

THE END OF THE CURVE COMPLEX

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ABSTRACT. Suppose that S is a surface of genus two or more, with exactly one boundary component. Then the curve complex of S has one end.

1. INTRODUCTION

We denote the compact, connected, orientable surface of genus g with b boundary components by $S_{g,b}$. The *complexity* of $S = S_{g,b}$ is $\xi(S) := 3g - 3 + b$. A simple closed curve α in S is *essential* if α does not cut a disk out of S . Also, α is *non-peripheral* if it does not cut an annulus out of S .

When $\xi(S) \geq 2$ the *complex of curves*, $\mathcal{C}(S)$, is the simplicial complex where vertices are isotopy classes of essential non-peripheral curves. The k -simplices are collections of $k + 1$ distinct vertices having disjoint representatives. We regard every simplex as a Euclidean simplex of side-length one. If α and β are vertices of $\mathcal{C}(S)$ let $d_S(\alpha, \beta)$ denote the distance between α and β in the one-skeleton $\mathcal{C}^1(S)$. It is a pleasant exercise to prove that $\mathcal{C}(S)$ is connected. It is an important theorem of H. Masur and Y. Minsky [8] that $\mathcal{C}(S)$ is Gromov hyperbolic.

Let $B(\omega, r) := \{\alpha \in \mathcal{C}^0(S) \mid d_S(\alpha, \omega) \leq r\}$ be the ball of radius r about the vertex ω . We will prove:

Theorem 5.1. *Fix $S := S_{g,1}$ for some $g \geq 2$. For any vertex $\omega \in \mathcal{C}(S)$ and for any $r \in \mathbb{N}$: the subcomplex spanned by $\mathcal{C}^0(S) - B(\omega, r)$ is connected.*

For such surfaces, Theorem 5.1 directly answers a question of Masur's. It also answers a question of G. Bell and K. Fujiwara [1] in the negative: the complex of curves need not be quasi-isometric to a tree. Theorem 5.1 is also evidence for a positive answer to a question of P. Storm:

Question 1.1. Is the Gromov boundary of $\mathcal{C}(S)$ connected?¹

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¹Added to v3: Leininger and I answered Storm's question in many cases [7]. The complete conjecture has been verified by D. Gabai [3]. The material in this note is of independent interest as it is self-contained and elementary.

Note that Theorem 5.1 is only evidence for, and not an answer to, Storm's question: for example, there is a one-ended hyperbolic space where the Gromov boundary is a pair of points. Finally, as we shall see in Remark 4.2, it is not obvious how to generalize Theorem 5.1 to surfaces with more (or fewer) boundary components.

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2. DEFINITIONS AND NECESSARY RESULTS

An important point elided above is how to define $\mathcal{C}(S)$ when $\xi(S) = 1$. The complex as defined is disconnected in these cases. Instead we allow a k -simplex to be a collection of $k + 1$ distinct vertices which have representatives with small intersection. For $S_{1,1}$ exactly one intersection point is allowed while $S_{0,4}$ requires two. In both cases $\mathcal{C}(S)$ is the famous *Farey tessellation*. Note that $\mathcal{C}(S_{0,3})$ is empty. We will not need to consider the other low complexity surfaces: the sphere, the disk, the annulus, and the torus.

A subsurface $X \subset S$ is *essential* if every component of ∂X is essential in S . We will generally assume that $\xi(X) \geq 1$. A pair of curves, or a curve and a subsurface, are *tight* if they cannot be isotoped to reduce intersection. We will generally assume that all curves and subsurfaces discussed are tight with respect to each other. We say a curve α *cuts* X if $\alpha \cap X \neq \emptyset$. If $\alpha \cap X = \emptyset$ then we say α *misses* X .

Following Masur and Minsky [9], we define the *subsurface projection* map π_X : this maps vertices of $\mathcal{C}(S)$ to collections of vertices of $\mathcal{C}(X)$. Fix a vertex $\alpha \in \mathcal{C}(S)$ and, for every component $\delta \subset \alpha \cap X$, form $N_\delta := \overline{\text{neigh}(\delta \cup \partial X)}$, a closed regular neighborhood of $\delta \cup \partial X$. Take $\pi_X(\alpha)$ to be the set of all vertices of $\mathcal{C}(X)$ which appear as a boundary component of some N_δ . If α misses X then $\pi_X(\alpha) = \emptyset$. Note if $\alpha \subset S$ is contained in X after tightening then $\pi_X(\alpha) = \{\alpha\}$.

As a useful bit of notation, if α and β both cut X , we set

$$d_X(\alpha, \beta) := \text{diam}_X(\pi_X(\alpha) \cup \pi_X(\beta))$$

with diameter computed in $\mathcal{C}^1(X)$. Masur and Minsky give an combinatorial proof [9, Lemma 2.2] that:

Lemma 2.1. *If α and β are disjoint and both cut X then $d_X(\alpha, \beta) \leq 2$. \square*

By *geodesic* in $\mathcal{C}(S)$ we will always be referring to a geodesic in the one-skeleton. Since $\mathcal{C}(S)$ is Gromov hyperbolic the exact position of the geodesic is irrelevant; we often use the notation $[\alpha, \beta]$ as if the geodesic was determined by its endpoints. We immediately deduce from Lemma 2.1:

Lemma 2.2. *Suppose that α, β are vertices of $\mathcal{C}(S)$, both cutting X . Suppose that $d_X(\alpha, \beta) > 2 \cdot d_S(\alpha, \beta)$. Then every geodesic $[\alpha, \beta] \subset \mathcal{C}(S)$ has a vertex which misses X . \square*

This is essentially Lemma 2.3 of [9].

Remark 2.3. There is a useful special case of Lemma 2.2: assume all the hypotheses and in addition that γ is the unique vertex of $\mathcal{C}(S)$ missing X . Then every geodesic connecting α to β contains γ .

In fact, γ is the unique vertex missing X exactly when $S - \text{neigh}(\gamma) = X$ or $S - \text{neigh}(\gamma) = X \cup P$ with $P \cong S_{0,3}$: a *pair of pants*.

Remark 2.4. Note that Lemma 2.2 is a weak form of the Bounded Geodesic Image Theorem [9, Theorem 3.1]. The proof of the stronger result appears to require techniques from Teichmüller theory.

We now turn to the *mapping class group* $\mathcal{MCG}(S)$: the group of isotopy classes of homeomorphisms of S . Note that the natural action of $\mathcal{MCG}(S)$ on $\mathcal{C}(S)$ is via isometries. We have an important fact:

Lemma 2.5. *If $\psi: S \rightarrow S$ is a pseudo-Anosov and α is a vertex of $\mathcal{C}^0(S)$ then $\text{diam}_S\{\psi^n(\alpha) \mid n \in \mathbb{Z}\}$ is infinite. \square*

It follows that the diameter of $\mathcal{C}(S)$ is infinite whenever $\xi(S) \geq 1$. A proof of Lemma 2.5, relying on Kobayashi's paper [5], may be found in the remarks following Lemma 4.6 of [9]. As a matter of fact, Masur and Minsky there prove more using train track machinery: any orbit of a pseudo-Anosov map is a quasi-geodesic. We will not need this sharper version.

Note that if $\psi: S \rightarrow S$ is a homeomorphism then we may restrict ψ to the curve complex of a subsurface $\psi|_X: \mathcal{C}(X) \rightarrow \mathcal{C}(\psi(X))$. This restriction behaves well with respect to subsurface projection: that is, $\pi_{\psi(X)} \circ \psi = \psi|_X \circ \pi_X$.

We conclude this discussion by examining *partial maps*. Suppose that $X \subset S$ is an essential surface, not homeomorphic to S . If $\psi: S \rightarrow S$ has the property that $\psi|_{S-X} = \text{Id}|_{S-X}$ then we call ψ a *partial map* supported on X . Note that if ψ is supported on X then the orbits of ψ do not have infinite diameter in $\mathcal{C}(S)$. Since ψ fixes ∂X and acts on $\mathcal{C}(S)$ via isometry, every point of an orbit has the same distance to ∂X in $\mathcal{C}(S)$. Nonetheless, Lemmas 2.2 and 2.5 imply:

Lemma 2.6. *Suppose $\psi: S \rightarrow S$ is supported on X and $\psi|_X$ is pseudo-Anosov. Fix a vertex $\sigma \in \mathcal{C}(S)$ and define $\sigma_n := \psi^n(\sigma)$. Then for any $K \in \mathbb{N}$ there is a power $n \in \mathbb{Z}$ so that $d_X(\sigma, \sigma_n) \geq K$. In particular, if $K > 4 \cdot d_S(\sigma, \partial X)$ then every geodesic $[\sigma, \sigma_n] \subset \mathcal{C}(S)$ contains a vertex which misses X . \square*

3. NO DEAD ENDS

We require a pair of tools in order to prove Theorem 5.1. The first is:

Proposition 3.1. *Fix $S = S_{g,b}$. For any vertex $\omega \in \mathcal{C}(S)$ and for any $r \in \mathbb{N}$: every component of the subcomplex spanned by $\mathcal{C}^0(S) - B(\omega, r)$ has infinite diameter.*

A more pithy phrasing is: the complex of curves has no *dead ends*. Proposition 3.1 allows us to push vertices away from ω while remaining inside the same component of $\mathcal{C}(S) - B(\omega, r)$. The proof is a bit subtle due to the behavior of $\mathcal{C}(S)$ near a non-separating curve.

Proof of Proposition 3.1. If $S = S_{0,3}$ is a pants then the curve complex is empty and there is nothing to prove. If $\mathcal{C}(S)$ is a copy of the Farey graph then the claim is an easy exercise. So we may suppose that $\xi(S) \geq 2$.

Now fix a vertex $\alpha \in \mathcal{C}(S) - B(\omega, r)$. Set $n := d_S(\alpha, \omega)$. Thus $n > r$. Our goal is to find a curve δ , connected to α in the complement of $B(\omega, n-1)$, with $d_S(\delta, \omega) = n+1$. Doing this repeatedly proves the proposition. Note that finding such a vertex δ is straight-forward if $r = 0$ and $n = 1$. This is because $\mathcal{C}(S) - \omega$ is connected and because, following Lemma 2.5, we know that the diameter of $\mathcal{C}(S)$ is infinite. Henceforth we will assume that $n \geq 2$; that is, ω cuts α .

Fix attention on a component X of $S - \text{neigh}(\alpha)$ which is not a pair of pants. So $\xi(X) \geq 1$ and, by the comments following Lemma 2.5, $\mathcal{C}(X)$ has infinite diameter. Since ω cuts α we find that ω also cuts X . Choose a curve β contained in X with $d_X(\beta, \omega) \geq 2n+1$. Note that $d_S(\alpha, \beta) = 1$. We may assume that β is either non-separating or cuts a pants off of S . (To see this: if β cannot be chosen to be non-separating then X is planar. As $\xi(X) \geq 1$ we deduce that X has at least four boundary components. At most two of these are parallel to α .) It follows from Lemma 2.2 that any geodesic from β to ω in $\mathcal{C}(S)$ has a vertex γ which misses X .

By the triangle inequality $d_S(\gamma, \omega)$ equals n or $n-1$. In the former case we are done: simply take $\delta = \beta$ and notice that $d_S(\beta, \omega) = n+1$. In the latter case $d_S(\beta, \omega) = n$ and we proceed as follows: replace α by β and replace X by $Z := S - \text{neigh}(\beta)$. We may now choose δ to

be a vertex of $\mathcal{C}(Z)$ with $d_Z(\delta, \omega) \geq 2n + 1$. As above, any geodesic $[\delta, \omega] \subset \mathcal{C}(S)$ has a vertex which misses Z . Since β is the *unique* vertex not cutting Z , our Remark 2.3 implies that $\beta \in [\delta, \omega]$. Thus $d_S(\delta, \omega) = n + 1$ and we are done. \square

4. THE BIRMAN SHORT EXACT SEQUENCE

We now discuss the second tool needed in the proof of Theorem 5.1. Following I. Kra [6] let $S^\circ = S_{g,1}$ and $S = S_g$ for a fixed $g \geq 2$. Let $\rho: S^\circ \rightarrow S$ be the quotient map crushing ∂S° to a point, say $x \in S$. This leads to the *Birman short exact sequence*:

$$\pi_1(S, x) \rightarrow \mathcal{MCG}(S^\circ) \rightarrow \mathcal{MCG}(S)$$

for $g \geq 2$. The map ρ gives the second arrow. The first arrow is defined by sending $\gamma \in \pi_1(S, x_0)$ to a mapping class ψ_γ . There is a representative of this class which is isotopic to the identity, in S , via an isotopy dragging x along the path γ . See Birman's book [2] or Kra's paper [6] for further details.

Fix an essential subsurface $X^\circ \subset S^\circ$ and let $X = \rho(X^\circ)$. If $\gamma \in \pi_1(S, x)$ is contained in X then ψ_γ is a partial map, supported in X° . We say that γ *fills* X if $\gamma \subset X$ and, in addition, every representative of the free homotopy class of γ cuts X into a collection of disks and peripheral annuli. For future use we record a well-known theorem of Kra [6]:

Theorem 4.1. *Suppose that $\xi(X^\circ) \geq 1$. If γ fills X then $\psi_\gamma|_{X^\circ}$ is pseudo-Anosov.* \square

Now note that, corresponding to the Birman short exact sequence, there is a “fibre bundle” of curve complexes:

$$\mathcal{F}_\tau \rightarrow \mathcal{C}(S^\circ) \rightarrow \mathcal{C}(S).$$

Here τ is an arbitrary vertex of $\mathcal{C}(S)$ and $\mathcal{F}_\tau := \rho^{-1}(\tau)$. The second arrow is given by ρ . The first is the inclusion of \mathcal{F}_τ into $\mathcal{C}(S^\circ)$.

Remark 4.2. If $|\partial S| \geq 2$ then collapsing one boundary component does not induce a map on the associated curve complexes. Thus, it is not clear how to generalize Theorem 5.1 to such surfaces. If ∂S is empty then it appears to be very difficult to find interesting mapping class group equivariant electrifications of $\mathcal{C}(S)$.

Using the Birman short exact sequence we obtain an action of $\pi_1(S, x)$ on the curve complex $\mathcal{C}(S^\circ)$. Behrstock and Leininger observe that:

Proposition 4.3. *The map $\rho: \mathcal{C}(S^\circ) \rightarrow \mathcal{C}(S)$ has the following properties:*

- *It is 1-Lipschitz.*
- *For any $\alpha \in \mathcal{C}(S^\circ)$, $\gamma \in \pi_1(S, x)$ we have $\rho(\alpha) = \rho(\psi_\gamma(\alpha))$.*
- *Every fibre \mathcal{F}_τ is connected.*

Remark 4.4. Behrstock and Leininger’s interest in the fibre \mathcal{F}_τ was to give a “natural” subcomplex of $\mathcal{C}(S)$ which is not quasi-convex: this is implied by the first pair of properties.

Remark 4.5. More of the structure of \mathcal{F}_τ is known. For example, since S is closed, the fibre \mathcal{F}_τ is either a single $\pi_1(S, x)$ -orbit or the union of a pair of orbits depending on whether τ is non-separating or separating. Furthermore, \mathcal{F}_τ is a tree. See [4] for a detailed discussion.

Proof of Proposition 4.3. Fix an essential non-peripheral curve α in S° . Note that $\rho(\alpha)$ is essential in S and so the induced map $\rho: \mathcal{C}(S^\circ) \rightarrow \mathcal{C}(S)$ is well-defined. If α and β are disjoint in S° then so are their images in S . Thus ρ does not increase distance between vertices and the first conclusion holds.

Now fix a curve $\alpha \subset S^\circ$ and $\gamma \in \pi_1(S, x)$. Note that ψ_γ is isotopic to the identity in S . Thus the images $\rho(\psi_\gamma(\alpha))$ and $\rho(\alpha)$ are isotopic in S . It follows that $\rho(\alpha) = \rho(\psi_\gamma(\alpha))$ as vertices of $\mathcal{C}(S)$, as desired.

Finally, fix $\tau \in \mathcal{C}(S)$. Let \mathcal{F}_τ be the fibre over τ . Pick $\alpha, \beta \in \mathcal{F}_\tau$. It follows that $a := \rho(\alpha)$ and $b := \rho(\beta)$ are both isotopic to τ and so to each other. We induct on the intersection number $\iota(\alpha, \beta)$. Suppose the intersection number is zero. Then α and β are disjoint and we are done. Suppose that the intersection number is non-zero. Since a and b are isotopic, yet intersect, they are *not* tight with respect to each other. It follows that there is a bigon $B \subset S - (a \cup b)$. Since α and β are tight in S° the point x must lie in B . Let $B^\circ := \rho^{-1}(\bar{B})$. Now construct a curve $\beta' \subset S^\circ$ by starting with β , deleting the arc $\beta \cap B^\circ$, and adding the arc $\alpha \cap B^\circ$. Isotope β' to be tight with respect to α . Now $\beta' \in \mathcal{F}_\tau$ because $\rho(\beta')$ is isotopic to $\rho(\beta)$ in S . Finally, $\iota(\alpha, \beta') \leq \iota(\alpha, \beta) - 2$. \square

5. PROVING THE THEOREM

We are now equipped to prove:

Theorem 5.1. *Fix $S^\circ := S_{g,1}$ for some $g \geq 2$. For any vertex $\omega \in \mathcal{C}(S^\circ)$ and for any $r \in \mathbb{N}$: the subcomplex spanned by $\mathcal{C}^0(S^\circ) - B(\omega, r)$ is connected.*

As above we use the notation $S^\circ = S_{g,1}$ and $S = S_g$ for some fixed $g \geq 2$. Also, we have defined a map $\rho: \mathcal{C}(S^\circ) \rightarrow \mathcal{C}(S)$ induced by collapsing ∂S° to a point, x . As above we use $\mathcal{F}_\tau = \rho^{-1}(\tau)$ to denote the fibre over τ .

Proof of Theorem 5.1. Choose α' and β' vertices of $\mathcal{C}(S^\circ) - B(\omega, r)$. By Proposition 3.1 we may connect α' and β' , by paths disjoint from $B(\omega, r)$, to vertices outside of $B(\omega, 3r)$. Call these new vertices α and β . We may assume that both α and β are non-separating because such vertices are 1-dense in $\mathcal{C}(S^\circ)$.

Choose any vertex $\tau \in \mathcal{C}(S)$ so that $d_S(\tau, \rho(\omega)) \geq 4r$. This is always possible because $\mathcal{C}(S)$ has infinite diameter. (See the remarks after Lemma 2.5.) It follows from Proposition 4.3 that $\mathcal{F}_\tau \cap B(\omega, r) = \emptyset$. We will now connect each of α and β to some point of \mathcal{F}_τ via a geodesic disjoint from $B(\omega, r)$. Since \mathcal{F}_τ is connected, by Proposition 4.3, this will complete the proof of Theorem 5.1.

Let $X^\circ := S^\circ - \alpha$ and take $X := \rho(X^\circ)$. Fix any point σ in \mathcal{F}_τ . If $\sigma = \alpha$ then α is trivially connected to the fibre. So suppose that $\sigma \neq \alpha$. Since α is non-separating deduce that σ cuts X° . Now, since $\xi(S^\circ) \geq 4$ we have $\xi(X^\circ) \geq 3$. Let $\gamma \in \pi_1(S, x)$ be any homotopy class so that ψ_γ is supported in X° and so that γ fills X . By Kra's Theorem (4.1) $\psi_\gamma|_{X^\circ}$ is pseudo-Anosov.

Since \mathcal{F}_τ is left setwise invariant by $\pi_1(S, x)$ (Proposition 4.3) the curves $\sigma_n := \psi_\gamma^n(\sigma)$ all lie in \mathcal{F}_τ . Since $\psi_\gamma|_{X^\circ}$ is pseudo-Anosov, Lemma 2.6 gives an $n \in \mathbb{Z}$ so that every geodesic $g = [\sigma, \sigma_n] \subset \mathcal{C}(S^\circ)$ has a vertex which misses X° . Since α is non-separating, as in Remark 2.3, it follows that α is actually a vertex of g .

We now claim that at least one of the two segments $[\sigma, \alpha] \subset g$ or $[\alpha, \sigma_n] \subset g$ avoids the ball $B(\omega, r)$. For suppose not: then there are vertices $\mu, \mu' \in g$ on opposite sides of α which both lie in $B(\omega, r)$. Thus $d_{S^\circ}(\mu, \mu') \leq 2r$. Since g is a geodesic the length along g between μ and μ' is at most $2r$. Thus $d_{S^\circ}(\omega, \alpha) \leq 2r$. This is a contradiction.

Thus we can connect α to a vertex of \mathcal{F}_τ (namely, σ or σ_n) avoiding $B(\omega, r)$. Identically, we can connect β to a vertex of \mathcal{F}_τ while avoiding $B(\omega, r)$. As noted above, this completes the proof. \square

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