READING ASSIGNMENT FOR GNF 101

SAUL SCHLEIMER

ABSTRACT. We give a sequence of mapping tori $M_n = M(\varphi_n)$, all with genus two fibre, such that φ_n is pseudo-Anosov, $\mathrm{vol}(M_n)$ tends to infinity with n, and yet all of the M_n have strongly irreducible Heegaard splittings of genus two or three. This shows that the main theorem of [1] cannot be altered to use translation distance in the pants complex.

1. Introduction

Here is a quick sketch of the construction: we will take a fixed genus two bundle over the circle and successively alter the monodromy via powers of a partial pseudo-Anosov map. As shown in the last homework assignment [3] there is a simple condition which ensures that the volumes increase without bound.

The resulting manifolds will all have a splitting of genus three. A bit of work will prove that this splitting, or a related genus two splitting, is strongly irreducible.

2. Construction of the examples

We first build the bundle M_0 and define a few natural submanifolds. Then we define the M_n's.

2.1. **Building $M = M_0$ and the subsurface T.** Let $\widetilde{M}$ be the convex polyhedron in $\mathbb{R}^3$ with vertices at the points

$$(\cos\left((2k+1)\pi/8\right), \sin\left((2k+1)\pi/8\right), \pm 1).$$

Here k ranges from zero to eight. That is, $\widetilde{M}$ is a regular octagon crossed with an interval. Note that the faces are disjoint from the x, y, and z-axes or meet them perpendicularly.

Construct M, a closed three-manifold, by identifying opposite vertical faces via translation and by gluing the top to the bottom via a one-eighth counterclockwise twist. Let $q : \widetilde{M} \to M$ be the natural quotient map. Let $\pi_F : M \to [-1, 1]/(-1 \sim 1)$ be the map which takes a point $p \in M$ to the z coordinate of $q^{-1}(p)$.

Date: December 7, 2002.

Thus M is a genus two surface bundle over the circle with periodic monodromy. Let $\widetilde{F}$ be the intersection of $\widetilde{M}$ with the x,y-plane. Let $F = q(\widetilde{F})$. We will think of the monodromy, φ, as a homeomorphism of both F and $\widetilde{F}$. Note that φ acts on $\widetilde{F}$ as an order eight counterclockwise rotation.

Fix an small ϵ-neighborhood of the union of the x, y, and z axes in $\mathbb{R}^3$. Set $\widetilde{V}$ equal to the intersection of this neighborhood with $\widetilde{M}$. Then $V = q(\widetilde{V})$ is a genus three handlebody embedded in M.

Claim 2.1. The surface $H = \partial V$ is a Heegaard splitting for M. In fact, H is a stabilization of one of the vertical splittings of the Seifert fibred space M.

Finally let $T = V \cap F$. Then T is homeomorphic to a once punctured torus. Let $V' \subset V$ be the intersection of a closed ϵ-neighborhood of F with V (with ϵ as above). Then V' is homeomorphic to $T \times I$ and $B = \overline{V \setminus V'}$ is homeomorphic to a three-ball. Thus $V' \cup B$ realizes V as $T \times I$ union a one-handle attached to either end of the $T \times I$.

Let $\gamma : T \to T$ be any homeomorphism of T which restricts to the identity map on ∂T. Define $V(\gamma)$ to be the three-manifold obtained by cutting V along T and regluing by the map γ. We have shown:

Claim 2.2. $V(\gamma)$ is a genus three handlebody.

2.2. **Checking M_n.** Let $W = \overline{M \setminus V}$. Fix $\gamma : T \to T$ a pseudo-Anosov automorphism of T restricting to the identity on ∂T. Define $M_n = M(\gamma^n \varphi) = W \cup V(\gamma^n)$. (This notation gives $M = M_0 = M(\varphi)$.) That is M_n is a genus two surface bundle with monodromy $\gamma^n \varphi$. We remark that the gluing map between W and $V(\gamma^n)$ is identical to the one between W and V.

It is easy to check:

Claim 2.3. The curves ∂T and $\varphi(\partial T)$ fill the surface F.

Thus by the arguments of [3], when n is sufficiently large, the maps $\varphi_n = \gamma^n \varphi : F \to F$ are pseudo-Anosov. (The map $\gamma|(F \setminus T)$ is the identity.) Also, by an earlier argument the volumes of M_n tend to infinity with n.

Confession: I don't really understand that last fact.

Let $H = \partial W = \partial V(\gamma^n)$. Note that H is a Heegaard splitting, as W and $V(\gamma^n)$ are handlebodies. I suspect, but cannot show, that H is strongly irreducible for n large. Instead we resort Lemma 3.2, stated and proved below. This will complete the construction.

3. The required facts about Heegaard splittings

We begin with a few general facts. Fix M, a closed orientable three-manifold. Suppose $H \subset M$ is a weakly reducible splitting of M which is not reducible. Following Casson-Gordon [2] there are disjoint collections of compressing disks $\mathbf{D}$ and $\mathbf{E}$ for the two sides of H. Note that $\mathbf{D}$ and $\mathbf{E}$ may be chosen any compressing curve in H is either parallel to a component of $\partial\mathbf{D} \cup \partial\mathbf{E}$ or intersects a component of $\partial\mathbf{D} \cup \partial\mathbf{E}$. (If not we could enlarge one of the two disk systems.) Compressing the splitting along these disks gives a surface G. The main result of [2] shows that at least one component of G is incompressible.

For every disk D in $\mathbf{D}$ let D' and D'' be the two subdisks of G resulting from compressing along D. Let $E', E'' \subset G$ be defined similarly. Call these disks D', E', etc, the *remnants* of $\mathbf{D}$ and $\mathbf{E}$.

Lemma 3.1. *Let G' be any incompressible component of G. Then G' meets exactly two remnants, one from $\mathbf{D}$ and one from $\mathbf{E}$.*

Proof. If G' meets only one remnant then G' may be isotoped into one of the handlebodies determined by H. It would follow that G' is compressible, a contradiction. Suppose G' contains a pair of remnants D'_1 and D'_2, say. Choose an embedded arc in G' connecting D'_1 to D'_2 which avoids all other remnants. Use this arc to perform a band sum of $\partial D'_1$ and $\partial D'_2$. The resulting curve is a compressing curve for H, is disjoint from $\partial\mathbf{D} \cup \partial\mathbf{E}$ and is not parallel to any component of $\partial\mathbf{D} \cup \partial\mathbf{E}$. This is a contradiction. □

We use the following fact, special to genus three splittings:

Lemma 3.2. *Suppose that M is an irreducible, atoroidal orientable closed three-manifold, which is not a lens space. Then any genus three splitting $H \subset M$ is either strongly irreducible or is stabilized. In the latter case H may be destabilized to obtain a strongly irreducible genus two splitting.*

Proof. If H is reducible then, as M is irreducible, H is stabilized. In this case we destabilize H to obtain H', a genus two splitting of M. Note that H' cannot by reducible as then M would be a lens space.

Suppose H (or H') is weakly reducible. As above choose disjoint collections of compressing disks $\mathbf{D}$ and $\mathbf{E}$ for the two sides of H (or H'). Compressing the splitting along these disks gives the surface G. At least one component of G is incompressible.

Now, any compression at all of H' will yield a torus. Hence H' must not have been weakly reducible. So if H was reducible we are done. The final case to consider is that G is obtained by compressing

H. As G cannot contain a torus component there is a incompressible component, G', with genus two. By Lemma 3.1 G' meets exactly two remnants, one from $\mathbf{D}$ and one from $\mathbf{E}$.

It follows that G' may be obtained from H by compressing along D_1 and E_1, say, and removing extraneous components. Let J be the surface obtained by compressing H along D_1 only. If J is disconnected then we have an immediate contradiction; E_1 must compress the genus two component of J and G' could not have had genus two. So J is connected. Also, ∂E_1 is not an essential separating curve of J. Nor can ∂E_1 be a nonseparating curve in J.

At the last ∂E_1 is a nonessential curve in J. Let C be the disk bounded by ∂E_1 in J. If C is disjoint from the remnants of D_1 then ∂E_1 bounded a disk in H and E_1 was not a compressing disk. If C meets only one of the remnants of D_1 then ∂E_1 is parallel to ∂D_1, a contradiction (H would be reducible). So C contains both remnants. Now compress J along E_1 and note that G' meets no remnant on D_1. This final contradiction completes the claim. □

4. Exercises

Please obtain the current homework from fellow student H. Masur. Note: no credit will be given for turning in a solution to the Poincare Conjecture. That assignment was due on Mon, 11 Nov 2002.

References

[1] David Bachman and Saul Schleimer. Surface bundles versus Heegaard splittings.
[2] A. J. Casson and C. McA. Gordon. Reducing Heegaard splittings. *Topology Appl.*, 27(3):275–283, 1987.
[3] Yair N. Minsky. Problem set 1. 2002.

Saul Schleimer, Department of Mathematics, UIC, 851 South Morgan Street, Chicago, Illinois 60607
E-mail address: saul@math.uic.edu