D – Description of Proposed Research

Heegaard splittings and the combinatorics of three-manifolds

1. Results from prior NSF support

Schleimer was the Principal Investigator for a NSF postdoctoral fellowship, award number 0102069. This grant provided two years of full time support and was spent at the University of Illinois, Chicago. In addition to running the three-manifold seminar and teaching duties the PI wrote approximately nine mathematical research papers, while collaborating with six mathematicians. The most important work done by the PI with NSF support was his work on Heegaard splittings and the recognition problem for the three-sphere.

The PI has shown in [28] that a three-manifold contains only finitely many *full* Heegaard splittings. This was the first generally applicable finiteness result – K. Johannson's earlier work [13] required that the manifold be Haken. With D. Bachman in [3] the PI investigated Heegaard splittings of surface bundles. Their work generalized results of M. Lackenby [16], H. Rubinstein [24], and K. Hartshorn [8] by giving a simple relationship between the monodromy of the surface bundle and the genus of *strongly irreducible* Heegaard splittings contained in the surface bundle.

The PI has also shown, in [27], that the three-sphere recognition problem lies in the complexity class **NP**. In order to do this he showed that an almost normal surface can be normalized in time at most polynomial in the logarithim of the weight of the almost normal surface in question. It seems likely that this result will have applications to other questions in algorithmic topology, such as the minimal Heegaard genus problem.

Schleimer also has an active interest in computer science. His work on document fingerprint selection, with A. Aiken and D. Wilkerson [29], is used in the engine of **MOSS** [1], an online plagiarism detector. Their paper has been widely cited and not just in the computer science literature – for example it appears in the second declaration of R. Davis in the SCO vs. IBM case [6]. Finally, some applications of this work have been patented by the Regents of the University of California [2].

2. Overview of the proposal

The algorithmic topology of Heegaard surfaces can be explored in a manner directly analogous to that of incompressible surfaces. This may be called the "Haken program for Heegaard splittings".

Recall that W. Haken's normal form for incompressible surfaces is extremely important to algorithmic three-manifold topology and to the study of three-manifolds in general. For example, the theory proves the sphere and torus decompositions which are a prerequisite for W. Thurston's geometrization. Also, normal surfaces give the first and sometimes only approach to algorithmic questions such as homeomorphism of Haken manifolds, recognition of Haken manifolds, counting incompressible surfaces in fixed genus, recognition of the unknot, and computing the genus of homology trivial knots. It is also possible to use Haken's theory directly to give non-existence results: see Thurston's discussion in Chapter 4 of [32] showing that the figure-eight knot complement is small.

The above rich theory for incompressible surfaces may be modified to discuss Heegaard surfaces. This is a very surprising idea. After all, Heegaard surfaces are as far as possible from being incompressible. Nevertheless, Rubinstein [25] gives the required foundational result: *strongly irreducible* Heegaard surfaces can be isotoped into *almost normal* form. Strong irreducibility was defined by A. Casson and C. Gordon [5] and provides the correct analogue of incompressibility.

We now see that for every result proved about incompressible surfaces there should be a corresponding question about strongly irreducible splittings. For example, translating the work of W. Jaco and U. Oertel [12] to this context, we have:

Question 2.1. How do almost normal strongly irreducible Heegaard splittings decompose as Haken sums?

More precisely I make the following:

Conjecture 2.2. *For any closed orientable triangulated three-manifold there is a finite collection of pairs $\{(H_i, B_i)\}_i$ so that any strongly irreducible Heegaard splitting many be isotoped to be an almost normal surface H which decomposes as $H = H_i + F_i$ where F_i is carried by the normal incompressible branched surface B_i.*

(We adopt the convention that the empty branched surface carries only the empty surface.) This is a translation of the work of W. Floyd and Oertel [7] to our situation.

However the conjecture currently appears to be out of reach.[1] In Section 4 we discuss several easier goals – in particular we plan to attack Problem 1 which is good evidence for Conjecture 2.2

The study of cut and paste constructions also leads naturally to a discussion of the *Hempel distance* introduced in [10]. In particular I am interested in algorithmic results:

Question 2.3. Is there an algorithm which, given a Heegaard diagram, computes the distance of the underlying Heegaard splitting?

This question is also quite difficult. For example, answering this would also solve the much simpler Question 4.2 (below) that been has been heavily studied due to its connection with the Poincare Conjecture. Nonetheless, computing reasonable upper bounds is within reach. In this line we discuss two more of my ongoing research programs in Section 5. Of particular interest is Problem 5. More speculative work on lower bounds is also mentioned – here we plan to focus on resolving Question 5.9.

3. Definitions

Recall that a *Heegaard splitting* of a closed orientable three-manifold M is an embedded surface $H \subset M$ which cuts M into a pair of handlebodies V and W. These are necessarily of the same genus. A disk D properly embedded in a handlebody V is *essential* if $\partial D \subset \partial V$ is not null-homotopic.

A *cut system* for a handlebody V of genus g is a collection of g disjoint essential disks which cut V into a three-ball. A Heegaard *diagram* of a Heegaard splitting of (H, V, W) is a choice of cut system for each of V and W.

A splitting $H \subset M$ is *reducible* if there are essential disks $D \subset V$ and $E \subset W$ with identical boundaries. Casson and Gordon extend this notion: $H \subset M$ is *weakly reducible* if there are essential disks $D \subset V$ and $E \subset W$ with disjoint boundaries. If H is not (weakly) irreducible we say H is *(strongly) irreducible.*

Thompson has further extended this notion: $H \subset M$ has the *disjoint curve property* if there are essential disks $D \subset V$ and $E \subset W$ and an essential simple closed curve $\alpha \subset H$ where $\partial D \cap \alpha = \alpha \cap \partial E = \emptyset$. If H does not have the disjoint curve property then we say H is *full.* This because, for any pair of essential disks in V and W, their boundaries fill the surface H

[1] The recent work of T. Li, [17] and [18], does not alter this fact, as discussed in the sequel.

A triangulation T of M is a disjoint union of tetrahedra, all modeled on the regular Euclidean three-simplex of sidelength one, together with face-pairings giving a three-manifold homeomorphic to M. We say an embedded surface $S \subset T$ is *normal* if S is transverse to the skeleta of T and inside every tetrahedron S consists of *normal disks*. See Figure 1.

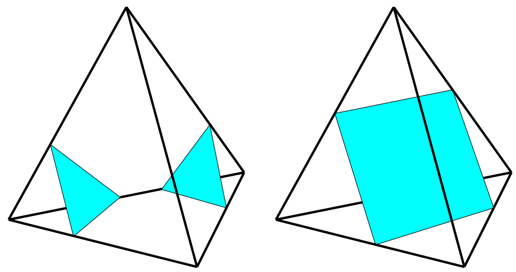

FIGURE 1. Two of the four normal triangles and one of the three quads.

We say an embedded surface $S \subset T$ is *almost normal* if S is transverse to the skeleta of T and inside every tetrahedron S consists of normal disks except there is exactly one tetrahedron also containing an *almost normal piece*. See Figure 2.

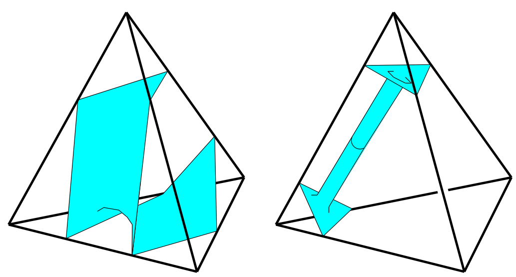

FIGURE 2. One of the three almost normal octagons and one of the 25 annuli.

4. CUT AND PASTE TECHNIQUES

In his 1978 survey paper [33] F. Waldhausen asked a sequence of questions about the structure of Heegaard splittings of three-manifolds. Rephrased in modern language they are:

Question 4.1. Does every three-manifold contain only finitely many irreducible Heegaard splittings?

Question 4.2. Is there an algorithm which, given a Heegaard diagram, decides whether or not the underlying splitting is irreducible?

Question 4.3. Is there an algorithm which, when given a Heegaard diagram, lists all irreducible splittings of the represented three-manifold?

Note that the answer Question 4.1 is "No"; manifolds with infinitely many irreducible splittings were first given by Casson and Gordon [4]. This construction, which uses pretzel knots with nice properties, has been generalized by M. Lustig and Y. Moriah [19] and T. Kobayashi [15].

The PI, together with Moriah and Sedgwick [23], has given new examples based on an essentially different construction. We find a (actually many) manifold containing a Heegaard surface H and an incompressible surface K, both of genus three. Higher genus splittings are obtained by taking many copies of K and performing a *Haken sum* with H. A small neighborhood of the intersection $H \cap K$ is removed and other annuli are added to form a closed surface as illustrated in Figure 3. After discovering these examples we then proved (also in [23]) that the examples of Casson and Gordon are also of this form.

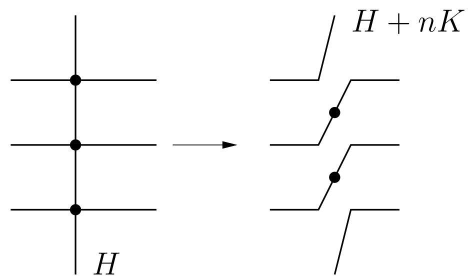

FIGURE 3. Cross sectional view before and after the Haken sum.

Note that, in general, the Haken sum includes the choices of new annuli. However in the presence of a background such as a triangulation, branched surface, or transverse orientation the Haken sum becomes canonical.

We remark that the decomposition of normal surfaces is one of their important features: on the one hand there is a connection to linear algebra, yielding finiteness results, on the other hand there is a connection to cut and paste constructions, yielding topological results.

In the same paper we asked if all high genus Heegaard splittings are obtained this way. Immediately after our paper was posted Li gave extremely strong evidence for a "yes" answer. In [17], for any atoroidal manifold he finds a finite collection of branched surfaces which carry all strongly irreducible Heegaard splittings of genus at least two. As these branched surfaces further only carry surfaces of negative Euler characteristic he deduces that any atoroidal manifold has only finitely many splittings in any genus. In [18], Li shows that an non-Haken manifold contains only finitely many irreducible splittings (as previously conjectured by Sedgwick in [23]). Thus Question 4.1 is in fact true if one assumes fixed genus or the absence of incompressible surfaces.

In line with the examples noted above, the I believe that I can solve:

Problem 1. Find a manifold M containing surfaces H, K, J so that $H + nK + mJ$ is a strongly irreducible Heegaard splitting for all positive integers n and m.

Such examples would definitely show that the results in Li's papers [17] and [18] are more general than those of [23]. To briefly outline the proposed construction: Double a high genus handlebody across its boundary. Decompose the interior of the handlebody into several regions which alternate between being I-bundles and being handlebodies. Also a curve in the boundary of the handlebody will be chosen. After twisting along this curve at least six times we will obtain a new manifold with the desired properties. See [23] for a detailed discussion of the $H + nK$ case.

We also note that Li uses compactness in an essential way in both of his arguments. Thus effective bounds are still lacking:

Problem 2. Show that, for any $H, K, H + K \subset M$ with $\text{genus}(K) > 1$, there is a positive integer n so that if $H + nK$ is a strongly irreducible Heegaard splitting then K is incompressible.

Theorem 1.1 of [23] draws a similar conclusion but requires a sequence of n's tending to infinity. Li's second paper [18] also requires arbitrarily large genus splittings. So Problem 2 points towards an effective version of "Haken's program for Heegaard splittings." As a brief indication of why Problem 2 is within reach recall that the proof of Theorem 1.1 in [23] examines a possible compressing disk D for K and discusses D's intersection with H and $H + nK$. It should be possible to apply complexity bounds similar to the ones found in [9] to control the size of the disk D. This, and a careful examination of cases, will solve Problem 2. However this is just the beginning of a richer theory.

For example, a carefully analysis of Li's work may solve:

Question 4.4. For any closed orientable three-manifold M there is a number $n > 0$ so that any strongly irreducible Heegaard splitting $H \subset M$ with genus greater than n has a two-sided incompressible surface as a Haken summand.

Even better from the algorithmic point of view would be:

Question 4.5. There is a computable function $f\colon \mathbb{N} \to \mathbb{N}$ so that if $H \subset (M, T)$ is a strongly irreducible Heegaard splitting with genus greater than $f(|T|)$ then H has a two-sided incompressible surface as a Haken summand.

Here T is a given triangulation of M and $|T|$ is the number of tetrahedra in T. It is difficult to imagine this result following from Li's work due to the compactness arguments he relies on. As is usual in normal surface theory, I suspect that this function is at worse exponential. As evidence for a positive answer to Question 4.5 we turn to my paper [28] on Thompson's disjoint curve property. There I investigated the "thick/thin" structure of an almost normal surface of sufficiently high genus which did not have the disjoint curve property. This, after a great deal of work, leads to an incompressible surface summand. From this a contradiction is deduced and we have:

Theorem 4.6. *Suppose (M, T) is a closed, orientable, triangulated three-manifold. Suppose $H \subset M$ is an almost normal Heegaard splitting with genus $g(H) > \exp(2^{16}|T|^2)$. Then H has the disjoint curve property.*

Here $\exp(x) = 2^x$. This theorem, when combined with [17], has an interesting corollary:

Theorem 4.7. *In any closed, orientable three-manifold there are only finitely many full Heegaard splittings, up to isotopy.*

Returning to the main theme: one way to resolve Question 4.5 is to carefully analyze the argument proving Theorem 4.6. Instead of assuming high genus and fullness assume only high genus and strong irreducibility. If the argument can be made direct (instead of relying on a contradiction) then a stronger theorem will be obtained. In particular answering Question 4.5 this way would give an effective version of Li's second paper [18].

An even more delicate analysis is required to deal with:

Conjecture 2.2. *For any closed orientable triangulated three-manifold there is a finite collection of pairs $\{(H_i, B_i)\}_i$ so that any strongly irreducible Heegaard splitting many be isotoped to be an almost normal surface H which decomposes as $H = H_i + F_i$ where F_i is carried by the normal incompressible branched surface B_i.*

What is called for here is an understanding of how a collection of incompressible surfaces meet a fixed Heegaard splitting – converse to the examples desired in Problem 1. Again, the thick/thin decomposition of [28] should give an approach.

We end this section by noting that the structure which Conjecture 2.2 imposes on the set of Heegaard splittings should have some implications for the Stabilization Conjecture [26]. As a first step in this direction we propose:

Problem 3. The Heegaard splitting $H + nK$, after one stabilization, destabilizes to any splitting $H + mK$ with $m \leq n$.

Note that this has been verified by Sedgwick [30] for Casson and Gordon's original examples in [4].

5. The Hempel distance

We now turn to Questions 2.3 and 4.2. These are quite difficult, as evidenced by the many failed attempts to find a recognition algorithm for the three-sphere and false proofs of the Poincare Conjecture. The common theme to many of these was to define a complexity measure of a Heegaard splitting (or some other combinatorial presentation) of the manifold in question. Then a finite collection of local simplification moves would be given and a proof assayed to show that these moves, when applied to splittings of the three-sphere, led to some standard splitting.

Successful proofs required a global simplification tool: thin position in the case of three-sphere recognition and (apparently) Ricci flow for the Poincare Conjecture. The catch is that neither has been computed algorithmically.

Similarly, *Hempel's distance* is a global complexity which is difficult to compute. Again this is not surprising as an algorithm to compute the Hempel distance of a splitting would resolve several long standing questions in the theory of Heegaard splittings. Before discussing these we require a few definitions:

Definition 5.1. Fix H a closed orientable surface of genus at least two. Define the *graph of curves* of H, $\mathcal{C}^1(H)$, to be the graph with vertices being isotopy classes of essential simple closed curves. There is an undirected edge for every pair of vertices which can be realized disjointly and all edges have length one.

Definition 5.2. If V is a handlebody with boundary H define the handlebody graph, $\mathcal{C}^1(V)$, to be subgraph with vertices those curves which bound disks in V.

Definition 5.3. If H is a Heegaard splitting separating M into handlebodies V and W then the *Hempel distance* of H, $d_{\mathcal{C}}(H)$, is the minimal distance in $\mathcal{C}(H)$ between $\mathcal{C}(V)$ and $\mathcal{C}(W)$.

The Hempel distance was introduced in [10]: a splitting H is irreducible if and only if $d_{\mathcal{C}}(H) > 0$, is strongly irreducible if and only if $d_{\mathcal{C}}(H) > 1$, and is full if and only if $d_{\mathcal{C}}(H) > 2$. J. Hempel has conjectured that the existence of a full splitting in M implies that M is hyperbolic. Indeed, this follows from the Geometrization Conjecture and the classification of Heegaard splittings of non-hyperbolic geometric manifolds.

Note that the graph of curves is locally infinite. It is immediately clear that algorithmic questions will be challenging. For example, it is non-trivial to give a method to determine distance between given vertices in the graph of curves.

A simple result in this direction is found by Hempel (see also Lemmas 2.1 and 4.11 of [22]):

Lemma 5.4. *If α and β are essential simple closed curves in H then*

$$d_{\mathcal{C}}(\alpha, \beta) \leq 2\log(|\alpha \cap \beta|) + 2.$$

This result can be greatly improved using the theory of *subsurface projections* developed by H. Masur and Y. Minsky [20], N. Ivanov [11], and others. We will return to this theme in Section 5.2

In light of the above it is not a surprise that that the following question is very difficult:

Question 2.3. Is there an algorithm which, when given a Heegaard diagram of $H \subset M$, computes $d_{\mathcal{C}}(H)$?

In particular, an answer to this also solves Waldhausen's second Question 4.2.

5.1. **Upper bounds.** A method for finding upper bounds on distance, refining work of Kobayashi [14], is given by Hartshorn [8]:

Theorem 5.5. *If $F \subset M$ is a two-sided incompressible surface and $H \subset M$ is a Heegaard splitting then $d_{\mathcal{C}}(H) \leq 2 \cdot \mathrm{genus}(F)$.*

To see this, use H to sweep out the manifold. Any generic level of the sweep-out intersects F in a collection of simple closed curves. As H moves the collection transforms via band-sums. An argument shows that the number of band-sums required is bounded by the topology of F. A question immediately arises:

Question 5.6. Find an upper bound for the distances of Heegaard splittings contained in a non-Haken manifold.

Note that Theorem 4.7 implies that for every non-Haken manifold there is *some* bound. Furthermore, recent work of J. Brock and J. Souto (in preparation) suggests that for any genus g there are constants a and b, given by a compactness argument, so that the related *pants distance* of a Heegaard splitting is bounded below by $\mathrm{vol}/a - b$ and bounded above by $a\mathrm{vol} + b$, where vol is the volume of the hyperbolic three-manifold. In any case the question of an effective bound persists. To be more specific:

Problem 4. Find an explicit constant K so that for any closed orientable triangulated three-manifold (M, T) and for any Heegaard splitting $H \subset M$ the inequality $d_{\mathcal{C}}(H) \leq K \cdot |T|$ holds.

As an idea of how to proceed, we may assume that $d_{\mathcal{C}}(H) \geq 2$ and apply the almost normalization theorem of Rubinstein [25] and M. Stocking [31]: with respect to any triangulation H may be isotoped to be an almost normal surface. Each tetrahedron of the triangulation is now divided into *core* and *product* pieces. Here a product piece is a component of a tetrahedron cut by H which is cobounded by exactly two normal disks of the same type. See Figure 4.

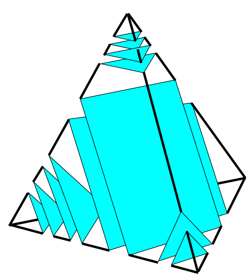

FIGURE 4. A tetrahedron cut by H. Note that there are two product blocks.

Recall that H cuts M into a pair of handlebodies V and W take V_I to be a regular neighborhood (in V) of all of the product blocks in V. Set $V_C = \overline{V \smallsetminus V_I}$ and define W_I, W_C similarly. Then V_I is an I-bundle over some surface and is relatively simple, topologically. On the other hand, V_C is build out of a bounded number of pieces (linear in $|T|$) each of uniformly bounded complexity. After we understand how these two submanifolds fit together we will find an essential disk $(D, \partial D) \subset (V, H)$ as do J. Hass *et al.*in [9], following Haken. Now, if ∂D had uniformly bounded complexity, and the same situation held in

W then we would be done by Lemma 5.4. In fact, the truth may be slightly more complicated: it seems difficult to bound the length of ∂D in the subsurface $H \cap V_I$. However, it seems likely that a bound on $\partial D \cap \partial(H \cap V_I)$ may be obtained. This, together with three applications of Lemma 5.4, should solve Problem 4 and thus give a first answer to Question 5.6. We remark that the appearance of the curves $\partial(H \cap V_I)$ is likely not just an ad hoc device. See Section 5.2 below.

There is a second important approach to Question 5.6 which I am pursuing with Marc Lackenby. The idea is to replace the incompressible surface of Hartshorn's theorem by another Heegaard splitting:

Problem 5. Suppose that H and K are Heegaard splittings of M and K is not a stabilization of H. Show that $d_{\mathcal{C}}(H) \leq 2 \cdot \operatorname{genus}(K)$.

Notice that the assumption on K cannot be removed: Hempel has found a sequence of splittings (of distinct manifolds) of fixed genus with increasing distance [10]. On the other hand, the assumption is extremely useful after making the standard reduction to the strongly irreducible case. Now isotope H and K close to disjoint spines. After labeling the components of $M \smallsetminus H$ by V and W we may assume that $K \subset V$. An argument proves that every compressing disk of H contained in V must meet K.

This should be thought of as the base case of an inductive process analogous to Stocking's proof that strongly irreducible Heegaard splittings can be made almost normal [31]. The spines of H replace the one-skeleton in Stocking's proof, and the sweep-out by surfaces isotopic to H replace the two-skeleton. Using the sweep-out by surfaces isotopic to K, and an inductive procedure, will put K into "almost normal position" with respect to the spine and sweep-out of H. This position of K should replace Hartshorn's incompressible surface.

As an application of Problem 5 we would have: If H is a genus g splitting of M with $d_{\mathcal{C}}(H) > 2g$ then H is a minimal genus splitting of M.

5.2. **Lower bounds.** This section describes work in progress with H. Masur. Finding lower bounds for the Hempel distance is more difficult than the upper bounds described above. More difficult still is computing the distance exactly. Nonetheless there are hints on how to proceed in a paper by Masur and Minsky [21]:

Theorem 5.7. *Suppose that V is a handlebody with genus at least two. Suppose that $H = \partial V$. Then the handlebody graph $\mathcal{C}(V)$ is quasiconvex inside of $\mathcal{C}(H)$.*

This is more meaningful in a broader context, also due to Masur and Minsky [22]:

Theorem 5.8. *The graph of curves $\mathcal{C}(H)$ is Gromov hyperbolic.*

We now apply this to Heegaard diagrams. Suppose that (H, D, E) is a Heegaard diagram, with D a cut system for V and E a cut system for W. There is a *hierarchy* (similar to a geodesic) from D to E, $\overline{DE}$, in $\mathcal{C}H$. By quasiconvexity combined with hyperbolicity, the hierarchy $\overline{DE}$ travels from D along $\mathcal{C}(V)$ then towards $\mathcal{C}(W)$ then along $\mathcal{C}(W)$ until it reaches E – see Figure 5. It may be possible to estimate the length of the hierarchy algorithmically. What is not known is how to estimate the distance between a point of $\overline{DE}$ and, say, $\mathcal{C}(V)$. If this was done, then we could compute the closest approach between $\mathcal{C}(V)$ and $\mathcal{C}(W)$. This would answer Question 2.3 as the closest approach between two handlebody graphs is the distance of the corresponding splitting.

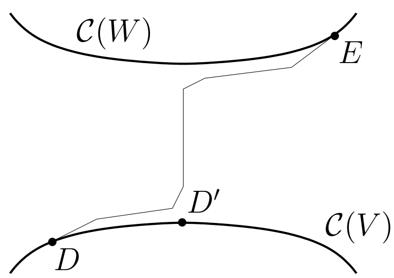

Figure 5. The hierarchy $\overline{DE}$ travels near the closest approaches between $\mathcal{C}(V)$ and $\mathcal{C}(W)$.

Suppose that D' is a cut system for V which is as close as possible (in $\mathcal{C}(H)$) to $\mathcal{C}(W)$. Then any hierarchy $\overline{DD'}$ between D and D' fellow travels $\overline{DE}$, at least while the latter remains close to $\mathcal{C}(V)$. The hope, then, is to analyze the curves making up $\overline{DD'}$. Any properties found in the curves of $\overline{DD'}$ should appear in the fellow-traveling segment of $\overline{DE}$.

As hinted above, in Section 5.1, a curve which does not bound a disk in V may appear in $\overline{DD'}$ if it is half of the boundary of an essential annulus in V. Specifically this annulus is part of the vertical boundary of an I-bundle V_I where $V = V_I \cup V_C$. It is easy to see how a disk in V may be altered by changing the way it meets V_I. This is closely related to the machinery of *subsurface projections*, which we now recall.

Suppose that T is a subsurface with essential boundary of the closed orientable surface S. Here we assume that $3 \cdot \text{genus}(T) + |\partial T| > 3$. (We are ignoring technical issues regarding annuli.) Let $\alpha \subset S$ be an essential curve, isotoped to meet ∂T *efficiently*: so that $S \smallsetminus (\partial T \cup \alpha)$ has no bigon components. Let the *subsurface projection* of α to T be $\pi_T(\alpha)$: the curves of $\partial \eta(\partial T \cup (\alpha \cap T))$ which are not peripheral in T. If α can be isotoped to lie in $S \smallsetminus T$ then $\pi_T(\alpha)$ is undefined.

Now if α and β are a pair of curves in S define $d_T(\alpha, \beta)$ to be the distance in $\mathcal{C}(T)$ between $\pi_T(\alpha)$ and $\pi_T(\beta)$, if both are defined. The Masur-Minsky theory states that a hierarchy between α and β can be essentially predicted from the set of large subsurface projections.

Returning to handlebodies we have:

Question 5.9. What are the essential subsurfaces of $H = \partial V$ which control the hierarchy $\overline{DD'}$?

Note that the answer "all" is incorrect because by Theorem 5.7 the graph $\mathcal{C}(V)$ is quasi-convex in $\mathcal{C}(H)$. It follows that a subsurface $T \subset H$ with $d_H(\mathcal{C}(V), \partial T)$ large does not appear in $\overline{DD'}$.

We require a more refined answer than this to obtain algorithmic results. There is an immediate list of subsurfaces T which must be allowed. Choose an I-bundle structure on V fibring over a connected surface with a single boundary component. Let T be the horizontal boundary of this I-bundle structure. Also, let α be the core curve of the vertical boundary (which is a single annulus). If D and D' have large subsurface projection to T then α must appear in the hierarchy $\overline{DD'}$. As a secondary output we find:

Lemma 5.10. *The subgraph $\mathcal{C}(V)$ is not quasi-isometrically embedded in $\mathcal{C}(H)$.*

It appears reasonable that other I-bundle structures may arise, as hinted at in Section 5.1. Surfaces $T \subset H$ bounding subhandlebodies $Z \subset V$. In any case, once Question 5.9 is resolved, an algorithm for Question 2.3 may be within reach. We first compute the distance between D and E, the given cut systems for V and W. We now wish to find D', a closest approach of $\mathcal{C}(V)$ to $\mathcal{C}(W)$. To do this recall that $\overline{DD'}$ and $\overline{DE}$ fellow-travel. The Masur-Minsky theory tells us that any sufficiently large subsurface projection for $\overline{DD'}$ will also appear in $\overline{DE}$. Thus we need only trace along $\overline{DE}$ until we find the first subsurface projection which is of the incorrect type to appear in $\overline{DD'}$.

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