without the positivity condition, a solution to (*) still gives a representation $\rho : \pi_1(M) \to PSL_2(\mathbf{C})$, and a developing map $D : \tilde{M} \to \mathbf{H}^3$ in the weak sense : satisfying the equivariance condition $D(\gamma m) = \rho(\gamma)D(m)$. However D will not be an immersion if the simplices are not consistently oriented; there will be "folds" along codimension one faces of simplices. In these cases, more work is needed to decide whether or not there is a hyperbolic structure on M with ρ as the holonomy representation.

It follows that solutions to the gluing equations parametrize developing maps rather than representations. There are two solutions to the gluing equations corresponding to each representation. (The two developing maps "spiral" in different directions around the geodesic preserved by $\rho(\pi_1(\partial M))$). The Dehn surgery coordinates parametrize developing maps rather than representations; (p, q) and $(-p, -q)$ give different developing maps corresponding to the *same* point of the character variety for representations $\pi_1(M) \to PSL_2(\mathbf{C})$. (**Caution :** The developing maps for (0,1) and (0,−1) have *different* (reducible) holonomy representations corresponding to the two different transversely hyperbolic foliations on $M_{(0,1)}$. However, the characters of these representations are the same.)

In Thurston's notation, the derivative of holonomy is given by $H'(l) = e^v = z^2(1-z)^2$, so (6) can be written

$$z(z-1) = e^{v/2} \qquad w(w-1) = e^{-v/2} \tag{7}$$

The connection between Thurston's notation and ours is given by the equation

$$-(e^{v/2}+e^{-v/2}) = -tr(\rho([a,b])) = \frac{\alpha^4-3\alpha^3+\alpha^2+4\alpha-2}{(\alpha-1)^2} \tag{8}$$

In particular, for reducible representations of $\pi_1(F)$, $tr(\rho([a,b])) = 2 = -2\cosh(v/2)$, so $z(z-1) = -1$ or

$$0 = \alpha^4-3\alpha^3-\alpha^2+8\alpha-4 = (\alpha-2)^2(\alpha^2+\alpha-1)$$

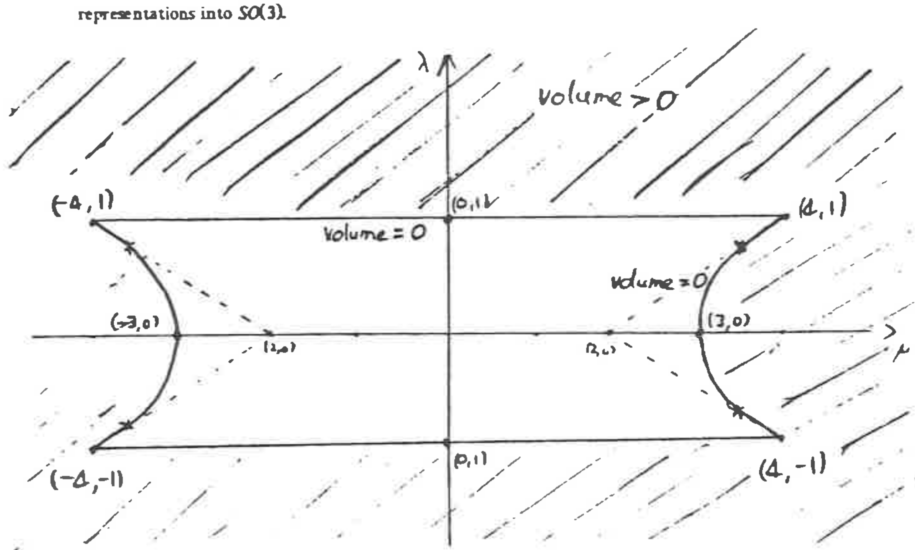
The first equation has solutions : $z = \frac{1+\sqrt5}{2}, \frac{1-\sqrt5}{2}$. The second has solutions : $\alpha = 2, \frac{-1-\sqrt5}{2}, \frac{-1+\sqrt5}{2}$ and corresponding $\beta = 2, \frac{-1+\sqrt5}{2}, \frac{-1-\sqrt5}{2}$.

For the solution $\alpha = 2$, the corresponding Dehn surgery coordinates are $(0,+1)$ $(z = -0.618...)$ and $(0,-1)$ $(z = +1.618...)$. In both cases the limiting holonomy describes a hyperbolic foliation on the torus bundle $M_{(0,1)}$; two different foliations are obtained corresponding to the two eigenvectors of the matrix Φ.

The solutions $\alpha = \frac{-1-\sqrt5}{2}, \frac{-1+\sqrt5}{2}$ occur when the Dehn surgery coordinates are $(\pm\frac{5+\sqrt5}{2}, \pm\frac{1+\sqrt5}{4}) = (\pm3.618033..., \pm0.809017...)$. The corresponding representations occur at the intersection point between the curves of representations into $SO(3)$ and into $PSL_2(\mathbb{R})$. We will discuss this further below.

29.3. The zero set of volume

By a computer study of the hyperbolic Dehn surgery space for M (for instance using the programs of Jeff Weeks described in [We]) one can easily determine approximately the zero set of volume in Dehn surgery coordinates. From this, the zero set appears to consist of straight line segments corresponding to representations into $PSL_2(\mathbb{R})$, and curves corresponding to representations into $SO(3)$.



(Solutions with all simplices flat (all cross ratios real) clearly give representations into $PSL_2(\mathbb{R})$ up to conjugacy. Solutions where the holonomy of each boundary torus consists of rotations are good *candidates* for orthogonal representations; but further work is needed to confirm this in any example.)

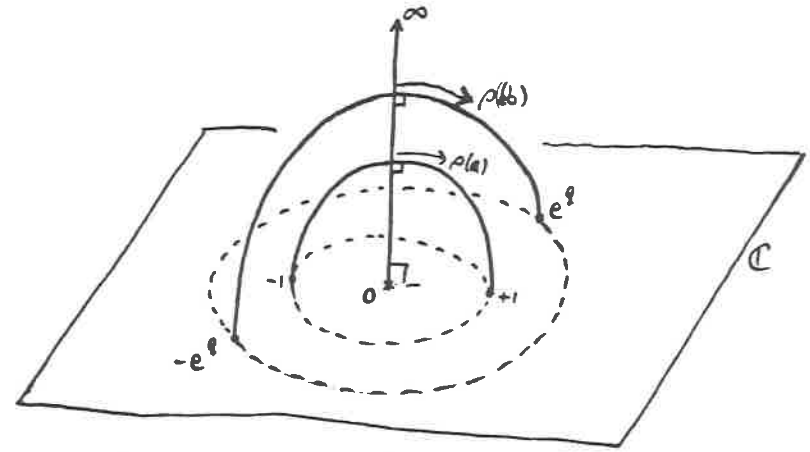
This suggests that exact zero set corresponds precisely to the representations conjugate to ones with image contained in $PSL_2\mathbb{R}$ or $SO(3)$. (Equivalently, the real points of the complex algebraic character variety : i.e. representations such that $tr\rho(\gamma)\in\mathbf{R}$ for all $\gamma\in\pi_1(M)$.) We now show that this is indeed the case; and give an exact description of the level set volume = 0.

For this we use an explicit form for the representations of $\pi_1(M)$. We can conjugate so that $\rho(a)=\begin{vmatrix}\cosh p & \sinh p\\ \sinh p & \cosh p\end{vmatrix}$ and $\rho(ab)=\rho(t)\rho(a)\rho(t)^{-1}=\begin{vmatrix}\cosh p & e^{-q}\sinh p\\ e^{q}\sinh p & \cosh p\end{vmatrix}$; then $\alpha=tr\rho(a)=tr\rho(ab)=2\cosh p$, $\quad\beta=tr\rho(b)=2\cosh^2 p-2\cosh q\sinh^2 p=\dfrac{\alpha}{\alpha-1}$. Hence

$$\cosh q=\frac{2\cosh^2 p-\beta}{2\sinh^2 p}=\frac{\frac{1}{2}\alpha^2-\beta}{\frac{1}{2}\alpha^2-2}=\frac{\alpha^2+\alpha}{\alpha^2+\alpha-2}.$$

(We won't need to find the exact form of $\rho(t)$, but it could be determined as follows. Since $\rho(t)$ takes the axis of $\rho(a)$ to the axis of $\rho(ab)$, $\rho(t)=\begin{vmatrix}e^{\frac{1}{2}q} & 0\\ 0 & e^{\frac{-1}{2}q}\end{vmatrix}\begin{vmatrix}\cosh r & \sinh r\\ \sinh r & \cosh r\end{vmatrix}$ and r can be determined from $\rho(tbt^{-1})=\rho(bab)$.)

Now the fixed points of $\rho(a)$ and $\rho(ab)$ in the sphere at infinity $\mathbf{C}\cup\infty$ are given by : $\{-1,1\}$ and $\{-e^q,e^q\}$ respectively. So the axes of $\rho(a)$, $\rho(ab)$ have the following arrangement in the upper half space model for $\mathbf{H}^3$.



It follows that the restriction $\rho|\pi_1(F)$ is conjugate to a representation into $PSL_2(\mathbf{R})$ if and only if

(1) $e^q\in\mathbf{R}$, i.e. $\alpha\in\mathbf{R}$.

Further, $\rho|\pi_1(F)$ is conjugate to an orthogonal representation if and only if

(2) $\alpha\in\mathbf{R}$ and $q\in i\mathbf{R}$ is pure imaginary.

The last condition is equivalent to requiring that $\cosh q=\dfrac{\alpha^2+\alpha}{\alpha^2+\alpha-2}$ is in the interval $[-1,1]$, and holds for α in the interval with enpoints given by $\dfrac{\alpha^2+\alpha}{\alpha^2+\alpha-2}=-1$, i.e. $\dfrac{-1-\sqrt{5}}{2}\leqslant\alpha\leqslant\dfrac{-1+\sqrt{5}}{2}$. Further, the real and orthogonal representations intersect at the points $\alpha=\dfrac{-1\pm\sqrt{5}}{2}$.

In fact, the conditions (1), (2) imply that $\rho:\pi_1(M)\to PSL_2(\mathbf{C})$ is conjugate into $PSL_2(\mathbf{R})$ or $SO(3)$ respectively. This follows easily from the fact that $\pi_1(F)$ is a normal subgroup of $\pi_1(M)$. (If $\rho|\pi_1(F)$ is an irreducible orthogonal representation, then there is a *unique* point p in $\mathbf{H}^3$ fixed by $\rho(\pi_1(F))$. Let $g\in\rho(\pi_1(M))$. Then for all $f\in\rho(\pi_1(F))$ $g^{-1}fg\cdot p=p$, so gp is fixed by $g^{-1}fg$ for all $f \in \rho(\pi_1(F))$. Since $\pi_1(F)$ is a normal subgroup of $\pi_1(M)$ it follows that the point gp is fixed by all of $\rho(\pi_1(M))$; hence $gp = p$. In the irreducible $PSL_2\mathbb{R}$ case, there is a *unique* plane $\mathbb{H}^2 \subset \mathbb{H}^3$ invariant under $\pi_1(F)$. Again, this is invariant under $\rho(\pi_1(M))$ by normality.)

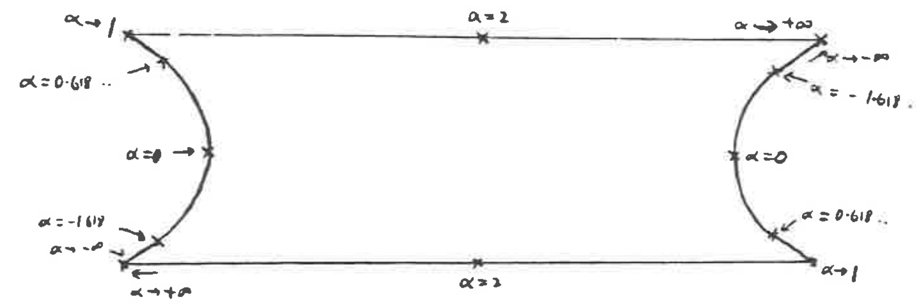
We conclude that the characters of real representations are given by the set of real solutions to equations (5).

These representations can be divided into three kinds:

(1) $1 < \alpha < \infty$

(2) $-\infty < \alpha \leqslant \dfrac{-1-\sqrt{5}}{2}$ or $\dfrac{-1+\sqrt{5}}{2} \leqslant \alpha < 1$

(3) $\dfrac{-1-\sqrt{5}}{2} \leqslant \alpha \leqslant \dfrac{-1+\sqrt{5}}{2}$. One can find the trace coordinates α corresponding to Thurston's Dehn surgery coordinates by analytic continuation from the complete solution. The curve corresponding to $\alpha \in \mathbb{R}$ is as shown below.



29.4. Searching for the boundary of hyperbolic Dehn surgery space

We now discuss the geometric significance of the representations with volume equal to zero. Using our previous results we will see, for example, that the zero set of volume is *locally* the exact boundary of the space $H(M)$ near each point with integer Dehn surgery coordinates (i.e. corresponding to an orbifold structure).

(1) Representations with $1 < \alpha < \infty$ and Dehn surgery coordinates $(m, 1)$, $-4 < m < 4$.

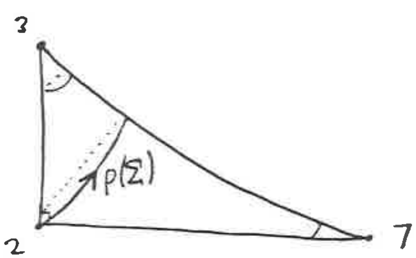
These give real representations corresponding to hyperbolic foliations. The foliations can be seen directly, since we have two positively oriented simplices flattening out simultaneously. It was shown explicitly in [Th] that this is part of the exact boundary of hyperbolic Dehn surgery space.

Special cases :

(a) (0 , 1) surgery on M gives a torus bundle over S^1 with Anosov gluing map with matrix $\Phi = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$. There are two hyperbolic foliations of the torus bundle $M_{(0,1)}$ coming from linear foliations of the torus by curves of irrational slope, parallel to the eigenvectors of Φ. Each of these hyperbolic foliations can be deformed to obtain nearby hyperbolic structures at all nearby points (x, y) with $y>1$ and nearby $PSL_2\mathbb{R}$ structures with Dehn surgery singularities for nearby points $(x, 1)$. (Compare section 24.)

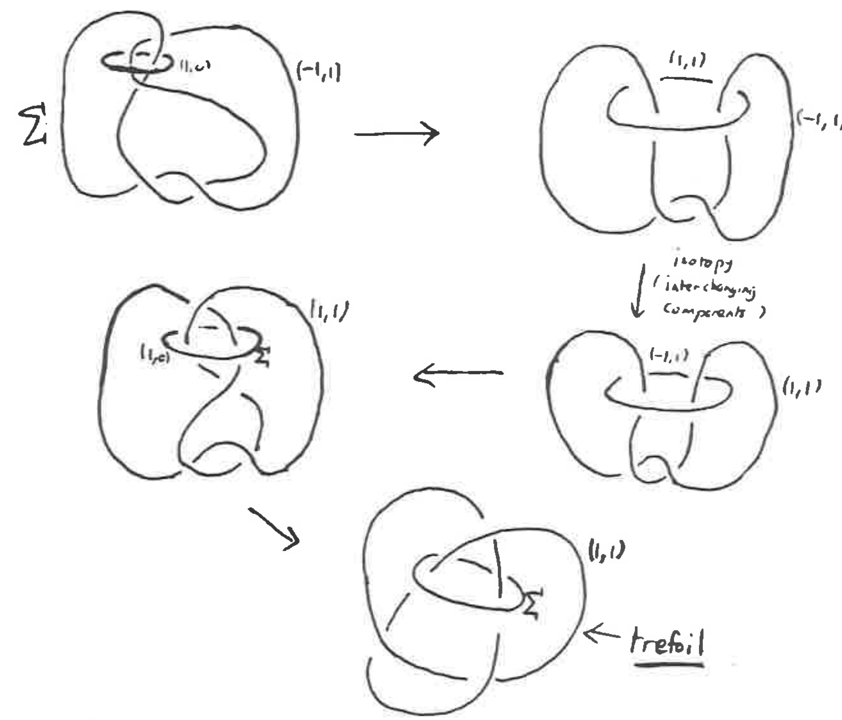
(b) (n , 1) surgery on M for $-4<n<4$ gives Seifert fibred spaces over the sphere with 3 cone points. (Compare section 21.)

Example : (-1 , 1) surgery on figure eight knot. This is the unit tangent bundle of the (2,3,7) spherical orbifold and the singular locus is the horizontal lift of geodesic through the order 2 cone point (double covering a geodesic in base):



To see this, one can use Kirby calculus to reduce to (1 , 1) surgery on trefoil knot with

singular locus transverse to the fibres as shown in the following diagram.



This shows that the result is Seifert fibred over a sphere with exceptional fibres of orders 2, 3 and 7; it is not hard to check that the Seifert notation is $(Oo0 : b = -1; (2, 1), (3, 1), (7, 1))$. Since the Euler class is $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}-1 = \frac{-1}{42}$ is the Euler characteristic of the $S^2_{2,3,7}$-orbifold this is the unit tangent bundle to this orbifold. It is easy to see that the singular locus projects to curve shown above.

(2) Representations with $-\infty < \alpha < \frac{-1-\sqrt{5}}{2} = -1.618\_$ with Dehn surgery coordinates on straight line from $(3.618\_, .809\_)$ to $(4, 1)$ and representations with $\frac{-1+\sqrt{5}}{2} = 0.618\_ < \alpha < 1$ with Dehn surgery coefficients on the straight line from $(-3.618\_, 0.809\_)$ to $(-4, 1)$.

These are also real representations since simplices are flat.

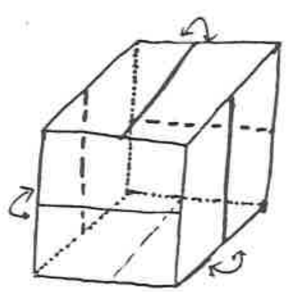
Questions: Do all these representations correspond to hyperbolic foliations with Dehn surgery singularities, possibly degenerating to a foliation with transverse Euclidean structure at $(3.618\_, .809\_)$? If so, can the foliations be deformed to hyperbolic structures?

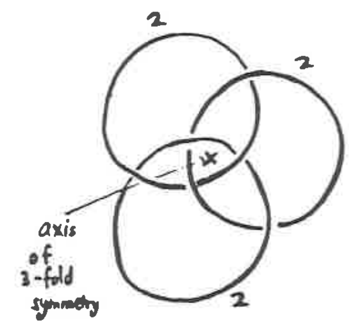
We hope to resolve these questions in the future.

(3) Orthogonal representations with $-1.618 < \alpha < .618\_$. The corresponding Dehn surgery coordinates lie on a curve from $(3.618\_, 0.809\_)$ passing through $(3,0)$ to $(3.618\_, -0.809\_)$.

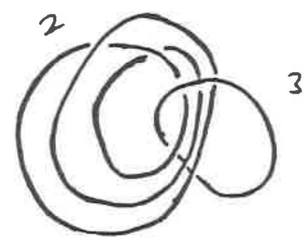
This curve can be determined exactly as an envelope of lines corresponding to orthogonal representations (compare proposition 5.9 and the following remarks).

Special case : The orbifold corresponding to $(3, 0)$ has a Euclidean structure obtained as follows. The Borromean rings labelled with 2's, has a well known Euclidean structure, obtained from a cube by folding up faces along axes as shown below :

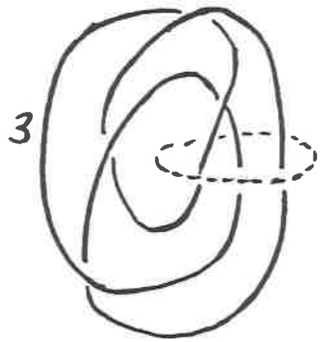




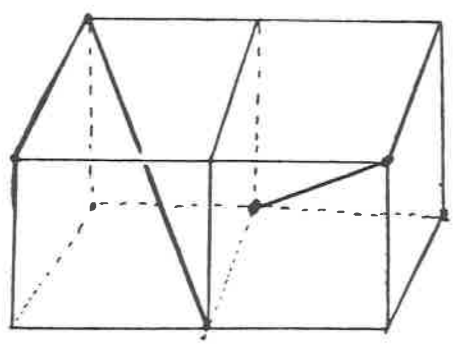
The Borromean rings have a 3-fold symmetry, giving the quotient orbifold :



The two components of the resulting link can be interchanged by an isotopy. Taking the 2-fold covering over the component labelled 2 gives a Euclidean orbifold structure on S^3 with the figure eight knot, labelled 3, as the singular locus.

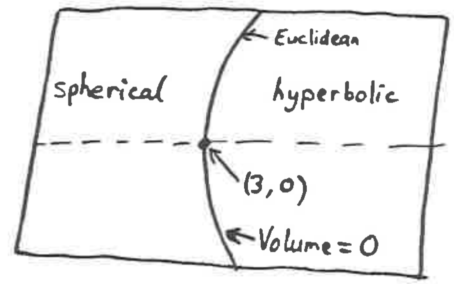


The corresponding Euclidean holonomy group is generated by 120° rotations about diagonals of two adjacent cubes as shown below :



The theory from section 20 implies that near this point, vol $= 0$ corresponds to Euclidean structures with Dehn surgery singularities, and is *locally* the boundary of hyperbolic Dehn surgery space. Furthermore, the Euclidean structure can also be approximated by spherical structures with Dehn surgery type singularities. If we consider the larger space

$GS(M)$ of all constant curvature geometric structures on M with Dehn surgery type singularities, then we locally obtain a manifold. The Dehn surgery coordinates give a local diffeomorphism from geometric structures in $GS(M)$, up to rescaling of metrics, to a neighbourhood of $(3, 0)$ in $\mathbb{R}^2$. The Euclidean structures correspond to the codimension one subspace, where *volume* $= 0$.



Question: Do all these orthogonal representations correspond to Euclidean structures with Dehn surgery type singularities, possibly degenerating at the endpoints?

One direct approach to this question would involve constructing a fundamental domain bounded by minimal surfaces. (Compare next section).

Remark : Cone manifold structures for the figure eight knot.

Hyperbolic, Euclidean, and spherical cone manifold structures on S^3 with the figure eight knot as the singular locus can be constructed explicitly as follows. Begin with a Dirichlet fundamental domain for the Euclidean cone manifold structure with cone angle $2\pi/3$; one third of a rhombic dodecahedron is a good choice. Then one can deform this polyhedra to obtain hyperbolic and spherical polyhedra, with different dihedral angles, which can be glued up to give hyperbolic and spherical cone manifold structures. In fact, using some care, such cone manifold structures can be obtained whenever the cone angle θ is in the open interval $(0, 4\pi/3)$. These cone structures are hyperbolic for $\theta < 2\pi/3$, Euclidean for $\theta = 2\pi/3$ and spherical for $2\pi/3<\theta<4\pi/3$. In particular, one can see directly the continuous change from hyperbolic to Euclidean to spherical structures near $\theta=2\pi/3$.

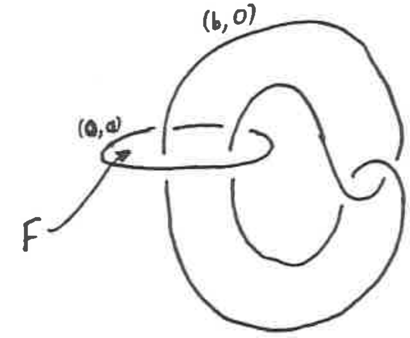
Further, there is no such cone manifold structure for $2\pi>\theta\geqslant 4\pi/3$. (This follows by looking at the possible orthogonal representations.) There is a limiting spherical cone structure on S^3 with cone angle $4\pi/3$, but the singular locus is no longer a figure eight knot : the cone locus bumps into itself in the limit. We expect that one could continue deforming this spherical cone structure continuously to obtain cone angles larger that $4\pi/3$ provided one is willing to allow the singular locus to change. The argument of Thurston in [Th3] shows that the singular locus cannot change combinatorial type in this way when the cone angle is $\leqslant\pi$.

30. Speculation and Open Problems

It seems likely that by a slight extension of ideas introduced here, one could show that the zero set of volume is *locally* the boundary of hyperbolic Dehn surgery space, for the figure eight knot complement. More precisely, this would mean that for every point on the zero set all nearby representations with positive volume correspond to hyperbolic structures with Dehn surgery type singularities. In general, the boundary of hyperbolic Dehn surgery space will be more complicated.

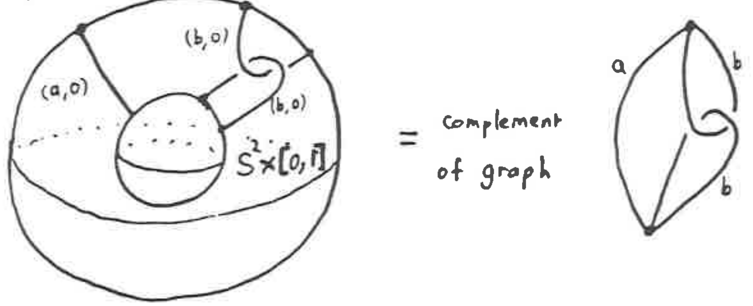
For example, there can be points in the boundary of the hyperbolic Dehn surgery space $H(M)$ where the holonomy representation has positive volumes. This behaviour can occur, for example, when some Dehn surgery on M gives an orbifold N containing an incompressible Euclidean suborbifold F such that a component of $N-F$ has a hyperbolic structure. The process is a 3-dimensional version of the degeneration that occurs going to the boundary of Teichmüller space for a surface, when a surface containing a geodesic with length going to infinity splits open to develop a cusp in the limit. (Compare section 19, p. 88.)

Another 3-dimensional example can be seen for $(0,a),(b,0)$ surgery on the Whitehead link complement M as $\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\to 1$. The limiting surgered orbifold contains an incompressible 3-punctured sphere F with cone angles of $2\frac{\pi}{a}, 2\frac{\pi}{b}, 2\frac{\pi}{b}$. When $\frac{1}{a}+\frac{2}{b}<1$ there are hyperbolic structures on $M_{(0,a),(b,0)}$ with representations "going to infinity" in $R(\pi_1(M), PSL_2\mathbb{C})/PSL_2\mathbb{C}$ as $\frac{1}{a}+\frac{2}{b}\to 1$. Choosing basepoints suitably, a geometric limit exists and is a "complete" hyperbolic structure on the cone manifold $M_{(0,a),(b,0)}-F$.



In this case, it is not hard to see that this complete structure can be deformed to give hyperbolic structures on $M_{(0,a),(b,0)}-F$, with $\frac{1}{a}+\frac{1}{b}+\frac{1}{b}>1$.

M split open along F:



In general, at least one other kind of degeneration can be expected to occur on the boundary of Dehn surgery spaces: the singular locus can "bump into itself" changing the combinatorial type of the singular locus. (This happens for spherical cone manifold structures on the figure eight knot; see the remarks at the end of the last section.)

Many interesting global questions are open. For example, is the space $H(M)$ connected, or star shaped? Is the map $H(M)\to H_1(\partial M;\mathbf{R})$ taking a hyperbolic structure to its Dehn surgery coordinate an injection? When is a representation $\rho:\pi_1(M)\to PSL_2(\mathbf{C})$ the holonomy of a hyperbolic structure in $H(M)$? (Or more generally : find some kind of geometric structure, with singularities, corresponding to ρ.) Another important problem is to find good general estimates on the size of the region of non-hyperbolic Dehn surgeries.

(1) One way to resolve these questions for fixed manifold M is to directly construct hyperbolic structures on M, corresponding to representations.

The approach, introduced by Thurston [Th1], of constructing hyperbolic structures by gluing together ideal hyperbolic simplices is very useful for studying examples. The programs developed Jeff Weeks (see [We]) implement this approach and give an automated method for studying the hyperbolic Dehn surgery spaces for links in the 3-sphere. However, not all

hyperbolic structures with Dehn surgery singularities can be obtained by gluing together *positively oriented* ideal hyperbolic simplices. For instance, a sequence of hyperbolic structures degenerating to a point cannot be obtained in this way.

Another obvious approach to constructing hyperbolic structures is by gluing together non-ideal hyperbolic polyhedra. With this method it should be possible to find the exact boundary of $H(M)$ along lines where the Dehn surgery coordinates have rational slope. Further, one can hope to pass continuously from hyperbolic to Euclidean to spherical cone manifold structures in this way. We have discussed examples where this can be done in chapter 4 (section 19), and in the remarks in the last section. The gluing equations in this approach are somewhat more complicated than those obtained using ideal simplices, but the method is quite feasible using a computer. However, only cone type singularities can be produced in this way (at least if all polyhedra are compact).

Another method, suggested recently by Thurston, is to glue together objects bounded by certain kinds of minimal surfaces, instead of polyhedra bounded by totally geodesic polygons. For instance, the faces of such a "minimal surface polyhedron" could be the minimal surface spanned by a right angled hexagon in $\mathbf{H}^3$ (or $\mathbf{E}^3$). With this approach it is possible to obtain arbitrary Dehn surgery type singularities and it seems likely that every structure in $H(M)$ could be obtained in this way.

A better understanding of foliations (with Dehn surgery type singularities) of 3-manifolds with transverse geometric structure would also help considerably in the problem of finding the boundary of Dehn surgery spaces. Computer studies using the programs of Weeks [We] suggest that such hyperbolic foliations occur frequently on the boundary of Dehn surgery spaces for knots and links in S^3.

(2) Another approach to these questions is to extend the work of Thurston in [Th3] and obtain a more global deformation theory. We outline some of the open questions.

(a) Local theory. For hyperbolic cone manifold M with cone angles $\leqslant\pi$, it is not known in general whether the space of representations $R(\pi_1(M), PSL_2\mathbb{C})$ is locally a manifold of dimension equal to the number of cusps of M. This is equivalent to a local rigidity theorem : there are no deformations of the hyperbolic structure M fixing the cone angles (or Dehn surgery coordinates). Alternatively, this is a cohomology problem : is $H^1(M, \partial M; E(\rho)) \to H^1(M; E(\rho))$ the zero map? See chapter 3 for further discussion. The same questions are open for cone angles greater than π; it is likely that some restriction on the combinatorial type of the singular locus is needed in this case. One possible approach to these problems would involve developing a theory of harmonic deformations (e.g. Hodge theory) for manifolds with cone-type and/or generalized Dehn surgery type singularities.

(b) Global theory. One of the main difficulties in extending the work of [Th3] to cone angles greater than π, is the possibility of changes in the combinatorial type of the singular locus as geometric structures are deformed. When cone angles are $\geqslant\pi$, the singular locus can bump into itself; Thurston shows in [Th3] that this cannot happen for cone angles less than π. It seems possible that one can continue deforming geometric structures when this bumping occurs, provided that singular locus is allowed to change.

The question of global rigidity for hyperbolic and other geometric structures with given cone angles or Dehn surgery coefficients also seems interesting.

Finally, we point out that many of the techniques developed here apply in arbitrary dimensions. There may well be interesting applications to problems in dimensions greater than three.

Appendix

1. Cohomology theory

In this appendix, we recall some standard definitions and properties of group cohomology and de Rham cohomology with coefficients in a flat vector bundle. See [Br], [R] and [B-T] for further details.

1.1. Group cohomology

Let Γ be a group. A Γ-module (or $\mathbb{Z}\Gamma$-module) is given by an abelian group V together with an action $\rho : \Gamma \to Hom_{\mathbb{Z}}(V)$ of Γ on V. Then the cohomology $H^*(\Gamma; V)$ of Γ with coefficients in V is defined as follows. Let $C^p = C^p(\Gamma, V)$ be the set of all maps $\Gamma^p \to V$ ($C^0 = V$). Define $d : C^p \to C^{p+1}$ by

$$df(x_1, \ldots, x_p) = \rho(x_1)f(x_2, \ldots, x_p)$$

$$+ \sum_{i=1}^{p-1} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_p) + (-1)^p f(x_1, \ldots, x_{p-1})$$

Then $d^2 = 0$ and $H^*(\Gamma, V)$ is the cohomology of the complex $\{C^p, d\}$. Thus $H^i = \frac{Z^i}{B^i}$, where $Z^i = ker(d : C^i \to C^{i+1})$ are the i-cocycles and $B^i = im(d : C^{i-1} \to C^i)$ are the i-coboundaries of Γ with coefficients in V.

In particular the maps d^0, d^1, d^2 are given by:

$$d^0 f(x) = \rho(x)f - f$$

$$d^1 f(x, y) = f(x) + \rho(x)f(y) - f(xy)$$

$$d^2 f(x, y, z) = \rho(x)f(y, z) - f(xy, z) + f(x, yz) - f(x, y)$$

Finally, we recall the definition of cup products in group cohomology Let U, V, W be Γ-modules, and make $U \otimes V$ into a Γ-module with the diagonal Γ action : $\gamma u \otimes v = \gamma u \otimes \gamma v$. Let $\mu : U \otimes V \to W$ be a Γ-module homomorphism. Then there is a cup product $\cup : H^p(\Gamma; U) \otimes H^q(\Gamma; V) \to H^{p+q}(\Gamma; U \otimes V)$ defined by :

$$a \cup b(x_1, \dots, x_{p+q}) = (-1)^{pq} \mu(a(x_1, \dots, x_p) \otimes x_1 \dots x_p b(x_{p+1}, \dots, x_{p+q}))$$

for $a \in C^p(\Gamma; U)$, $b \in C^q(\Gamma; V)$.

1.2. De Rham cohomology

The de Rham cohomology with coefficients in the flat vector bundle E is the cohomology of the complex $\{\Omega^*, d_E\}$, where $\Omega^i = \Omega^i(M, E)$ is the vector space of E valued i-forms on M, and the differential $d_E : \Omega^i \to \Omega^{i+1}$ arises from the flat connection on E. More precisely, let $e_1, \dots, e_k$ be *flat* local sections of $E(\rho)$ giving a basis for the fibre at each point of an open set $V \subset M$. Then an element ω of Ω^i can be represented locally as

$$\sum_j \omega_j e_j$$

where ω_j is an (R-valued) i-form. Then d_E is given by

$$d_E \omega = \sum_j d\omega_j e_j$$

where d is the usual exterior derivative. (See [B-T] or [R] for more details.)

We will also be using relative de Rham cohomology $H^*(M, N; E)$ where N is a closed submanifold of M and E is flat vector bundle on M. This is just the cohomology of the complex $\Omega^*(M, N; E) = ker(\Omega^*(M; E) \xrightarrow{restrict} \Omega^*(N; E))$ consisting of E-valued forms on M vanishing on N. Then there is an exact sequence :

$$\dots \longrightarrow H^i(M, N) \longrightarrow H^i(M) \longrightarrow H^i(N) \longrightarrow H^{i+1}(M, N) \longrightarrow \dots$$

1.3. Interpreting H^*

The elements of $H^0(M; E(\rho))$ are exactly the global flat sections of E and can be interpreted geometrically as global Killing vector fields on M. $H^0(\Gamma; g_{Ad\rho})$ is the space of Γ-invariants, $\{x \in g \mid Ad\rho(\gamma).x = 0$ for all $\gamma \in \Gamma\}$, and is Lie algebra of the centralizer of $\rho(\Gamma)$ in G.

Elements of $H^1(M; E(\rho)) \cong H^1(\Gamma; Ad\rho)$ represent tangent vectors to spaces of representations. (See chapter 1 and 3.)

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