BRIDGE PRESENTATION AND DISTANCE OF KNOTS IN THE THREE-SPHERE

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Abstract. For any positive integers t and b we find a tunnel number t knot in the three-sphere which has no (t', b')-decomposition for any $t' \leq t$ and $b' \leq b$. This result was known only for $(t, b) = (1, 2)$. Our technique relies on finding knot complements with high distance splittings.

1. Introduction

In this paper we address the problem of generating knots with high *distance*, in the sense of J. Hempel [7]. Note that it is easy to find three-manifold/knot pairs (M, K) where the exterior $E(K) = M - n(K)$ admits a Heegaard splitting of high distance. Fix any splitting (V, W) of high distance: that these exist is due to Hempel, refining a technique of T. Kobayashi [9]. Remove an unknotted solid torus from V to obtain a compression body V_0. Clearly the distance of the Heegaard splitting (V_0, W) is at least the distance of (V, W). See Section 2 for definitions.

The problem becomes more challenging if the ambient manifold M is specified beforehand. For example, the above argument cannot be used if M does not admit any high distance splitting. The case of the three-sphere is of particular interest: here every Heegaard splitting is isotopic to the standard one. Hence the *disk complexes* of V and W have distance zero and indeed infinite-diameter intersection. This makes finding an appropriate compression body more difficult. Nevertheless, by refining Kobayashi's techniques we prove:

Theorem 3.1. *For any integers $g > 1, n \geq 0$ there is knot $K \subset S^3$ and a genus g splitting $S \subset E(K)$ having distance greater than n.*

Now we consider another measure of complexity of a knot: $K \subset M$ has a (g, b)-*decomposition* if K may be isotoped to have exactly b bridges with respect to a genus g Heegaard splitting of M. When $b = 0$ we further insist that K be a *core* of one of the handlebodies.

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For example, the unknot in S^3 has a $(0,1)$-decomposition and is the unique such. By definition, a 2-bridge knot in S^3 has a $(0,2)$-decomposition. After a moment's reflection it is possible to find $(1,1)$ and $(2,0)$-decompositions for such knots.

Any knot K with tunnel number $t = t(K)$ (again, see Section 2) has a $(t+1,0)$-decomposition. The corresponding splitting surface of $E(K)$ has minimal genus. We note (Lemma 6.5) that K has a $(t,1)$-decomposition if and only if K is *μ-primitive*. It has been known for about ten years that there are knots which are *not* μ-primitive. For example, see the papers of Moriah and Rubinstein [12], Morimoto, Sakuma, and Yokota [15], and Eudave-Munoz [5]. In each case, proving the knots in question are not μ-primitive is a highly non-trivial task. Notice that the knots described in [12] have $(t,2)$-decompositions. The knots in [15] are all tunnel number one knots and have $(1,2)$-decompositions. The knots in [5] are all tunnel number one and some are known to be $(1,2)$ while the rest have unknown optimal $(1,b)$-decomposition.

Recently J. Johnson and A. Thompson have shown, improving on earlier work of Johnson's in [8], that for every n there is a tunnel number one knot which are not $(1,n)$. Independently, M. Eudave-Munoz has claimed the existence of tunnel number one knots in S^3 which are not $(1,2)$. His examples are either $(1,3)$ or $(1,4)$ knots but exactly which is not yet known.

Using the work of M. Scharlemann and M. Tomova (see [17] and [19]) and using Theorem 3.1 above, we prove:

Theorem 5.3. *For any positive integers t and b there is a knot $K \subset S^3$ with tunnel number t so that K has no (t,b)-decomposition.*

The question of determining the (g,b)-decomposition of a knot in S^3 has risen in other contexts as well. For example Kobayashi and Reick ask, as Question 1.9 of [10], whether there are knots in S^3 with a $(1,n)$-decomposition for $n \geq 3$. Theorem 5.3 answers in the affirmitive. Additional discussion is deferred to Section 6.

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2. Preliminaries

Two-dimensional. Fix S a compact connected orientable surface. Suppose that α and β are simple closed curves in S.

Definition 2.1. The curves α and β are *tight* if no component of $S - (\alpha \cup \beta)$ is a *bigon*: a disk B with $|B \cap \alpha| = |B \cap \beta| = 1$.

Let $\mathcal{PML}(S)$ be the space of projectively measured laminations on S. References on $\mathcal{PML}$ include Casson and Bleiler's book [3], Penner and Harer's book [16], and Bonahon's article [2].

We note that tightness may also be obtained for measured laminations; this is somewhat difficult to see combinatorially but can be proved using a hyperbolic structure on S and geodesic realizations of the laminations in question.

A discussion of train tracks may also be in order.

Three-dimensional. Recall that a *handlebody* V is homeomorphic to a closed regular neighborhood of a finite, connected, polygonal graph Γ embedded in $\mathbb{R}^3$. The image of Γ in V is called a *spine* for the handlebody. Suppose now that $\Gamma' \subset \Gamma$ is a subgraph of the spine of V and take $V_0 = V - n(\Gamma')$. Then V_0 is a *compression body*. The boundary of V_0 is naturally partitioned: $\partial V_0 = \partial_+ V_0 \cup \partial_- V_0$ where the positive boundary $\partial_+ V_0$ equals ∂V and the negative boundary $\partial_- V_0$ equals $\partial V_0 - \partial_+ V_0$.

A *Heegaard splitting* of a 3-manifold M is a decomposition $M = V \cup_S W$ where V, W are compression bodies so that $S = \partial_+ V = \partial_+ W = V \cap W$. The surface S will be called the *Heegaard surface* of the Heegaard splitting.

Complexes. Let $\mathcal{C}_S$ be the one-skeleton of Harvey's *curve complex* [6]. That is,

Definition 2.2. Fix S a closed connected orientable surface of genus at least two. Then $\mathcal{C}_S$ is the graph whose vertices are isotopy classes of essential simple closed curves and whose edges connect distinct classes with disjoint representatives.

We remark that $\mathcal{C}_S$ is connected. We place a metric $d(\cdot, \cdot)$ on $\mathcal{C}_S$ by setting the length of every edge to be one. For subsets $X, Y \subset \mathcal{C}_S$ we define $d(X, Y) = \min\{d(x, y) \mid x \in X,\ y \in Y\}$.

Definition 2.3. Fix V, a compression body. A curve $\alpha \subset S = \partial_+ V$ is a *meridian* of V if α bounds a disk in V.

We have a variant of a definition due to McCullough [11]:

Definition 2.4. The subcomplex $\mathcal{D}_V \subset \mathcal{C}_S$ spanned by meridians is called the *disk complex*.

The following definition is due to Hempel [7]:

Definition 2.5. Suppose that M is given a Heegaard splitting $V \cup_S W$. Define the *distance* of the splitting to be

$$d(V, W) = d(\mathcal{D}_V, \mathcal{D}_W).$$

We will need the following lemma, essentially due to Hempel [7], who in turn adapted an argument of Kobayashi [9].

Lemma 2.6. *Suppose that $X, Y \subset \mathcal{C}_S$ and that $\overline{X}, \overline{Y}$ are the closures in $\mathcal{PML}(S)$. Fix a pseudo-Anosov map Φ with stable and unstable laminations $\lambda^{\pm}$. Assume that $\lambda^- \notin \overline{Y}$ and $\lambda^+ \notin \overline{X}$. Then $d(X, \Phi^n(Y)) \to \infty$ as $n \to \infty$.* □

This statement is virtually identical to that of Theorem 2.4 in Abrams-Schleimer [1] and the same holds for the proof.

Multi-curves. Recall that *pair of pants* is another name for the three-holed sphere. A *seam* in a pair of pants is an essential properly embedded simple arc connecting distinct boundary components. A *wave* has all the same properties except the last: it connects one boundary component to itself. There is a matching between waves and seams determined by which pairs have non-trivial geometric intersection. We call such seam/wave pairs *dual*.

Definition 2.7. Suppose that S is a compact orientable surface of genus two or more. A *multi-curve* $\mathcal{M} \subset S$ is a collection of essential simple closed curves which are disjoint, non-parallel, and not boundary parallel. If $\mathcal{M}$ is maximal in size we call it a *pants decomposition*.

The seams (waves) of a multi-curve $\mathcal{M}$ are exactly the seams (waves) of the pairs of pants in $S - \mathcal{M}$. Suppose $\beta \subset S$, a simple closed curve, and $\mathcal{M}$ are tight. Then β has a seam (wave) with respect to $\mathcal{M} \subset S$ exactly when β has a seam (wave) in one of the pairs of pants of $S - \mathcal{M}$. Fix now some multi-curve $\mathcal{M} \subset S$.

Definition 2.8. We say that a curve or lamination λ is *traverses all seams* of $\mathcal{M}$ if

- λ and $\mathcal{M}$ are tight and
- λ runs over every seam of every pair of pants in $S - \mathcal{M}$.

No requirement is placed on the non-pants components of $S - \mathcal{M}$.

"Traverses all seams" is a slight generalization of Kobayashi's notion of "full-type", given in [9]. This is because we allow curve systems where he only allows pants decompositions. We do this in order to work with compression bodies.

Definition 2.9. Fix a compression body V with $S = \partial_+ V$. A *cut system* $\mathcal{M} \subset S$ is a maximal multi-curve which contains only meridians of V.

A standard outermost bigon argument proves the following:

Lemma 2.10. *Suppose V is a compression body and $\mathcal{M}$ is a cut system. Suppose α is a meridian of V and $\mathcal{M}$ and α are tight. Then either α is parallel to a component of $\mathcal{M}$ or α has a wave with respect to $\mathcal{M}$.* □

Again following Kobayashi [9], we conclude:

Corollary 2.11. *Suppose V is a compression body and $\mathcal{M}$ is a cut system. Assume that $|\mathcal{M}| \geq 2$. Then no lamination $\lambda \in \overline{\mathcal{D}_V}$ traverses all seams of $\mathcal{M}$.*

Proof. Since $\mathcal{M}$ has at least two components, $S - \mathcal{M}$ contains at least one pair of pants. Suppose now, for a contradiction, that λ traverses all seams of $\mathcal{M}$. Fix $\beta_i \in \mathcal{D}_V$, a sequence of meridians converging to λ. Passing to a subsequence we may assume that none of the β_i are parallel into $\mathcal{M}$. It follows from Lemma 2.10 that all of the β_i have a wave in some pants of $S - \mathcal{M}$. Passing to subsequences again we may assume that all of the β_i have the same wave in the same pair of pants. Since the β_i become increasingly parallel to λ, for sufficiently large index I the curve β_I runs parallel to the given wave and to the dual seam. It follows that β_I is not simple. This is the desired contradiction. □

Decompositions. We now present a well-known generalization of *bridge position.*

Definition 2.12. Suppose $\mathcal{A} \subset V$ is a disjoint collection of properly embedded arcs in a compression body V where $\partial\mathcal{A} \subset \partial_+ V$. We say $\mathcal{A}$ is *unknotted* if $\mathcal{A}$ can be properly isotoped, rel boundary, into ∂V.

We now have a definition due to Doll [4]:

Definition 2.13. Suppose that $M = V \cup_S W$ and K is a knot in M. The knot K is in *bridge position* with respect to S if:

- K is transverse to S,
- $K \cap S \neq \emptyset$, and
- both $K \cap V$ and $K \cap W$ are unknotted.

In this situation, if $g = g(S)$ and $b = |K \cap V|$, we say that K admits a (g, b)*-decomposition.*

We adopt a non-standard version of this when $b = 0$. Recall that, if $K \subset M$ then the *knot exterior* is $E(K) = M - n(K)$.

Definition 2.14. Suppose that $K \subset M$. Any Heegaard splitting of $E(K)$, of genus g, gives a $(g, 0)$-decomposition of K.

There is a well-known way to add structure to a $(g, 0)$-decomposition:

Definition 2.15. Fix a closed orientable manifold M and suppose $K \subset M$ is a knot. A disjoint collection of arcs $\{\tau_i\}$ properly embedded in $E(K)$ is a *tunnel system* for K if $E(K) - n(\cup\tau_i)$ is a handlebody. The size of a smallest such collection is $t(K)$, the *tunnel number* of K.

Thus the tunnel number of K and the minimal Heegaard genus of $E(K)$ measure the same quantity. We end this introductory material with some standard observations: If K has a (g, b)-decomposition then K also has $(g, b + 1)$ and $(g + 1, b)$-decompositions. It is also easy to check that, as long as $b \geq 1$, any knot with a (g, b)-decomposition also has a $(g + 1, b - 1)$-decomposition.

Thus our interest lies in moving in the opposite direction: turning a (g, b)-decomposition into a $(g - 1, b')$-decomposition, with b' as small as possible. Theorem 5.3 exactly says if $b = 0$ then we cannot bound b' by a function of g, even for knots in S^3.

3. High distance

The purpose of this section is to prove the following theorem:

Theorem 3.1. *For any integers $g > 1, n \geq 0$ there is knot $K \subset S^3$ and a genus g splitting $S \subset E(K)$ having distance greater than n.*

Equip $S^3 = V \cup_S W$ with the standard genus g Heegaard splitting. Let $\mathcal{C}_S$, $\mathcal{D}_V$, and $\mathcal{D}_W$ be the corresponding curve and disk complexes. Let $D \subset V$ be a disk cutting V into a solid torus X and a handlebody Y with genus $g-1$. Take K_0 to be the core of X. Thus $V_0 = V - n(K_0)$ is a compression body and $V_0 \cup W$ equals $E(K_0)$.

We must find a sequence of compression bodies $V_n \subset V$ each homeomorphic to V_0 so that

$$d(\mathcal{D}_{V_n}, \mathcal{D}_W) \to \infty$$

as $n \to \infty$. The corresponding knots $K_n \subset S^3$ will satisfy the conclusion of the theorem.

To this end, set $\delta = \partial D$. As shown in Figure 1 extend δ to a pants decomposition $\mathcal{P}$ whose boundary curves are all meridians of V. Likewise, as shown in Figure 1, choose a pants decomposition $\mathcal{Q}$ whose boundary curves are meridians of W.

Set $S_0 = \partial Y \cap S$. It is the once-punctured genus $g-1$ surface to the right of δ. Let $\mathcal{P}_0 = \mathcal{P} \cap S_0$ be the pair of pants decomposition of S_0 induced by $\mathcal{P}$. The important feature of our chosen decompositions is that the curves of $\mathcal{P}_0$ traverse every seam of $\mathcal{Q}$ in S_0.

The construction of V_n consists of several steps:

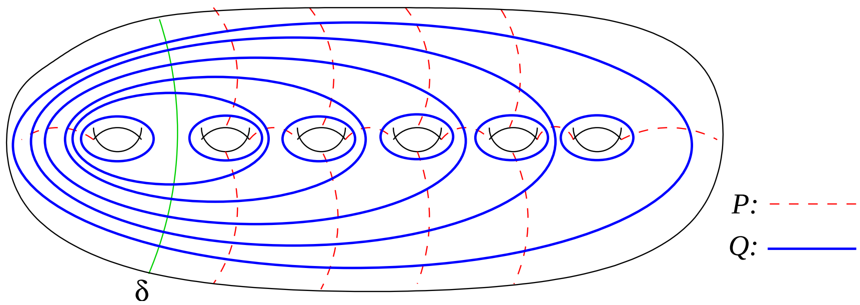

FIGURE 1. The pants decompositions $\mathcal{P}$ of V and $\mathcal{Q}$ of W. Each component of $\mathcal{Q}$ (except the littlest and the biggest) has a symmetric "partner" on the underside of V.

Step 1. Find a meridian curve $a \in \mathcal{D}_V$ such that

$a1$: a traverses all seams of $\mathcal{Q}$ and
$a2$: a traverses all seams of $\mathcal{P}_0$.

Step 2. Use the curve a to construct a pseudo-Anosov map $\Phi\colon S \to S$ which extends over V and whose stable lamination λ^+ and unstable lamination λ^- have

$\lambda 1$: λ^+ traverses all seams of $\mathcal{Q}$ and
$\lambda 2$: λ^- traverses all seams of $\mathcal{P}_0$.

Step 3. Show that $d(\mathcal{D}_W, \Phi^n(\mathcal{D}_{V_0})) \to \infty$ as $n \to \infty$.

Conclusion. Since Φ extends over V we may define $V_n = \Phi^n(V_0)$. This is again a compression body embedded in V. Thus (V_n, W) is a Heegaard splitting of a knot exterior $E(K_n)$ where $K_n = \Phi^n(K_0) \subset S^3$. As desired in the conclusion of the theorem we have:

$$d(\mathcal{D}_{V_n}, \mathcal{D}_W) \to \infty$$

4. PROOF OF THEOREM 3.1

We now carry out the steps outlined above.

Step 1. We will first find a useful curve $\gamma \subset S_0$. We will then use two copies of γ to build the desired meridian a. So consider a train track τ modeled on $\mathcal{P}_0$ and described in Figure 2.

Two important features are:

(1) τ contains $\mathcal{P}_0 - \delta$ and additional branches. These branches contain all seams of $\mathcal{P}_0$, with the exception of the two seams incident to δ.

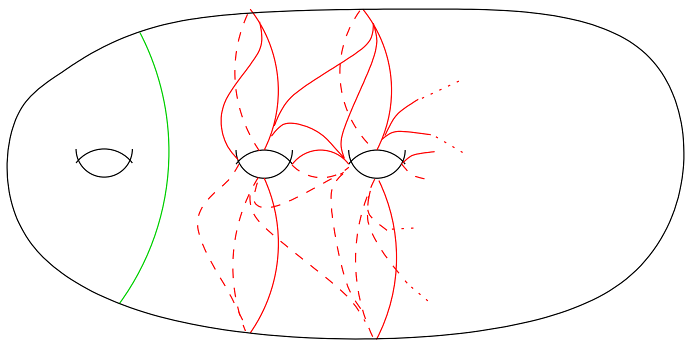

FIGURE 2. The train track in S_0. Only the first few segments are drawn.

(2) The entire picture is invariant under a 180° rotation about the horizontal line meeting V in $g+1$ arcs. This will be important when dealing with the pair of pants (of $\mathcal{P}_0$) adjacent to δ.

Let γ be a simple closed curve carried by τ with the property that it traverses every branch at least twice and goes around each component of $\mathcal{P}_0 - \delta$ at least once. (One can obtain γ by starting with an arational measured lamination carried on τ, and then taking a nearby rational approximation.) Note that γ has the following properties:

(1) It traverses every seam of $\mathcal{P}_0$, again excluding the two seams incident to δ.
(2) It traverses every seam of Q, excluding the four seams in X. This is because γ follows every curve of $\mathcal{P}_0 - \delta$ and these traverse the relevant seams of Q.

To build the meridian a we add two loops to the train track τ as shown in Figure 3. Each loop begins inside one of the switches of τ nearest to X, enters X, goes once around the meridian disk, and then returns to the same switch. We define an integer measure on this new train track τ': Let μ be the measure on τ that defines γ. Hence 2μ defines two copies of γ. Now subtract 2 from the weight that 2μ puts on the branch of τ closest to δ, and put a weight of 1 on each of the new loops. This measure on τ' defines a. By construction a satisfies conditions $(a1)$ and $(a2)$.

Step 2. Choose two meridians $b, c \in \mathcal{D}_V$ so that b and c together fill S. Let τ_a and τ_b and τ_c denote the Dehn twists about a and b and c respectively. Set

$$\Phi_0 = \tau_b \circ \tau_c^{-1}.$$

It follows from Thurston's construction [18] that Φ_0 is a pseudo-Anosov. Since Φ_0 is a composition of Dehn twists along meridian disks it extends

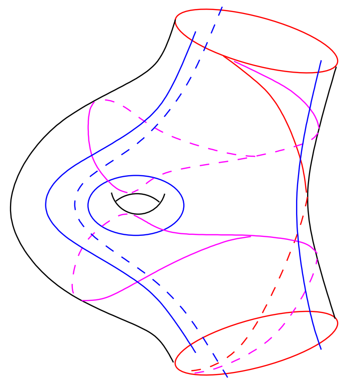

FIGURE 3. The meridian a is constructed via a band-sum. The surface here shows the leftmost 1-holed torus and pair of pants from Figure 2 – the curve δ separating them has not been drawn, and the shape is somewhat distorted for convenience. The loops are the purple arcs. Note that each loop takes care of two of the seams among the four pictured (blue) curves of Q. To check that all intersections are essential, note that there are no bigons in the picture.

to V. Define

$$\Phi_N = \tau_a^N \circ \Phi_0 \circ \tau_a^{-N}.$$

Note that Φ_N also extends to V. The stable and unstable laminations $\lambda_N^\pm$ of Φ_N are just $\tau_a^N(\lambda^\pm)$. Since a meets $\lambda^\pm$, as $N \to \infty$ the laminations $\lambda_N^\pm$ converge to $[a]$ in $\mathcal{PML}(S)$. Hence eventually both laminations satisfy conditions $(a1)$ and $(a2)$. Take $\Phi = \Phi_N$ for such a large N and take $\lambda^\pm = \lambda_N^\pm$. Thus conditions $(\lambda 1)$ and $(\lambda 2)$ are satisfied.

Step 3. We must show that $d(\mathcal{D}_W, \Phi^n(\mathcal{D}_{V_0})) \to \infty$ with n. Step 2 above proves that $\lambda^\pm$ are of full-type with respect to $\mathcal{Q}$ and $\mathcal{P}_0$. Thus, by Corollary 2.11, $\lambda^- \notin \overline{\mathcal{D}}_{V_O}$ and $\lambda^+ \notin \overline{\mathcal{D}}_W$. The desired conclusion now follows from Lemma 2.6. This completes the proof of Theorem 3.1.

5. Ruling out simple decompositions

In order to prove Theorem 5.3 we rely first on a Theorem of Scharlemann and Tomova [17] and then on a considerable refinement due to Tomova [19].

Theorem 5.1. *Let M be a three-manifold and suppose that S and Q are Heegaard splittings of M. Suppose that the genus of Q is less than the genus of S. Then the distance of the Heegaard splitting S is at most $2g(Q)$.* $\square$

Theorem 5.2. *Let $K \subset S^3$ be a knot and S be a Heegaard splitting surface of the exterior of K. Suppose that Q is Heegaard splitting surface of S^3 so that*

- *K is in bridge position with respect to Q and*
- *the genus of Q is less than the genus of S.*

Then the distance of the Heegaard splitting S is at most $2 - \chi(Q - K)$. □

We are now in postion to prove Theorem 5.3:

Theorem 5.3. *For any positive integers t and b there is a knot $K \subset S^3$ with tunnel number t so that K has no (t, b)-decomposition.*

Proof. Fix t and b. Choose $K_n \subset S^3$, as constructed in Theorem 3.1, so that the associated splitting $E(K_n) = V \cup_S W$ has genus $t + 1$. We choose n sufficiently large to ensure that the distance $d(V, W)$ is greater than $2t + 2b + 2$.

We first prove that the tunnel number $t(K_n)$ equals t. Clearly $t(K_n) \leq t$. Suppose $t(K_n)$ is strictly less. Then let Q be the splitting associated to some minimal tunnel system. We find that $g(Q) < g(S)$ and so, by Theorem 5.1, the distance satisfies $d(V, W) \leq 2g(Q) < 2t+2$, a contradiction.

We now consider the possibility of a (t, b)-decomposition of K_n. Let Q be the associated splitting of S^3. Then the genus of Q is less than that of S. Applying Theorem 5.2 we find that $d(V, W) \leq 2 - \chi(Q - K) = 2 - (2 - 2g(Q) - 2b) = 2t + 2b$, again a contradiction. □

6. Conjectures, questions, and applications

6.1. **Other manifolds.** To the general fun I will add the following conjecture:

Conjecture 6.1. *Fix M, an orientable compact manifold. Suppose that $M = V \cup_S W$ is a twice stabilized Heegaard splitting. For any n there is a core curve $K \subset V$ so that the splitting $E(K) = (V - n(K)) \cup_S W$ has distance greater than n.*

Yee-hah!

6.2. **Addetivity.** It seems that Theorem 5.3 is relevant to the question of addetivity of tunnel number of knots under connected sum. We first need a few definitions:

Definition 6.2. Let $K \subset S^3$ be a knot and (W, V) a Heegaard splitting of $S^3 - N(K)$ with $\partial(S^3 - N(K)) \subset V$. Let μ denote a simple closed curve on $\partial_- V$. We say that μ is *primitive* if there is a vertical annulus

A in the compression body V such that $\partial A = \mu \cup \gamma$ where $\gamma \subset \partial_+ V$ meets an essential disk D of W in a single point.

Definition 6.3. We say that a knot $K \subset S^3$ has a primitive meridian if $S^3 - N(K)$ has a minimal genus Heegaard splitting with a primitive meridian. The knot K is *not μ-primitive* if there is no Heegaard splitting for $E(K)$ which has a primitive meridian.

Definition 6.4. Given a knot K in a 3-manifold M, a *tunnel system* for K is a collection $\mathcal{T} = \{t_1, ..., t_n\}$ of unknotted arcs such that $M \smallsetminus N(\mathcal{T} \cup K)$ is a handlebody. It will be called a *minimal tunnel system* if n is minimal over all possible tunnel systems. We will then say that the knot K has tunnel number n and denote $t(K) = n$. Tunnel number can be similarly defined for a general 3-manifold.

This following lemma is well-known; we give the the proof for the sake of completeness.

Lemma 6.5. *Let $K \subset M$ be a knot with $t(K) = t$. Then K has a $(t, 1)$-decomposition if and only is K is μ-primitive.*

Proof. Let (V, W) be the Heegaard splitting of genus t of M which realizes the $(t, 1)$-decomposition. Hence $K = t_1 \cup t_2$ where $t_1 \subset V, t_2 \subset W$ are unknotted arcs. Consider a regular neighborhood of $t_2 \subset W$. We can think of it as a a 1-handle containing t_2. Remove it from W and add it to V to obtain a handlebody V' of genus $t + 1$. Since t_2 was unknotted the drilled out manifold W' is also a handlebody. The cocore disk of the 1-handle meets an essential disk of W' in a single point. The one which is determined by the given isotopy of t_2 into $\partial_+ W$. Hence when $K \subset V'$ is removed we get a compression body V''. Thus (V'', W') is a Heegaard splitting of minimal genus $g = t + 1$ and by the construction is μ-primitive.

If (V, W) is a minimal genus Heegaard splitting for $M - N(K)$ with V being the compression body then add a solid torus neighborhood of K to V to obtain a handlebody V'. It has an essential disk D which meets K in a single point and since (V, W) is μprimitive and essential disk E of W in a single point. If we remove a regular neighborhood $N(D)$ of D from V' and add it to W along an annulus as a 2-handle we obtain handlebody V'' and W'' and hence a Heegaard splitting $(V'', W')'$ of genus $g - 1 = t(K)$ for M. Now $K \cap N(D)$ is and arc $t_2 \subset W''$ and $K \cap V''$ is an arc t_1 and they are clearly unknotted. So we have a $(t, 1)$-decomposition for $K \subset M$. $\square$

If the knots $K_1, K_2 \subset S^3$ have tunnel number $t(K_1), t(K_2)$ respectively then we have the following conjecture by Morimoto (see [14]):

Conjecture 6.6. *If K_1 and K_2 are knots in S^3 then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if neither of K_1 and K_2 are μ-primitive.*

Remark 6.7. The conjecture is known to be true for tunnel number one knots [13] and for m-small knots [14].

However there is some evidence for the converse:

(1) It is a theorem (see [10]) that if there exists a knot $K \subset S^3$ such that both K and $2K = K \# K$ are not μ-primitive then the above Conjecture 6.6 is false. Where $nK = K \# K \#, \dots, \# K$, n times.
(2) It is also a theorem [10] that if a knot K has a (t, n)-decomposition then the knot nK is μ-primitive. Since all known examples of knots which are not $(t, 1)$ are $(t, 2)$ they fail as candidates for the above theorem.

Note that (1) follows from (2) if we add the additional condition that mK is not μ-primitive for $m \leq n-1$ and $n \geq 3$: Assume Conjecture 6.6 is true. Since nK is μ-primitive we have:

$$t((n+1)K) = t(nK) + t(K) =$$

$$nt(K) + (n-1) + t(K) = (n+1)t(K) + (n-1)$$

However, for $m < n$ both mK and $(n+1)K$ are not μ-primitive so we have:

$$t((n+1)K) = t(mK) + t((n+1-m)K) =$$

$$mt(K) + (m-1) + (n+1-m)t(K) + (n-m) =$$

$$(n+1)t(K) + (n-1+1) = (n+1)t(K) + n$$

This a contradiction. We give a proof for (2) below.
Hence:

Question 6.8. (1.9 of [10]): Are there knots which are (t, n) For $n \geq 3$?

Remark 6.9. Theorem 5.3 gives a positive answer to Question 6.8. However showing that a knot K which is a non-trivial connected sum has a $(t(K), b)$-decomposition must be very tricky since these knots contain essential tori so all their Heegaard splittings are distance at most 2 by [7].

Though the above Remark 6.9 shows that we are far away from finding a counterexample to the Morimoto Conjecture using these techniques it is the belief of the second author that the conjecture is in fact false. A better conjecture would be:

Conjecture 6.10. *If K_1 and K_2 are prime knots in S^3 then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if neither of K_1 and K_2 are μ-primitive.*

Theorem 6.11. *(2) If the knot K has a (t,n)-decomposition then $nK = K\#, ...\#K$ n-times, has a Heegaard splitting of genus $nt(K)+n$ which is μ-primitive.*

Proof. Let $K \subset S^3$ be a knot with a $(t(K),n)$ decomposition. Then there is a Heegaard splitting (V_1,V_2) of $E_{S^3}(K)$ of genus $t(K)+n$ so that $\partial E_{S^3}(K) \subset V_1$. It is obtained by taking n 1-handles which are regular neighborhoods of the n arcs $\{t_1,\dots,t_n\}$ from one handlebody W_2 in the $(t(K),n)$ decomposition and adding them to the other handlebody W_1. Then removing a smaller regular neighborhood of K from the modified W_1 to obtain a compression body V_1 and a handlebody V_2 both of genus $t(K)+n$.

The boundary of the cocore disks of the tunnels now determine a collection of n curves $\{\gamma_1,\dots,\gamma_n\} \subset \partial V_2$ which are by definition primitive. They have the additional property that the essential disks $\{D_1,\dots,D_n\} \subset V_2$ that each curve $\{\gamma_1,\dots,\gamma_n\} \subset \partial V_2$ intersect in a single point can be chosen to be pairwise disjoint.

Now consider $(U_1^1,U_2^1),\dots,(U_1^{n-1},U_2^{n-1})$ $n-1$ copies of a minimal genus Heegaard splitting for $E_{S^3}(K)$ so that $\partial E_{S^3}(K) \subset U_1$. For each i cut U_1^i along a vertical annulus A^i so that $\partial A^i = \alpha_1 \cup \alpha_2$ with $\alpha_1 \subset \partial E_{S^3}(K)$. This operation leaves two images A_1^i A_2^i of the annulus on the resulting handlebody. Similarly cut V_1 along $n-1$ vertical annuli B_i corresponding to the cocore disks of the tunnels $\{t_1,\dots,t_{n-1}\}$. Attach a copy of U_1^i to V_1 by identifying the images of A_1^i and B_1^i and A_2^i and B_2^i to obtain a compression body.

Attach a copy of U_2^i to V_2 by identifying the image of an annulus neighborhood of α_2 in ∂U_2^i with the annulus neighborhood of γ_i. Since γ_i is primitive we obtain a handlebody of genus $t(K)+n+(n-1)(t(k)+1)-(n-1) = nt(K)+n$ which determines a Heegaard splitting (H_1,H_2) of $E_{S^3}(nK)$. The meridian corresponding to the t_n arc is clearly primitive from the construction meeting the essential disk D_n of H_2 in a single point. □

6.3. **Other manifolds.** What can be said about other manifolds? That is, for which 3-manifolds M is the following statement true:

> For any $g > 2$ and $n > 0$ there exists a knot K in M such that $M - K$ has a genus g Heegaard splitting with distance greater than n.

I would conjecture that it's false for M a connected sum of $S^2 \times S^1$'s, and true if M and g such that M has a genus g splitting of distance at least 3. (Is 3 the right number?)

For a connected sum of $S^2 \times S^1$'s: If we start with the standard splitting, then M is the double of V so $\mathbb{D}_V = \mathbb{D}_W$, and no version of the construction can work: $\mathbb{D}_{V_n}$ is always in $\mathbb{D}_V$, and so can't be a large distance from $\mathbb{D}_W$. A general splitting is (is this right?) a stabilization of the standard splitting. What can we say in this case?

6.4. **Geometry.** Given the construction in this paper, what can we tell about the hyperbolic geometry of this knot complement? For example, it seems that the distance lower bound we get should mean that the complement of the cusp in $S^3 \smallsetminus K$ would still contain a "deep" handlebody. In particular it should hold that:

Conjecture 6.12. *For any $D > 0$ there is a hyperbolic knot K in S^3 such that the D-neighborhood of the standard (Margulis tube) cusp of K does not cover all of $S^3 - K$.*

And if this is true, is it interesting? Do we know other ways of building knots with this property?

A way to analyze the geometry of this: Consider a fixed z_n as in the construction. Now use $h_n = \tau_{z_n}^m$ as before, but let $m \to \infty$ while fixing n. The geometric limit (using Dehn filling theorem) is the manifold with two rank 2 cusps obtained from drilling both K (i.e. the torus $h_n(T_n)$) and z_n. One can try to study this manifold. It has a cover corresponding to the fundamental group of W (now no longer the whole fundamental group) which has a cusp associated to z_n. Now remember that $z_n = \Phi^n(z_0)$ – so there should be a nice limit of this as $n \to \infty$. I think it actually suffices to use Kleineidam-Souto to extract this limit, and the structure of it is as in Hossein's thesis. That is, the geometric limit as $z_n \to \infty$ is a handlebody with a simply-degenerate end.

That tells us that in the original manifold (provided m_n is chosen large enough) there will be a large handlebody that looks a lot like a big compact core of this degenerate handlebody. As $n \to \infty$ we get larger and larger pieces like this and hence we get Conjecture 6.12.

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