On manifolds with infinitely many strongly irreducible Heegaard splittings

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Abstract

In this paper we show that if a 3-manifold M which has infinitely many strongly irreducible Heegaard splittings of arbitrarily high genus all of the form $H + nK$ i.e., taking the Haken sum of a given surface h with n copies of another given surface K, then the surface K is incompressible. This is true for all known examples of such manifolds. We further use this result to obtain many more new examples of such manifolds M.

Keywords Heegaard splittings, strong irreducibility, Haken sum, Incompressible

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1 Introduction

The first examples of 3-manifolds with infinitely many strongly irreducible were obtained by Casson and Gordon in 1985 using a result of Paris [Pa]. Their goal was to show that Waldhausen's conjecture (see [Wa]) that manifolds had only finitely many unstabilized Heegaard splittings was false. Their examples are obtained by surgery on certain pretzel knots and they are actually strongly irreducible. Their result is unpublished but a proof is given in [MS]. Soon afterwards in 1988 in an unpublished paper Kobayashi obtained such manifolds and his examples involve surgery on similar pretzel knots.

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His method for showing that they are strongly irreducible is some what different. In later work the first author and Martin Lustig (see [LM]) showed that indeed this phenomena is actually quite common and one can obtain manifolds with infinitely many strongly irreducible Heegaard splittings of arbitrarily large genus by surgery on a very large class of knots, much larger than pretzel knots.

In this paper we want consider the reverse question: What do we know about a manifold M that has an infinite collection of strongly irreducible Heegaard splittings? We will show that if a manifold M has infinitely many strongly irreducible Heegaard splittings of arbitrarily large genus which are all obtained in a certain form (described below) then M contains a closed incompressible surface i.e., M is Haken. To make a precise statement we need the following notation:

Given any two closed orientable surfaces in a closed and orientable 3 manifold M we will denote by $H + nK$ the Haken sum of H and n parallel copies of the surface K. If K has negative Euler characteristic then the genus of $H+nK$ will be arbitrarily large as the Euler characteristic is additive under Haken sum. For precise definitions of Haken sum see Section 2. We claim:

Theorem 1.1. *Let M be a closed orientable and irreducible 3-manifold and let H and K be two given closed and orientable surfaces in M. If $H_n = H + nK$, are strongly irreducible Heegaard splittings for M, for all $n = 1, \ldots,$ and K is not a 2-torus then the surface K is incompressible.*

Remark 1.2. In particular we would like to stress that all known examples are of this form.

Remark 1.3. The case when K is a 2-torus is of a different nature and will be discussed in Section 3.

Theorem 1.1 was conjectured by the the Eric Sedgwick who in fact also conjectured the following stronger statement:

Conjecture 1.4. *Let M be a closed orientable and irreducible 3-manifold with infinitely many strongly irreducible Heegaard splittings of arbitrarily large genus. Then M is Haken.*

Remark 1.5. The Heegaard splittings discussed here must all be of distance two, since it was shown by the second author that any 3-manifold M has at most only finitely many Heegaard splittings of distance three (see [Sc]).

We will show, in Section 4, that all the known examples of manifolds with infinitely many strongly irreducible Heegaard splittings of arbitrarily large genus (i.e., the examples of Casson-Gordon, Kobayashi and Lustig-Moriah) are indeed of the form $H + nK$. Finding the actual surface K of the construction turned out to be extremely difficult. Using Theorem 1.1 we conclude that they are all Haken. Thought it can be proved directly that these manifolds are Haken the proof requires some non-trivial theorems e.g. [FM], [Wu].

Using the ideas from the proof of the theorem we can generate new manifolds with infinitely many strongly irreducible Heegaard splittings of arbitrarily large genus. This is done in Section 5.

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2 Preliminaries

Let M be a closed orientable 3-manifold. A *Heegaard splitting* for M is a decomposition for $M = V \cup W$ where V and W are handlebodies and $V \cap W = \partial V = \partial W = S$. We call S the *Heegaard surface*. A Heegaard splitting will be called *reducible* if there is a curve $\gamma \subset S$ which bounds essential disks $D_1 \subset V$ and $D_2 \subset W$. If the 2-sphere $D_1 \cup D_2$ bounds a 3-ball in M we say that the Heegaard splitting is *stabilized*. If there are disjoint essential disks in V and W we say that the Heegaard splitting is *weakly reducible* and if every essential disk on V meets every essential disk in W (necessarily in their boundary) we say that the Heegaard splitting is *strongly irreducible*.

Given two surfaces H and K in M in general position their intersection, if not empty, is a collection Γ of simple closed curves $\cup\gamma$. The regular neighborhood of each curve $\gamma \in \Gamma$ is a a solid torus $T(\gamma)$ and $\partial T(\gamma) - (H \cup K)$ is composed of four open annuli $A_1(\gamma) \cup A_2(\gamma) \cup A_3(\gamma) \cup A_4(\gamma)$ enumerated cyclically. These annuli can be separated into opposite pairs $A_+(\gamma) = A_1(\gamma) \cup A_3(\gamma)$ and $A_-(\gamma) = A_2(\gamma) \cup A_4(\gamma)$. We can now define the *Haken sum* of H and K as follows:

Definition 2.1. For each curve $\gamma \subset \Gamma$ choose a value $e(\gamma) \in \{+, -\}$ and consider the following surface

$$H + K = (H \cup K) - (\cup_\gamma T(\gamma)) \cup (\cup_\gamma A_{e(\gamma}(\gamma))$$

Definition 2.2. We define the surface $H + nK$ to be the Haken sum of the surface H and n copies of the surface$K$ (see Fig. 1).

Figure 1

Remark 2.3. This surface $H + K$ as described is not canonical and depends on the choice of $e(\gamma)$. Given more structure on M such as a triangulation one can obtain a canonical construction.

In our situation we are assuming that the manifold M contains infinitely many strongly irreducible Heegaard splitting which are of the form $H + nK$. Hence the ambiguity of the possible choices of $e\gamma)$ are already dealt with as part of the input information.

We further need:

Definition 2.4. Given a surface S and a simple closed curve γ on S a *wave* with respect to γ is an arc ω such that $\omega \cap \gamma = \partial\omega$. Further, ω meets γ from the same side and is not isotopic into γ in S.

Remark 2.5. Note that this is a some what more general definition of a wave than the standard one.

We also need the following lemmas:

Lemma 2.6. *If $K \subset$ is a compressible surface then there is a compressing disk D so that ∂D is a separating curve on K or K is a 2-torus.*

Proof. Consider the two copies D^+ and D^- of the disk D in the boundary of a regular neighborhood. If D is non-separating on K then there is a curve α outside the regular neighborhood connecting D^+ and D^-. Now band sum D^+ to D^- along α to obtain a separating disk D'. If K is not a torus we can always choose α so that the resulting disk D' is essential.

□

Remark 2.7. The intersection $H \cap K \subset K$ is a collection of simple closed curves. Each curve in the collection can be marked with a " $+$ " or " $-$ " sign depending on the configuration of the Haken sum. If D is a compressing disk for K then the collection of n copies of K will intersect D in a collection Γ of n concentric curves parallel to ∂D. These concentric curves will be connected by l arcs $\cup\delta_k$ from the fixed intersection $D \cap H$. Now the intersection $\cup\delta_k \cap \Gamma$ is a collection of points on D each with a " $+$ " or " $-$ " sign attached. These points will disappear after we perform the Haken sun $H + nK$ to obtain the intersection $D \cap H_n$.

Lemma 2.8. *Assume the notation of Remark 2.7 and that n is much larger than l. If the number of points in $\cup\delta_k \cap \Gamma$ marked by " $+$ " signs is equal to the number of points marked by " $-$ " signs then the intersection $D \cap H_n$ will be a collection of at least $n - 2l$ concentric curves contained in an annulus A in D plus at most l simple closed curves (not necessarily concentric in the disk $D - A$ plus l arcs located between the annulus A and ∂D.*

Proof. To see this recall that the parallel curves of $nK \cap D$ occur in a fixed annulus A so that ∂A is the union of the intersections of the first and last copies of K with D. Let $D_1 = D - A$ start on a point on one of the concentric curves of $nK \cap D$ and travel in, say, a counter clockwise direction. We can choose the convention so that every time we meet a " $+$ " sign we go towards D_1 and when we meet a " $-$ " we go towards ∂D. If a component of $H_n \cap D$ meets D_1 it must meet one of the l arcs of intersection of $H \cap D$ and each of the fixed l arcs on D_1 can be umet at most once. Hence there can be at most l non-concentric simple closed curves in D_1. Similarly the $2l$ arcs of $H \cap \partial D$ must be connected by l arcs. Now since we assume that n is "much" larger than l there are plenty of concentric curves which do not meet the disk D_1 at all. As the number of the $+$'s and $-$'s on the curve $K \cap D$ is equal we must end at the same level after winding around once and this situation repeats itself for all curves of $nK \cap D$ which are sufficiently "interior" on the annulus A.

□

Corollary 2.9. *There are many concentric annuli $A_1, \ldots, A_{n-l-1} \subset A \subset D$ so that $\partial A_i, i = 1, \ldots, n - l - 1$ are curves on H_n.*

□

3 Results

Proof. (Of Theorem 1.1). Assume in contradiction that the surface K is compressible with compressing disk D. We can assume by Lemma 2.6 that D is separating so that $cl(K - N(D)) = K' \cup K''$. Because ∂D is separating all curves of $H \cap K'$ (resp. $H \cap K''$) which meet ∂D are waves with respect to the disk D. Hence along ω the Haken sum looks as indicated in Figure 1. That is, if we assign a " $+$ " and a " $-$ " to each intesection point of $\partial D \cap K$ depending on the Haken sum configuration, the wave ω pairs a $''+''$ sign with a $''-''$ sign. Hence there is an equal number of $''+''$ signs and $''-''$ signs. It follows that Lemma 2.8 can be applied and that the intersection $H_n \cap D$ is as described.

Let A_i be one of the annuli from Corollary 2.9 set $\partial A_i = \gamma_i \cup \gamma_{i+1}$. We want to show that for a large n the boundary curves γ_i of the annuli which are away from ∂A (i.e., there is an integer k such that $n > k >> l > 0$ and $k < i < n - l - k - 1$, are essential in the Heegaard surface H_n.

More precisely:

(1) γ_i is essential for all $i = k+1, \ldots, n-l-k-1$.

(2) The annuli A_i are not boundary parallel in the handlebodies V_n and W_n.

(1) Let α'_j be the curves of $H \cap K'$ and α''_j be the curves of $H \cap K''$. They are a collection of both arcs and simple closed curves. Choose a point $*$ on $\partial D = \partial K' = \partial K''$ and let β'_j (resp. β''_j) be a collection of arcs with end points in $*$ so that $K' - \cup \alpha'_j \cup \beta'_j$ (resp. $K'' - \cup \alpha''_j \cup \beta''_j$) is a collection of disks. We can define such curves for each of the copies K'_i, K''_i of the surface $K_i = K'_i \cup K''_i$. and they will be denoted by $\alpha'_{j,i}$ (resp. $\alpha''_{j.i}$) and $\beta'_{j,i}$ (resp. $\beta''_{j,i}$). Where we use j as an index to enumerate the curves and the index i to enumerate the copy (level) on which they are.

After Haken summing H and nK each of the α'_j (α''_j) will induce $2n$ copies along vertical annuli in the Haken summed surface H_n. The curves $\beta'_{j,i}$ ($\beta''_{j,i}$) will "lift" to the surface H_n . In other words they will change "level" between the copies of nK each time they meet a curve $\alpha'_{j,i}$ ($\alpha''_{j,i}$). If we think of the curve $\beta'_{j,i}$ ($\beta''_{j,i}$) as paths then we define the *shift* of $\beta'_{j,i}$ ($\beta''_{j,i}$) to be the difference in the i index between the beginning point and the end point of the path $\beta'_{j,i}$ ($\beta''_{j,i}$). Note that since the beginning and end point of the curves $\beta'_{j,i}$ ($\beta''_{j,i}$) occurs on different copies of the point $*$ the shift is well defined. Also note that for each j the shift of $\beta'_{j,i}$ ($\beta''_{j,i}$) is fixed as we vary the index i. There are several cases:

(a) The shifts for all the lifted curves of the $\beta'_{j,i}$ ($\beta''_{j,i}$) are 0. In this case the surface H_n is not connected and hence cannot be a Heegaard splitting. So this case cannot occur.

(b) Suppose that all shifts in, say, the K' side are 0 and some shift in the K'' side is non zero. Since D is a compressing disk for K the surfaces K' and K'' are not disks. Thus γ_i cannot bound a disk on the K' side as K'_i is a copy of K'. If γ bounds a disk E on the K'' side then cap off the surface on the K' side with this disk. We get a closed surface which therefore must be H_n but its genus is too small. So this case cannot happen as well.

(c) Assume that there are non-zero shifts r on the K' side and s on the K'' side. Let $t = l.c.m.(r, s)$ and let δ' be the union of paths $\beta'_{j,i}, \ldots \beta'_{j,i+t/r}$. They exist because n is very large. Let δ'' be the union of paths

$\beta''_{j,i}, \ldots \beta''_{j,i+t/s}$. This union is a well defined path as for a fixed i all $\beta'_{j,i}$ and $\beta''_{j,i}$ begin and end at the same point. The loop $\delta = \delta' \cup \delta''$ is curve on H_n which intersects γ_i is a single point.

Hence γ_i is essential for all i.

(2) Assume now in contradiction that some annulus A_i is boundary parallel on the K' side. Then there must be some arc $\beta'_{j,i}$ which has a shift of 1. Thus $\beta'_{j,i+1}$ also has a shift of 1. Hence traveling along $\beta'_{j,i} \cup \beta'_{j,i+1}$ we can get to γ_{i+2}. This is a contradiction.

Let $cl(M - H_n) = V_n \cup W_n$, where V_n, W_n are handlebodies. The curve $\gamma_i = A_{i-1} \cap A_i$ is the boundary of a compressing sub-disk of D and by the "no nesting lemma" (see [Sch]) must bound a disk D' either in V or in W, say, W. We can assume that A_i is in W (otherwise just reverse the order of the indices). In this case the annulus A_{i-1} cannot be compressible in V as then H_n will be a reducible Heegaard splitting in contradiction. So the annulus A_{i-1} is an essential annulus in the handlebody V and boundary compresses in V to give an essential disk E' which is disjoint from D'. Hence H_n is a weakly reducible Heegaard splitting in contradiction.

Thus our assumption that the compressing disk D for K exist must be wrong and K is incompressible.

□

If K is a 2- torus T then since the Euler characteristic of the Haken sum is equal to the sum of the Euler characteristics (i.e., $\chi(H+T) = \chi(H) + \chi(T)$) we obtain a sequence of Heegaard splittings all of the same genus. Hence we have a variation on the Waldhausen conjecture, namely:

Conjecture 3.1. *For any given genus g an atoroidal 3-manifold M has only finitely many non-isotopic irreducible Heegaard splittings of genus g.*

This conjecture is true for Haken manifolds by work of K. Johansen (see [Jo]). For non-Haken manifolds it had been claimed by H. Rubinstein and W. Jaco but no proof is available. If we remove the required condition that the manifold be atoroidal then there are examples by K. Morimoto and M. Sakuma of tunnel number one toroidal knots, each of whose exteriors admits infinitely many isotopoy classes of genus two Heegaard splittings (see [MS1]). Since these Heegaard splittings are all of genus two they are all strongly irreducible. Hence the situation in this case is clearly more delicate and we will need more terminology and assumptions.

A surface F in a triangulated 3-manifold Mwill be called *normal* if it meets any tetrahedron in the triangulation in triangles and quadrilaterals who's boundary meets edges in a single point only. A triangulation for a closed and orientable M will be called *0-efficient* if the only embedded normal 2-spheres are vertex linking. A closed orientable 3-manifold M with a 0-efficient triangulation is irreducible not RP^3 and the triangulation has a single vertex or it is S^3 and the triangulation has two vertices.. We know by Jaco-Rubinstein [RJ] that each orientable 3-manifold M which is not S^3, RP^3 or $L(3,1)$ can be triangulated with a 0-efficient triangulation.

What we can prove is:

Theorem 3.2. *Suppose M is a closed and irreducible 3-manifold with a 0-efficient triangulation. Let $H_n = H + nT$, be strongly irreducible Heegaard splittings for all $n = 1, \ldots$. Suppose further that H and T are normal with respect to the triangulation and that $H_n \neq H_m, n \neq m$. Then T is incompressible.*

Proof. The manifolds S^3 and $L(3,1)$ RP^3??? have a unique stabilized Heegaard splitting in each genus so they are not relevant. Assume in contradiction that T is compressible then as M is irreducible T bounds a solid torus V or a cube with a knotted hole.

Consider the first case. Let $\Gamma = H \cap T$.

(a) Each curve $\gamma \in \Gamma$ is inessential in T. Choose n to be larger than the twice the cardinality c of the largest nested collection of curves. This number c will be called the *depth of* Γ. In this case copies of T which are distance bigger than $\frac{c}{2}$ from the first and last copy of T cannot be connected by that Haken sum and hence $H_n = H + nT$ is disconnected and not a Heegaard surface.

(b) Some curve $\gamma \in \Gamma$ is essential in T. Let $\Gamma_e^* \subset \Gamma$ be the collection of all essential curves in Γ. They must all be parallel in T. Let Γ_e^+ denote those curves on which the Haken sum gluing is positive and Γ_e^- denote curves with negative gluing. Let $k = ||\Gamma_e^+| - |\Gamma_e^-||$. Hence H_n differs from H_{n+1} by a $\frac{1}{k}$ of a Dehn twist along T. Thus H_{n+k} is homeomorphic to H_n. In fact since T bounds a solid torus V H_{n+k} is isotopic to H_n. This contradicts the assumption that we have infinitely many different Heegaard splittings.

In the second case if T bounds a cube with a knotted hole then compress T to obtain a 2-sphere S. Since M is irreducible, S bounds a 3- ball and this ball contains T. We may assume that S does not bound a 3-ball to the other side, as then M would be the S^3. In fact we can normalize S because T being normal acts as a barrier in the 3-ball side so S cannot pass through it when normalized. Nor can S be vertex linking, because a vertex linking sphere does not contain a normal cube with knotted hole. The normalized 2-sphere S is demonstrates that the triangulation of M is not 0-efficient in contradiction..

□

Recall that for any given M and a given triangulation the number of fundamental sufaces is finite. Denote them by $K_1, \ldots, K_m$. By Rubinstein [Ru] and Stocking [St] any strongly irreducible Heegaard splitting can be isotoped onto an almost normal surface. Hence any strongly irreducible Heegaard surface H_n can be written as a sum $H + a_{n.1}K_1, + \ldots, +a_{n.m}K_m$. As Haken sum is commutative we can assume up to taking subsequences of H_n that for all $i = 1, \ldots, m$ the sequences $a_{n,i}$ are non bounded.

Corollary 3.3. *If there is a sub-sequence H_{n_j} of H_n so that $H_{n_j} = H + n_j(a_{n_j.1}K_1, + \ldots, +a_{n_j.m}K_m)$ Then M is Haken.*

□

4 examples I

In this Section we show that all known examples of manifolds with infinitely many strongly irreducible Heegaard splittings have the property that these Heegaard splittings are of the form $H+nK$ for some closed orientable surfaces $H, K \subset M$. These examples include manifolds of the following two types:

(a) Let $M = K(\frac{1}{k})$ be obtained by $\frac{1}{k}, k \in \mathbb{Z}, |k| \geq 6$ surgery on a pretzel knot $K(p_1, \ldots, p_r) \subset S^3$, where $p_i, i = 1, \ldots, r$ and $r \geq 5$ are odd.

(b) Let $M = K(\frac{1+k\,a(K)}{k})$, where $k \in \mathbb{Z}$, $|k| \geq 6$, be obtained by $\frac{1+k\,a(K)}{k}$ surgery on K and K is the knot defined as follows: Let T be a generalized trellis and let $K = K(A) \subset S^3$ be a knot carried by T with twist matrix A. Assume that all coefficients $a_{i,j}$ of A satisfy $|a_{i,j}| \geq 3$ and that there is an interior pair of edges $(e_{i,j}, e_{i,h})$ of T with twist coefficients $|a_{i,j}|, |a_{i,h}| \geq 4$ and $a(K) = 2\Sigma a_{i,j}|a_{i,j} \in A$.

For precise definitions of the terminology we refer the reader to [MS], [LM] and [Ko]. Note that Kobayahi's method proves that $M(l)$ has infinitely many strongly irreducible Heegaard splittings for $l > 1$ which is somewhat better than the results by [CG] and [LM].

Claim 4.1. *Each of the manifolds $M = K(\frac{1}{k})$ and $M = K(\frac{1+k\,a(K)}{k})$, of examples (a) and (b), has an infinite collection of strongly irreducible Heegaard splittings of the form $H + nK$ for some closed surfaces H and K.*

Proof. The Heegaard surfaces of manifolds of type (a) are obtained by taking the boundary of a regular neighborhood of a Seifert surface for the knot K. The crucial property of the knots of type (a) and (b) is that the pair (S^3, K) contains a 2-sphere which meets the knot in four points. This 2-sphere bounds two 3-balls the "interior" ball will be denoted by B_1 and the "exterior" by B_2. We can twist the knots around using the 2-sphere so that $K \cap B_2$ is fixed and only $K \cap B_1$ changes. By considering these different projections for each of the knots we obtain a sequence of Seifert surfaces. Now we take the boundary of a regular neighborhood of these surfaces to obtain a surface Σ_m of genus $2s + 2m$ where s is the genus of the original Seifert surface and m is the number of twists. The knot K can be embedded in these Heegaard surfaces is such a way so that the surface $\Sigma_m - N(K)$ is incompressible in the union of the two handlebodies $M - \Sigma_m$. The boundary components of $\Sigma_m - N(K)$ specify a specific framing on $N(K)$. Since we are twisting around the 2-sphere we obtain different surfaces however the framing determined by their boundary stays constant. Fix some triangulation for $M - N(K)$. When considering the Haken coordinates of these surfaces in that triangulation we see that the coefficients corresponding to the boundary are equal for all $\Sigma_m - N(K)$. Thus the surfaces obtained by taking the Haken difference between any two of them will be a closed surface. We will denote the closed surface which is the difference between Σ and Σ_1 by K and rename Σ as H. As the difference between any two consecutive surfaces is contained in the interior 3-ball it follows that the difference $\Sigma_2 - \Sigma_1$ is also K and the same is true for $\Sigma_{m+1} - \Sigma_m$ for all m. Hence all surfaces obtained in this way are of the form $H + nK$.

The only difference between the examples of type (b) and type (a) is that the Heegaard surfaces are not a regular boundary of a Seifert surface but they could also be the twisted I-bundle over a non-orientable bounded surface. Hence the same argument as above applies to those Heegaard surfaces as well to show that they are of the form $H + nK$.

$\square$

5 examples II

6 References

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