HEEGAARD SPLITTINGS OF AMALGAMATED 3-MANIFOLDS

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Abstract. This paper is very abstract.

Keywords: Heegaard Splitting, Incompressible Surface

1. Introduction

This is the greatest paper ever written.

2. Definitions

In this section, we give some of the standard definitions that will be used throughout the paper.

A loop γ embedded in the interior of a surface F is called *essential* if it does not bound a disk in F. If F is embedded in a 3-manifold, M, a *compressing disk* for F is a disk, $D \subset M$, such that $F \cap D = \partial D$, and such that ∂D is essential on F. If we identify a thickening of D in $\overline{M \setminus F}$ with $D \times I$ then to *compress F along D* is to remove $(\partial D) \times I$ from F and replace it with $D \times \partial I$.

A properly embedded surface is *incompressible* if there are no compressing disks for it. A properly embedded, separating surface is *strongly irreducible* if there are compressing disks for it on both sides, and all compressing disks on one side intersect all compressing disks on the other.

Suppose F is either an orientable, (not necessarily connected) surface embedded in $\mathbb{R}^3$, or a point. In the former case F separates $\mathbb{R}^3$ into X and Y. In the latter case we let $X = \mathbb{R}^3$ and $Y = \emptyset$. Let Σ denote a connected complex obtained from F by attaching arcs in X with endpoints on F. Let C denote the closure of a neighborhood of Σ. Any manifold W homeomorphic to C is called a *compression body.* The image of Σ under such a homeomorphism is a *spine* of W. The image of $\partial C \cap Y$ is denoted $\partial_- W$ and the image of $\partial C \cap X$ is $\partial_+ W$.

A surface, F, in a 3-manifold, M, is a *Heegaard surface for M* if F separates M into two compression bodies, W, and W', such that $F = \partial_+ W = \partial_+ W'$.

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3. Labelling sweepouts

In this section we prove the main technical lemma from which all subsequent results will follow.

Lemma 3.1 (Scharlemann [1])**.** *Let H be a strongly irreducible Heegaard surface, and γ be an essential curve on H. If γ bounds a disk then it bounds a compressing disk for H.*

Definition. Two surfaces H and F embedded in a 3-manifold are *almost transverse* is their only non-transverse intersection point is a saddle.

Note that if H and F are almost transverse, and D is a compressing disk for a component of $H \setminus F$, then it follows from our definitions in the previous section that ∂D does not go through the saddle point of $H \cap F$.

Lemma 3.2. *Let M be a compact, irreducible, orientable 3-manifold whose boundary, if non-empty, is incompressible. Suppose $M = V \cup_H W$, where H is a strongly irreducible Heegaard surface. Suppose further that M contains an incompressible, orientable, closed, non-boundary parallel surface F. Then H may be isotoped to be almost transverse to F, with every component of $H \setminus F$ incompressible in the respective submanifold of $M \setminus F$, except for at most one strongly irreducible component.*

Remarks 3.3.
1. After applying the lemma every loop of $H \cap F$ must be essential on both surfaces. Otherwise there is such a loop that bounds a compressing disk D for a component H' of $H \setminus F$. As H' is thus not incompressible it must be strongly irreducible. But there is no compressing disk on the opposite side of H' which meets D.
2. In the case where $F \cong \mathbb{T}^2$ it will follow from the proof that H may actually be isotoped to be transverse to F, while satisfying the conclusion of the lemma.

Proof of Lemma 3.2. Choose spines Σ_V of V and Σ_W of W.

Claim. The surface F meets both Σ_V and Σ_W.

Proof. Suppose $F \cap \Sigma_V = \emptyset$. Then F lies in a compression body homeomorphic to W. As the only incompressible surfaces in W are components of $\partial_- W$, we conclude that F is boundary parallel in M. This violates the hypotheses of Lemma 3.2. □

It is a standard result that there is a continuous map $\Phi : H \times I \to M$ such that

- $H(0) = \Sigma_V$,
- $H(1) = \Sigma_W$, and
- the restriction of Φ to the open interval $(0, 1)$ is a smooth homeomorphism onto the complement of $\Sigma_V \cup \Sigma_W$.

Here we are denoting the image of $H \times t$ under Φ as $H(t)$. The map Φ is a *sweepout* of M. Let $V(t)$ and $W(t)$ denote the compression bodies bounded by $H(t)$ (where $\Sigma_V \subset V(t)$).

Perturb F so that $\pi \circ \Phi^{-1}|_F$ is Morse on $F \setminus (\Sigma_V \cup \Sigma_W)$, where π denotes projection onto the second factor. Let $\{t_i\}_{i=0}^n$ denote the set of critical values of $\pi \circ \Phi^{-1}|_F$. It follows from Claim 3 that $t_0 = 0$ and $t_n = 1$. We now label each subinterval (t_i, t_{i+1}) with the letters $\mathbb{V}$ and/or $\mathbb{W}$ by the following scheme. If, for some $t \in (t_i, t_{i+1})$, there is a compressing disk for $H(t)$ in $V(t)$ whose boundary is disjoint from F then label this subinterval with the letter $\mathbb{V}$. Similarly, if there is a compressing disk in $W(t)$ whose boundary is disjoint from F then label with the letter $\mathbb{W}$.

Claim 3.4. If the subinterval (t_i, t_{i+1}) is unlabelled then the conclusion of Lemma 3.2 follows.

Proof. Suppose $t \in (t_i, t_{i+1})$. First, we claim that all curves of $H(t) \cap F$ are essential on both or inessential on both. If not then, as F is incompressible, there is a loop $\delta' \subset H(t) \cap F$ that is inessential on F but essential on $H(t)$. The loop δ' bounds a subdisk D' of F. Let δ denote a loop of $H(t) \cap D'$ which is innermost among all loops which are essential on $H(t)$. Then δ bounds a subdisk D of D'. Furthermore, every loop of $D \cap H(t)$ is inessential on both surfaces. Hence, as M is irreducible, we may remove all such loops by a sequence of isotopies. We conclude that δ bounds a compressing disk for $H(t)$. Finally, δ may be pushed off of F on $H(t)$, violating the assumption that (t_i, t_{i+1}) is unlabelled.

As M is irreducible we may isotope $H(t)$ to remove those loops of $H(t) \cap F$ which are inessential on both surfaces, without effecting those loops of $H(t) \cap F$ which were essential on both. We now claim that after such an isotopy any essential loop of $H(t) \setminus F$ is essential on $H(t)$. We prove the contrapositive. Let γ be a loop which bounds a disk $E \subset H(t)$. All curves of $E \cap F$ must be inessential on both surfaces, and hence there are now none. We conclude $E \subset M \setminus F$, and hence γ is inessential on $H(t) \setminus F$.

Finally, we claim that the components of $H(t) \setminus F$ are incompressible in the respective submanifolds of $M \setminus F$. Suppose H' is a compressible component. Then there is an essential loop $\gamma \subset H'$ which bounds a compressing disk for H'. By the preceding remarks γ is essential on

$H(t)$ as well. By Lemma 3.1 the loop γ bounds a compressing disk for $H(t)$, which must be in $V(t)$ or $W(t)$. This now contradicts the fact that (t_i, t_{i+1}) is unlabelled. □

Claim 3.5. If the subinterval (t_i, t_{i+1}) has both of the labels $\mathbb{V}$ and $\mathbb{W}$ then Lemma 3.2 follows.

Proof. Suppose $t \in (t_i, t_{i+1})$. We begin as in the proof of Claim 3.4 by asserting that all curves of $H(t) \cap F$ are either inessential or essential on both. If not, then as in the aforementioned proof there is a loop $\delta \subset H(t) \cap F$ which bounds a compressing disk for $H(t)$. Suppose δ bounds a compressing disk in $V(t)$. Since (t_i, t_{i+1}) has the label $\mathbb{W}$ there is a loop γ on some component of $H(t) \setminus F$ which bounds a disk in $W(t)$. But then $\delta \cap \gamma = \emptyset$ contradicts the strong irreducibility of H.

As in the proof of Claim 3.4 it now follows that we may isotope $H(t)$, preserving the set of loops of $H(t) \cap F$ which are essential on both, so that any loop which is essential on $H(t) \setminus F$ is also essential on $H(t)$.

Let H' be the component of $H(t) \setminus F$ which contains the loop γ from above. By strong irreducibility any essential loop of $H(t) \setminus F$ which bounds a compressing disk in $V(t)$ must meet γ, and hence must be on H'. Furthermore, since the subinterval (t_i, t_{i+1}) has the label $\mathbb{V}$, there is at least one such loop ρ. By identical reasoning we conclude that any essential loop of $H(t) \setminus F$ which bounds a compressing disk in $W(t)$ must meet ρ, and hence must also be on H'. We conclude that there are no loops on any other component of $H(t) \setminus F$ which bound compressing disks, and hence they are all incompressible in the respective submanifolds of $M \setminus F$. Furthermore, it follows from the strong irreducibility of H that all loops bounding disks on opposite sides of H' must intersect, and from the fact that both labels appear that such loops exist. We conclude that H' is strongly irreducible. □

Claim 3.6. If the labelling of (t_{i-1}, t_i) is different from that of (t_i, t_{i+1}) then the critical value t_i corresponds to a saddle tangency between $H(t_i)$ and F.

Proof. The only other option is a center tangency, which cannot introduce a new compression for $H(t)$ whose boundary is on a component of $H(t) \setminus F$. □

Claim 3.7. The subinterval $(0, t_1)$ is labelled $\mathbb{V}$ and the subinterval $(t_{n-1}, 1)$ is labelled $\mathbb{W}$.

Proof. For sufficiently small ϵ the surface $H(\epsilon)$ looks like the frontier of a neighborhood of Σ_V. By Claim 3 the surface F meets Σ_V transversely. Hence, F contains small compressions for $H(\epsilon)$ in $V(\epsilon)$. We can push these compressions off F, giving compressions with boundary on a component of $H(\epsilon)\setminus F$ in $V(\epsilon)$. Hence, the label of $(0,t_1)$ is $\mathbb{V}$. A symmetric argument completes the proof of the claim. □

Following Claims 3.4 and 3.5 we may assume that every subinterval has a label, and that it is unique. It then follows from Claim 3.7 that there is some critical value t_i where the labelling changes from $\mathbb{V}$ to $\mathbb{W}$. By Claim 3.6 this critical value must correspond to a saddle tangency. Our goal now is to show that all components of $H(t_i) - F$ are incompressible in the respective submanifolds of $M\setminus F$, and hence Lemma 3.2 follows.

First, we claim that every loop of $H(t_i) \cap F$ is either essential or inessential on both surfaces. If not, then as in the proof of Claim 3.4 there is a loop δ of $H(t_i) \cap F$ which bounds a compressing disk for $H(t_i)$. Near the loops of $H(t_i) \cap F$ the surface $H(t_i)$ is identical to the surfaces $H(t_i - \epsilon)$ and $H(t_i + \epsilon)$. If δ bounds a compressing disk in $W(t_i)$ then we push δ off of F and see that there is a loop on a component of $H(t_i-\epsilon)\setminus F$ that bounds a compressing disk in $W(t_i-\epsilon)$. This violates the fact that the subinterval (t_{i-1}, t_i) does not have the label $\mathbb{W}$. Similarly, if δ bounds a compressing disk in $V(t_i)$ then it follows that there is a loop on a component of $H(t_i + \epsilon)\setminus F$ bounding a compressing disk in $V(t_i + \epsilon)$, violating the fact that the subinterval (t_i, t_{i+1}) does not have the label $\mathbb{V}$.

As in the proof of Claim 3.4 it now follows that we may isotope $H(t_i)$, preserving the set of essential loops of intersection with F, to remove those loops of intersection that are inessential. Now, let E be a compressing disk for a component H' of $H(t_i)\setminus F$. It follows from Lemma 3.1 and the fact that we have removed all of the inessential loops of $H(t_i)\cap F$ that E is also a compressing disk for $H(t_i)$. Furthermore, as ∂E is essential on H' it misses the saddle point, and is hence present on components of both $H(t_i - \epsilon)\setminus F$ and $H(t_i + \epsilon)\setminus F$. If $E \subset W(t_i)$ then this violates the fact that (t_{i-1}, t_i) does not have the label $\mathbb{W}$. On the other hand, if $E \subset V(t_i)$, then we contradict the fact that (t_i, t_{i+1}) does not have the label $\mathbb{V}$.

We conclude that the components of $H(t_i)\setminus F$ are incompressible in the respective submanifolds of $M\setminus F$, as asserted by the lemma. □

We now use the above result to establish the following lemma.

Lemma 3.8. *Let M be a compact, irreducible, orientable 3-manifold whose boundary, if non-empty, is incompressible. Suppose $M = X \cup_F Y = V \cup_H W$, where F is incompressible, orientable, connected, closed, and non-boundary parallel and H is a Heegaard surface. Then either*

H is an amalgamation of splittings of X and Y or there are properly embedded surfaces $H_X \subset X$ and $H_Y \subset Y$ with boundaries on F such that at least one of the following holds:

1. *The surfaces H_X and H_Y are incompressible, non-boundary parallel, and satisfy $\partial H_X = \partial H_Y$ and $\chi(H_X) + \chi(H_Y) \geq \chi(H)$.*
2. *After possibly exchanging X and Y the surface H_X is incompressible and non-boundary parallel, the surface H_Y is strongly irreducible, $\partial H_X = \partial H_Y$ and $\chi(H_X) + \chi(H_Y) \geq \chi(H)$.*
3. *The surfaces H_X and H_Y are incompressible, non-boundary parallel, and satisfy $\partial H_X \cap \partial H_Y = \emptyset$ and $\chi(H_X) + \chi(H_Y) - 1 \geq \chi(H)$.*

Remark 3.9. If H is assumed to be strongly irreducible then we will show that each of the above inequalities can be replaced by equalities.

Proof. By [**?**] we may *untelescope* the Heegaard splitting H. That is, there is a sequence $\{H_i\}_{i=0}^{2n}$ of pairwise disjoint, closed surfaces in M such that

- $\partial M = H_0 \cup H_{2n}$ (if $\partial M = \emptyset$ then $H_0 = H_{2n} = \emptyset$),
- for each odd i, the surface H_i is a strongly irreducible Heegaard splitting of the submanifold cobounded by H_{i-1} and H_{i+1}, and
- for each i between 1 and $2n-1$ the surface H_i is obtained from H by some number of compressions.

In addition, it is shown in [**?**] that for each even i the surface H_i is incompressible in M. We will call the set of surfaces with even index *thin levels* and the set with odd index *thick levels*.

Isotope F to meet the set of thin levels of $\{H_i\}$ in a minimal number of curves. Suppose first that for some i, the surface F is parallel to a component of the thin level H_{2i}. Then the components of $\{H_i\}$ which meet X form an untelescoped Heegaard splitting of X, and the components which meet Y form and untelescoped Heegaard splitting of Y. Telescoping (the operation which is the inverse of untelescoping) each now produces Heegaard splittings of X and Y whose amalgamation is H. Hence, the conclusion of Lemma 3.8 follows.

Now suppose F intersects the thin level H_{2i}. Then F divides H_{2i} into subsurfaces $H_X \subset X$ and $H_Y \subset Y$. We claim that H_X is incompressible in X and H_Y is incompressible in Y. If not, then there is some compressing disk D for H_X (say) in X. As H_{2i} is incompressible in M, ∂D bounds a disk E in H_{2i}. As M is irreducible we can now do a sequence of isotopies to remove all curves of $E \cap F$, reducing the number of times F meets the set of thin levels.

Since F meets all thin levels minimally it also follows that neither H_X nor H_Y are boundary parallel. Finally, since $H_{2i} = H_X \cup H_Y$, and H_{2i} is obtained from H be some number of compressions, we have $\chi(H_X) + \chi(H_Y) \geq \chi(H)$. Hence, Case (1) of the conclusion of Lemma 3.8 follows.

We are now reduced to the case where F misses all thin levels, and is parallel to none. Hence, F is completely contained in a submanifold with incompressible boundary which has a strongly irreducible Heegaard splitting, obtained from H by some number of compressions. It suffices, then, to prove Lemma 3.8 in the case where H is strongly irreducible.

Use Lemma 3.2 to isotope H so that it is almost transverse to F, and so that every component of $H \setminus F$ is incompressible, except for possibly one strongly irreducible component. If H is actually transverse to F then let $H_X = H \cap X$ and $H_Y = H \cap Y$, and we are done. The remaining case is that H meets F transversally, except at a single saddle tangency at a point $p \in H$. Isotope H by pushing the point p slightly into Y, to obtain the surface H'. Hence, H' transverse to F. Furthermore, any compressing disk for $H_X = H' \cap X$ is a compressing disk for $H \cap X$, so there must be none. We conclude H_X is a properly embedded, incompressible surface in X. Similarly, by pushing p slightly into X we may obtain from H a properly embedded, incompressible surface $H_Y \subset Y$.

As H and F are orientable, it follows that $H_X \cap F$ may be made disjoint from $H_Y \cap F$. Furthermore, the only essential difference between $H_X \cup H_Y$ and H is a pair of pants, having Euler characteristic negative one. Hence, Case (3) of the conclusion of Lemma 3.8 now follows. □

4. Amalgamating small manifolds

For convenience of the argument we will need to define relative compression bodies. Let F be an orientable surface, possibly with boundary components, and possibly disconnected. Let C be the manifold obtained by forming the $F \times I$ and attaching one handles to the surface $F \times 1$. Then C is a *relative compression body*. We label the boundary as follows: the negative boundary is $\partial_- C = F \times 0$, the vertical boundary is $\partial_V C = \partial F \times I$, and the positive boundary is $\partial_+ C = \partial C - \partial_- C - \partial_V C$. The vertical boundary is a collection of annuli. It is important to note that a given manifold may admit many relative compression body structures. For example, if F is a surface with boundary and $C = F \times I$, then C can be thought of as a relative compression body with $\partial_- F = F \times 0$, or C can be thought of as a handlebody with $\partial_- F = \emptyset$. In fact, given a relative compression body C, it is always possible to think of C as a (non-relative) compression body by *promoting* all non-closed components of $\partial_- C$ and all components of $\partial_V C$ to positive boundary.

We can define a *relative Heegaard splitting*, as the union of two relative compression bodies, identified along their positive boundaries. The splitting will be considered *non-trivial* if neither relative compression body is a product, i.e., both compression bodies have 1-handles.

Lemma 4.1. *Let X be a manifold that admits a strongly irreducible non-trivial relative Heegaard splitting $X = C_1 \cup C_2$. Then $\partial_- C_1$ and $\partial_- C_2$ are incompressible in X.*

Proof. An examination of the proof of the Haken Lemma (see [**?**]) will reveal that it applies directly to the case of relative Heegaard splittings. In particular, if either $\partial_- C_1$ or $\partial_- C_2$ has compressible boundary, then there is a compressing disk D for the boundary component that meets the splitting surface is a single closed loop. The loop decomposes the compressing disk into a vertical annulus in one compression body, say C_1, and a disk $D_2 \subset C_2$. Since C_1 is not a product, we can find a compressing disk D_1 for $\partial_+ C_1$ that is disjoint from the annulus, hence disjoint from D_2. The pair (D_1, D_2) is a weak reducing pair for the splitting, a contradiction. $\square$

Lemma 4.2. *A connected, small, manifold with compressible boundary is a compression body.*

Proof. Let X be a connected small manifold with compressible boundary. In an optimistic fashion, denote a compressible boundary component by $\partial_+ X$ and all other components by $\partial_- X$. Since $\partial_+ X$ is compressible it bounds a compression body (which is not a product) $C \subset X$ so that $\partial_+ C = \partial_+ X$. Choose C to be maximal in this regard. Precisely, choose C so that $\sum(1 - \chi(S))$ is minimal, where the sum is taken over all non-sphere components $S \subset \partial_- X$. Since X is irreducible, we can also choose C so that $\partial_- C$ contains no spheres.

Let S be a component of $\partial_- C$. S is incompressible in C. By maximality of C, S is also incompressible in $X - C$. As X is small, S must in fact be peripheral, and since C is not a product, it is parallel in $X - C$ to a component of $\partial_- X$. The (possibly disconnected) surface $\partial_- C$ separates the components of $\partial_- X$ from $\partial_+ X$, so in fact each component of $\partial_- X$ is in fact parallel to a component of $\partial_- C$. The parallelism yields an isotopy between X and C. X is therefore a compression body. Note that only one boundary component, $\partial_+ X$ is compressible. $\square$

Proposition 4.3. Let F be a non-peripheral incompressible surface that is properly embedded in a small manifold X. Then $h(X) \leq 1 - \chi(F)$. If X has a single boundary component or F meets every boundary component of X, then this applies to the tunnel number: $t(X) \leq 1 - \chi(F)$.

Proof. Let $\partial_1 X$ denote those boundary components of X that meet F and $\partial_2 X$ denote those boundary components which do not meet F.

Let $X_1 = \overline{N(F \cup \partial_1 X)}$ and $X_2 = X - int(X_1)$. This decomposes X into $X = X_1 \cup_{F'} X_2$, where F' is the closed surface that is the common boundary of X_1 and X_2. See Figure **??**. Note that $\partial_1 X$ and F are contained in X_1 and $\partial_2 X$ is contained in X_2. While X_1 is always connected, X_2 will contain two components if F separates.

Since X is a small manifold each component of F' must be either:

(1) compressible into X_1,
(2) peripheral to a component of $\partial_1 X$,
(3) compressible into X_2, or
(4) peripheral to a component of $\partial_2 X$.

Claim. No component of F' can be compressible into X_1.

If so, we could choose a compressing disk for this component which is disjoint from the incompressible surface F, and this component would be compressible into the product $X_1 - N(F)$, a contradiction.

Claim. No component of F' is peripheral into $\partial_1 X$.

If this occurred, X_1 would be contained in a product neighborhood of a boundary component. This in turn implies that F was peripheral.

Claim. X_2 is small.

Suppose that X_2 contains a closed essential surface G. Since X is small, G is either compressible in X or peripheral to a boundary component in $\partial_1 X$. Since F is incompressible, any compressing disk $D \subset X$ for G can be modified into a compressing disk D' for G for which $\partial D' = \partial D$ and so that D' does not intersect F. As before, this implies that the interior of D' can be modified to be disjoint from X_1 and G is therefore compressible in X_2. Since G is essential in X_2, if it is peripheral in X, it is peripheral to a component of $\partial_1 X$. Again this implies that F is contained in product neighborhood of ∂_X, contradicting the fact that F is not peripheral.

Each component of F' is therefore compressible into X_2 or peripheral to a component of $\partial_2 X$. In either case, by Lemma 4.2 or by parallelism, $F' = \partial_+ X_2$, where X_2 is either one or two compression bodies.

It is now straightforward to build a handle system for X (tunnel system in the case that $\partial_1 X = \partial X$). Choose τ, a minimal collection of arcs that are properly embedded in F and that cut F into a single disk D. The collection τ contains $1 - \chi(F)$ arcs. Moreover, τ is a handle system that induces a Heegaard splitting, $X = C_1 \cup C_2$, where $C_1 = \overline{N(\partial_1 X \cup \tau}$ and $C_2 = \overline{X - C_1}$. Clearly C_1 is a compression body. C_2 is a compression body because it is formed by attaching a handle (the cocore of D to the positive boundary of the compression body X_2. □

Proposition 4.4. Let F be a non-peripheral, bi-compressible, strongly irreducible surface that is properly embedded in a small manifold X. Then $h(X) \leq 1 - \chi(F)$. If X has a single boundary component, then this applies to the tunnel number: $t(X) \leq 1 - \chi(F)$.

Proof. We may apply the previous theorem if X also contains an non-peripheral incompressible surface with boundary and Euler characteristic less than F. We may therefore assume that F is a separating surface; if not we may compress F to obtain such an incompressible surface. As before we will let $\partial_1 X$ denote those boundary components of X that meet F and $\partial_2 X$ denote those boundary components which do not meet F.

By compressing F maximally to both sides, we define a relative Heegaard splitting of a submanifold $X' \subset X$, $X' = C_1 \cup C_2$. Since we have compressed maximally, the negative boundary components of C_1 and C_2 are incompressible outside X'. They are incompressible inside X' by Lemma 4.1. Each component of $\partial_- C_i, i = 1, 2$ is therefore peripheral. In particular, this implies that X' is isotopic to X.

As in the earlier theorem, this structure defines a handle system for X. Choose τ, a minimal collection of arcs that are properly embedded in F and that cut F into a single disk D. Now, τ is a handle system for X that induces the Heegaard splitting, $X = C'_1 \cup C'_2$, where $C'_1 = \overline{N(\partial_1 X \cup \tau}$ and $C'_2 = \overline{X - C_1}$. Clearly C'_1 is a compression body. C'_2 is a compression body because it can be obtained by first promoting the vertical and non-closed negative boundary components of C_1 and C_2 and then joining the positive boundary of these (honest) compression bodies with a 1-handle (the cocore of D).

The handle number of X is therefore $h(X) \leq 1 - \chi(F)$. □

Corollary 4.5. *Let M be a compact, orientable 3-manifold with incompressible boundary. If M can be obtained by gluing two small manifolds along an incompressible surface, $M = X \cup_F Y$, then the following statements hold:*

(1) $g(M) \geq 1/2(h(X) + h(Y))$
(2) *if* M *is closed,* $g(M) \geq 1/2(t(X) + t(Y))$
(3) $g(M) \geq 1/2(g(F) + g(Y)) - g(F)$.

Proof. Let H be a minimal genus splitting of M. If H is an amalgamation of splitting of X and Y, then the result holds trivially. Otherwise, by Lemma 3.8 we can construct properly embedded non-boundary parallel surfaces $H_X \subset X$ and $H_Y \subset Y$ so that each is either incompressible or strongly irreducible. Furthermore, $\chi(H_X) + \chi(H_X) \geq \chi(H) = 2 - 2g(M)$, or equivalently, $g(M) \geq 1/2(2 - \chi(H_X) - \chi(H_Y))$.

By either Proposition 4.3 or Proposition 4.4, X and Y admit handle systems that are attached to components of F and so that the number of handles is less than $1 - \chi(H_X)$ and $1 - \chi(H_Y)$, respectively. The statements in the theorem follow. □

Remark 4.6. In both cases above, the splittings of X and Y that are constructed have low distance. The 1-handles are the spanning arcs of a surface and cut that surface into a disk D. In the resulting splitting the disk D will meet each of the cocores exactly twice.

5. 3-Manifolds with no strongly irreducible Heegaard splittings

Using some almost normal trickery and a nice result of Eric and Bus, we blow people's minds.

"How do you do that?", I hear you cry. Well, it goes like this:

Theorem 5.1. *Suppose that* X *and* Y *are* knot manifolds *and* $\varphi\colon \partial X \to \partial Y$ *is* sufficiently complicated. *Then the manifold* $M(\varphi) = X \cap_\varphi Y$ *contains no strongly irreducible Heegaard splittings.*

A *knot manifold* is a compact, orientable, irreducible three-manifold with a single, toroidal, incompressible boundary component. To make the theorem precise we must also define *sufficiently complicated*.

Fix, once and for all, triangulations on X and Y. Let $\Delta(X)$ be the set of isotopy classes of essential curves as follows: the class of γ is in $\Delta(X)$ if and only if γ is a component of the boundary of some normal or almost normal surface in X. Note that $\Delta(X)$ is finite, by a result of Jaco and Sedgwick [**?**]. We define $\Delta(Y)$ similarly.

Recall now the definition of the *Farey graph*, $\mathcal{F}(X)$. The vertex set of $\mathcal{F}(X)$ is all isotopy classes of essential simple closed curves in ∂X. Two vertices are connected by an edge if and only if they have geometric intersection number one.

Suppose $\varphi\colon \partial X \to \partial Y$ is given. Now set $d_{\mathcal{F}}(\varphi) = d_{\mathcal{F}}(\Delta(X), \varphi^{-1}(\Delta(Y)))$ equal to the minimal number of edges required in a path connecting

$\Delta(X)$ to $\varphi^{-1}(\Delta(Y))$ inside of $\mathcal{F}(X)$. The map φ is *sufficiently complicated* if $d_{\mathcal{F}}(\varphi)$ is greater than or equal to two.

Remark 5.2. Note that, as $\Delta(X)$ and $\Delta(Y)$ are finite, "most" elements of $\mathcal{MCG}(\mathbb{T}^2) \cong SL(2,\mathbb{Z})$ are sufficiently complicated, in the above sense. In particular any sufficiently large power of a Anosov map is sufficiently complicated. The same holds for all but a finite number of Dehn twists.

Before giving the proof of Theorem 5.1 we must discuss boundary compressions. Suppose $G \subset N^3$ is a two-sided surface properly embedded in a compact, orientable, irreducible three-manifold N. We suppose further that ∂N is incompressible in N. Suppose $D \subset N$ is a boundary compression for G. Define $\partial_+ D = D \cap G$ and $\partial_- D = D \cap \partial N$.

Definition. The boundary compression D is *honest* if $\partial_- D$ is essential as a properly embedded arc in $\partial N \setminus \partial G$. If D is not honest it is *dishonest*.

Suppose now that $\partial N \cong \mathbb{T}^2$. We now define the *banding*, $\hat{D}$, of an honest boundary compression D for G. Note that $\partial_- D$ meets distinct boundary components of ∂G, as G is orientable. These components of ∂G cobound an annulus $A \subset \partial N$ such that $\partial_- D \subset A$. Take D', D'' parallel copies of the disk D and form the disk $D''' = D' \cup D'' \cup (A \setminus \eta(\partial_- D))$. Isotope D''' to be disjoint from ∂N while maintaining $\partial D''' \subset G$. The resulting disk is the desired banding $\hat{D}$ of D.

The following fact is well-known:

Lemma 5.3. *If D is a boundary compression for G and $\partial N = \mathbb{T}^2$ then G is either compressible or G contains boundary parallel annulus.*

Proof. If D is honest then band to obtain $\hat{D}$. If $\hat{D}$ is not a compressing disk then G is a boundary parallel annulus.

On the other hand, if D is a dishonest compression of $G \subset N$ then there is a subarc $\alpha \subset \partial G$ so that $\partial_- D \cap \alpha = \partial\alpha = \partial\partial_- D$. Also, this arc α is isotopic to $\partial_- D$ inside ∂N relative to $\partial\alpha$. So $\alpha \cup \partial_- D$ bound a subdisk of ∂N, say E. Also interior$(E) \cap G = \emptyset$ since $\partial_- D$ is not an essential arc in $\partial N \setminus \partial G$. So $D \cup E$ is a compressing disk for G. □

We are now ready for the proof:

Proof of Theorem 5.1. Suppose that X and Y are triangulated knot manifolds, as above. Fix a gluing $\varphi \colon \partial X \to \partial Y$. Suppose that $H \subset M(\varphi) = X \cup_\varphi Y$ is a strongly irreducible Heegaard splitting surface. Let $F \cong \mathbb{T}^2$ be the image of ∂X inside of $M(\varphi)$.

Now apply Lemma 3.2 and Remark 3.3 to the pair H and F in $M(\varphi)$.

Let H_X be a component of $H \cap X$ which is incompressible and not a boundary parallel annulus, if such exists. If no such component exists take H_X to be the non-boundary parallel component of $H \cap X$. In this case H_X is strongly irreducible. (Not all components of $H \cap X$ are boundary parallel annuli as then we could isotope H into Y, a contradiction.) Choose H_Y similarly and note that, by Lemma 3.2, not both of H_X and H_Y are strongly irreducible. Note that ∂H_X is the same slope as $\varphi^{-1}(\partial H_Y)$.

Suppose that H_X and H_Y are both incompressible. As $\partial X \cong \partial Y \cong \mathbb{T}^2$ it follows from Lemma 5.3 that H_X and H_Y are also boundary-incompressible. So H_X and H_Y may be normalized with respect to the given triangulations [**?**]. It follows that $\Delta(X)$ and $\varphi^{-1}(\Delta(Y))$ intersect and thus φ is not sufficiently complicated.

Suppose now that H_X is incompressible and thus boundary-incompressible. Suppose that H_Y is a strongly irreducible surface. Recall that by *strongly irreducible* we mean a two-sided surface which compresses on both sides and all pairs of compressing disks on opposite sides must meet.

There are two possibilities for H_Y: either, applying work of the first author [**?**], the surface H_Y is properly isotopic to an almost normal surface or there is a disjoint pair of honest boundary compressions on opposite sides of H_Y.

In the former situation $\partial H_X \in \Delta(X)$ and $\partial H_Y \in \Delta(Y)$. Thus φ is not sufficiently complicated and we are done. So suppose the latter: let $D, E \subset Y$ be disjoint, honest boundary compressions on opposite sides of H_Y.

Note that the bandings $\hat{D}$ and $\hat{E}$ are both compressing disks for H_Y (otherwise H_Y would either not be connected, or H_Y would be a boundary-parallel annulus.)

We will now cut away a part of H_Y to find an incompressible surface H'_Y in Y with boundary meeting the boundary of H_Y at most once. Proceed as follows.

Let $K \cong \mathbb{T}^2 \times I$ be a closed collar of ∂Y inside of Y. Isotope H_Y so that $H_Y \cap K$ is a regular neigborhood of ∂H_Y. Isotope D so that $D \cap K$ is a regular neighborhood of $\partial_- D = D \cap \partial Y$ in D and do the same for E. Now we may use D to guide an isotopy of H_Y in Y which fixes $H_Y \smallsetminus \eta(\partial_+ D)$ pointwise and moves $\partial_+ D$ into K. We do the same for E. Now, if there are four distinct curves of ∂H_Y meeting $D \cup E$ then $\hat{D}$ and $\hat{E}$ are disjoint compressions or H_Y contains a boundary parallel

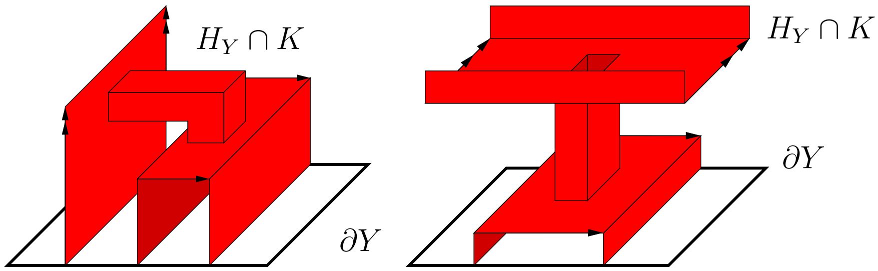

FIGURE 1. The intersection of H_Y and K. Note that the left-hand picture omits other components of $H_Y \cap K$.

annulus component. The first contradicts strong irreducibility and the second contradicts the connectedness of H_Y.

There remains two possibilities for the resulting surface $H_Y \cap K$. Both are shown in Figure 1.

Note that, in either case, the surface $H_Y \cap K$ compresses on both sides via the disks $\hat{D}$ and $\hat{E}$. It follows from strong irreducibility that the surface $H'_Y = H_Y \setminus \text{interior}(K)$ is incompressible inside of $Y \setminus \text{interior}(K)$. Thus, by Lemma 5.3 either H'_Y is also boundary incompressible or H'_Y is a boundary parallel annulus.

Consider the latter possibility – if H'_Y is a boundary parallel annulus then it follows that all of H_Y can be isotoped into $K \subset Y$, the collar of ∂Y. As all other components of $H \cap Y$ were boundary-parallel annuli we may properly isotope all of $H \cap Y$ into K. Thus $H \subset X \cup_\varphi Y$ may be isotoped into X, a contradiction.

Now consider the former possibility – H'_Y is both incompressible and boundary incompressible in $Y \setminus K$. Now, $Y \setminus K$ is homeomorphic to Y and thus there is an incompressible, boundary incompressible surface in Y and this surface has boundary slope meeting ∂H_Y at most once (again, see Figure 1). It follows that φ was not sufficiently complicated, and we have finished the proof of Theorem 5.1. □

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