LAYERED-TRIANGULATIONS OF 3–MANIFOLDS

WILLIAM JACO AND J. HYAM RUBINSTEIN

ABSTRACT. A family of one-vertex triangulations of the genus-g-handlebody, called layered-triangulations, is defined. These triangulations induce a one-vertex triangulation on the boundary of the handlebody, a genus g surface. Conversely, any one-vertex triangulation of a genus g surface can be placed on the boundary of the genus-g-handlebody in infinitely many distinct ways; it is shown that any of these can be extended to a layered-triangulation of the handlebody. To organize this study, a graph is constructed, for each genus $g \geq 1$, called the L_g graph; its 0–cells are in one-one correspondence with equivalence classes (up to homeomorphism of the handlebody) of one-vertex triangulations of the genus g surface on the boundary of the handlebody and its 1–cells correspond to the operation of a diagonal flip (or $2 \leftrightarrow 2$ Pachner move) on a one-vertex triangulation of a surface. A complete and detailed analysis of layered-triangulations is given in the case of the solid torus ($g = 1$), including the classification of all normal and almost normal surfaces in these triangulations. An initial investigation of normal and almost normal surfaces in layered-triangulations of higher genera handlebodies is discussed. Using Heegaard splittings, layered-triangulations of handlebodies can be used to construct special one-vertex triangulations of 3-manifolds, also called layered-triangulations. Minimal layered-triangulations of lens spaces (genus one manifolds) provide a common setting for new proofs of the classification of lens spaces admitting an embedded non orientable surface and the classification of embedded non orientable surfaces in each such lens space, as well as a new proof of the uniqueness of Heegaard splittings of lens spaces, including S^3 and $S^2 \times S^1$. Canonical triangulations of Dehn fillings called triangulated Dehn fillings are constructed and applied to the study of Heegaard splittings and efficient triangulations of Dehn fillings. It is shown that all closed 3–manifolds can be presented in a new way, and with very nice triangulations, using layered-triangulations of handlebodies that have special one-vertex triangulations of a closed surface on their boundaries, called 2–symmetric triangulations. We provide a quick introduction to a connection between layered-triangulations and foliations. Numerous questions remain unanswered, particularly in relation to the L_g-graph, 2–symmetric triangulations of a closed orientable surface, minimal layered-triangulations of genus-g-handlebodies, $g \geq 1$ and the relationship of layered-triangulations to foliations.

Contents

1. Introduction

This work began with an interest in managing triangulations of Dehn fillings. Specifically, suppose X is a compact 3–manifold with boundary a torus; $\mathcal{T}$ is a one-vertex triangulation of X; and α is a slope on ∂X. We wanted a one-vertex triangulation $\mathcal{T}(\alpha)$ of the Dehn filled manifold $X(\alpha)$ that restricts to $\mathcal{T}$ on X; i.e., $\mathcal{T}(\alpha)$ is an extension of $\mathcal{T}$. Furthermore, we needed to understand how normal surfaces in the triangulation $\mathcal{T}(\alpha)$ could meet the solid torus of the Dehn filling. It turned out that what we are now calling layered-triangulations of a solid torus are exactly what we needed. As we refined our understanding of these triangulations, we realized that layered-triangulations of the solid torus lead to nice triangulations of all lens spaces, again called layered-triangulations, as well as provide an excellent tool to study Heegaard splittings of Dehn fillings. Moreover, layered-triangulations of the solid torus and of lens spaces (genus 1 Heegaard splittings) raise the issue of extending these ideas to triangulations of genus-g-handlebodies and nice triangulations of 3–manifolds having genus g Heegaard splittings. We define layered-triangulations of higher genus handlebodies analogously to that for the solid torus and lay a foundation for further study of these triangulations and their use leading to layered-triangulations of all 3–manifolds.

We provide a thorough study of layered-triangulations of the solid torus and of lens spaces. An understanding of layered-triangulations of the solid torus is necessary in our work on 1–efficient triangulations [13]; however, we have carried it much further. Once we realized that layered-triangulations of lens spaces provide a common framework in which to recover many of the interesting results about lens spaces, using quite elementary techniques, we carried out the various details. This reinforced our belief that nice triangulations of 3–manifolds can be used for a better understanding of the topology of 3–manifolds and indeed might provide a combinatorial tool for a better understanding of their geometry.

In Sections 4 and 5 we consider the mechanics of layered-triangulations of the solid torus and classify the normal and almost normal surfaces in these triangulations. A way to visualize this situation is to think of the infinity of possibilities of placing a one-vertex (two-triangles, minimal) triangulation of the torus on the boundary of the solid torus. Two such triangulations on the boundary of the solid torus are considered equivalent if there is a homeomorphism of the solid torus taking one to the other. We have that the equivalence classes of one-vertex triangulations on the boundary of the solid torus are in one-one correspondence with the rational numbers $p/q, 0 < p < q$, p and q relatively prime, including the forms $0/1, 1/1$. We next consider extending these triangulations to triangulations of the solid torus; layered-triangulations provide such extensions. We call a layered-triangulation extending a p/q-triangulation on the boundary of the solid torus a p/q–layered-triangulation (of the solid torus). We conjecture that the unique minimal p/q–layered-triangulation is the minimal such extension of the p/q–triangulation on the boundary, in the sense of using the minimal number of tetrahedra. This conjecture is one of the compelling open problems. A graph having vertices corresponding to an equivalence class of one-vertex triangulations on the boundary of the solid torus and having edges corresponding to the elementary move from one such triangulation to another via a "diagonal flip" can be defined, called the L-graph. We then have that layered-triangulations of the solid torus correspond to certain paths and minimal layered-triangulations correspond to certain minimal arcs in the L-graph. Finally, we show that a minimal layered-triangulation of a solid torus has only finitely many connected normal and almost normal surfaces; we classify them completely.

In Sections 6 and 7 we define layered-triangulations of lens spaces and classify the normal and almost normal surfaces in minimal layered-triangulations of lens spaces. All lens spaces have layered-triangulations. A very effective method to study them is to start with a minimal p/q–layered-triangulation of the solid torus and then use any one of the three possible ways to identify the two triangles on the boundary; each identification gives a lens space. We provide the combinatorics giving the lens spaces obtained from a p/q–layered-triangulation; and, conversely, for the lens space $L(X, Y)$, we give the solution as to what layered-triangulation of the solid torus will give $L(X, Y)$ upon identifying its boundary faces. What is quite pleasing is the simplicity provided for understanding the normal and almost normal surfaces in minimal layered-triangulations of lens spaces. The only orientable embedded, normal surfaces are small neighborhoods of edges in the 1–skeleton; the triangulations have only one-vertex and so all edges are simple closed curves. There are no almost normal octagonal surfaces, except in the one-tetrahedron layered-triangulation of S^3, the two-tetrahedra layered-triangulations of $\mathbb{R}P^3$ and $S^2 \times S^1$, and one (the

"bad" one) of the two distinct two-tetrahedra minimal layered-triangulations of $L(3,1)$. Finally, the only normal non orientable surface embedded in a minimal layered-triangulation of a lens space is the unique minimal genus and incompressible one, except in the two-tetrahedra layered-triangulations of $S^2 \times S^1$, which admits an embedded, normal Klein bottle, and of $\mathbb{R}P^3$, which admits an embedded normal non orientable surface of genus 3 in addition to $\mathbb{R}P^2$. We leave unanswered the analogous question to that above regarding the minimal triangulation extending a p/q-triangulation on the boundary of the solid torus; namely, is the minimal layered-triangulation of a lens space the minimal triangulation of a lens space. We conjecture it is (this has been independently conjectured by S. Matveev in [22] but using the language of simple spines and handle decompositions).

In Section 8, we provide applications of these methods. We recover the earlier work of G. Bredon and J. Wood [3], giving the classification of lens spaces admitting embedded, non orientable surfaces and classify such surfaces in these lens spaces. Similarly, our methods lend themselves to the study of Heegaard splittings. We recover the earlier work of F. Bonahon and J.P. Otal [2], classifying Heegaard splittings of lens spaces, including the cases of S^3 and $S^2 \times S^1$ done by F. Waldhausen [34]. In particular, in a minimal layered-triangulation of a lens space, distinct from S^3, an almost normal surface is a Heegaard surface if and only if it is the vertex-linking 2–sphere with thin edge-linking tubes and an almost normal tube along a thick edge or the vertex-linking 2–sphere with thin edge-linking tubes and an almost normal tube at the same level as a thin edge-linking tube. There are no normal Heegaard surfaces in a minimal layered-triangulation of a lens space distinct from S^3. The one-tetrahedron triangulation of S^3 has the vertex-linking 2–sphere as a normal Heegaard surface and, necessarily, has an octagonal almost normal 2–sphere as a Heegaard surface. Finally, we use these triangulations to give examples of 0– and 1–efficient triangulations, as well as examples that are neither. In fact, we classify those layered-triangulations of lens spaces that are 0–efficient and those that are 1–efficient. All lens spaces, except $\mathbb{R}P^3$ and $S^2 \times S^1$, admit infinitely many 0–efficient triangulations but a given lens space admits only finitely many 1–efficient triangulations.

While these facts about lens spaces are not new, we encourage the reader to consider the methods and the use of nice-triangulations for understanding the topology and geometry of 3–manifolds. These triangulations also provide examples for students wanting to see real normal and almost normal surfaces in real triangulations; however, we believe possibly their most important feature is that they might guide us in using layered-triangulations in the study of manifolds of higher genus than one.

In Subsection 8.4, we define triangulated Dehn fillings. These are canonical triangulations for Dehn fillings of a knot-manifold. We use these for consideration of Heegaard splittings of Dehn fillings. A reader familiar with work done in [25, 33, 29, 31, 30] will immediately recognize how layered-triangulations of a solid torus and almost normal, strongly irreducible Heegaard splittings fit together in a study of Heegaard splittings of Dehn fillings. We provide a detailed proof of the results in [25, 33], using our combinatorial methods to provide a contrast with the more difficult topological methods. In fact, we are able to give a complete analysis of the intersection between a normal or almost normal Heegaard surface and a solid torus with a layered-triangulation without any restrictions regarding the nature of the

intersection curves on the boundary of the solid torus. We complete this subsection with conditions as to when a 0–efficient triangulation of a knot-manifold extends to a 0–efficient triangulated Dehn-filling. We conclude that minimal triangulations of knot-manifolds, which must be 0–efficient, extend to 0–efficient triangulated Dehn fillings of that manifold in all but finitely many cases. A similar result is true for 1–efficient triangulations but we do not give that here. It will appear in [13].

In Sections 9 and 10 we look at layered-triangulations of handlebodies and of 3–manifolds in general. These sections, we believe, give exciting new ways to study 3–manifolds via Heegaard splittings with the aid of nice triangulations and may be the most important sections in this paper.

As in the above situation with the solid torus, we consider one-vertex (minimal) triangulations of the genus g closed, orientable surface on the boundary of the genus-g-handlebody. We consider two such triangulations equivalent if there is a homeomorphism of the handlebody throwing one to the other (including orientation reversing homeomorphisms). A graph having vertices corresponding to an equivalence class of one-vertex triangulations on the boundary of the genus-g-handlebody and having edges corresponding to the elementary move from one such triangulation to another via a "diagonal flip" can be defined, called the L_g-graph. This graph admits an extension to a higher dimensional complex, just as the complex used by J. Harer [10] and L. Mosher [26] in studying the mapping class group. We believe the L_g graph is a very important graph to study, as well as its extension and relationship to the Harer complex.

We define layered-triangulations of a handlebody in a manner analogous to the definition in the genus one case. We show that any one-vertex-triangulation on the boundary of the genus-g-handlebody can be extended to a layered-triangulation of the genus-g-handlebody. It follows that layered-triangulations of the genus-g-handlebody correspond to paths in the L_g graph. If $[\tau]$ is an equivalence class of a one-vertex triangulation on the boundary of a genus-g-handlebody, we call a layered-triangulation extending τ, a $[\tau]$-layered-triangulation. A minimal such layered-triangulation extending the class $[\tau]$ is called a minimal $[\tau]$-layered triangulation. Since the L_g graph is not simply connected, it is suspected that there are many distinct minimal $[\tau]$-layered-triangulations of the genus-g-handlebody. We examine the normal surfaces in a minimal (smallest number of tetrahedra possible) four-tetrahedra triangulation of the genus-2-handlebody. According to Ben Burton, using REGINA [4], there are 196 distinct minimal triangulations of the genus-2-handlebody; we do not know if all are layered. The embedded normal surfaces in higher genera handlebodies are much richer than what we discovered in genus one; for example, in the particular four-tetrahedra triangulation of the genus-2-handlebody we consider, there is an infinite family of distinct normal meridional disks. However, we suspect that while the task may be much harder, we should be able to understand normal and almost normal surfaces in these triangulations.

Given a genus g Heegaard splitting of a 3–manifold, we can select any one-vertex triangulation on the genus g Heegaard surface, extend this triangulation to each of the handlebodies in the splitting, and gain a one-vertex triangulation of the 3–manifold, which we call a layered-triangulation. Just as above for lens spaces, we can modify this view to achieve a new and very curious presentation for 3–manifolds. The description we want also leads to an interesting phenomenon of minimal triangulations of the genus g closed surface. In the case of genus one, we

started with a layered-triangulation of the solid torus and observed that if we use any one of the three possible orientation reversing identifications of the two faces in the boundary, we get a lens space and a layered-triangulation of that lens space. Such identifications of the two faces in the boundary can equally well be viewed as an identification of the two faces with the one-triangle Möbius band. We have three possibilities because the edges in the one-vertex triangulation of the torus all have a symmetry about them that allows a simplicial involution, keeping the edge fixed. The quotient of this involution is the one-triangle Möbius band. For genus g, we do not have such 2–symmetry about all edges in a minimal (one-vertex) triangulation; in fact, some of these triangulations have no 2–symmetry at all about an edge. However, we do have lots of these triangulations with 2–symmetry, which can be seen by considering the genus g orientable surface, S, as a two-sheeted branched cover over a compact 2–manifold, B, having one boundary component and Euler characteristic $1 - g$, with the branching locus the boundary of B. Then a minimal (one-vertex) triangulation of B lifts to a 2–symmetric minimal (one-vertex) triangulation of S. Hence, every closed 3–manifold can be obtained from a layered-triangulation of a handlebody having a 2–symmetric minimal triangulation on its boundary by identifying the faces in the boundary of the triangulation via the associated simplicial involution. We call such a presentation of a 3–manifold a triangulated Heegaard splitting. We suspect that the most fruitful analysis of 3–manifolds coming from such presentations will be in using minimal [τ]–layered-triangulations of handlebodies, where τ is 2–symmetric. There are numerous open and very compelling questions regarding these constructions. In Section 10, we examine many of these questions and explore some possibilities in examples.

Finally, we suggest on first reading, one might skim through Sections 2-7. Followed by starting a close reading in Section 8 and referring back to earlier sections for possible definitions and clarification. The applications of layered-triangulations in Section 8 give the model for some of our interest in nice triangulations. Sections 9 and 10 leave more questions than answers. In fact, layered-triangulations of handlebodies, an understanding of L_g-graphs, and the applications of triangulated Heegaard splittings to the study of 3–manifolds is entirely unexplored ground. We have not had time to carry our investigations further in this area but find it very compelling.

We have taken a very long time to write up this work, which was done between 1996-98. Related work, with the knowledge of our work, has been published in [15] and in [24] and [8]. Also, Ben Burton has put his particular spin on some of this work in his thesis [5] and used it in the development of the program REGINA [4]. His view makes for interesting reading complementary to this. We would like to thank David Letscher and Eric Sedgwick, as well as Ben Burton, for many interesting discussions regarding this material.

2. Triangulations, normal and almost normal surfaces

We remark that our triangulations are more likely to be encountered in the literature under the name pseudo-triangulations. Furthermore, since the closure of our tetrahedra are not embedded, we need to add a technical caveat to the definition of both normal and almost normal surfaces in these triangulations. We feel these slight modifications are well worth the effort, as we find these triangulations to be nicely suited for algorithms and the study of the topology of 3–manifolds. The

reader is referred to our paper on 0–efficient triangulations [14] for a more complete discussion of these notions.

Suppose $\boldsymbol{\Delta} = \tilde{\Delta}_1 \amalg \tilde{\Delta}_2 \amalg \ldots \amalg \tilde{\Delta}_n \amalg \ldots$ is a disjoint union of tetrahedra. If σ_i and σ_j are distinct faces of Δ_i and Δ_j, respectively, then we call an affine isomorphism $\phi_{i,j} : \sigma_i \leftrightarrow \sigma_j$ a *face pairing* or *face identification*. While σ_i and σ_j are always distinct faces, it is possible $i = j$. Since a face pairing is an affine isomorphism we think of the pairing going in either direction. A family of face pairings induces an equivalence relation on $\boldsymbol{\Delta}$. We denote the identification space determined by this relation $\boldsymbol{\Delta}/\boldsymbol{\Phi}$ and the associated projection map by $\rho : \boldsymbol{\Delta} \to \boldsymbol{\Delta}/\boldsymbol{\Phi}$. Typically, the identification space $\boldsymbol{\Delta}/\boldsymbol{\Phi}$ is not a 3–manifold. If we require the family of face pairings to be *monogamous*; i.e., if a face is in a face pairing, then it is in only one face pairing, then $\boldsymbol{\Delta}/\boldsymbol{\Phi}$ will be a 3–manifold at all points except possibly at the image of certain edges and certain vertices. If we take a bit more care regarding orientation of identified edges, then the identification space will be a 3–manifold at all points, except possibly at the image of the vertices. This will then provide us with a satisfactory combinatorial structure. In the case $\boldsymbol{\Delta}/\boldsymbol{\Phi}$ is homeomorphic to a manifold M, we set $\mathcal{T} = (\boldsymbol{\Delta}, \boldsymbol{\Phi})$ and say $\mathcal{T}$ is a triangulation of M. If $\boldsymbol{\Delta}/\boldsymbol{\Phi}$ minus the image of the vertices is a 3–manifold, then it is homeomorphic to the interior of a compact 3–manifold X; in this case, we say $\mathcal{T}$ is an ideal triangulation of $\overset{\circ}{X}$, the interior of X. A vertex is then called an *ideal vertex*. In either of the cases that $\mathcal{T}$ is a triangulation or an ideal triangulation, we have the interiors of the simplices in the tetrahedra of $\boldsymbol{\Delta}$ embedded. Thus we use the terms vertex, edge, face and tetrahedron for the images under the identification map of a vertex, edge, face or tetrahedron.

We remark that the probability of getting a 3–manifold, even after avoiding the problems in faces and about edges, goes to zero as the number of tetrahedra in $\boldsymbol{\Delta}$ increases without bound. Finally, since the second derived subdivision of such a triangulation of a 3–manifold gives a PL-triangulation in the classical sense, we use classical PL terminology such as "small regular neighborhood" to mean a suitable small regular neighborhood in some derived PL-subdivision.

We shall assume the reader is familiar with the basics of normal and almost normal surface theory. However, observe that since our tetrahedra, faces and edges are not embedded (only their interiors are embedded), it is not necessarily the case that a normal surface in M is a surface that meets the tetrahedron in M in a collection of normal triangles and normal quadrilaterals but rather a normal surface in M is a surface that meets the tetrahedra in M so that the lifts of the intersection of the surface with the tetrahedra in M are normal triangles and normal quadrilaterals in the tetrahedra of $\boldsymbol{\Delta}$. We exhibit this in the two one-tetrahedron triangulations of S^3 in Figure 1. However, this is a technicality that cause no problems and will be implicitly understood.

3. One-vertex triangulations

A closed surface with non positive Euler characteristic admits a one-vertex triangulation (a triangulation of the 2–sphere must have at least three vertices while one of $\mathbb{R}P^2$ must have at least two vertices). One of the consequences of the work in [14] is that irreducible, orientable 3–manifolds have one-vertex triangulations. Also, in Section 9, we provide a proof that all closed, orientable 3–manifolds (including reducible ones) admit a one-vertex triangulation. (Such results also follow from the
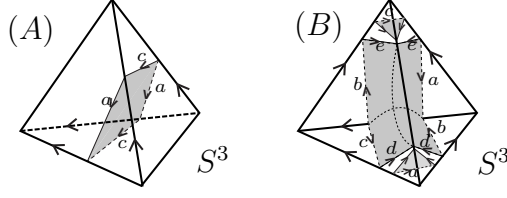
FIGURE 1. A normal torus (A) in the one-tetrahedron, two-vertex triangulation of S^3 and an almost normal 2–sphere (B) in the one-tetrahedron, one-vertex triangulation of S^3.

existence of special spines for closed, orientable 3–manifolds [21, 6].) A one-vertex triangulation of a closed surface is minimal. On the other hand, by using $2 \leftrightarrow 3$ Pachner moves [27], it is easy to see that any 3–manifold that has a one-vertex triangulation has infinitely many; so, in particular, a fixed 3–manifold can have a one-vertex triangulation with an arbitrarily large number of tetrahedra. In [14], it is shown that a minimal triangulation of an irreducible, orientable 3–manifold, except $S^3, \mathbb{R}P^3$ and $L(3,1)$, has only one vertex. (S^3 has 2 one-tetrahedron triangulations, 1 with one vertex and 1 with two vertices; $\mathbb{R}P^3$ has 2 two-tetrahedra (minimal) triangulations, 1 with one vertex and 1 with two vertices; and $L(3,1)$ has 3 two-tetrahedra (minimal) triangulations, 2 with one vertex and 1 with two vertices. We do not know if minimal triangulations of reducible 3–manifolds have just one vertex, but we suspect this also is true. Finally, we point out a very interesting problem; namely, given a triangulation of a closed 3–manifold, can it be decided if the triangulation is minimal? An affirmative solution to the Homeomorphism Problem for 3–manifolds enables an algorithm to decide if a given triangulation of a manifold is minimal; the Homeomorphism Problem follows from the Thurston Geometrization Theorem, which recently has been announced by G.Perelman [28]. However, this certainly strikes us as a circuitous resolution of the problem. From the census given in [22, 23] or the census by Ben Burton given by REGINA [4], we have a list of distinct minimal triangulations of closed, orientable 3–manifolds up to seven tetrahedra ([23]), extended to nine tetrahedra ([20]), and a census extending these to bounded 3–manifolds, non orientable 3–manifolds and ideal triangulations of the interior of some compact 3–manifolds, up to seven tetrahedra ([5]).

We have the following observations about one-vertex triangulations of closed surfaces.

3.1. **Lemma** ([15]). *In a one-vertex triangulation of a closed surface every trivial, normal curve is vertex-linking.*

3.2. **Lemma** ([15]). *In a one-vertex triangulation of the torus, two normal curves are normally isotopic if and only if they are isotopic.*

The previous Lemma is not true for normal curves in one-vertex triangulations of higher genera (genus greater than one) surfaces. Typically, in a one-vertex triangulation of a closed orientable surface other than the torus, there are infinitely many distinct normal curves in an isotopy class. An easy way to see this is to consider two disjoint non trivial simple closed curves ℓ and ℓ' that co-bound an annulus A containing the vertex (ℓ and ℓ' are isotopic but for an orientable surface different from the torus, they are *not* isotopic via an isotopy missing the vertex). We also

assume A does not separate the surface. For any simple closed curve meeting A in a single non trivial spanning arc, we can make "finger moves" within A over the vertex; thereby, generating an infinite family of simple closed curves that are isotopic but no two are isotopic via an isotopy missing the vertex. These curves can be normalized to normal simple closed curves all of which are in the same isotopy class but no two are normally isotopic. On the other hand, we do have the following partial generalization for higher genera surfaces.

3.3. Lemma. *In a one-vertex triangulation of a closed surface, two curves are isotopic via an isotopy missing the vertex if and only if they are normally isotopic.*

This can be proved using the same techniques as in [15].

If Q is a quadrilateral, possibly with some edge identifications, then we can subdivide Q into two triangles, without adding vertices, by adding one of the two possible diagonals as an edge. The operation of going from one of these triangulations of Q to the other is called a *diagonal flip*. Now, suppose F is a surface and $\mathcal{P}$ is a triangulation of F. If e is an interior edge of $\mathcal{P}$, then there are triangles σ and β meeting along e. If $\sigma \neq \beta$, the union $Q = \sigma \cup \beta$ is a quadrilateral, possibly with some identifications of boundary edges, with diagonal e and anti-diagonal the edge e'. If we swap the diagonal e for the diagonal e', then we have a new triangulation $\mathcal{P}'$ of F, which we say is obtained from $\mathcal{P}$ by a *diagonal flip*. We have the following well-known lemma. There is a proof in [26]

3.4. Lemma. *Suppose S is a closed surface and $\mathcal{P}$ and $\mathcal{P}'$ are one-vertex triangulations of S. Then there is a sequence of triangulations $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_n = \mathcal{P}'$ of S, where $\mathcal{P}_{i+1}$ differs from $\mathcal{P}_i$ by a diagonal flip and (possibly) an isotopy.*

Now, suppose M is a compact 3–manifold with nonempty boundary, $\mathcal{T}$ is a triangulation of M, and $\mathcal{P}$ is the triangulation induced on ∂M by $\mathcal{T}$. Furthermore, suppose e is an edge in $\mathcal{P}$ and there are two distinct triangles σ and β in $\mathcal{P}$ meeting along the edge e. Let $\tilde{\Delta}$ be a tetrahedron distinct from the tetrahedra in $\mathcal{T}$ and let $\tilde{e}$ be an edge in $\tilde{\Delta}$. Let $\tilde{\sigma}$ and $\tilde{\beta}$ be the faces of $\tilde{\Delta}$ that meet along $\tilde{e}$. Now, if we identify $\tilde{e}$ with e and extend this to face identifications from $\tilde{\sigma} \to \sigma$ and $\tilde{\beta} \to \beta$, we get a new triangulation $\mathcal{T}'$ of the 3–manifold M so that the triangulation $\mathcal{P}'$, induced by $\mathcal{T}'$ on ∂M, is obtained from $\mathcal{P}$ via a diagonal flip in the quadrilateral $\sigma \cup \beta$. There are two ways to attach $\tilde{\Delta}$, depending on the direction we attach $\tilde{e}$ to e (we have specified that $\tilde{\sigma}$ goes to σ and $\tilde{\beta}$ goes to β); however, the resulting triangulations are isomorphic. We call this operation a *layering on the triangulation $\mathcal{T}$ along the edge e of $\mathcal{P}$* or simply *a layering on $\mathcal{T}$ along the edge e*. The operation on $\mathcal{P}$ is called a Pachner or bi-stellar move of type $2 \leftrightarrow 2$ on $\mathcal{P}$ along the edge e (as well as a diagonal flip). See Figure 2.

4. Layered-triangulations of a solid torus

We consider very special one-vertex triangulations of the solid torus, which we call "layered-triangulations." We show that given a solid torus and a one-vertex triangulation on its boundary, there is an algorithm to construct a layered-triangulation of the solid torus that restricts to the given one-vertex triangulation on its boundary. In fact, for any positive integer N there is such an extension to a layered-triangulation of the solid torus having more than N tetrahedra; however, there is a unique (up to combinatorial isomorphism) extension of a one-vertex triangulation on the boundary of a solid torus to a minimal layered-triangulation of
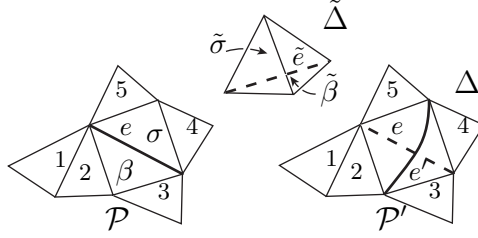
FIGURE 2. Layering along an edge e results in a diagonal flip taking the triangulation $\mathcal{P}$ to the triangulation $\mathcal{P}'$.

the solid torus. It remains an open question as to whether the minimal layered-triangulation of a solid torus that extends a one-vertex triangulation on the boundary of the solid torus is the minimal such extension. We discuss this and related problems about extending triangulations from the boundary of a 3–manifold to triangulations of the 3–manifold in later sections.

4.1. **One-vertex triangulations on the boundary of a solid torus.** An isotopy class of a simple, closed, normal curve, γ, in a one-vertex triangulation of a torus (which is the same as the normal isotopy class of γ) determines a unique unordered triple of nonnegative integers $\{y_1, y_2, y_3\} = \{|\gamma \cap e_1|, |\gamma \cap e_2|, |\gamma \cap e_3|\}$, where $y_i = |\gamma \cap e_i|$ is the number of times the normal curve γ meets the edge e_i, $i = 1, 2, 3$ of the triangulation. This includes the triples $\{0, 1, 1\}$ and $\{1, 1, 2\}$. In any case, we have for one of these numbers, say y_k, that $y_k = y_i + y_j$ and with only the exceptions $\{0, 1, 1\}$ and $\{1, 1, 2\}$, the integers y_i, y_j (hence, also y_k) are relatively prime. It follows, in all cases, that precisely one of the terms y_i, y_j or y_k is even, as is the sum $y_1 + y_2 + y_3$. Note these are not the normal coordinates of the normal curve in the one-vertex triangulation; however, there is an easy translation between normal coordinates and these triples (see [15]). We use standard terminology and call an isotopy class of a nontrivial simple closed curve in a torus a *slope*. If γ is the unique normal curve representing a slope, then we will often just use γ for both the slope and the normal curve.

For a solid torus, there is a unique slope in its boundary so that any curve, with that slope, bounds a disk. We call this slope the *meridional slope* and a disk it bounds is called a *meridional disk*. So, if $\mathcal{T}_\partial$ is a one-vertex triangulation on the boundary of a solid torus, there is a unique normal curve representing the meridional slope and thus a unique triple of nonnegative integers $\{p, q, p+q\}$ determined by the intersections of the (normal) meridional slope with the edges of the triangulation $\mathcal{T}_\partial$. We allow the triples $\{0, 1, 1\}$ and $\{1, 1, 2\}$; the first being when the meridional slope is the slope of an edge and the second is the slope of the anti-diagonal, if we designate the edge meeting the curve in two points the diagonal. The triple $\{p, q, p+q\}$ is completely determined once we specify p and q and except for $p = q = 1$, we may assume notation is such that $0 \le p < q$ and, except for $p = 0, q = 1$, p and q are relatively prime. We will call such a one-vertex triangulation on the boundary of a solid torus a p/q–*triangulation* on the boundary of the solid torus, including $0/1$ and $1/1$ for the triples $\{0, 1, 1\}$ and $\{1, 1, 2\}$, respectively. Notice that while there is a unique one-vertex triangulation of the torus, there are infinitely many ways that a one-vertex triangulation of the torus can sit upon the boundary of a solid torus, differing in how the edges of

the triangulation meet the meridional slope. The following lemma shows that the association of a rational p/q with a triangulation on the boundary of the solid torus is unique, up to homeomorphism of the solid torus.

4.1. Lemma. *Suppose $\mathcal{T}_\partial$ and $\mathcal{T}'_\partial$ are p/q– and p'/q'–triangulations, respectively, on the boundary of the solid torus $\mathbb{T}$. There is a homeomorphism of $\mathbb{T}$ taking the triangulation $\mathcal{T}_\partial$ to the triangulation $\mathcal{T}'_\partial$ if and only if $p/q = p'/q'$.*

Proof. Suppose $\mathcal{T}_\partial$ and $\mathcal{T}'_\partial$ are one-vertex triangulations on the boundary of the solid torus $\mathbb{T}$. Denote their edges by e_1, e_2, e_3 and e'_1, e'_2, e'_3, respectively. Let μ denote the meridional curve on the boundary. Since μ is unique up to isotopy, it determines a unique normal isotopy class in $\mathcal{T}_\partial$ and in $\mathcal{T}'_\partial$. As above, $\mathcal{T}_\partial$ determines a unique triple $\{y_1, y_2, y_3\}$, where $y_i = |e_i \cap \mu|$ and similarly, $\mathcal{T}'_\partial$ determines a unique triple $\{y'_1, y'_2, y'_3\}$, where $y'_i = |e'_i \cap \mu|$. (These triples are the triples $\{p, q, p+q\}$ and $\{p', q', p'+q'\}$, respectively, given in the hypothesis. We have chosen this notation here to avoid ambiguity as we proceed with the proof.)

Now, if there is a homeomorphism of the solid torus $\mathbb{T}$ taking $\mathcal{T}_\partial$ to $\mathcal{T}'_\partial$, then by the uniqueness of the normal isotopy class of the meridian, we have $\{y_1, y_2, y_3\} \equiv \{y'_1, y'_2, y'_3\}$ as unordered triples.

So, conversely, suppose $\{y_1, y_2, y_3\} \equiv \{y'_1, y'_2, y'_3\}$ and notation has been chosen so that $y_i = y'_i, i = 1, 2, 3$ and $y_3 = y_1 + y_2$ $(y'_3 = y'_1 + y'_2)$.

Choose orientation on e_1, e_2 and μ so $\mu = y_2 e_1 + y_1 e_2$. Choose a longitude, say λ, on the boundary of $\mathbb{T}$ so $\lambda = x_2 e_1 + x_1 e_2$, where x_1, x_2 satisfy $x_1 y_2 - x_2 y_1 = 1$. It follows that
$$e_1 = x_1 \mu - y_1 \lambda,$$
$$e_2 = -x_2 \mu + y_2 \lambda$$
and so,
$$e_3 = e_2 - e_1 = (-x_1 - x_2)\mu + (y_1 + y_2)\lambda.$$

Similarly, we can choose orientation on e'_1 and e'_2 and μ so $\mu = y_2 e'_1 + y_1 e'_2$ and there exists x'_1, x'_2 such that we can have $\pm\lambda = x'_2 e'_1 + x'_1 e'_2$ and $x'_1 y_2 - x'_2 y_1 = 1$. Notice, we want $x'_1 y_2 - x'_2 y_1 = 1$, therefore, we have the ambiguity $\pm\lambda$. Hence,
$$e'_1 = x'_1 \mu \mp y_1 \lambda,$$
$$e'_2 = -x'_2 \mu \pm y_2 \lambda$$
and so,
$$e'_3 = e'_2 - e'_1 = (-x'_1 - x'_2)\mu \pm (y_1 + y_2)\lambda.$$

A Dehn twist about μ, repeated n times, takes $\mu \to \mu$ and $\lambda \to n\mu + \lambda$. So,
$$e_1 \to x_1 \mu - y_1 n\mu - y_1 \lambda = (x_1 - y_1 n)\mu - y_1 \lambda$$
$$e_2 \to -x_2 \mu + y_2 n\mu + y_2 \lambda = (-x_2 + y_2 n)\mu + y_2 \lambda.$$

Recall, we have $x_1 y_2 - x_2 y_1 = x'_1 y_2 - x'_2 y_1$; so, since y_1 and y_2 are relatively prime, it follows that $(x_1 - x'_1) = y_1 n$ for some integer n. Similarly, we get $(x_2 - x'_2) = y_2 n$ for the same integer n. We conclude that for this n, we have
$$x'_1 = x_1 - y_1 n$$
and
$$-x'_2 = -x_2 + y_2 n.$$
Therefore
$$e_1 \to x'_1 \mu - y_1 \lambda,$$

$$e_2 \to -x_2'\mu + y_2\lambda,$$

and

$$e_3 \to (-x_1' - x_2')\mu + (y_1 + y_2)\lambda.$$

Hence, either there is a Dehn twist taking $\mathcal{T}_\partial$ to $\mathcal{T}_\partial'$ or a Dehn twist followed by an orientation reversing homeomorphism of $\mathbb{T}$ leaving μ fixed and reversing orientation of the longitude λ taking $\mathcal{T}_\partial$ to $\mathcal{T}_\partial'$.   □

Since there are different ways in which a one-vertex triangulation of a torus can sit upon the boundary of a solid torus, it is natural to wonder if any such triangulation can be extended (without adding vertices) to a triangulation of the solid torus. Below, we show that any one-vertex triangulation on the boundary of a solid torus can be extended to a (one-vertex) triangulation of the solid torus. A proof of this appears in [15] following our work. As pointed out above, we leave open the interesting and closely related question of given a (one-vertex) triangulation on the boundary of a solid torus, what is the minimal number of tetrahedra needed to extend this triangulation to a triangulation of the solid torus? More generally, it can be shown that any triangulation on the boundary of a compact 3–manifold can be extended without adding vertices to a triangulation of the 3–manifold; so, in particular, all vertices of such an extension are in the boundary of the 3–manifold. In this more general context, we also have the question of what is the minimal number of tetrahedra needed to extend a given triangulation on the boundary.

4.2. **Layered-triangulations of a solid torus.** If $\mathcal{T}$ is a one-vertex triangulation of a solid torus $\mathbb{T}$, then the boundary torus has a triangulation with two faces, three edges and one vertex. If e is one of the edges in the induced triangulation on the boundary, then we can add a tetrahedron $\tilde{\Delta}$, which is disjoint from the lifts of tetrahedra in $\mathcal{T}$, by layering along the edge e; the result after layering is still a solid torus with a triangulation having one more tetrahedron. We denote the new solid torus by $\mathbb{T}' = \mathbb{T} \cup_e \Delta$, where Δ is the image of $\tilde{\Delta}$, and denote the new triangulation by $\mathcal{T}' = \mathcal{T} \cup_e \tilde{\Delta}$. As we mentioned above, once the edge e is designated, there are several possibilities for how we attach the tetrahedron $\tilde{\Delta}$ along e and its two adjacent faces; however, in this case, the resulting triangulations are all isomorphic.

First, we make some observations about some special small triangulations of the solid torus. Consider the tetrahedron $\tilde{\Delta}$ pictured in Figure 3. There are three ways to layer the back two faces of the tetrahedron $\tilde{\Delta}$ onto the one-triangle Möbius band. In parts (A) and (B) the tetrahedron is layered along the interior (orientation reversing edge) on the one-triangle Möbius band, the labels and arrows give the identifications. In these cases, the boundary torus has a one-vertex triangulation consisting of the two front faces of the tetrahedron. All four vertices are identified to a single vertex and the induced edge identifications are indicated in Figure 3, Parts A and B, respectively. Combinatorially, these triangulations are the same (exchange the vertices labeled b and c). This triangulation of the solid torus will be referred to as *the one-tetrahedron solid torus*. The numbers 1, 2 and 3 labeling the edges indicate the number of times the edge meets the boundary of the meridional disk; thus, this is an extension of the $1/2$–triangulation on the boundary of the solid torus. In the last case, Figure 3, Part C, we show a *creased 3–cell* obtained by a single layering of a tetrahedron along the boundary edge of the one-triangle Möbius band; again the labels and arrows give the identification. The Möbius band and the creased 3–cell both have the homotopy type of a solid torus (the
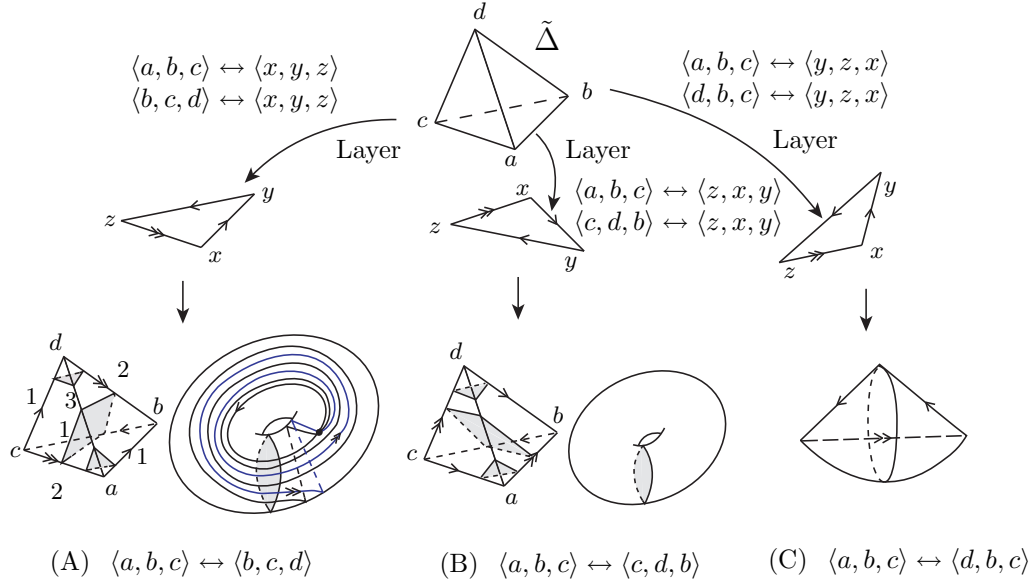
FIGURE 3. One-tetrahedron solid torus and creased 3–cell (layering of a tetrahedron on a one-triangle Möbius band).

Möbius band, in particular, plays an important role in our theory); we think of each as a degenerate layered-triangulation of the solid torus. In the case of the one-triangle Möbius band, we have a (degenerate) extension of a $1/1$–triangulation on the boundary of the solid torus and in the case of the creased 3–cell, we have a (degenerate) extension of a $0/1$–triangulation on the boundary of the solid torus. Note, if we layer another tetrahedron along the edge labeled 1 on the boundary of the creased 3–cell, we do get a two-tetrahedron triangulation of the solid torus extending the $0/1$–triangulation on the boundary. This example was shown to us by Eric Sedgwick. See Figure 5. Layering onto the one-triangle Möbius band is useful in using layered triangulations for Dehn fillings and in our generalization of layered-triangulations of a solid torus to analogous triangulations of handlebodies.

Inductively, we define a triangulation $\mathcal{T}_t$ to be a *layered-triangulation of the solid torus with t–layers* if

(1) $\mathcal{T}_0$ is the one-triangle Möbius band,
(2) $\mathcal{T}_1$ is either the one-tetrahedron solid torus or the creased 3–cell, each of which is obtained by a layering of a tetrahedron along an edge of T_0
(3) $\mathcal{T}_t = \mathcal{T}_{t-1} \cup_e \tilde{\Delta}_t$ is a layering along the edge e of a layered-triangulation $\mathcal{T}_{t-1}$ having $t - 1$ layers, $t \geq 1$. See Figure 4.

In Figure 5, we give examples of what we might call "exceptional" layered-triangulations of the solid torus. Part A is a two-tetrahedron minimal extension of the $1/1$-triangulation on the boundary; and Part B is a three-tetrahedron layered-triangulation extending the $0/1$-triangulation on the boundary. The latter is *not* the minimal extension of the $0/1$–triangulation on the boundary, which actually is the two-tetrahedron layered-triangulation, given in Part C. Notice in Parts A and B of Figure 5, the edge labeled 3 is of index two. Generally, in a triangulation with
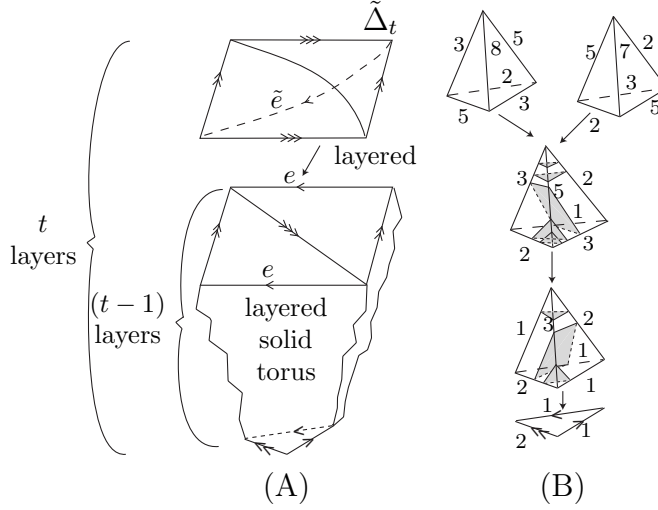
FIGURE 4. A layered-triangulation of a solid torus. In the example, (B), the numerical values along an edge denote the number of times the meridional disk meets the edge.

an edge of index two, we can reduce the number of tetrahedra by crushing the two tetrahedra containing the common edge to two faces. However, in this case the crushing does not give a 3–manifold. In Part B it gives the Möbius band and in Part C it gives the creased 3–cell. These are supporting observations for thinking of the one-triangle Möbius band and the creased 3–cell as layered-triangulations, extending the $1/1$– and the $0/1$–triangulations on the boundary of a solid torus, respectively.

Except in the degenerate cases, a layered-triangulation of a solid torus, with t layers, has one vertex, which is in the boundary torus, $t + 2$ edges, three of which are in the boundary torus, and $2t + 1$ faces, two of which are in the boundary.

Figure 4(B) gives some specific examples of layered-triangulations of the solid torus. The numbers along the edges indicate the number of times the edge meets the meridional disk of the solid torus.

Thick and thin edges. There is a special edge in a layered-triangulation of a solid torus. At the base of a layered-triangulation of a solid torus is the Möbius band. Its edges meet the meridional disk 1 and 2 times. The edge that meets the meridional disk 1 time is called a *thick edge* in the sense that no small neighborhood of this edge has boundary a normal torus - the shrinking of the boundary of a small regular neighborhood of this edge never normalizes and sweeps across the entire solid torus. On the other hand, every other edge has arbitrarily small neighborhoods with normal boundaries. (Even if they meet the meridional disk just once; see, for example, the edge labeled $1'$ in Part B of Figure 5.) All edges other than the unique thick edge are called *thin edges*. If a thin edge is in the interior of the solid torus, then there are arbitrarily small neighborhoods of the edge having boundary a normal once-punctured torus; if the thin edge is in the boundary of the solid torus, then there are arbitrarily small neighborhoods of the edge having boundary a normal annulus. It is possible that the thick edge is in the boundary.

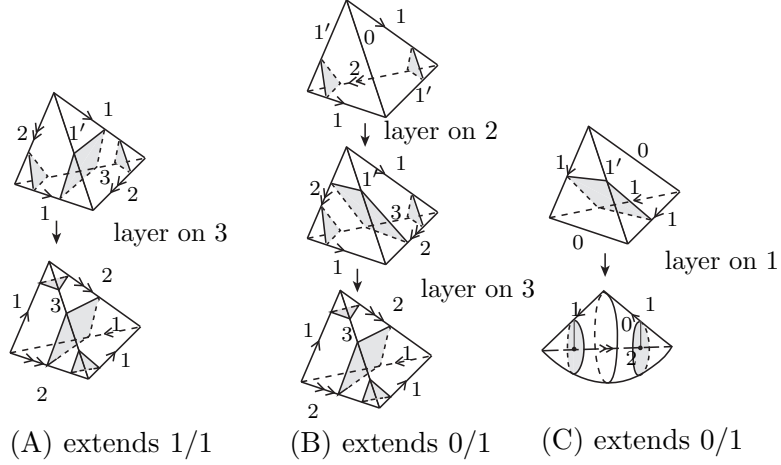(A) extends 1/1   (B) extends 0/1   (C) extends 0/1

FIGURE 5. (A) Two-tetrahedron (minimal) layered-triangulation of solid torus extending 1/1; (B) Three-tetrahedron layered-triangulation of solid torus extending 0/1; (C) Two-tetrahedron (minimal) layered-triangulation of solid torus extending 0/1

There is another edge in a layered-triangulation of a solid torus that is distinguished. If $\mathcal{T}$ is a layered-triangulation of a solid torus, then the unique edge on the boundary having valence one (meeting only one tetrahedron) is called the *univalent edge*. There is a unique univalent edge for the one-triangle Möbius band (the edge labeled "2") and for the creased 3–cell (the edge labeled "0").

Slopes and layering. Suppose $\mathcal{T}$ is a one-vertex triangulation of a solid torus $\mathbb{T}$ and $\mathcal{T}_\partial$ is the triangulation induced by $\mathcal{T}$ on the boundary of $\mathbb{T}$. Let e_1, e_2, e_3 denote the three edges of the triangulation $\mathcal{T}_\partial$ in the boundary of $\mathbb{T}$. Then a layering along an edge of $\mathcal{T}_\partial$, say for example along the edge e_3, gives a solid torus $\mathbb{T}' = \mathbb{T} \cup_{e_3} \Delta$, with a one-vertex triangulation $\mathcal{T}'$. If $\mathcal{T}'_\partial$ is the triangulation on the boundary of $\mathbb{T}'$, induced by $\mathcal{T}'$, then the layering has the effect of designating the edge e_3 in $\mathcal{T}_\partial$ as the diagonal and "flipping the diagonal" in going from the triangulation $\mathcal{T}_\partial$ to $\mathcal{T}'_\partial$. We have edges e_1, e_2, e'_3 of the triangulation $\mathcal{T}'_\partial$ in the boundary of $\mathbb{T}'$.

Now, suppose γ is a slope on $\partial\mathbb{T}$. The slope γ is determined by its intersection with the edges e_1 and e_2, which form a basis for the first homology of $\partial\mathbb{T}$. For a layering along the edge e_3, we still have e_1 and e_2 edges in the boundary of the resulting torus, $\mathbb{T}' = \mathbb{T} \cup_{e_3} \Delta$; and they also form a basis for the first homology of $\partial\mathbb{T}'$. Furthermore, they meet γ exactly as they did before layering. So, if γ' is the unique slope in the boundary of $\mathbb{T}'$ that meets e_1 and e_2 the same as γ, we say the slope γ *pushes through* the layering to the slope γ'.

The set of edges $\{e_1, e_2, e_3\}$ in $\mathcal{T}_\partial$ is replaced by the edges $\{e_1, e_2, e'_3\}$ in the one-vertex triangulation $\mathcal{T}'_\partial$ on the boundary of the solid torus $\mathbb{T}'$. For any orientation we choose on e_1 and e_2, we may orient e_3 so that either $e_3 = e_1 + e_2$ or $e_3 = e_1 - e_2$, with respect to homology. Thus, e'_3 can be oriented so that $e'_3 = e_1 - e_2$, if $e_3 = e_1 + e_2$; or $e'_3 = e_1 + e_2$, if $e_3 = e_1 - e_2$. Choose an orientation on γ and orient e_1 and e_2 so that the oriented intersection numbers $\langle \gamma, e_1 \rangle = y_1$ and $\langle \gamma, e_2 \rangle = y_2$. Thus, if $e_3 = e_1 + e_2$, then $y_3 = y_1 + y_2$; and if $e_3 = e_1 - e_2$, then $y_3 = |y_1 - y_2|$. After

layering on e_3, e_1 and e_2 remain in the boundary and the intersection number of the new slope, γ', with e_1 and e_2 remains y_1 and y_2, respectively. It follows that in layering a tetrahedron on the boundary of $\mathbb{T}$ along the edge e_3 of $\mathcal{T}$, the slope γ pushes through the added tetrahedron to a unique slope γ' in the boundary of $\mathbb{T}'$ and the unique triple of numbers associated with γ gives the unique triple of numbers associated with γ' as

$$\{y_1, y_2, |y_1 - y_2|\} \rightarrow \{y_1, y_2, y_1 + y_2\}, \quad y_3 = |y_1 - y_2|$$

and

$$\{y_1, y_2, y_1 + y_2\} \rightarrow \{y_1, y_2, |y_1 - y_2|\}, \quad y_3 = y_1 + y_2.$$

Example. Suppose $\mathcal{T}$ is a one-vertex triangulation of a solid torus, γ is a slope on the boundary of the solid torus and $\{3, 8, 11\}$ is the unique triple associated with γ. If we layer $\mathcal{T}$ along the edge meeting γ three times, then the slope γ "pushes through" to the slope γ' with triple $\{19, 8, 11\}$; similarly layering along the edge meeting γ eight times, the slope $\{3, 8, 11\}$ "pushes through" to $\{3, 14, 11\}$; and layering along the edge meeting γ eleven times, the slope $\{3, 8, 11\}$ "pushes through" to $\{3, 8, 5\}$.

Slope of an edge. If $\mathcal{T}$ is a layered-triangulation of a solid torus, then each edge in $\mathcal{T}$ is "unknotted," in the sense that it is isotopic into the boundary torus; however, there are infinitely many curves in the boundary to which it is isotopic through the solid torus – each differing by a Dehn twist about a curve representing the meridional slope. On the other hand, notice that if e is an edge in the triangulation $\mathcal{T}$, then either e is in the boundary, and so determines a *preferred slope*, or at some unique level of the layering, the edge e becomes an interior edge. If this is the case, then there is a unique "push through" of the slope of the edge e, using the layered-triangulation, which determines a *preferred slope* on the boundary of the layered solid torus for the edge e. In Figure 6 we provide an example of the preferred slope on the boundary for an interior edge e.

We call such a slope on the boundary, the *slope of the edge e* or, more generally, *the slope of an edge*. Hence, for a solid torus with a fixed layered-triangulation there are a finite number of special slopes on the boundary of the solid torus, the (preferred) slopes of the edges of the layered-triangulation. In particular, we have such a preferred slope in the boundary determined by the unique "thick edge", and so have a *preferred longitude*. Since there is a unique meridian slope, a layered-triangulation of a solid torus induces a *preferred framing* (meridian/longitude coordinate system) on the boundary torus. Below we will see that there is another very special longitude on the boundary. Also, in Figure 6, we exhibit that it is possible that two (or more) edges of a layered-triangulation give the same slope on the boundary (in Figure 6 the interior edge e and the boundary edge f have the same slope). We use the phrase "two edges have the same slope," should this happen. It also may be the case that an edge has the same slope as the meridian; if this is the case, we say "an edge has the slope of the meridian". However, for minimal and nearly-minimal layered-triangulations (defined below), distinct thin edges give distinct slopes and an edge can not have the slope of the meridian except for the minimal layered-triangulation extending $0/1$.
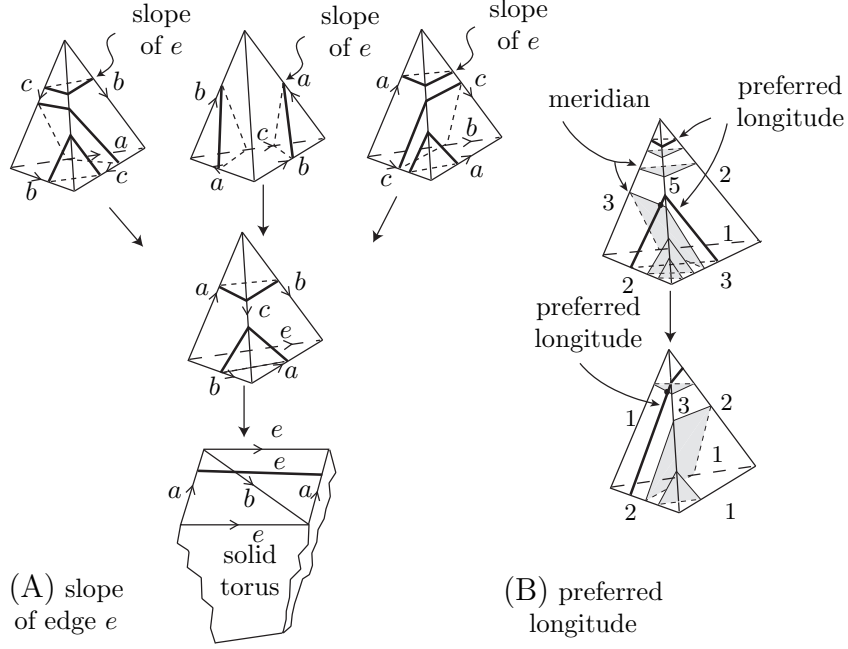
FIGURE 6. (A) gives slope of edge e on boundary; (B) gives slope of thick edge, the preferred longitude, on the boundary.

Existence: Extending triangulations from the boundary of a solid torus. We wish to show that a one-vertex triangulation on the boundary of a solid torus can be extended to a layered-triangulation of the solid torus. In general, given any triangulation on the boundary of a compact 3–manifold, it extends to a triangulation of the 3–manifold; i.e., there is a triangulation of the 3–manifold so that its restriction to the boundary is the given triangulation. In fact, the triangulation of the 3–manifold can be chosen so as not to add any vertices. The distinction here is that any one-vertex triangulation on the boundary of a solid torus extends to a *layered-triangulation* of the solid torus.

To see this, suppose $\mathcal{P}$ is a triangulation of a 2–manifold S and γ is a normal curve in S (with respect to $\mathcal{P}$), we call the cardinality of $\gamma \cap \mathcal{P}^{(1)}$ the *length of the curve* γ, where $\mathcal{P}^{(}i)$ denotes the i–skeleton of $\mathcal{P}$, and denote it by $L(\gamma)$. If S is a torus, $\mathcal{P}$ is a one-vertex triangulation of S with edges e_1, e_2 and e_3, and γ is a normal simple closed curve in S, then choosing the orientation on γ and e_1, e_2, e_3, as above, $L(\gamma)$ is the sum of the intersection numbers of γ with the three edges in $\mathcal{P}$: $L(\gamma) = y_1 + y_2 + y_3$, where $y_i = \langle \gamma, e_i \rangle, i = 1, 2, 3$. So, for S a torus and $\mathcal{P}$ a one-vertex triangulation of S, if we perform a diagonal flip along an edge e in the triangulation $\mathcal{P}$, other than the one with highest intersection number with γ, then the curve γ has length strictly greater in the resulting triangulation than it had in the triangulation $\mathcal{P}$. Alternatively, by doing a diagonal flip along the edge of $\mathcal{P}$ having the highest intersection number with γ, then in the resulting triangulation, γ has length strictly less than its length in $\mathcal{P}$, unless, with respect to some ordering of the edges, we have $y_1 + y_2 = |y_1 - y_2|$ (i.e. one intersection coordinate is zero).

The condition $y_1 + y_2 = |y_1 - y_2|$ means γ is, in fact, disjoint from some edge and is therefore the normal representative of that edge; γ has intersection numbers $0, 1, 1$ and $L(\gamma) = 2$. A diagonal flip on either of the edges labeled "1" does not change the intersection numbers of γ in the resulting triangulation from those in $\mathcal{P}$.

Following our work, these observations were used in [15], to prove the following.

4.2. **Proposition.** *Suppose $\mathcal{P}$ is a one-vertex triangulation of a torus T and suppose μ is an arbitrary slope in T. Then there is a layered-triangulation $\mathcal{T}$ of the solid torus and an isomorphism from $\mathcal{P}$ to $\mathcal{T}_\partial$, the triangulation induced by $\mathcal{T}$ on the boundary of the solid torus, taking μ to the meridional slope.*

This result was applied in [15] to Dehn fillings of cusped manifolds (compact, irreducible 3–manifolds having a single boundary component an incompressible torus). If a cusped manifold M is endowed with a one-vertex triangulation $\mathcal{T}$, α is any slope in ∂M, and $M(\alpha)$ is a Dehn filling of M along the slope α, then by Proposition 4.2, there is an extension of the triangulation $\mathcal{T}$ to a layered-triangulation of the solid torus (depending on α) resulting in a triangulation $\mathcal{T}(\alpha)$ of $M(\alpha)$ that restricts to $\mathcal{T}$ on M. We discuss this more below where we consider Dehn fillings.

The statement of Proposition 4.2 is one interpretation of extending a one-vertex triangulation on the boundary of a solid torus to a triangulation of the solid torus and is directly applicable to Dehn fillings. In the following theorem we provide a different proof from that given in [15], a proof that adds some interesting combinatorial information about layered-triangulations of solid tori.

4.3. **Theorem.** *Suppose $\mathcal{T}_\partial$ is a one-vertex triangulation on the boundary of a solid torus. Then $\mathcal{T}_\partial$ can be extended to a layered-triangulation of the solid torus.*

Proof. Let $p, q, p+q$ be the unique triple of intersection numbers of the meridional slope with the edges of the triangulation $\mathcal{T}_\partial$. We can assume the triple is not $0, 1, 1$ or $1, 1, 2$ and that $0 < p < q$. If we view the Euclidean algorithm for the pair q, p as a subtraction rather than a division algorithm, we have:

$$q = (q - p) + p$$
$$q - p = (q - 2p) + p$$
$$\vdots$$

where the general step is

$$r_{i-2} - (a_i - 1)r_{i-1} = r_{i-1} + r_i, \quad r_i < r_{i-1}$$
$$\vdots$$

and, finally

$$r_{n-1} = (r_{n-1} - 1) + 1$$
$$r_{n-1} - 1 = (r_{n-1} - 2) + 1$$
$$\vdots$$
$$2 = 1 + 1.$$

Reversing the steps and starting with layering on the Möbius band, we pass from a triple $\{y_1, y_2, |y_1 - y_2|\}$ to the triple $\{y_1, y_2, y_1 + y_2\}$, eventually ending in the triple $\{p, q, p+q\}$. It follows that we can layer on the Möbius band to get a layered-triangulation, $\mathcal{T}'$, of a solid torus having the p/q–triangulation on its boundary. By

Theorem 4.1, there is a homeomorphism from our original solid torus to the solid torus we have built by layering, taking the triangulation $\mathcal{T}_\partial$ isomorphically onto the triangulation induced by $\mathcal{T}'$ on the boundary. Thus $\mathcal{T}'$ is an extension of $\mathcal{T}_\partial$ to a layered-triangulation of the solid torus.

In the cases where $\mathcal{T}_\partial$ is a $0/1$– or $1/1$–triangulation on the boundary, then we have the two-tetrahedra extensions of these triangulations given in Figure 5 and by applying Theorem 4.1, we arrive at the desired conclusion in these cases. $\qquad\square$

Since $\mathcal{T}_\partial$ can be extended to a layered-triangulation without adding vertices, we can take the layered-triangulation extending $\mathcal{T}_\partial$ with the smallest number of tetrahedra. We call such a triangulation a *minimal layered-triangulation* of the solid torus extending $\mathcal{T}_\partial$. If a layered triangulation of the solid torus extends the p/q–triangulation on the boundary, we call it a *p/q–layered-triangulation* and if it is the minimal such layered-triangulation, we call it the *minimal p/q–layered-triangulation*. We shall try to use "layered" in this way to distinguish the triangulation on the boundary from that of the solid torus. We also find it convenient to refer to an extension of a one-vertex triangulation on the boundary of a solid torus to a layered triangulation of the solid torus as an $\{x, y, z\}$–layered-triangulation, particularly, when we do not wish to make a distinction as to the relative magnitude of the intersection numbers.

The L–graph: Classification of layered-triangulations of a solid torus. From Theorem 4.1, equivalence classes (up to homeomorphism of the solid torus) of one-vertex triangulations on the boundary of the solid torus are in one-one correspondence with reduced rational numbers $p/q, 0 \le p < q$, including $0/1$ and $1/1$. The rational number is uniquely determined by intersection numbers of the meridional slope with the edges of the triangulation in the boundary of the solid torus. Furthermore, if we add a tetrahedron by layering along an edge of a one-vertex triangulation on the boundary of a solid torus, we have a new solid torus and a new one-vertex triangulation on its boundary. If the triangulation on the boundary before layering has associated rational p/q, then after layering, the new triangulation on the boundary has associated rational one of $p/p + q$, $q/p + q$, or $p/q - p$, if $p < q - p$ or $q - p/p$, if $q - p < p$. We call these the *adjacency relations*.

From these adjacency relations, we construct a graph called the *L–graph*. The 0–cells of the L–graph are rational numbers $p/q, 0 \le p < q$, with p, q relatively prime and they include the values $0/1$ and $1/1$. There is a 1–cell between two 0–cells (rationals) if and only if one 0–cell is p/q and the other satisfies one of the adjacency relations given above. In Figure 7, we exhibit some of the L–graph. All 0–cells except $1/1$ have index three and the L–graph minus the closed edge loop at $0/1$ is a tree. The 0–cells $1/1$ and $0/1$ are each a bit of an anomaly.

We can think of a 1–cell in the L–graph going between 0–cells labeled, say p/q and $p/p + q$, as a tetrahedron in a layered-triangulation of the solid torus, where two opposite edges of the tetrahedron that correspond to the attaching edge and the univalent edge meet the meridional disk of the solid torus in q and $2p+q$ points; the remaining two edges meet the meridional disk in p and $p+q$ points. See Figure 4(B). We have the following connection between layered-triangulations of the solid torus and paths in the L–graph.

4.4. **Proposition.** *Suppose p/q is a 0–cell in the L–graph (i.e., a one-vertex triangulation on the boundary of a solid torus), then paths in the L–graph starting at*
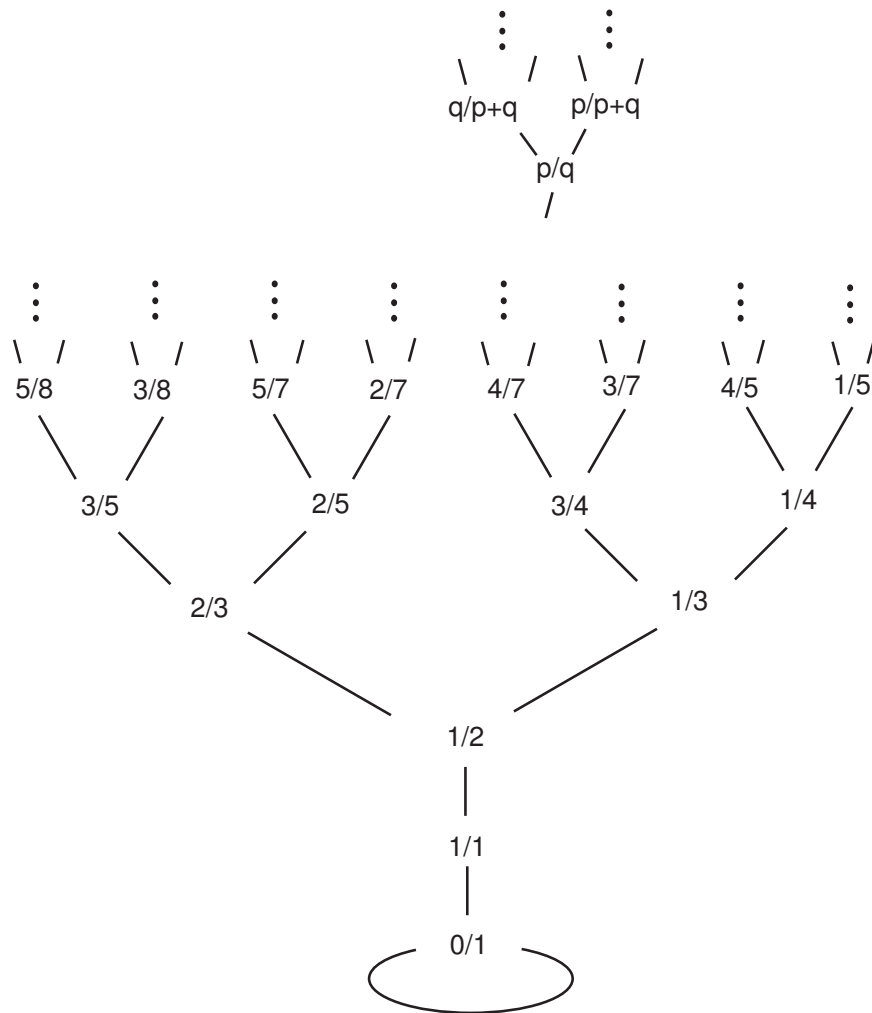
FIGURE 7. The start of the L–graph. The 0–cells are in one-one
correspondence with one-vertex triangulations on the boundary of
the solid torus.

*p/q and ending at 1/1 are in one-one correspondence with layered-triangulations of
the solid torus, extending the p/q–triangulation on the boundary.*

By using the one-triangle Möbius band as a (degenerate) triangulation of the
solid torus extending the $1/1$–triangulation, and using the creased 3–cell as a (de-
generate) triangulation of the solid torus extending the $0/1$–triangulation, then the
number of 1–cells in a path from p/q to $1/1$ in the L–graph is the same as the
number of tetrahedra in the corresponding layered-triangulation of the solid torus,
extending the p/q–triangulation on the boundary. The layered-triangulation cor-
responding to the (unique) shortest path in the L–graph from p/q to $1/1$ is the
minimal p/q–layered-triangulation. Again, we point out that the minimal layered-
triangulations extending the $1/1$– and $0/1$–triangulations are degenerate. If one is

actually wanting a triangulation of the solid torus in these two cases, then one can use the two-tetrahedron triangulations for both or the three-tetrahedron layered-triangulation for 0/1 given in Figure 5. Notice that because of the loop in the L–graph at the 0–cell 0/1, there is an infinite family of non homotopic paths (keeping end points fixed) corresponding to cycles about this loop. The homotopy classes of these loops give rise to infinitely many interesting p/q–layered-triangulations; however, these will play essentially no role in our study; as we will see below that these do not give 0–efficient triangulations.

There is another observation that is useful. Namely, we can extend a path in the L–graph that ends at 1/1 by a closed loop that goes to 1/2 and then back to 1/1. This is the same as opening the one-tetrahedron solid torus or the creased 3–cell along the Möbius band and inserting a Dehn filling along the curve with intersection numbers $1, 1, 2$ by adding the two-tetrahedron layered-triangulation of the solid torus extending 1/1. Observe that the $1, 1, 2$ curve is completely determined on the boundary; in particular, if the edge labeled 2 is the univalent edge upon opening, then we have opened the creased 3–cell and the original path approached 1/1 from 0/1 in the L–graph. We refer to this as *opening at* 1/1. With this new terminology, the two-tetrahedron layered-triangulation extending the 1/1–triangulation is an opening of the Möbius band at 1/1 and the three-tetrahedron layered-triangulation of the solid torus extending the 0/1–triangulation is an opening of the creased 3–cell at 1/1. Also, a path in the L–graph starting at 0/1 and looping between 0/1 and 1/1 and eventually terminating at 1/1 is a creased 3–cell; i.e., has the homotopy type of a solid torus but is not a 3–manifold. If the path transverses the loop at 0/1 or goes from 1/1 to 1/2 at any time, then we do have a solid torus. Finally, the number of tetrahedra for the minimal layered-triangulation of the solid torus extending the triangulation p/q on the boundary (with the exceptions of 0/1 and 1/1) is $(\sum a_i) - 1$, where a_i is the ith partial quotient in the continued fraction expansion $p/q = (a_0; a_1; \ldots; a_n)$.

Layered-triangulations are very nice ways to extend one-vertex triangulations on the boundary of the solid torus and the minimal layered-triangulations exhibit some great properties. We have the following very compelling conjecture about triangulations of the solid torus. We have tried but been unable to confirm this conjecture.

Conjecture. *The minimal triangulation of the solid torus that extends the p/q–triangulation on the boundary is the minimal p/q–layered-triangulation of the solid torus.*

A similar issue also comes up later as to whether layered-triangulations of lens spaces are the minimal triangulations of lens spaces. This question appears in the work of S. Matveev [22]. Of course, we can ask the same question regarding a minimal extension among all extensions of any triangulation from the boundary of a 3–manifold to the 3–manifold.

In a minimal layered-triangulation of the solid torus, with the exception of the two-tetrahedron triangulation extending 1/1, the univalent edge *always* corresponds to the edge meeting the meridional disk the greatest number of times. In the two-tetrahedron layered-triangulation extending 1/1, the univalent edge meets the meridional disk just once. In the two-tetrahedron layered-triangulation extending 0/1, the univalent edge is labeled 1 and satisfies our observation; whereas, in the three-tetrahedron layered-triangulation extending 0/1, the univalent edge does not

meet the meridional disk at all (corresponds to the edge labeled 0). See Figure 5. In each of these examples, there are two edges labeled 1; one is the thick edge, the other is thin. We also observed above that in these two layered-triangulations, the first layering is along the edge labeled 3, which is the univalent edge. In all other minimal layered-triangulations, the layering is *never* along the univalent edge. More, on the implications of layering along univalent edges, will be discussed below in our classi- fication of normal and almost normal surfaces in layered-triangulations of the solid torus and of lens spaces and in the discussions of 0– and 1–efficient triangulations. These are simple, yet very important, observations that make the understanding of minimal layered-triangulations much easier than arbitrary (layered) triangulations of the solid torus. In particular, we have that for a minimal layered-triangulation of a solid torus, the *slope of an edge*, which determines a unique isotopy class on the boundary and therefore a unique triple $r, s, r + s$, *always* meets the univalent edge the greatest number of times (of course, except for the slope of the univalent edge itself).

Later, we define, in an analogous way, layered-triangulations of handlebodies.

5. Normal and almost normal surfaces in layered-triangulations of a solid torus

We can classify all normal and all almost normal surfaces in a *minimal* layered- triangulation of the solid torus. In order to apply our study to efficient triangula- tions, we also classify all genus 0 and genus 1 normal surfaces in a (general) layered triangulation of the solid torus.

Examples:

We first give some examples of normal surfaces in minimal layered-triangulations of a solid torus and develop some terminology for the various families of such normal surfaces. See Figures 8-11

A. *Vertex-linking disk.* The unique vertex-linking surface in a layered-triangulation of a solid torus is a disk. Notice the general pattern for generating the in- duced triangulation of a vertex-linking disk, Figure 8.
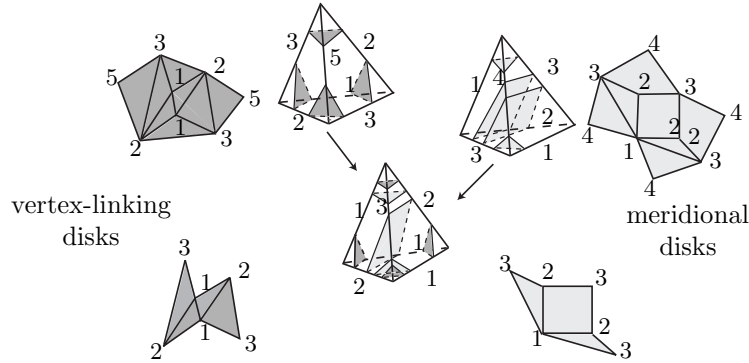

Figure 8. Examples of normal vertex-linking and meridional disks.

B. *Meridional disk.* There is a *unique* normal meridional disk in a layered-triangulation of a solid torus. While there may be other normal embedded disks having trivial boundary in the boundary of the solid torus, there always is only one normal meridional disk. Again, there is a general pattern for generating the induced cell-decomposition (must have normal quadrilaterals) of the meridional disk. See Figure 8. The growth (number) of quadrilaterals in the induced cell decomposition of the meridional disk in a minimal layered-triangulation of the solid torus, as a function of the number of layers, is like that of a Fibonacci sequence and so is exponential.
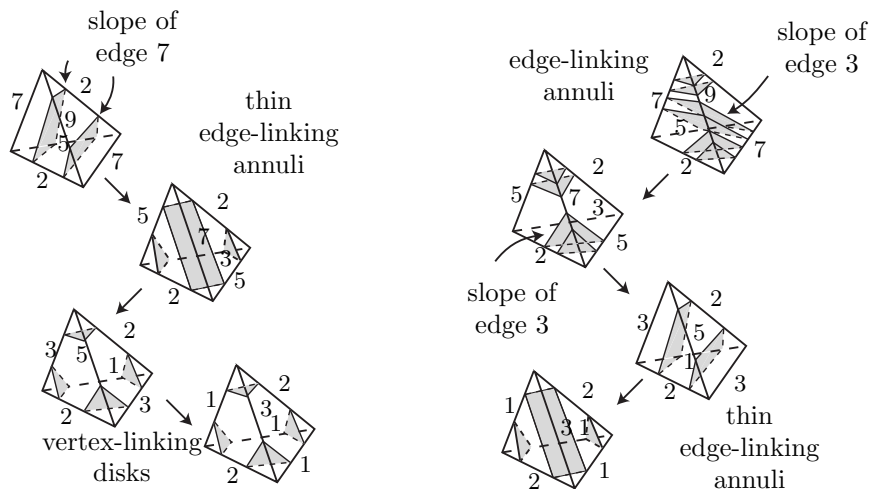
FIGURE 9. Examples of normal edge-linking annuli.

C. *Edge-linking annuli.* There are two models of edge-linking annuli. One is a *thin* edge-linking annulus and is a normal surface determined by taking the boundary of a small regular neighborhood of a thin edge in the boundary of the layered solid torus. Figure 9 gives such an example where the thin edge-linking annulus is about the edge 7 in the boundary of the minimal 2/7–layered-triangulation of the solid torus. The other example, in Figure 9, gives a normal annulus linking an interior edge; specifically, the edge 3 in the interior of 2/7–layered-triangulation. Note that the boundaries of these annuli have slopes equal to the slopes of the edges they link (7 and 3, respectively).

D. *Vertex-linking disk with thin edge-linking tubes.* In Figure 10 there is an example of the vertex-linking disk with a 1-handle attach. A 1–handle is formed by taking a thin edge-linking tube about the edge 3 in the minimal 2/7–layered-triangulation (as well as already in the minimal 2/5–layered-triangulation of the solid torus). There can be numerous thin edge-linking tubes attached to the vertex-linking disk.

E. *Edge-linking annuli with thin edge-linking tubes.* In Figure 10 there also is an example of an edge-linking annulus with a 1–handle attached. We
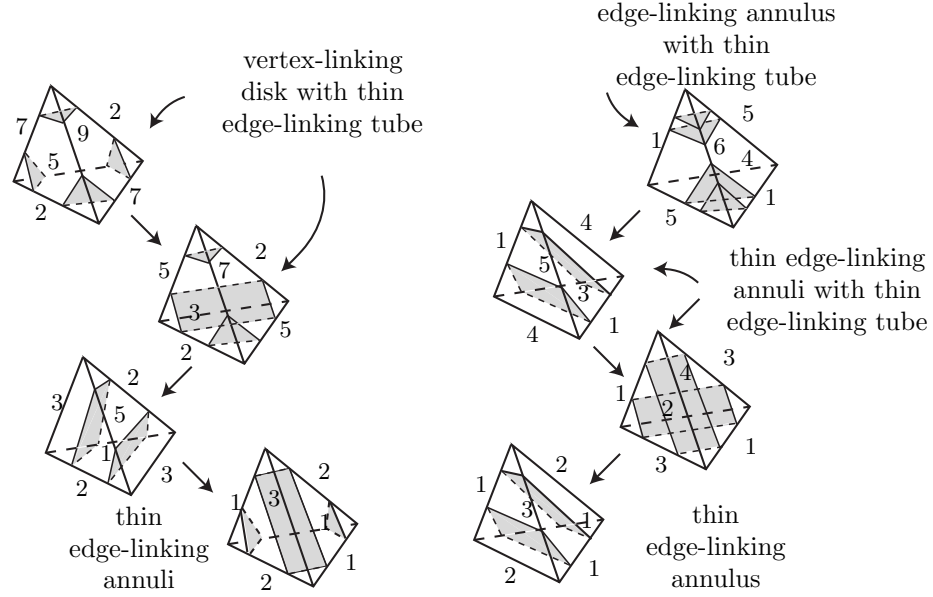
FIGURE 10. Examples of a normal vertex-linking disk and a normal edge-linking annulus, each with a thin edge-linking tube; on the left the tube is about the edge labeled 3 and on the right it is about the edge 2.

already have this at level 1/3, where there is a thin edge-linking annulus about the edge 4 with a 1–handle determined by a thin edge-linking tube about the edge 2. The annulus is still thin edge-linking at level 1/4; however, at level 1/5, we have an edge-linking annulus with boundary the slope of the edge 4; it is edge-linking but *not* thin edge-linking.

F. *Nonorientable surfaces.* In Figure 11 we give three examples of embedded (incompressible) nonorientable surfaces in the minimal 1/5–layered-triangulation of the solid torus. One is a Möbius band and has boundary slope the slope of the edge 2; one is a punctured Klein bottle and has boundary slope the slope of the edge 4; and one is a punctured genus three (3 crosscaps) nonorientable surface and has boundary slope the slope of the edge 6. Note that the double of such a surface is an edge-linking annulus with thin edge-linking tubes. Furthermore, in the double there must be a tube linking every even ordered edge prior to the edge of the edge-linking annulus. In general, there is a distinct nonorientable surface for each "even order" edge and its boundary has slope equal to the slope of the edge it is associated with. There are infinitely many nonorientable (incompressible) properly embedded surfaces in a solid torus; however, for a fixed layered-triangulation, there are only a finite number that are isotopic to a normal surface. This displays one of the discriminating properties of layered-triangulations.

5.1. **Normal surfaces in minimal layered-triangulations of a solid torus.**
We show that the model examples given in the previous section represent *all* possible
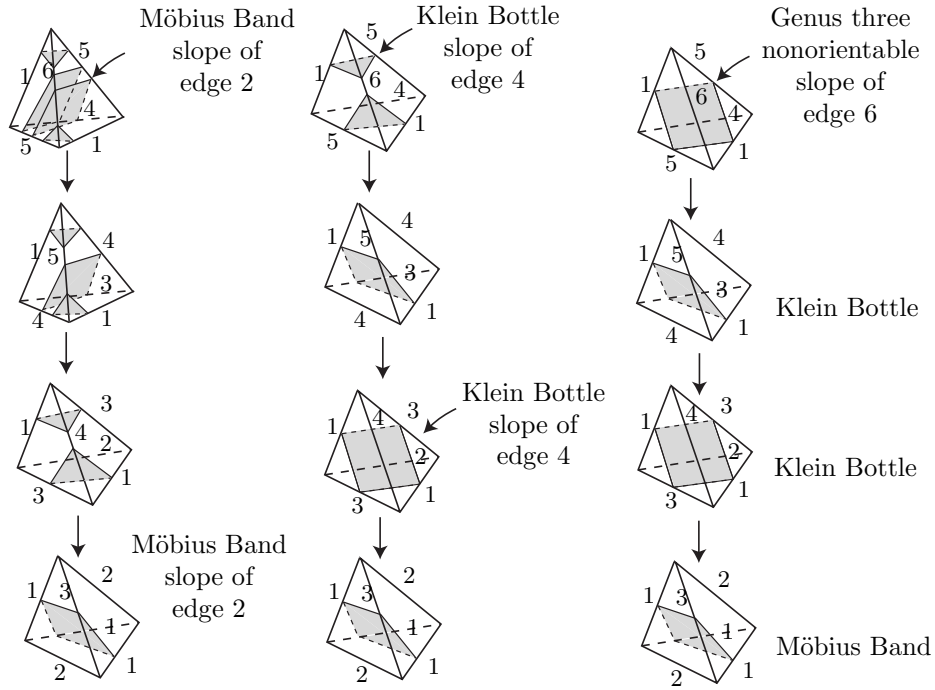
FIGURE 11. Examples of nonorientable normal surfaces: a Möbius Band, a Klein Bottle and a nonorientable surface of genus three. Each has a single boundary curve having the slope of an even edge.

types of embedded normal surfaces in minimal layered-triangulations of a solid torus.

The classification of normal surfaces in minimal layered-triangulations of a solid torus is made by induction on the number of tetrahedra in the layering. We start with the classification of normal surfaces in the one-tetrahedron solid torus, which begins our induction step, and is the minimal layered-triangulation extending the 1/2–triangulation on the boundary. We also need to consider, separately, the 2 two-tetrahedron minimal layered-triangulations extending the 1/1– and the 0/1–triangulations on the boundary.

5.1. Lemma. *The connected, embedded normal surfaces in the one-tetrahedron triangulation of the solid torus are:*

(1) *Vertex-linking disk,*
(2) *Meridional disk,*
(3) *Thin edge-linking annulus with boundary slope the slope of the edge* 3,
(4) *Möbius band with boundary slope the slope of the edge* 2, *and*
(5) *Thin edge-linking annulus with boundary slope the slope of the edge* 2, *which is also the double of the Möbius band.*

Proof. This is an easy exercise in using normal coordinates and face matching equations. There are 7 variables and 3 matching equations. We give the equations and their solutions in Figure 12. Also, one can use quadrilateral coordinates. In this case, there are three variables and no edge equations (there are no interior edges).

$$(a) : x_1 + y_3 = x_2 + y_2$$

$$(b) : x_2 + y_1 = x_4 + y_1$$

$$(c) : x_4 + y_2 = x_3 + y_3$$

(1,1,1,1,0,0,0)   (1,0,1,0,0,1,0)   (0,1,0,1,0,0,1)   (0,0,0,0,1,0,0)   (0,0,0,0,2,0,0)
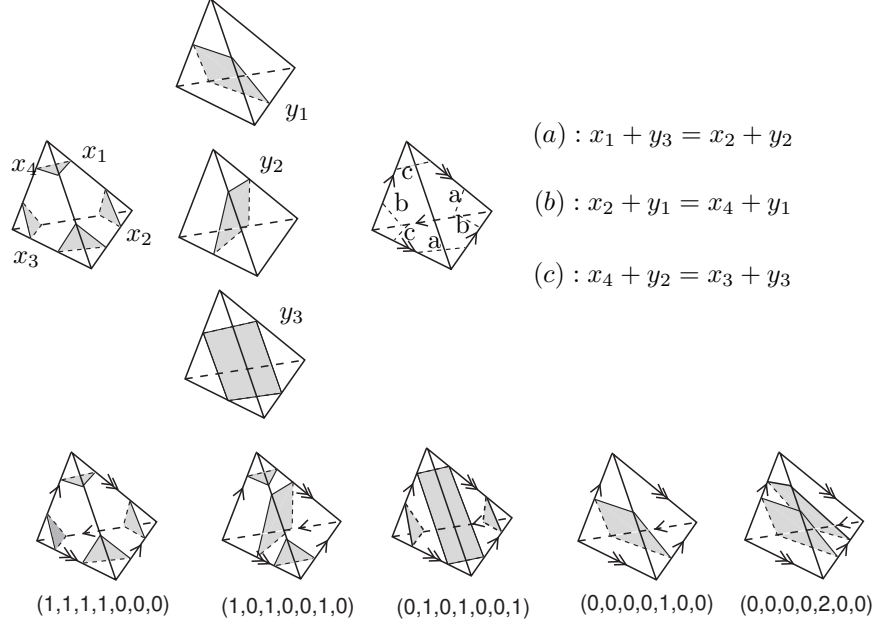
FIGURE 12. Embedded normal surfaces in the one-tetrahedron solid torus.

There is precisely one fundamental solution for each quadrilateral; these solutions and the vertex-linking disk give the embedded fundamental solutions. Solution (5) is the double of solution (4). □

We now use the method of quadrilateral equations to classify the normal surfaces in the two-tetrahedron solid torus extending the 0/1–triangulation on the boundary of the solid torus.

5.2. **Lemma.** *The connected, embedded normal surfaces in the two-tetrahedron triangulation of the solid torus extending the 0/1–triangulation are:*

(1) *Vertex-linking disk,*
(2) *Thin edge-linking annulus about the thin edge labeled 1 in the boundary,*
(3) *Möbius band with boundary slope the slope of the interior edge labeled 2,*
(4) *Double of the Möbius band, which is an edge-linking annulus about the interior edge labeled 2,*
(5) *Meridional disk,*
(6) *Vertex-linking disk with a thin edge-linking tube about the interior edge labeled 2,*
(7) *Thin edge-linking annulus about the thin edge labeled 1 in the boundary with a thin edge-linking tube about the interior edge labeled 2,*
(8) *Punctured Klein bottle with boundary slope the slope of the edge labeled 0,*
(9) *Thin edge-linking annulus about the edge labeled 0 in the boundary,*
(10) *Edge-linking annulus with a tube about the edge labeled 2 and with boundary slope that of the edge labeled 0, which is the double of the punctured Klein bottle.*

$(e):$ $x_2 + y_3 = x_2 + y_2$
$y_2 = y_3$

(3) $\quad y_1 = 0 \quad y_4 = 0$
$\quad\quad y_5 = 1 \quad y_6 = 0$

(7) $\quad y_1 = 1 \quad y_4 = 1$
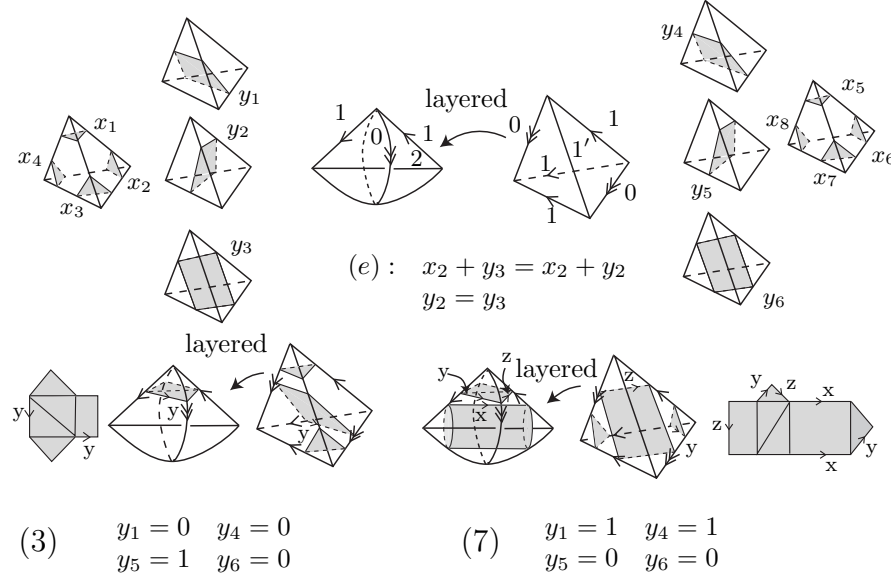$\quad\quad y_5 = 0 \quad y_6 = 0$

FIGURE 13. Embedded normal surfaces in the two-tetrahedron layered extension of $0/1$. (3) is the Möbius band with boundary the slope of the edge 2; and (7) is a thin edge-linking annulus with a thin edge-linking tube about the edge 2.

Proof. Since there are only two tetrahedra, using quadrilateral coordinates there are 6 variables (the 6 quad types) and only 1 equation, coming from the one interior edge of the triangulation. Using the notation from Figure 13, the variables can be denoted $y_1, y_2, y_3, y_4, y_5,$ and y_6; the one equation is $y_2 = y_3$. Thus if $y_2(y_3) \neq 0$, we have a singular surface. So, we can organize the possibilities as: $y_1 = 0$ or $y_1 \neq 0$ and at most one of $y_4 \neq 0, y_5 \neq 0$ or $y_6 \neq 0$. The easiest way to find the coordinates for the normal triangles is from the matching equations. $\square$

Before we do the general classification, we establish some terminology generalizing that used above. Suppose $e_1, \ldots, e_n$ are distinct interior edges of a layered-triangulation of a solid torus and the boundary of an arbitrarily small regular neighborhood of the wedge of the $e_i's$ is a normal surface. Not every wedge of distinct interior edges has this property and, in particular, none of the $e_i's$ are the thick edge. The normal, regular neighborhood can be thought of combinatorially as removing the collection of triangles that meet the $e_i's$ from the vertex-linking disk and replacing them with the quadrilaterals about the $e_i's$. We call such a surface a *vertex-linking disk with thin edge-linking tubes* or, simply, a *vertex-linking disk with tubes*. Notice that each of the tubes is "unknotted" in the solid torus in the sense that its core simple closed curve (an interior edge) is isotopic in the solid torus to a curve in the boundary of the solid torus. Examples are given above in the layered-triangulation extending the $2/7$–triangulation (Example D and Figure 10) and in item (6) of Lemma 5.2 in the classification of surfaces in the two-tetrahedra triangulation of the solid torus extending the $0/1$–triangulation.

The next general form is the edge-linking annulus. Suppose we have a layered-triangulation of a solid torus, extending the p/q-triangulation on the boundary;

hence, there is a corresponding path in the L–graph beginning at p/q and ending at $1/1$. For any 0–cell r/s in this path, a segment of the path from $1/1$ to r/s is called an *initial segment*. This initial segment determines a subcomplex of the original layered-triangulation and is a layered-triangulation of a solid torus extending the r/s-triangulation on the boundary of a solid torus. (See Figure 10 above for the initial segment corresponding to the vertex $1/3$ in the minimal layered-triangulation extending the $1/5$–triangulation.) Now, if e is a thin edge in the boundary of a layered-triangulation of a solid torus, the boundary of an arbitrarily small regular neighborhood of e is a normal surface. It can be thought of combinatorially as removing the collection of triangles that meet e from the vertex-linking disk and replacing them with the quadrilaterals about e. We call such a surface a *thin edge-linking annulus*. Next, suppose $\mathcal{T}$ is a layered-triangulation of a solid torus and $\mathcal{T}'$ is an initial segment of $\mathcal{T}$. If A' is a thin edge-linking annulus in $\mathcal{T}'$, then we can push the boundary of A' through the layered-triangulation of $\mathcal{T}$ to get a new normal, properly embedded annulus A in $\mathcal{T}$ (again see C above and Figure 10). We call such a normal surface an *edge-linking annulus*, as opposed to a *thin* edge-linking annulus.

Now, suppose e is an edge in the boundary of a layered-triangulation of a solid torus and $e_1, \ldots, e_n$ are edges in the interior of the layered solid torus so that the boundary of an arbitrarily small regular neighborhood of the wedge of e and the edges $e_i, 1 \leq i \leq n$ is a normal surface. Such a normal surface can be combinatorially described as removing the collection of triangles that meet e and any e_i from the vertex-linking disk and replacing them with the quadrilaterals about e and each e_i. We call such a surface a *thin edge-linking annulus with thin edge-linking tubes* or, simply, a *thin edge-linking annulus with tubes* (see Examples E and F above and in Figure 10 and Figure 11, as well as item (9) of Lemma 5.2).

Finally, suppose $\mathcal{T}$ is a layered-triangulation of a solid torus and $\mathcal{T}'$ is an initial segment of $\mathcal{T}$ and A' is a thin edge-linking annulus with tubes in $\mathcal{T}'$. If we push A' through the layered-triangulation $\mathcal{T}$ to get the normal surface A, then A is an *edge-linking annulus with tubes* (as opposed to a *thin* edge-linking annulus with tubes). Notice that the boundary slope of the boundary of the edge-linking annulus (with tubes) is the slope of an edge; it is the slope of the edge of the thin edge-linking annulus A'. Also, there are no tubes possible between $\partial A'$ and ∂A. An example is given in Figure 10; in that example we can take A' in the initial segment corresponding to the $1/3$–layered-triangulation, where A' is the thin edge-linking annulus about the edge labeled 4 with a (thin edge-linking) tube about the edge labeled 2. The surface A is the "push through" of A' and it is an edge-linking annulus with a thin edge-linking tube in the $1/5$–layered-triangulation. We sometimes just use "edge-linking annulus with tubes" including the possibility of a thin edge-linking annulus with tubes; however, they are different in nature. The thin edge-linking annulus with tubes is the boundary of a regular neighborhood of a wedge of edges in the layered-triangulation; whereas, the more general edge-linking annulus with tubes may not be. A thin edge-linking annulus also has boundary slope the slope of an edge in the boundary; whereas, an edge-linking annulus that is not a thin edge-linking annulus has boundary slope the slope of an edge from the interior of the layered-triangulation. There is another very important fact about edge-linking annuli. In a layered-triangulation that is not minimal, there may be distinct edges determining the same slope on the boundary of the solid torus. In

such a case there can be distinct (not normally isotopic) edge-linking annuli with the same boundary slope. Even though the annuli are not normally isotopic, they are isotopic. This phenomenon can not happen in a minimal layered-triangulation of a solid torus. Any edge-linking annulus, A, is parallel in the solid torus to an annulus in the boundary containing the vertex. We call this annulus in the boundary of the solid torus the *companion annulus* (to A). The annulus complementary to the companion annulus in the boundary of the layered solid torus is called the *complementary annulus* (to A).

We now have the classification of normal surfaces in minimal layered-triangulations of the solid torus.

5.3. **Theorem.** *A connected, embedded normal surface in a minimal layered-triangulation of the solid torus is normally isotopic to one of:*

    (1) *The vertex-linking disk,*
    (2) *A vertex-linking disk with tubes,*
    (3) *The (unique) meridional disk,*
    (4) *An edge-linking annulus, having boundary slope the slope of the edge,*
    (5) *An edge-linking annulus with tubes, having boundary slope the slope of an edge, or*
    (6) *An incompressible, nonorientable surface, having boundary slope the slope of an even ordered edge.*

Proof. We shall use induction for our proof and assume we are starting with the one-tetrahedron solid torus. Furthermore, we will consider the minimal $1/1$–layered-triangulation separately; hence, we never layer on the univalent edge.

From Lemma 5.1, we know exactly the connected, embedded normal surfaces in the one-tetrahedron triangulation of the solid torus. These satisfy our conclusions above. Our induction hypothesis is then that in a minimal layered-triangulation $\mathcal{T}_t$ with t tetrahedra, ($t \geq 1$), the list (1)–(6) in the conclusion of Theorem 5.3 is the complete list of connected, embedded normal surfaces; furthermore, each slope of an edge corresponds to a unique edge. This is a central feature of a minimal layered-triangulation. Let $\mathcal{T}_{t+1} = \mathcal{T}_t \cup_e \tilde{\Delta}_{t+1}$ be a minimal layered-triangulation having $t + 1$ tetrahedra. Furthermore, suppose the minimal layered-triangulation $\mathcal{T}_t$ extends the p/q–triangulation on the boundary of the solid torus. Note also, we may assume that the edge e is *not* a univalent edge.

If F_{t+1} is a normal surface in $\mathcal{T}_{t+1}$, then it meets $\mathcal{T}_t$ in a normal surface, say F_t. Hence, each component of F_t must fall into one of the categories (1)–(6) in the conclusion of Theorem 5.3. Furthermore, we have F_{t+1} is obtained from F_t by adding cells that lift to normal triangles and quads in $\tilde{\Delta}_{t+1}$. There are two possibilities.

Case A. We push F_t through Δ_{t+1}. Then F_{t+1} is homeomorphic with F_t and either F_{t+1} is a vertex-linking disk with tubes and has trivial boundary, F_{t+1} is a push through of an edge-linking annulus with tubes and has boundary the slope of an edge, or F_t is the meridional disk and F_{t+1} is also the meridional disk. In the last two cases, we have that if the boundary of F_t does not have the slope of the univalent edge, then it meets the univalent edge maximally and therefore the boundary of F_{t+1} meets the univalent edge maximally. If the boundary of F_t has the slope of the univalent edge, then it is still true that the slope of the boundary

of F_{t+1} meets the univalent edge maximally. A push through in a minimal layered-triangulation can not have boundary with slope equal to that of the univalent edge.
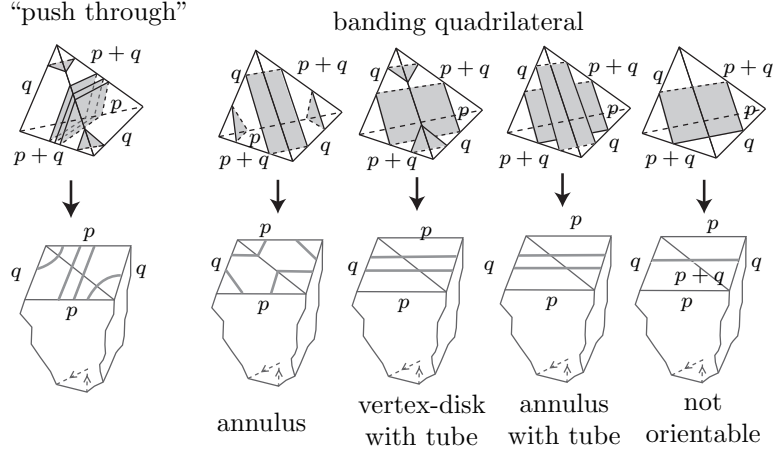

FIGURE 14. Examples of "pushing through" a surface and of adding a banding quadrilateral.

Case B. We add "banding quads" to F_t in $\tilde{\Delta}_{t+1}$. A *banding quadrilateral* is a normal quadrilateral in $\tilde{\Delta}_{t+1}$ separating the edge that becomes the univalent edge in $\mathcal{T}_{t+1}$ from its opposite edge, which is the edge in $\tilde{\Delta}_{t+1}$ attached to e (see Figure 14 and [15]).

In this case, we need to consider the various possibilities. First, in order to add a banding quad, the boundary components of F_t must be either vertex-linking curves or have the slope of the edge e, along which we are attaching the tetrahedron $\tilde{\Delta}_{t+1}$. Hence, by induction a component of F_t is a vertex-linking disk (possibly with thin edge-linking tubes), a thin edge-linking annulus (possibly with thin edge-linking tubes), or a nonorientable surface. In this last case, the edge e meets the meridional disk an even number of times. Since we are assuming F_{t+1} is connected, we have that F_t must be connected and in the case F_t is an annulus (possibly with thin edge-linking tubes) a single banding quad must attach to distinct boundary components of F_t.

If the boundary of F_t is the vertex-linking curve in $\mathcal{T}_t$, then we can add at most one banding quad. Furthermore, in this case by our induction hypothesis, F_t is either the vertex-linking disk in $\mathcal{T}_t$ or it is the vertex-linking disk with thin edge-linking tubes. In the first case, adding the banding quad (along with two normal triangles, which push through the remaining part of the boundary of F_t), we get that F_{t+1} is a thin edge-linking annulus with boundary slope that of the univalent edge in $\mathcal{T}_{t+1}$. If F_t is the vertex-linking disk with tubes, then adding the banding quad (along with two normal triangles that push through the rest of the boundary of F_t), we get that F_{t+1} is a thin edge-linking annulus with tubes and with boundary slope that of the univalent edge in $\mathcal{T}_{t+1}$.

If the slope of the boundary of F_t is the same as the edge e in the boundary of $\mathcal{T}_t$, then again by our induction hypothesis, we have two possibilities: the boundary of F_t is a single curve and F_t is the unique nonorientable surface with boundary slope

the edge e in the boundary of $\mathcal{T}_t$ (and this edge has even order) or the boundary of F_t has two curves.

In the first situation, we have only one possibility for adding a banding quad and we have that F_{t+1} is the unique nonorientable surface with boundary slope the slope of the univalent edge in $\mathcal{T}_{t+1}$ (and the univalent edge has even order for its intersection number with the meridional disk). The genus of F_{t+1} is one more than the genus of F_t.

If F_t has two curves in its boundary, then it is either a *thin* edge-linking annulus or a *thin* edge-linking annulus with tubes and, in either case, with boundary slope that of the edge e. We can add one band (and push through the remaining part of the boundary of F_t), then F_{t+1} is a vertex-linking disk with tubes (it must have at least a tube about the edge e). We can add two bands, then F_{t+1} is a thin edge-linking annulus with tubes (it must have at least a tube about the edge e) and with boundary slope that of the univalent edge of $\mathcal{T}_{t+1}$. This completes the induction step.

This leaves only determining the normal surfaces in the two-tetrahedron, minimal, non degenerate layered-triangulation of the solid torus extending the 1/1–triangulation. See Figure 5. We can easily adapt the preceding induction proof to this simple case of a layering on the one-tetrahedron solid torus. By pushing surfaces in the one-tetrahedron torus through, we get the vertex-linking disk, meridional disk, a Möbius band with boundary slope that of the edge 2 in the boundary, its double, which is a thin edge-linking annulus about the edge 2, and an edge-linking annulus with boundary slope that of the interior edge 3. We can band the vertex-linking disk in the one-tetrahedron solid torus , getting a thin edge-linking annulus on the univalent edge. We can also band on the thin edge-linking annulus about the edge 3 in the one-tetrahedron solid torus. Using just one band, we get a vertex-linking disk with a tube about the interior edge 3; using two banding quads, we get a thin edge-linking annulus about the univalent edge along with a tube about the interior edge 3. This completes the proof of Theorem 5.3. ☐

Notice we indicated three methods to find normal surfaces in layered-triangulations: normal coordinates and face-matching equations, quadrilateral coordinates and edge-matching equations, and an induction argument, using the layered structure of the triangulation. The latter is the method of choice for layered-triangulations.

We have from the proof of Theorem 5.3 that:

- There are no closed normal surfaces in a minimal layered-triangulation of a solid torus (this remains true even if the layered-triangulation is not minimal - see below).
- There is a unique normal meridional disk in a minimal layered-triangulation of a solid torus and the slope of its boundary meets the univalent edge a maximum number of times. It remains true that there is a unique meridional disk even if the layered-triangulation is not minimal. We think of the nontrivial normal spanning arc in the one-triangle Möbius band and the wedge of two triangular cones in the creased 3–cell as "meridional disks" for the (degenerate) 1/1– and 0/1–triangulations, respectively.
- There is a unique trivial normal disk, the vertex-linking disk, in a minimal layered-triangulation (however, this is not necessarily true if the layered-triangulation is not minimal - see below).

- The boundary slope of any normal surface other than the meridional disk and the vertex-linking disk is a slope of an edge and therefore, meets the univalent edge a maximum number of times (unless the boundary slope is the slope of the univalent edge itself).
- There are infinitely many incompressible non orientable surfaces embedded in a solid torus; however, a layered-triangulation only has finitely many normal non orientable surfaces, one for each even order edge in the layered-triangulation. Furthermore, the slope of the boundary of a non-orientable normal surface is the slope of the even ordered edge with which it can be associated. Below we give a recursive function that determines the genus of a normal non-orientable surface in a minimal layered-triangulation of the solid torus.
- We repeat that in the case of an edge-linking annulus with tubes, all of the tubes are about edges coming in the layering before the edge about which the annulus links.
- If λ and γ are two slopes on a torus, let $d(\lambda, \gamma)$ denote the distance between them. For F a vertex-linking disk with thin edge-linking tubes about edges having slopes λ and γ, then $d(\lambda, \gamma) \geq 2$; similarly, if F is an edge-linking annulus about the edge e with slope λ and F has thin edge-linking tubes about the edges e' and e'' with slopes γ' and γ'', respectively, then $d(\lambda, \gamma')$, $d(\lambda, \gamma'')$ and $d(\gamma', \gamma'')$ are all ≥ 2.

5.2. **Genus 0 and 1 normal surfaces in a (general) layered-triangulation of a solid torus.** Below it will be necessary for us to understand genus 0 and genus 1 normal surfaces embedded in a (general) layered-triangulation of a solid torus in order to study 0–efficient and 1–efficient layered-triangulations of lens spaces.
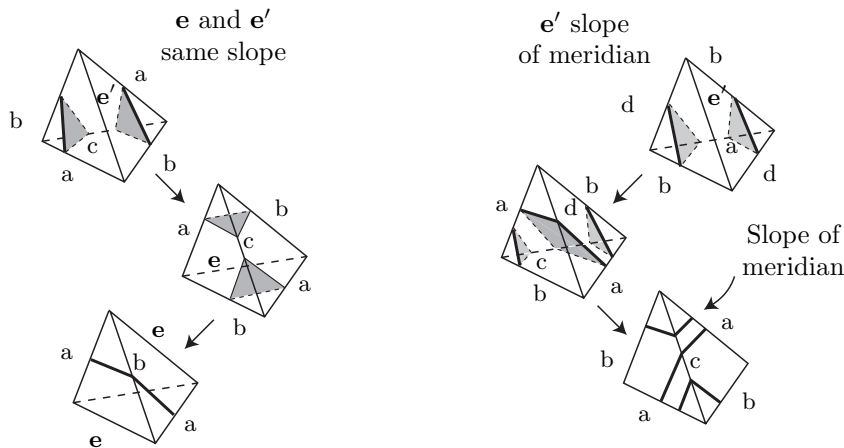

FIGURE 15. Examples of two edges (e and e') with the same slope; and an edge (e') with the slope of a meridional disk.

We are only interested in orientable surfaces of genus 0 and genus 1; as might be expected, some of these occur as the doubles of embedded normal Möbius bands or

embedded normal punctured Klein bottles; we will point out these possibilities in our classification of genus 0 and genus 1 embedded normal surfaces.

Examples. First, we give some examples of genus 0 and genus 1 normal surfaces in general layered-triangulations of a solid torus; some are new and are not seen in minimal layered-triangulations.

Genus 0 *normal surfaces*: A genus 0 normal surface is obtained in $\mathcal{T}_{t+1}$ by pushing through a genus 0 normal surface from $\mathcal{T}_t$ or by adding a banding quad in $\tilde{\Delta}_{t+1}$ to a disk or to two copies of a disk.

(1)–(3) We have the possibility of any of the genus 0 normal surfaces that can occur in *minimal* layered-triangulations and are classified in Theorem 5.3. (1) the *vertex-linking disk*, (2) the *meridional disk*, and (3) *edge-linking annuli* (both thin edge-linking and non-thin edge-linking annuli).

(4) In addition, we have the possibility for two new genus 0 normal surfaces, if the edge e in $\mathcal{T}_t$, along which we attach $\tilde{\Delta}_{t+1}$, has the same slope as the meridian. To see this, suppose the edge e has the same slope as the meridian. Let P_t be two copies of the meridian disk (P_t has two components). First, add a banding quad in $\tilde{\Delta}_{t+1}$ that runs between the distinct components of P_t; such a banding quad, along with two triangles in $\tilde{\Delta}_{t+1}$, gives an embedded normal disk in $\mathcal{T}_{t+1}$ that is trivial. Its boundary is the vertex-linking curve in the boundary of $\mathcal{T}_{t+1}$ but it is not the vertex-linking disk. Hence, it is (4A) a *non vertex-linking trivial disk*. See Figure 16(A).
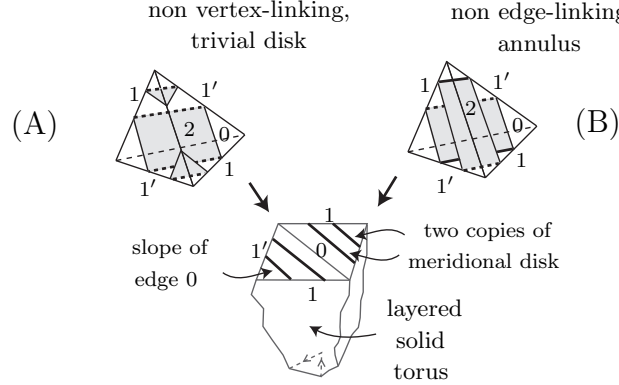
FIGURE 16. Examples of a non vertex-linking, trivial, normal disk and a non edge-linking, normal annulus.

Secondly, and in the previous situation, if we attach two banding quads in $\tilde{\Delta}_{t+1}$ to P_t, then we get (4B) a *non edge-linking annulus* in $\mathcal{T}_{t+1}$. Notice that if we take only one copy of the meridional disk in $\mathcal{T}_t$ and attach one banding quad in $\mathcal{T}_{t+1}$, then we get a Möbius band and the non edge-linking annulus is the double of this Möbius band. Also, note that the slope of its boundary corresponds to an edge slope of an edge meeting the meridional disk 2 times. See Figure 16 (B). Specific examples of (4A) and (4B) can be constructed in a four tetrahedron layered-triangulation of the solid torus extending the 1/1–triangulation and corresponding to a path in the L–graph starting at 1/1, going to 0/1 back through 1/1 to 1/2 and ending at

1/1. The boundary of a non edge-linking annulus also has the slope of an edge; it is analogous to an edge-linking annulus but is a banding of a non vertex-linking trivial disk and so we do not think of it as an edge-linking annulus, as we do when we band a vertex-linking disk.

(5) We have still another possibility for a new genus 0 normal surface, if there are two edges having the same slope. Suppose the edges e and e' in $\mathcal{T}_t$ have the same slope and, say e appears earlier in the layering than e'. Then if we take an edge-linking annulus about e, then at the level that e' first appears, the edge-linking annulus about e has boundary slope that of the edge e'. We can then continue to shove this annulus through the layered triangulation, getting an edge-linking annulus unlike those above. We call this (5) a *fat edge-linking annulus*. An example can be obtained in the three-tetrahedron layered-triangulation of the solid torus corresponding to an extension of the $1/2$–triangulation for the path $1/2, 1/1, 1/2, 1/1$ in the L–graph. See the first three layers in Figure 18 and for edges take $e = 3$ and $e' = 3'$.

We have from Theorem 5.4 that these examples give all the five types of genus 0 normal surfaces in a general layered-triangulation of a solid torus. We now consider genus 1 normal surfaces.

Genus 1 normal surfaces: A genus 1 normal surface is obtained in $\mathcal{T}_{t+1}$ by pushing through a genus 1 normal surface from $\mathcal{T}_t$ or by adding a banding quad in $\tilde{\Delta}_{t+1}$ to a genus 0 surface in $\mathcal{T}_t$ and reducing the number of boundary components (raising the genus) or adding a banding quad in $\tilde{\Delta}_{t+1}$ to a genus 1 surface in $\mathcal{T}_t$ and increasing the number of boundary components (keeping genus 1).

(6) We have as possible genus 1 normal surfaces any of those that appear in minimal layered-triangulations. These can be (6A) a *vertex-linking disk with a thin edge-linking tube*, which gives a once-punctured torus with trivial boundary curve, or (6B) an *edge-linking annulus with a thin edge-linking tube*. The last example gives a twice punctured torus.

(7) Using the above possibility of a non vertex-linking trivial disk, we obtain new genus 1 surfaces. The simplest of these is a (7A) *non vertex-linking trivial disk with a thin edge-linking tube*. In this case we have a non vertex-linking trivial disk with boundary the trivial vertex-linking curve; hence, at some later stage, we can add a thin edge-linking tube. On the other hand, if we have such a non vertex-linking trivial disk with a thin edge-linking tube at some stage in a layering, then we can band it and get (7B) a *non edge-linking annulus with a thin edge-linking tube*. As with edge-linking annuli, these annuli have boundary slope the slope of an edge. We do not give a different name for these, as we did for edge-linking annuli, depending on whether the boundary slope is that of an edge in the boundary or not. Examples (7A) and (7B) can happen only if an edge has the slope of the meridian. Specific examples can be obtained for (7A) and (7B) in a six-tetrahedron layered-triangulation of the solid torus extending the $2/5$–triangulation, using the path $2/5, 2/3, 1/2, 1/1, 0/1, 1/1$. In this example, we have a non vertex-linking trivial disk in the initial three layers $(1/1, 0/1, 1/1)$; through the next three layers, we can attach a thin

edge-linking tube about the edge 3. Thus in this six-tetrahedron layered-triangulation of the solid torus extending the 2/5–triangulation, we have non vertex-linking trivial disk with a thin edge-linking tube about the edge 3 (7A) and also we have a non edge-linking annulus, with boundary slope that of the edge 7, with a thin edge-linking tube about the edge 3. See Figure 17.
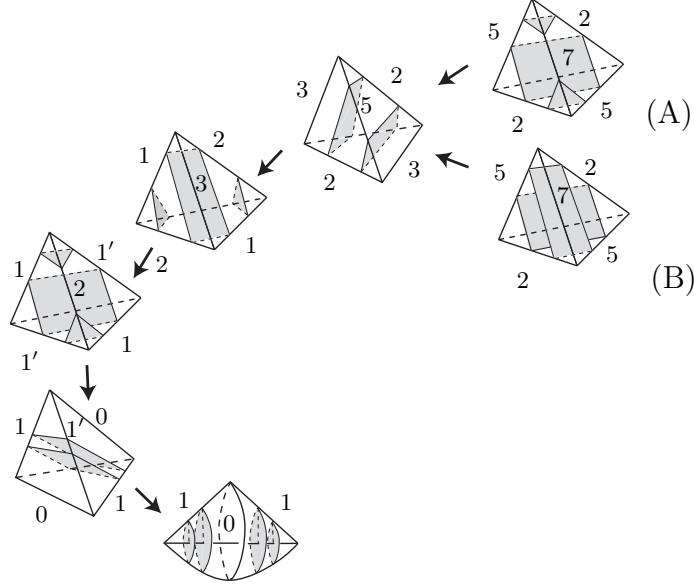
FIGURE 17. Examples of a non vertex-linking trivial normal disk with a thin edge-linking tube (A) and a non edge-linking normal annulus with a thin edge-linking tube (B).

(8) We also have another possibility for new genus 1 normal surfaces, if the edge e in $\mathcal{T}_t$, along which we attach $\tilde{\Delta}_{t+1}$, has the same slope as another edge. To see this, suppose the edge e has the same slope as the edge e', which is necessarily an interior edge of $\mathcal{T}_t$. Let P_t be an edge-linking annulus about e'; then P_t is not a thin edge-linking annulus but does have boundary slope the same as the boundary edge e in $\mathcal{T}_t$. Thus, we can add a banding quad in $\tilde{\Delta}_{t+1}$, which necessarily runs between distinct components of ∂P_t. Attaching such a banding quad and two triangles in Δ_{t+1} to P_t, we get (8A) a *vertex-linking disk with a non edge-linking tube*. The tube links two edges with the same slope; we think of these tubes as "fat" tubes. This can happen only if another edge has the same slope as e. The resulting surface is a punctured-torus. If we attach two banding quads, then we get (8B) an *edge-linking annulus with a non edge-linking tube*; and the surface is a twice-punctured torus. Also, we can have an edge-linking annulus with a non edge-linking tube and have still another edge with the same slope as that of the boundary of the annulus; thus we get a fat edge-linking annulus with a non edge-linking tube. Specific examples of both (8A) and (8B) can be obtained in a four-tetrahedron layered-triangulation of the solid

torus corresponding to an extension of the 1/1–triangulation for the path $1/1, 1/2, 1/1, 1/2, 1/1$ in the L–graph. See Figure 18. Having edges with the same slope leads to two other possibilities. For example, before we see the two edges with the same slope, we can have a vertex-linking disk with a thin edge-linking tube; then we have two edges with the same slope. We then have (8C) a *fat annulus with a thin edge-linking tube*. Also, we could have a vertex-linking disk with a non edge-linking tube and then see two more edges with the same slope, getting (8D) a *fat annulus with a non edge-linking tube*. Examples can be obtained from the example in Figure 18 by adding a couple more layers.
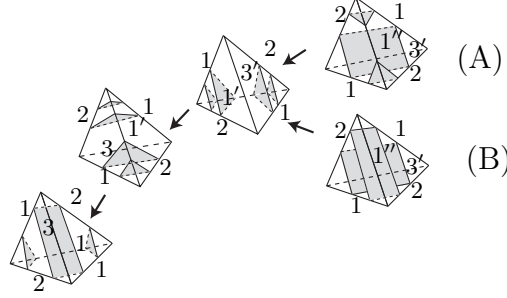

FIGURE 18. Examples of (A) a vertex-linking disk with a non edge-linking ("fat") tube and (B) an edge-linking annulus with a non edge-linking ("fat") tube .

(9) Here we have both an edge with slope that of a meridian disk as in Example (7) and two edges with the same slope, as in Example (8). Hence, we have (9A) a *non vertex-linking disk with a non edge-linking tube* or (9B) a *non edge-linking annulus with a non edge-linking tube*. An example can be easily constructed by using the initial segment with four tetrahedron extending the 1/2–triangulation in Figure 17, from this we can get an example of a non vertex-linking trivial disk; then we layer on these four tetrahedra the last three tetrahedra given in Figure 18 ending with a seven tetrahedra layering. Using the two options (A) and (B) in Figure 18, we get examples of (9A) and (9B), respectively.

For the classification of genus 0 and genus 1 normal surfaces, it is possible that we have a creased 3–cell at the base of our layered-triangulation and it also is possible that we layer along the univalent edge in the boundary. In fact, it could be possible to have a path in the L–graph beginning at some point, going through 1/1 to 0/1 and then simply oscillating back and forth between 0/1 and 1/1, finally ending at 1/1. Such a path gives a terribly degenerate triangulation; in fact, one in which some initial segment does not even give a 3–manifold. Such an initial segment can be crushed to the Möbius band. We will assume that the only initial segment that is not a 3–manifold is when we have a one-tetrahedron creased 3–cell. In Lemma 5.1, we listed all normal surfaces in the one-tetrahedron solid torus; and in Lemma 5.2, we listed the connected normal surfaces in the two-tetrahedron minimal triangulation extending the 0/1–triangulation. In these cases, there are only genus 0 and genus 1 normal surfaces. We consider the three-tetrahedron

layered-triangulation extending the 0/1-triangulation on the boundary as part of the general induction step. With this information, we can begin our induction argument, building upon the surfaces in these small examples.

5.4. **Theorem.** *A connected, embedded genus* 0 *or genus* 1 *normal surface in a (general) layered-triangulation of a solid torus is normally isotopic to one of:*
Genus 0*:*

   (1) *Vertex-linking disk,*
   (2) *meridional disk (unique),*
   (3) *an edge-linking annulus with boundary slope the slope of an edge,*
   (4) *(A) a trivial, non vertex-linking disk or (B) a non edge-linking annulus with boundary slope the slope of an edge (only if an edge has the slope of the meridional disk) or*
   (5) *a fat edge-linking annulus (only if two edges have the same slope).*

Genus 1*:*

   6 *(A) a vertex-linking disk with a thin edge-linking tube or (B) an edge-linking annulus with a thin edge-linking tube,*
   7 *(A) a non vertex-linking trivial disk with a thin edge-linking tube or (B) a non edge-linking annulus with a thin edge-linking tube (only if an edge has the slope of the meridian),*
   8 *(A) a vertex-linking disk with a non edge-linking tube or (B) an edge-linking annulus with a non edge-linking tube or (C) a fat edge-linking annulus with a thin edge-linking tube, or (D) a fat edge-linking annulus with a non edge-linking tube (in all cases only two edges have the same slope),*
   9 *(A) a non vertex-linking trivial disk with a non edge-linking tube or (B) a non edge-linking annulus with a non edge-linking tube (only if an edge has the slope of the meridian and two edges have the same slope).*

The proof uses induction, following the same ideas as in the proof of Theorem 5.3. We change the topological type of a surface by attaching a banding quad; however, a banding quad attached to a surface having boundary the vertex-linking curve gives an annulus or an annulus with tubes and a banding quad attached to an annulus (or an annulus with tubes) must run between distinct boundary components and so raises genus. See [15] for a detailed proof.

If we ever have the meridional slope also being the slope of an *interior* edge, then, except for the two-tetrahedra extension of the 0/1–triangulation, we do *not* have a minimal layered-triangulation. On the other hand, we may construct a layered-triangulation that is not a minimal layered-triangulation, and does not have the meridional slope as the slope of any edge.

As mentioned above, if we do not have a minimal layered-triangulation, there is a vertex on the path corresponding to the layered-triangulation in the L–graph where we have layered along the univalent edge. If we layer along the univalent edge, we either get two thin edges with the same slope or we pick up the exceptional longitude (defined below) as a slope of an edge. For example, in the two-tetrahedron layered-triangulation extending the 1/1–triangulation and the three-tetrahedron one extending the 0/1–triangulation, we do not have two thin edges with the same slope; however, we do have the exceptional longitude having the slope of the edge labeled 1′ in Figure 5. If we layer along the univalent edge and do have two edges with the same slope, then these slopes will necessarily meet the meridional disk the

same number of times. We try to distinguish the edges by using the same integer but adding primes as superscripts. Just because edges meet the meridional disk the same number of times does not mean they determine the same slope on the boundary. For example, at the vertex labeled $0/1$ in the L–graph, we have an edge (cycle) with both its end points at $0/1$. If we take a layered-triangulation corresponding to a path in the L–graph that transverses this cycle a number of times, we get a number of edges meeting the meridional disk just once but they determine different slopes (longitudes) on the boundary. We summarize the conclusions we need later to classify 0– and 1–efficient layered-triangulations of lens spaces in the following corollaries. Most notably is that the number of boundary components of a genus 0 or a genus 1 normal surface is either one or two and we have a normal disk or normal annulus possibly with a single tube.

5.5. **Corollary.** *A connected, embedded, normal, genus 0 surface in a layered-triangulation of a solid torus is either the unique meridian disk, a trivial disk with boundary the vertex-linking curve, or an annulus with boundary slope the slope of an edge.*

5.6. **Corollary.** *A connected, embedded, normal, genus 1 surface in a layered-triangulation of a solid torus is either a disk with a tube (a punctured-torus) with boundary the vertex-linking curve or an annulus with a tube (a twice-punctured torus) with boundary slope the slope of an edge.*

5.3. **Almost normal surfaces in minimal layered-triangulations of a solid torus.** In this section we classify the almost normal surfaces in a minimal layered-triangulation of a solid torus that extends a given one-vertex triangulation on its boundary.

We begin by giving some examples of almost normal octagonal surfaces in minimal layered-triangulations of a solid torus. There are, of course, various almost normal tubed surfaces formed from normal surfaces in these triangulations.

Examples.

(1) Three different almost normal octagonal annuli, two with boundary slope a "longitude" in the minimal $2/3$–layered-triangulation of the solid torus and one with boundary slope the slope of the thin edge 2 in the minimal $1/4$–layered-triangulation. See Figure 19. Notice that (A) has slope the preferred longitude, which is a slope of an edge, while (B) has slope also a longitude but not what we are calling the slope of an edge. The third (C) has slope the same as a thin edge and is isotopic to an edge-linking normal annulus about the edge 2. Examples (A) and (B) are not isotopic to edge-linking normal annuli. Example (B) is a unique exception; all other octagonal annuli have boundary slope the slope of an edge. We will refer to such an octagonal annulus in a layered solid torus as an *exceptional* octagonal annulus and its boundary is called an *exceptional longitude.*

(2) An almost normal, octagonal annulus with a thin edge-linking tube in the minimal $1/4$–layered-triangulation. See Figure 20. Often it will be the case that an almost normal annulus with tubes is isotopic to a normal annulus with tubes; however, this example is not. The slope of the boundary of the almost normal annulus is that of the edge 3. The tubes must come before the octagon in the layering.
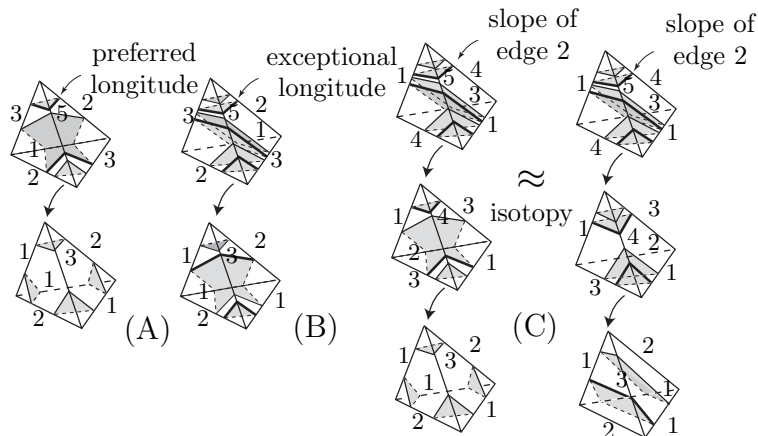
FIGURE 19. Examples of (A) an almost normal octagonal annulus with slope the preferred longitude; (B) an almost normal octagonal annulus with slope the exceptional longitude; and an almost normal octagonal annulus with slope that of the edge 2.
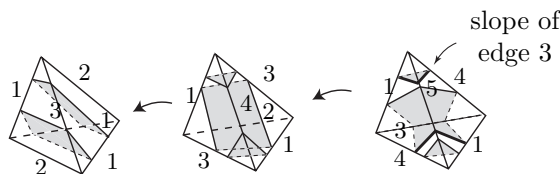
FIGURE 20. Example of an almost normal octagonal annulus with a tube and boundary slope that of the edge 3.

(3) Two (interesting) almost normal octagonal disks with trivial boundary slope: one in the three-tetrahedra layered-triangulation extending the 0/1–layered-triangulation (Figure 21(A)); and the other in the minimal two-tetrahedra triangulation of the solid torus extending the 0/1–triangulation (Figure 21(B)). The first is formed by using an almost normal octagon to band two copies of the meridional disk together. The second is a "push through" of an almost normal octagonal disk embedded in the creased 3–cell. Notice that the latter almost normal octagon may be viewed as the solution $y_2 = y_3$ in Lemma 5.2. In general, almost normal octagons can be found as solutions to the normal equations (or quad equations) having two distinct quads in the same tetrahedron. Both normal octagonal disks isotopically shrink in one direction to the vertex-linking disk. These octagonal disks lead to almost normal octagonal 2–spheres in examples of 0–efficient triangulations of S^3.

We now give the classification of almost normal surfaces in a minimal layered-triangulation of a solid torus. In many of these cases the almost normal surface is isotopic to a normal surface; we note this feature in such examples. Neither of the longitudinal, octagonal annuli are isotopic to a normal annulus; whereas, other octagonal annuli (typically) are isotopic to a normal edge-linking one. We also observe that an almost normal tube along a thin edge is (typically) isotopic
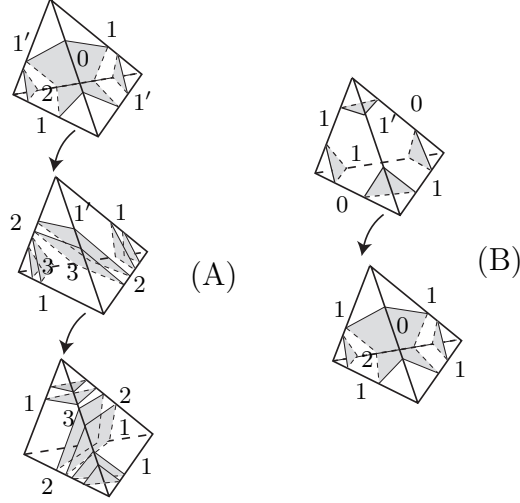
FIGURE 21. Examples of almost normal octagonal disks with trivial boundary slope.

over the edge to a thin edge-linking tube. The minimal (two-tetrahedron) triangulation and the three-tetrahedron layered-triangulation of the solid torus extending the 0/1–triangulation both admit an almost normal octagonal disk, with boundary the vertex-linking curve, which is isotopic to the vertex-linking disk. These do not fit into the general pattern; however, layered-triangulations extending 0/1 lead to "layered" triangulations of the 3–sphere and these octagonal disks, which appear, become part of an almost normal octagonal 2-sphere, which is necessary in certain triangulations of the 3–sphere. Finally, there are a number of interesting non orientable, almost normal octagonal surfaces in layered-triangulations of the solid torus; we avoid these.

5.7. **Theorem.** *A connected, embedded, orientable, almost normal surface in a minimal layered-triangulation of the solid torus is normally isotopic to one of the following surfaces.*

(1) *If octagonal:*
- *An almost normal, octagonal annulus with boundary slope the exceptional longitude,*
- *An almost normal, octagonal annulus with boundary slope a slope of an edge, possibly the preferred longitude,*
- *An almost normal, octagonal annulus with thin edge-linking tubes and having boundary slope the slope of an edge (possibly the longitude), or*
- *An almost normal, octagonal disk with boundary the vertex-linking curve, (only in the minimal layered-triangulation extending 0/1).*

(2) *If tubed:*
- *The vertex-linking disk (possibly) with thin edge-linking tubes and an almost normal tube,*
- *An edge-linking annulus (possibly) with thin-edge-linking tubes and an almost normal tube,*

- *Two copies of the meridional disk with an almost normal tube from one to the other (not in the product region between them),*
- *A vertex-linking disk (possibly) with thin edge-linking tubes and an almost normal tube between it and a distinct and disjoint normal surface, or*
- *An edge-linking annulus (possibly) with thin edge-linking tubes and an almost normal tube between it and a distinct and disjoint normal surface.*

We note, in general, both normal annuli and almost normal octagonal annuli have boundary slope the slope of an edge, with the single exception the unique almost normal octagonal annulus with boundary slope the exceptional longitude; however, all such annuli, including the exceptional longitudinal one, are parallel into an annulus in the boundary containing the vertex. In particular, we have *companion* and *complementary* annuli defined for almost normal octagonal annuli.

Proof. We defer the consideration of the three-tetrahedron layered-triangulation of the solid torus extending $0/1$ and the minimal triangulations of the solid torus extending $1/1$ and $0/1$ until after we have handled the general case.

We first consider the case where the almost normal surface is octagonal. There are three possible octagons in a tetrahedron. The octagons are distinguished by the opposite edges in the tetrahedron that they meet twice (there are three such pair of edges). See Figure 22. This distinction is particularly significant in a minimal layered-triangulation.
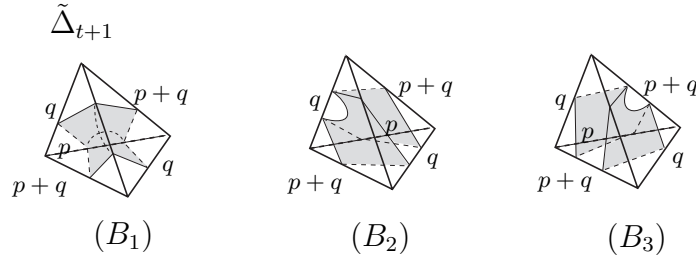
FIGURE 22. Examples of octagons in the $(t+1)$-layer.

In the one-tetrahedron solid torus, we have an octagonal annulus when the octagon meets the univalent edge (and its opposite edge) twice. This is the exceptional octagonal annulus having boundary slope a longitude. The other two possibilities give octagonal, once-punctured Klein bottles.

So, let us assume our statement is true for a minimal layered-triangulation of a solid torus having t layers and let $\mathcal{T}_{t+1} = \mathcal{T}_t \cup_e \tilde{\Delta}_{t+1}$ be a minimal layered-triangulation having $t+1$ tetrahedra. Furthermore, suppose the minimal layered-triangulation $\mathcal{T}_t$ extends the p/q–triangulation on the boundary of the solid torus.

As in the proof of Theorem 5.3, we are only considering minimal layered-triangulations; so, we have that the attaching edge e is either the edge labeled p or the edge labeled q and $\mathcal{T}_{t+1}$ extends either the $(q/p + q)$–triangulation or $(p/p + q)$–triangulation on the boundary of the solid torus.

If F_{t+1} is an almost normal octagonal surface in $\mathcal{T}_{t+1}$, then it meets $\mathcal{T}_t$ either in a normal surface F_t, each component of which satisfies the conclusion of Theorem 5.3,

or in an almost normal surface F_t each component of which satisfies the conclusion of this theorem, except that none of the components of F_t can be a surface with an almost normal tube. Thus there are two possibilities for F_{t+1}:

A. F_{t+1} is obtained from F_t, which is an almost normal octagonal surface, by adding normal triangles and quads in $\tilde{\Delta}_{t+1}$.

B. F_{t+1} is obtained from F_t, which is normal, by adding an octagon along with possibly some triangles in $\tilde{\Delta}_{t+1}$.

In case A, we may assume that we have a component of F_t that is an almost normal octagonal surface and therefore by induction is an almost normal octagonal annulus, possibly with thin edge-linking tubes, and with boundary the slope of an edge; or F_t is the exceptional, longitudinal almost normal octagonal annulus. Recall the slope of the boundary of the exceptional, longitudinal almost normal octagonal annulus behaves as a slope of an edge but is never the slope of an edge (unless it is the exceptional octagonal annulus in the two-tetrahedron 1/1–layered-triangulation, whose boundary has slope that of the univalent edge, as we shall see below). So, it follows that we can only "push through" the boundary of F_t. Hence, F_t is connected and F_{t+1} is homeomorphic to F_t and its boundary is either the slope of an edge (possibly the preferred longitude) or F_{t+1} is the exceptional octagonal annulus.

In case B, we must consider adding an octagon, along with possibly some triangles, in $\tilde{\Delta}_{t+1}$ to F_t. The situation is symmetric by attaching along the edge p or q in the boundary of $\mathcal{T}_t$; so, say we are attaching along the edge that is labeled p.

There are the three possibilities for an octagon in $\tilde{\Delta}_{t+1}$, shown in Figure 22; we designate the cases as B_1, B_2 and B_3.

In case B_1, the octagon meets the univalent edge in the boundary of $\mathcal{T}_{t+1}$ twice, and therefore its opposite edge, which is the attaching edge p, twice. We analyze the possibilities by considering how we might add triangles, along with the octagon, in $\tilde{\Delta}_{t+1}$. If there are no triangles, then the boundary of F_t would meet the edge p twice, which would be a maximal number of times; however, this is possible only if T_t is the two-tetrahedron minimal layered-triangulation of the solid torus extending the 1/1–triangulation, which it is not. If we add triangles meeting the edge p, then the edge p would have to be the univalent edge in $\mathcal{T}_t$, which it is not since we have a minimal layered-triangulation distinct from those extending 0/1 and 1/1. If we add triangles not meeting the edge labeled p, then we have a vertex-linking curve as a curve in the boundary of F_t. Since we have F_{t+1} connected, then the only possibility is adding two triangles, F_t is the vertex-linking disk with possibly some thin edge-linking tubes, making F_{t+1} an almost normal octagonal annulus with possibly some thin edge-linking tubes and with boundary slope the slope of the edge p.

In case B_2, the octagon meets the edge q, twice. If we do not add triangles in $\tilde{\Delta}_{t+1}$, then, as above, F_t would meet the edge q a maximal number of times, which is not possible in our situation. If we add triangles, then any added triangle meets the edge q; again this would force F_t to meet the edge q a maximal number of times. Hence, this situation is impossible.

In case B_3, the octagon meets the univalent edge of $\mathcal{T}_t$, the edge $p + q$, twice. In this case, if we do not add any triangles, then the single octagon component in

$\tilde{\Delta}_{t+1}$ meets the boundary of $\mathcal{T}_t$ in a slope that is possibly the boundary of a normal surface. However, the normal surface must have *connected* and nontrivial boundary and meet the univalent edge precisely two times. It follows that the only possibility for F_t would be a nonorientable surface. We have excluded such a possibility. If we add triangles at every corner, then again we would have neither F_t nor F_{t+1} connected. If we add triangles that meet the edge labeled p, then the surface F_t would have a vertex-linking curve and so could not be connected; but then the octagonal surface would only band on the vertex-linking disk of F_t and F_{t+1} would not be connected. Finally, the only possibility is to add triangles that do not meet the edge labeled p. In this case we can get a permissible slope for a surface F_t; however, F_t must have connected boundary and so must be nonorientable. Notice that in this case, adding the octagon in $\tilde{\Delta}_{t+1}$ increases the number of cross caps in F_{t+1} to two more than in F_t.

Our conclusions are that if F_{t+1} is orientable and octagonal, then in Case (A) the surface F_t is itself an almost normal octagonal surface and we "push through" F_t to get F_{t+1}; in Case (B_1) the surface F_t is normal and we add an octagon and two triangles in $\tilde{\Delta}_{t+1}$ to F_t, which makes F_{t+1} an octagonal annulus (possibly) with thin edge-linking tubes. The almost normal octagonal annulus F_{t+1} can be moved by an isotopy to a normal annulus, unless the attaching edge for $\tilde{\Delta}_{t+1}$ is the preferred longitude or there is a thin edge-linking tube in F_t that meets the attaching edge.

Now, we consider the situation where the almost normal surface is a tubed surface and is obtained either from a connected normal surface by adding a tube along an edge (Case (C_1)) or is obtained from two disjoint connected normal surfaces (not necessarily distinct) by adding a tube from one surface to the other (Case (C_2)). Since we have classified all normal surfaces in a minimal layered-triangulation, we only need consider the possibilities for adding an almost normal tube to these normal surfaces.

C_1. Adding an almost normal tube to a connected normal surface.

We make some observations. If the normal surface is the vertex-linking disk and we add an almost normal tube along any edge except the unique "thick edge" or an edge in the boundary, then the surface is isotopic to a normal surface with a thin edge-linking tube; the edge-linking tube is about the edge along which we added the almost normal tube. If the normal surface is a vertex-linking disk with thin edge-linking tubes, we can not add an almost normal tube along an edge that has a thin edge-linking tube. If we add an almost normal tube along an edge other than the unique "thick edge" or an edge in the boundary, then we have the possibility of an isotopy of the almost normal tube over the edge to get a normal surface with a thin edge-linking tube about that edge. However, there is an obstruction; if there is a thin edge-linking tube in the surface having a quadrilateral that meets the edge adjacent to the almost normal tube. If this is the case, then we can not just shove the almost normal tube over the edge to get a thin edge-linking tube without destroying the existing thin edge-linking tube. If the normal surface is the meridional disk and we add an almost normal tube, we get a non orientable surface. If the normal surface is an edge-linking annulus, having boundary slope the slope of an edge. We can not add a tube along the edge the annulus is linking. However, we may add the tube along any edge in the initial segment before we come to the edge

the annulus is linking. We can move such an almost normal tube over the edge by an isotopy to get a thin edge-linking tube or the almost normal tube is adjacent to the unique "thick edge" or an edge that meets a quad in the edge-linking annulus. So, the possibilities can be summarized as an edge-linking annulus having boundary slope the slope of an edge and an almost normal tube along an edge. We also can have the almost normal tube added after the edge that the annulus is linking. In this case, the tube can be moved by an isotopy to be along an edge in the boundary of the solid torus (the argument is inductive and is the same as that below). If the normal surface is an edge-linking annulus with tubes, having boundary slope the slope of an edge. As in the preceding, we can not add a tube along the edge the annulus is linking or along any of the edges having thin edge-linking tubes. The almost normal tube may be added before the edge of the edge-linking annulus. If the edge along which it is added does not meet a quad of a thin edge-linking tube or a quad in the edge-linking annulus, then we can move the almost normal tube over the edge by an isotopy to get a thin edge-linking tube. Again, it is possible the almost normal tube is added after the edge that the annulus links. We then have an edge-linking annulus with thin edge-linking tubes and having boundary slope the slope of an edge and an almost normal tube along an edge in the boundary of the solid torus.

$\mathbf{C_2}$. Adding an almost normal tube between two distinct normal surfaces.

If two distinct normal surfaces are connected via an almost normal tube, then up to isotopy, we show that we may assume the almost normal tube is along an edge in the boundary of the solid torus. To do this, we use induction.

The induction begins easily enough for the one-tetrahedron solid torus - all edges are in the boundary.

So, we assume the statement is true for a layered-triangulation of length t and consider the minimal layered-triangulation, $\mathcal{T}_{t+1}$, having $t+1$ tetrahedra. If we have two connected normal surfaces, F_{t+1} and F'_{t+1} in $\mathcal{T}_{t+1}$, then each must meet $\mathcal{T}_t$ in a connected normal surface, say F_t and F'_t, respectively. If an almost normal tube is added to connect F_t and F'_t, then, by the induction hypothesis, it is along the boundary. This tube will be along the boundary of $\mathcal{T}_{t+1}$ unless it is along the edge of the layering. However, a tube along this edge must be either between parallel normal triangles or parallel normal quadrilaterals or between a triangle and a quadrilateral. In any case, it is isotopic to an almost normal tube along an edge in the boundary of the solid torus with the triangulation $\mathcal{T}_{t+1}$. Thus we see that by an isotopy we can have the almost normal tube along an edge in the boundary.

Finally, we consider the almost normal surfaces in the special cases for the minimal extensions of the triangulations $1/1$ and $0/1$.

$\mathbf{D}$. Almost normal surfaces in the two-tetrahedra $1/1$–layered-triangulation.

In the case of the two-tetrahedron extension of $1/1$, we can use quadrilateral equations to see two octagonal solutions: one with an octagon in the one-tetrahedron solid torus, which meets the univalent edge twice, and another with an octagon in the second tetrahedron in the layering, also having two points on the univalent edge. Thus we have just the two possibilities shown in the *first two layers* in Figure 23(D_1) and (D_2): (D_1) is an octagonal annulus with boundary slope the exceptional longitude, which happens in this case to be the slope of the thin edge

labeled $1'$, and (D_2) is an octagonal annulus with boundary slope the slope of the edge labeled 3. The latter can be moved via an isotopy to an edge-linking annulus about the edge labeled 3.

The almost normal tubed surfaces in the two-tetrahedron extension of $1/1$ can be easily listed from the normal surfaces. There are no surprises.

E. Almost normal surfaces in the small layered-triangulations extending the $0/1$–triangulation.

We consider both the three-tetrahedron layered-triangulation and the minimal two-tetrahedron triangulation of the solid torus extending $0/1$.

For the three-tetrahedron triangulation, we have the layering of a tetrahedron onto the two-tetrahedron layered-triangulation extending $1/1$; the almost normal surfaces of which were analyzed in the preceding paragraphs.

First, there are the almost normal octagonal surfaces in minimal $1/1$–layered-triangulation that can be pushed through. See Figure $23D_1$ and D_2. These give an almost normal octagonal annulus with boundary slope the exceptional longitude, labeled $1'$, and an almost normal annulus with boundary slope the slope of the edge 3. So, we need only consider the possibilities if we add an octagonal disk in the third layer. In this situation, we have the same possibilities as in the induction argument above. The first we consider is when the octagonal disk in the third layer meets the univalent edge, labeled 0, and the attaching edge, labeled 2, twice. Just as above, in this situation, we can attach the octagonal disk to the vertex-linking curve. The possibilities are that we can attach the octagonal disk to either the vertex-linking disk, getting an almost normal octagonal annulus with boundary slope the slope of the edge 2 (see Figure $23E_1$), or to the vertex-linking disk with a thin edge-linking tube about the edge 3, getting an almost normal octagonal annulus with a thin edge-linking tube and, again, having boundary slope the slope of the edge 2 (see Figure $23E_2$).
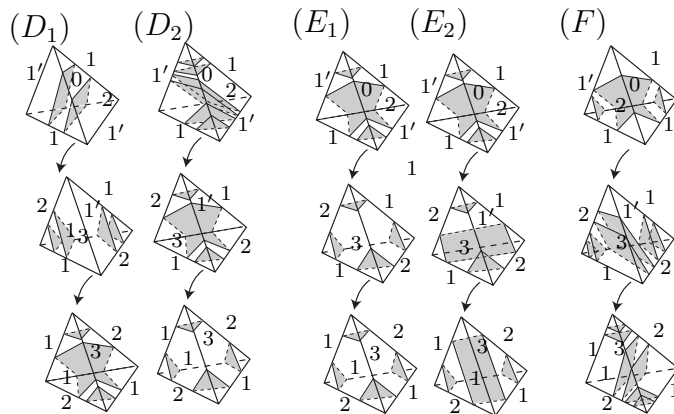

FIGURE 23. Almost normal octagonal surfaces in the three-tetrahedron, layered extension of the $0/1$–triangulation. In D_1 and D_2 are almost normal octagonal surfaces in the two-tetrahedron extension of the $1/1$–triangulation, which can be seen in the initial segments.

Continuing, we have a possibility here that we did not have in the general induction argument above. The univalent edge in the $1/1$–layered-triangulation is *not* the edge meeting the meridional disk a maximal number of times; thus we get a *new* almost normal surface by attaching the octagonal disk and two triangles meeting the edge 2 to two copies of the meridional disk in the $1/1$–layered-triangulation. See Figure 23F. This gives us an almost normal, trivial, non vertex-linking disk having boundary the vertex-linking curve. It is a banding (by the octagon) of two copies of the meridional disk. We can also band just one copy of the meridional disk by not adding any triangles in the third layer; however, this gives an almost normal Möbius band. The other two possibilities for adding an octagonal disk in the third layer can be analyzed as above in the induction step. There are no orientable surfaces possible in these two cases.

For the minimal two-tetrahedron triangulation extending $0/1$, we go back to the quadrilateral solutions in the proof of Theorem 5.2 and Figure 13. There is only one quadrilateral equation (from the interior edge 2). It is $y_2 = y_3$. Otherwise, we have that y_4, y_5 and y_6 can take on independent values.

For $y_2 = y_3 = 1$ and $y_i = 0$ otherwise, we have a solution with triangular coordinates, $x_2 = x_4 = x_6 = x_5 = x_7 = 1$, which is an almost normal, trivial, octagonal disk with boundary the vertex-linking curve. For $y_2 = y_3 = 1, y_4 = 1$ and $y_1 = y_5 = y_6 = 0$, we have a solution with triangular coordinates, $x_2 = x_4 = x_6 = x_8 = 1$, which is a longitudinal, almost normal octagonal annulus having boundary slope that of the univalent edge labeled $1'$ in the boundary, a longitude. Note that here this is not an exceptional longitude and is the slope of an edge. For $y_2 = y_3 = 1, y_5 = 1$ and $y_1 = y_4 = y_6 = 0$, we have a non orientable solution (an almost normal Möbius band with boundary slope that of the edge 2). For $y_2 = y_3 = 1, y_6 = 1$ and $y_1 = y_4 = y_5 = 0$, we have a solution with triangular coordinates $x_2 = x_4 = 2$ and $x_5 = x_7 = x_6 = x_8$, which gives us disjoint copies of the meridional disk and the almost normal, trivial, octagonal disk with boundary the vertex-linking curve. Notice that two such surfaces also are in the three-tetrahedron minimal layered-triangulation of the solid torus extending $0/1$ and are similarly disjoint there. This takes care of all possibilities for an octagonal disk in the first tetrahedron.

For $y_4 = y_5 = 1, y_i = 0$, otherwise, and $y_4 = y_6 = 1, y_i = 0$, otherwise, we have nonorientable solutions. For $y_5 = y_6 = 1, y_i = 0$, otherwise, we have a solution with triangular coordinates $x_1 = x_3 = x_2 = x_4 = x_5 = x_7$, which is an almost normal, exceptional, longitudinal octagonal annulus.

Thus in the two-tetrahedron, minimal triangulation of the solid torus extending $0/1$ we have an almost normal trivial octagonal disk, an almost normal octagonal annulus with boundary slope the exceptional longitude, and an almost normal octagonal annulus with boundary slope that of the new longitude that is an edge in the boundary (not the preferred longitude, also the slope of an edge in the boundary). The almost normal tubed surfaces can be listed off by using the possibilities for normal surfaces from Theorem 5.2.   □

5.4. **Nearly-minimal, layered-triangulations of a solid torus.** As is evident, one of the key features of layered-triangulations is the ability to classify the normal and almost normal surfaces in such a triangulation. If we have two thin edges with the same slope or have an edge with the slope of the meridian, we get more complex normal surfaces than in the case of a minimal layered-triangulation. It follows

that layered-triangulations of the solid torus, in which the meridional slope is not a slope of an edge and no two *thin* edges have the same slope, have the same structure for normal surfaces as do minimal layered-triangulations. We say a layered-triangulation of a solid torus is *nearly-minimal* if the meridional slope is not a slope of an edge and no two thin edges have the same slope. For layered-triangulations of the solid torus a minimal layered triangulation is nearly-minimal, except for the two-tetrahedra layered-triangulation extending the $0/1$–triangulation, where the edge labeled 0 has slope that of the meridian.

Examples.

(1) The opening along $1, 1, 2$ of a minimal layered-triangulation is nearly-minimal.
(2) A nearly-minimal but non minimal layered-triangulation extending $1/6$. Start with the minimal layered-triangulation extending $1/6$. At the point $1/3$ we add the path to $3/4$ and back to $1/3$. This increases the layered-triangulation by two tetrahedra, it has a layering onto the univalent edge of the $3/4$–layered-triangulation, the edge labeled 7, and has two edges having the same slope. One is the thick edge and the other is the new edge meeting the meridional disk just once. This is a nearly-minimal layered-triangulation. Notice that we get a different one for each point $1/n$ between, and including $1/6$ and $1/2$; that is, in this case five different nearly-minimal, non minimal layered-triangulations. However, one can not repeat these at more than one point or extend beyond the points $n/n + 1$. Since the edge 1 is the thick edge, there are not two thin edges having the same slope.

6. Layered-triangulations of lens spaces

In this section we define layered-triangulations of a lens spaces, as well as minimal and nearly-minimal layered-triangulations of lens spaces. We show that every lens space has a layered-triangulation (in fact, infinitely many); each lens space, except $\mathbb{R}P^3$ and $S^2 \times S^1$, has a 0–efficient layered-triangulation with an arbitrarily large number of tetrahedra; and each lens space, having a 0–efficient triangulation, except S^3, has only a finite number of 1–efficient layered-triangulations. In particular, the (unique) minimal layered-triangulation of these lens spaces is 1–efficient.

Definition. Having layered triangulations of the solid torus, then the obvious method for getting similar triangulations of a lens space is to exploit the genus one Heegaard decompositions of lens spaces. In doing this we include the degenerate solid tori, the creased 3–cell and the one-triangle Möbius band, as factors in a Heegaard decomposition of a lens space. For this we need to specify what we mean by a simplicial attachment of two layered-triangulations of the solid torus along their boundaries when one of the factors is a degenerate solid torus. If both factors are non degenerate, the attaching map is a simplicial isomorphism. If one factor is non degenerate and the other is a creased 3–cell, then the simplicial map is determined by the edge e on the boundary of the non degenerate layered-triangulation of the solid torus identified with the univalent edge on the boundary of the creased 3–cell; we call this a *pinching* about the edge e. If one factor is non degenerate and the other is the one-triangle Möbius band, then the simplicial map is determined by the edge e on the boundary of the non degenerate layered-triangulation of the solid torus identified with the boundary edge of the Möbius band; we call this a

folding along the edge e. Finally, we have the possibilities where both solid tori are degenerate. If both are creased 3–cells, the double does not give a 3–manifold and the one remaining simplicial attachment gives one of the 2 two-tetrahedra, one-vertex triangulations of S^3. If one of the factors is a creased 3–cell and the other a one-triangle Möbius band, then we do not get a manifold if we take the univalent edge to the boundary of the Möbius band, leaving only one possibility which gives us the one-tetrahedron, one-vertex triangulation of the 3–sphere. In Figure 24, we give the options, just described, for a simplicial attachment of two layered-triangulations of the solid torus. Typically, a layered-triangulation of a lens space can be viewed in a number of different ways as a union of two layered solid tori.
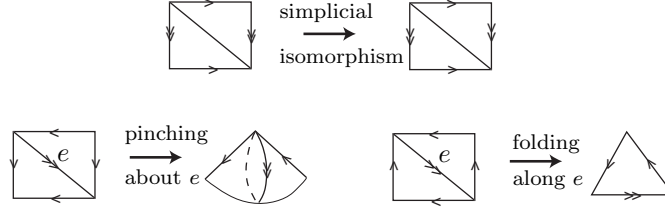
FIGURE 24. Simplicial attaching maps from the boundary of a non degenerate layered-triangulation of the solid torus to the boundary of a layered-triangulation of the solid torus, including the degenerate creased 3–cell and the Möbius band.

Examples.
(1) Suppose we have a one-tetrahedron solid torus. There are three ways to identify the boundary torus to the one-triangle Möbius band. For any one of these, we get a lens space with a one-tetrahedron triangulation. The results are one-tetrahedron triangulations for the lens spaces $L(1,1) = S^3$, $L(4,1)$, and $L(5,2)$. These are layered (and minimal) triangulations. The example that gives S^3 can also be viewed as an identification of the two faces of the creased 3–cell. All of these examples are 1–efficient triangulations.
(2) Up to isomorphism, there is only one triangulation of a 3–manifold obtained by attaching two creased 3–cells along their boundaries. We get a two-tetrahedron, one-vertex triangulation of S^3, which is 1–efficient.
(3) There are 4 two-tetrahedra solid tori: the two-tetrahedra triangulations of the solid torus extending the $0/1, 1/1, 1/3$ and $2/3$ triangulations on the boundary of the solid torus.

   First, we consider the identifications of the two faces in the boundary of the two-tetrahedron extension of $0/1$. We have: $\mathbb{R}P^3 = L(2,1) \equiv \{0,1,1'\} \leftrightarrow \{2,1,1\}$ (we fold over the edge labeled 0); $S^3 = L(1,0) \equiv \{0,1,1'\} \leftrightarrow \{1,2,1\}$; and $S^3 = L(1,1) \equiv \{0,1,1'\} \leftrightarrow \{1,1,2\}$, where we use the label $1'$ to denote the thin (univalent) edge rather than the thick edge. By reversing our view in the first example giving S^3, we see that it is the same as attaching a creased 3–cell to the boundary of the one-tetrahedron solid torus with the edge labeled 0 in the creased 3–cell attached to the thick edge in the one-tetrahedron solid torus. The last example is the same

as two creased 3–cells identified along their boundaries. These last two examples are non isomorphic two-tetrahedron triangulations of S^3, which has 5 non isomorphic two-tetrahedron triangulations. These two are 1–efficient.

For the two-tetrahedron triangulation of the solid torus extending $1/1$, we have: $L(3,1) \equiv \{1',1,2\} \leftrightarrow \{2,1,1\}$; $L(3,1) \equiv \{1',1,2\} \leftrightarrow \{1,2,1\}$; and $S^2 \times S^1 = L(0,1) \equiv \{1',1,2\} \leftrightarrow \{1,1,2\}$, where, again, we use $1'$ to denote the thin (univalent) edge. These two triangulations of $L(3,1)$ are not isomorphic and give two of the three non isomorphic triangulations of $L(3,1)$. The first one of these is not 0–efficient; we are folding over the univalent edge and the triangulation is the same as attaching the creased 3–cell to the one-tetrahedron solid torus with the edge labeled 0 in the creased 3–cell identified to the edge labeled 3 in the one-tetrahedron solid torus (we have an edge bounding a disk). The second is 1–efficient.

For the two-tetrahedron triangulation of the solid torus extending $1/3$, we have: $L(7,3) \equiv \{1,3,4\} \leftrightarrow \{2,1,1\}$; $L(5,1) \equiv \{1,3,4\} \leftrightarrow \{1,2,1\}$; and $\mathbb{R}P^3 = L(2,1) \equiv \{1,3,4\} \leftrightarrow \{1,1,2\}$. This is one of the 2 two-tetrahedron triangulations of $\mathbb{R}P^3$ and is isomorphic to the one above.

For the two-tetrahedron triangulation of the solid torus extending $2/3$, we have: $L(8,3) \equiv \{2,3,5\} \leftrightarrow \{2,1,1\}$; $L(7,2) \equiv \{2,3,5\} \leftrightarrow \{1,2,1\}$; and $S^3 = L(1,2) \equiv \{2,3,5\} \leftrightarrow \{1,1,2\}$. This triangulation of $L(7,2) = L(7,3)$ is equivalent to that above and the last triangulation of S^3 is isomorphic to the one above with $S^3 = L(1,0) \equiv \{0,1,1\} \leftrightarrow \{1,2,1\}$.

(4) If we have a layered solid torus, then we can attach a creased 3–cell to its boundary via a simplicial map. This is the same as adding a 2– and 3–handle to the solid torus, where the 2–handle is attached along the slope of an edge in the boundary of the layered solid torus. For example, if we have the one-tetrahedron solid torus, an extension of the $1/2$–triangulation on the boundary of the solid torus, then attaching the creased 3–cell gives: $L(1,0) = S^3$ for the edge labeled 0 in the boundary of the creased 3–cell attached to the edge labeled 1; $L(2,1) = \mathbb{R}P^3$ for the edge 0 attached to the edge 2; and $L(3,1)$ for the edge 0 attached to the edge 3. These give examples of two-tetrahedra, layered-triangulations for S^3, $\mathbb{R}P^3$ and $L(3,1)$. These two-tetrahedra layered-triangulations for $\mathbb{R}P^3$ and $L(3,1)$ are minimal but neither is 0–efficient. The one for S^3 is 1–efficient (this distinction will be discussed below). Recall that except for these minimal triangulations of $\mathbb{R}P^3$ and $L(3,1)$, a minimal triangulation of an irreducible 3–manifold must be 0–efficient.

(5) A more typical example of a layered-triangulation of a lens space is given as $\{9,7,2\} \leftrightarrow \{5,3,8\} = L(62,27)$, meaning any layered-triangulation of a solid torus extending $2/7$ identified with any layered-triangulation of a solid torus extending $3/5$ so that $9 \leftrightarrow 5, 7 \leftrightarrow 3$ and $2 \leftrightarrow 8$ gives the lens space $L(62,27)$.

Existence. The next lemma follows from the fact that we can extend any one-vertex triangulation on the boundary of a solid torus to a layered-triangulation of the solid torus, along with the fact that lens spaces have a genus one Heegaard splitting.

6.1. Theorem. *Every lens space admits a layered-triangulation.*

Proof. Start with a genus one Heegaard splitting; fix a one-vertex triangulation of the splitting torus. By Theorem 4.2, we can extend the one-vertex triangulation of the splitting torus to a layered-triangulation of the solid torus on each side.   □

Combinatorics of layered-triangulations of lens spaces. As mentioned above, a fixed layered-triangulation of a lens space, typically, can be viewed as a simplicial attachment along the boundaries of two layered-triangulations of the solid torus in a number of ways. On the other hand, a layered-triangulation of a lens space can be viewed as being formed from a layered-triangulation of a single solid torus along with one of the three possible ways of simplicially attaching the boundary torus to the one-triangle Möbius band; i.e., folding along one of the three edges in the boundary. This gives us a very nice way to address the combinatorics of what lens spaces we get from a given layered-triangulation of the solid torus and conversely, given the relatively prime integers X, Y, we provide an algorithm to construct a layered-triangulation of the lens space $L(X, Y)$.

The combinatorics we develop work equally well for determining the lens space obtained by folding along one of the three edges in the boundary of any one-vertex triangulation of a solid torus. So, to this end, suppose $\mathcal{T}$ is a one-vertex-triangulation of the solid torus extending the p/q–triangulation on the boundary torus. There are three edges in the boundary and we can choose a labeling and an orientation of the edges e_1, e_2, e_3 so that the meridional disk meets these edges in p, q and $p+q$ points, respectively. The one-triangle Möbius band can be considered as a degenerate solid torus having "meridian slope" the nontrivial spanning arc and "longitude" the interior edge in the one-triangle triangulation. It follows that the meridian of the Möbius band lifts to a normal curve in the boundary of the solid torus meeting one edge twice and each of the other two edges in the boundary once each. We can think of the edge being met twice as the edge in the boundary of the solid torus along which we are folding, see Figure 24. It is the edge identified with the boundary edge in the one-triangle Möbius band. The other two edges in the boundary of the solid torus are identified to the single interior edge in the Möbius band, which functions as a longitude for the Möbius band.

We have that e_1 and e_2 form a basis for the homology of the boundary torus and $e_3 = e_1 - e_2$. Thus the meridional slope, μ', for the one-vertex triangulation of the solid torus can be expressed in terms of the basis $\{e_1, e_2\}$ as $\mu' = qe_1 + pe_2$. There are three possibilities for identifying the two faces in the boundary torus, depending on which edge, e_1, e_2, or e_3, is the edge along which we are folding.

Type I. Folding along e_1. Then the meridional slope for the Möbius band lifts to the curve $\mu = e_1 - 2e_2$; $\lambda = e_2$ and $\hat{\lambda} = e_3 = e_1 - e_2$ are lifts of the longitude in the Möbius band. It follows that $d(\mu, \mu')$, the distance (or number of intersections) between μ and μ', is $p + 2q$; whereas, $d(\lambda, \mu') = q$ and $d(\hat{\lambda}, \mu') = p + q$. Thus using the longitude λ, we conclude that we have the lens space $L(p+2q, q)$; and using the longitude $\hat{\lambda}$, we conclude that we have the lens space $L(p + 2q, p + q)$. Of course, these are the same lens space; $p + q$ is the additive inverse of q, mod $(p + 2q)$.

Type II. Folding along e_2. This is exactly the same as Type I with a change in the role of p and q. For this case, the meridional slope for the Möbius band lifts to the curve $\mu = 2e_1 - e_2$; $\lambda = -e_1$ and $\hat{\lambda} = e_3 = e_1 - e_2$ are lifts of the longitude

in the Möbius band. It follows that $d(\mu, \mu') = 2p + q$; whereas, $d(\lambda, \mu') = p$ and $d(\hat{\lambda}, \mu') = p + q$. Thus using the longitude λ, we conclude that we have the lens space $L(2p + q, p)$; and using the longitude $\hat{\lambda}$, we conclude that we have the lens space $L(2p + q, p + q)$. As above, these are the same lens space.

Type III. Folding along e_3. Now, the meridional slope for the Möbius band lifts to the curve $\mu = e_1 + e_2$; and $\lambda = e_1$ and $\hat{\lambda} = -e_2$ are the lifts of the longitude in the Möbius band. We have in this case that $d(\mu, \mu') = |q - p|$; whereas, $d(\lambda, \mu') = p$ and $d(\hat{\lambda}, \mu') = q$. Thus using the longitude λ, we conclude that we have the lens space $L(|q - p|, p)$; and using the longitude $\hat{\lambda}$, we conclude that we have the lens space $L(|q - p|, q)$. These are the same lens space as p is equivalent to q, $\mod (|q - p|)$.

Conversely, suppose we wish to construct a layered-triangulation for the lens space $L(X, Y)$.

If $0 \leq Y \leq X/2$, then choose a layered-triangulation of the solid torus extending the one-vertex triangulation on the boundary whose edges, e_1, e_2, e_3, meet the meridional slope in $X - 2Y, Y, X - Y$ points, respectively, and fold over the edge e_1 (with intersection number $X - 2Y$).

If $X/2 \leq Y \leq X$, then choose a layered-triangulation of the solid torus extending the one-vertex triangulation on the boundary whose edges, e_1, e_2, e_3, meet the meridional slope in $X - Y, 2Y - X, Y$ points, respectively, and fold over the edge e_2 (with intersection number $2Y - X$).

If $0 < X < Y$, then choose a layered-triangulation of the solid torus extending the one-vertex triangulation on the boundary whose edges, e_1, e_2, e_3, meet the meridional slope in $Y, Y - X, 2Y - Y$ points, respectively, and fold over the edge e_3 (with intersection number $2Y - X$). More generally, for any $X > 0, Y \geq 0$, choose a layered-triangulation of the solid torus extending the one-vertex triangulation on the boundary whose edges, e_1, e_2, e_3, meet the meridional slope in $Y, X + Y, X + 2Y$ points, respectively, and fold over the edge e_3 (with intersection number $X + 2Y$). Notice that in each of these last two cases, we are folding over the edge with highest intersection number with the meridional slope; hence, these triangulations admit a reduction in the number of tetrahedra.

Note that if we have a minimal layered-triangulation of a solid torus, then the univalent edge is the furthest from the meridional slope (meets the meridional slope the greatest number of times) and if we were to fold over the univalent edge, then, typically, we could reduce the number of tetrahedra in the triangulation. Hence, the preferred situation is that we have a p/q–layered-triangulation and we fold over one of the edges labeled p or q. The following is an easy mnemonic. If we have a p/q-triangulation on the boundary of the solid torus and we fold over the edge labeled p; i.e., the lift of the meridian for the one-triangle Möbius band meets the edge labeled p two times, then set down the array

$$
\begin{array}{cc}
2 & 1 \\
p & q
\end{array}
$$

cross multiply and add, getting $2q + p$. This is X in the resulting lens space $L(X, Y)$ and $Y = q$ is the value in the column with 1. If we use as the longitude the edge labeled $p + q$ rather than the edge labeled q, then we have the array

$$
\begin{array}{cc}
2 & 1 \\
p & p+q
\end{array}
$$

and in this case we take the difference $2p + 2q - p$, after cross multiplying, and have $X = 2q + p$; but now we have $Y = p + q$, which is the value in the column with 1.

Example. Suppose we have a 4/5–triangulation on the boundary of the solid torus. If we obtain a lens space by folding over the edge labeled 5, we have the arrays

$$\begin{array}{cc} 2 & 1 \\ 5 & 4 \end{array} \quad \text{or} \quad \begin{array}{cc} 2 & 1 \\ 5 & 9 \end{array}$$

depending on a choice of longitude. In the first array, by cross multiplying and adding, we see that we get the lens space $L(13, 4)$; whereas, in the second array, we cross multiply and take the difference and have the lens space $L(13, 9)$.

If we fold over the edge labeled 4 and use as longitude the edge labeled 5, we have an array

$$\begin{array}{cc} 2 & 1 \\ 4 & 5 \end{array}$$

and get the lens space $L(14, 5)$.

If we have a layered-triangulation of a lens space obtained by folding along an edge in the boundary of a p/q–layered-triangulation, then there is a symmetry; namely, once we have folded over, then we have a layered-triangulation of a solid torus in the reverse direction and have obtained the layered-triangulation of the lens space by folding along an edge in its boundary. We describe the combinatorial relationship of looking at a layered-triangulation of a lens space by considering it as obtained by folding at either end. The layered-triangulation of the solid torus in the reverse direction is a p'/q'–layered-triangulation for some p', q'. We call these two views of the same layered-triangulation of a lens space as *reversed layerings*.

Since a layered-triangulation extending p/q on the boundary can be viewed as a path in the L–graph, it has inside it the path of the minimal layered-triangulation extending p/q. This provides us with an algorithm to determine p'/q' from p/q. We show this in a couple of examples.

First we establish some notation. If we have a layered-triangulation extending a one-vertex triangulation on the boundary where the intersection numbers of the meridional slope with the three edges is the triple $\{x, y, z\}$, we use $\{x, \underline{y}, z\}$ to denote that the edge in the triangulation of the boundary meeting the meridional slope y times is the univalent edge for our layering. If we have obtained a lens space by folding over the edge labeled x, then we can think of the simplicial identification $\{x, y, z\} \leftrightarrow \{2, 1, 1\}$ of the boundary of the solid torus with the one-triangle Möbius band, where we have the edge labeled x identified with the boundary of the Möbius band and the edges y and z identified with the interior edge. Now, using the above notation with y the univalent edge, we have $\{x, \underline{y}, z\} \leftrightarrow \{2, \underline{1}, 1\}$ or $\{x, y', z\} \leftrightarrow \{2, 3, 1\}$, where $y' = x + z$ or $y' = |x - z|$.

Examples:

(a) $\{\underline{1}, 2, 1\} \leftrightarrow \{\underline{14}, 3, 11\} = L(25, 11)$
$\{3, 2, \underline{1}\} \leftrightarrow \{8, 3, \underline{11}\}$
$\{\underline{3}, 2, 5\} \leftrightarrow \{\underline{8}, 3, 5\}$
$\{7, 2, \underline{5}\} \leftrightarrow \{2, 3, \underline{5}\}$
$\{7, 2, 9\} \leftrightarrow \{2, \underline{3}, 1\}$
$\{7, 16, 9\} \leftrightarrow \{2, 1, 1\} = L(25, 9)$

(b) $\{\underline{1}, 2, 1\} \leftrightarrow \{\underline{5}, 3, 8\} = L(13, 8)$
$\{\underline{3}, 2, 1\} \leftrightarrow \{\underline{11}, 3, 8\}$
$\{1, 2, \underline{1}\} \leftrightarrow \{5, 3, \underline{8}\}$
$\{\underline{1}, 2, 3\} \leftrightarrow \{\underline{5}, 3, 2\}$
$\{5, 2, 3\} \leftrightarrow \{1, \underline{3}, 2\}$
$\{5, 8, 3\} \leftrightarrow \{1, 1, 2\} = L(13, 5)$

More generally, suppose we have two solid tori and layered-triangulations of each. If we identify the boundaries of the two solid tori via a simplicial isomorphism, we have a layered-triangulation of a lens space. The above method provides us with an algorithm to determine the lens space obtained by knowing the original layered-triangulations. We exhibit this in the following example.

(c) Suppose we have $\{9, 7, 2\} \leftrightarrow \{3, 7, 4\}$, where the notation means that we have identified the boundaries of two solid tori, one has a layered-triangulation extending $2/7$, the other has a layered-triangulation extending $3/4$ and the identification takes the edges $9 \leftrightarrow 3, 7 \leftrightarrow 7$ and $2 \leftrightarrow 4$.

$$
\begin{array}{lcl}
\ldots\ldots\ldots\ldots\ldots & & \{31, 11, 20\} \leftrightarrow \{1, 1, 2\} = L(42, 11) \\
\ldots\ldots\ldots\ldots\ldots & & \{9, 11, 20\} \leftrightarrow \{3, 1, 2\} \\
\ldots\ldots\ldots\ldots\ldots & & \{9, 11, \underline{2}\} \leftrightarrow \{3, 1, \underline{4}\} \\
\{\underline{9}, 7, 2\} \leftrightarrow \{\underline{3}, 7, 4\} & \equiv & \{9, \underline{7}, 2\} \leftrightarrow \{3, \underline{7}, 4\} \\
\{\underline{5}, 3, 2\} \leftrightarrow \{\underline{11}, 15, 4\} & & \ldots\ldots\ldots\ldots\ldots \\
\{1, 3, 2\} \leftrightarrow \{19, \underline{15}, 4\} & & \ldots\ldots\ldots\ldots\ldots \\
\{1, 1, 2\} \leftrightarrow \{19, 23, 4\} = L(42, 19) & & \ldots\ldots\ldots\ldots\ldots
\end{array}
$$

Of course, we have that our combinatorial description of the lens space obtained by attaching two layered solid tori gives in one reduction the lens space description as L(X,Y) and by *reversing* the description we get $L(X, Y')$, where $Y \cdot Y' \equiv \pm 1 mod(X)$.

We summarize our observations in the following two propositions.

6.2. Proposition. *Any lens space can be formed from identifications of Type I (or Type II).*

6.3. Proposition. *If the lens space $L(X, Y) = \{x, y, z\} \leftrightarrow \{1, 1, 2\}$, then the reverse identification $\{1, 1, 2\} \leftrightarrow \{x', y', z'\} = L(X, Y')$ and $Y \cdot Y' \equiv \pm 1 mod(X)$.*

A layered-triangulation of a lens space is said to be a *minimal* layered-triangulation if it has the minimal number of tetrahedra among all layered-triangulations of the lens space. Clearly, if we have a minimal layered-triangulation of a lens space, then it is obtained by folding along an edge in the boundary of a minimal layered-triangulation of a solid torus and, in general, the edge along which the boundary is folded is not the univalent edge (the only exception is the one-tetrahedron layered-triangulation of S^3 obtained by folding the one-tetrahedron solid torus along the univalent edge. Later, by using the uniqueness of Heegaard splittings of lens spaces, we will see that if a layered-triangulation of a lens space is obtained by folding the boundary of a minimal p/q–layered-triangulation of the solid torus along an edge other than the univalent edge, then the triangulation is a minimal layered-triangulation of the lens space. At this point, we could employ the classification of lens spaces and the last proposition of the previous section for this conclusion; however, uniqueness of Heegaard splittings gives us the standard classification of lens spaces. It seems to us that these results should follow from the combinatorics above but this is not clear. Finally, with the exception of $L(3, 1)$ there is a unique minimal layered-triangulation of a lens space.

6.4. Proposition. *Every lens space has a minimal layered-triangulation.*

There is another curious question which we have not been able to answer. Namely, in general, is a minimal triangulation of a lens space a layered-triangulation? There are three exceptions, S^3, $\mathbb{R}P^3$ and $L(3,1)$; each has a minimal triangulation that is not layered. S^3 has a one-tetrahedron layered-triangulation and also has a one-tetrahedron, two-vertex triangulation. Both $\mathbb{R}P^3$ and $L(3,1)$ have minimal, two-tetrahedra triangulations that are not layered. They also have two-tetrahedra layered-triangulations. S^3 has 5 distinct two-tetrahedra triangulations, 3 of which are not layered. Triangulations of these three manifolds also provide the exceptions in the reduction of an arbitrary triangulation to a 0–efficient one [14].

Conjecture. *With the exceptions of S^3, $\mathbb{R}P^3$ and $L(3,1)$, a minimal triangulation of a lens space is a minimal layered-triangulation.*

This has also been conjectured by S. Matveev [22].

As a final observation, every layered triangulation of a lens space L(p,q) is invariant under the canonical involution. The quotient space of this involution is the 3-sphere and the fixed set of the involution projects to a 2-bridge knot or link. Therefore we can view the classification of minimal layered-triangulations as a version of the classification of 2-bridge knots and links.

7. NORMAL AND ALMOST NORMAL SURFACES IN LAYERED-TRIANGULATIONS OF LENS SPACES

We first classify the orientable normal and almost normal surfaces in a minimal layered-triangulation of a lens space; then we classify the nonorientable normal surfaces. The last uses our layered-triangulations for a new proof of earlier work by G. Bredon and J. Wood, [3]. From now on we shall refer to the two-tetrahedra layered-triangulation of $L(3,1)$ that is obtained by folding over along the univalent edge in the two-tetrahedra $1/1$–layered-triangulation as the *bad* minimal layered-triangulation of $L(3,1)$. Recall that we also get $L(3,1)$ if we fold over along the thick edge; however, this latter triangulation of $L(3,1)$ does not admit the unnecessary embeddings of normal and almost normal surfaces as does the former.

Theorem. *The embedded, orientable, normal and almost normal surfaces in a minimal layered-triangulation of a lens space are:*

If normal:

- *the vertex-linking 2–sphere, possibly, with thin edge-linking tubes, except*
- *a non vertex-linking normal 2–sphere for the two-tetrahedra layered-triangulations of $\mathbb{R}P^3$, $S^2 \times S^1$, and the bad minimal layered-triangulation of $L(3,1)$.*

If almost normal and not normal:

- *an almost normal tubed surface obtained either by adding an almost normal tube to the vertex-linking 2–sphere (possibly) with thin edge-linking tubes, or by adding an almost normal tube between two such surfaces, except*
- *an almost normal octagonal 2–sphere, in the minimal layered-triangulations of S^3, $\mathbb{R}P^3$, and the bad minimal layered-triangulation of $L(3,1)$, and*
- *an almost normal octagonal torus in the bad two-tetrahedra layered-triangulation of $L(3,1)$.*

Proof. A layered-triangulation of a lens space is obtained from a layered-triangulation of a solid torus by folding along its boundary. Hence, a normal or almost normal surface in the layered-triangulation of the lens space, must come from a normal or

almost normal surface embedded in the layered-triangulation of the solid torus after some identifications in its boundary. In particular, the boundary of the surface in the solid torus must be invariant under the folding. It follows that if the edges in the boundary of a layered solid torus are denoted e_1, e_2, e_3 and if we fold over along the edge e_3, say, then the number of intersections of the normal or almost normal surface in the layered solid torus with the edges labeled e_1 and e_2 must be the same. Hence, following the classification of normal and almost normal surfaces in layered solid tori, the only possible curves, given by their intersection number with e_1, e_2 and e_3, respectively, are: $1, 1, 0$; $1, 1, 2$; and $2, 2, 2$. These very restricted possibilities are the reason for the limited types of normal and almost normal surfaces in layered-triangulations of lens spaces.

We shall defer consideration of the normal and almost normal surfaces in the small, exceptional triangulations obtained by folding along an edge in the boundaries of the minimal $0/1$–, $1/1$– and $1/2$–layered-triangulations until we have done the general case. Otherwise, for a minimal layered-triangulation of a solid torus:

- An orientable normal surface whose boundary curves have intersection numbers $1, 1, 0$ is a thin edge-linking annulus about the edge e_3 (possibly) with thin edge-linking tubes. Hence, upon folding the boundary of the solid torus, such a surface gives a vertex-linking 2-sphere with thin edge-linking tubes.
- An orientable normal surface whose boundary curves have intersection numbers $1, 1, 2$ can only happen if e_3 is the univalent edge; however, in a minimal layered-triangulation of a lens space, we do not fold over along the univalent edge.
- An orientable normal surface whose boundary curves have intersection numbers $2, 2, 2$ is a vertex-linking disk (possibly) with some thin edge-linking tubes. Hence, upon folding the boundary of the solid torus, such a surface gives the vertex-linking 2–sphere, possibly, with some thin edge-linking tubes. For intersection $2, 2, 2$ we have the same surfaces under consideration no matter which edge we fold along.

Thus, to complete our proof, we only need to consider lens spaces obtained by folding the boundary of the one-tetrahedron solid torus extending $1/2$, or the two two-tetrahedra solid tori extending $1/1$ or $0/1$. The normal surfaces in these layered-triangulations are given in Lemmas 5.1, 5.2, and in the case of $1/1$ in the final paragraph of the proof of Theorem 5.3.

For the minimal $1/2$–layered-triangulation:

This is the one-tetrahedron solid torus. Folding along the edge 3 (the univalent edge), we get a one-tetrahedron triangulation of S^3. There is a thin edge-linking annulus about the edge 3 having boundary with intersection numbers $1, 1, 0$ that gives the vertex-linking 2-sphere with a thin edge-linking tube about the edge 3. There is an almost normal octagonal annulus having boundary with intersection numbers $1, 1, 2$ (the slope of the exceptional longitude) that gives an almost normal octagonal 2–sphere. The vertex-linking curve with intersection numbers $2, 2, 2$ bounds only the vertex-linking disk and gives the vertex-linking 2–sphere. Hence, the exception in this case is an almost normal octagonal 2–sphere.

Folding along the edge 2, we get the one-tetrahedron triangulation of $L(4, 1)$. The boundary of a normal Möbius band has intersection numbers $1, 0, 1$ and gives

us a Klein bottle, which we consider below; its double gives the vertex-linking 2-sphere with a thin edge-linking tube about the edge 2. There are no surfaces having boundary with intersection numbers $1, 2, 1$. The intersection $2, 2, 2$ gives the vertex-linking 2–sphere.

Folding along the edge 1 (the thick edge), we get the one-tetrahedron triangulation of $L(5, 2)$. There are no normal surfaces with intersection numbers $0, 1, 1$ or $2, 1, 1$. The intersection $2, 2, 2$ gives the vertex-linking 2–sphere.

For the minimal 0/1–layered-triangulation:

Folding along the univalent (thin) edge 1 or the thick edge 1, gives two distinct two-tetrahedra triangulations of S^3. These are not minimal; however, the examples are interesting so we briefly list their orientable normal and almost normal surfaces.

First, we consider the case of folding along the univalent (thin) edge. For boundary with intersection $0, 1, 1$, we have a thin edge-linking annulus about the univalent edge and this same annulus with a thin edge-linking tube about the edge 2; these give the vertex-linking 2–sphere with a thin edge-linking tube about the univalent edge and this same surface with a second tube about the edge 2. There also is an almost normal octagonal annulus having boundary slope the same as the univalent edge. This gives an almost normal octagonal torus; however, we do not list this among the conclusions of the theorem because this is not a minimal layered triangulation for S^3. For intersection numbers $2, 2, 2$ we have the vertex-linking disk and the vertex-linking disk with a thin edge-linking tube about the edge 2; these give the vertex-linking 2–sphere and the vertex-linking 2–sphere with a thin edge-linking tube, respectively. We also have a almost normal octagonal disk with trivial boundary curve; this gives an almost normal octagonal 2–sphere.

If we fold along the thick edge, we only get surfaces with the slope $2, 2, 2$ and we have the vertex-linking 2-sphere, the vertex-linking 2–sphere with a thin edge-linking tube on the edge 2 and an almost normal octagonal 2–sphere.

Folding along the edge labeled 0, we get $\mathbb{R}P^3$. Here we are concerned with surfaces having boundary slope one of $2, 2, 2$ or $1, 1, 0$; there are no surfaces with boundary slope $1, 1, 2$. For $2, 2, 2$ we get the vertex-linking 2–sphere, the vertex-linking 2–sphere with a thin edge-linking tube along the edge 2 and an almost normal octagonal 2–sphere. For $1, 1, 0$, we have the meridional disk in the solid torus, which leads to the projective plane in $\mathbb{R}P^3$ and its double is a normal 2–sphere. The twice punctured Klein bottle, also with boundary intersection $1, 1, 0$, leads to a non orientable surface having genus 3; its double is a genus two orientable surface, which is the vertex-linking 2–sphere with thin edge-linking tubes about edges 0 and 2.

For the minimal 1/1–layered-triangulation::

Folding along the edge 2 is the same as doubling the one-tetrahedron solid torus and we get $S^2 \times S^1$. Here we have all three possible slopes. For $2, 2, 2$, we get the vertex-linking 2–sphere and the vertex-linking 2–sphere with a thin edge-linking tube on the edge 3. For $1, 1, 2$, we have the meridional disk which gives the essential 2–sphere; and for $1, 1, 0$, the Möbius band doubles to a Klein bottle and its double is the vertex-linking 2–sphere with a thin edge-linking tube about the edge 2. There are no almost normal octagonal surfaces.

Folding along either the thick edge or the univalent edge, we get the two distinct minimal layered-triangulations of $L(3, 1)$. In the first case, folding along the thick

edge 1, we only have the one slope $2, 2, 2$ and thus we have the vertex-linking 2–sphere and the vertex-linking 2–sphere with a thin edge linking tube about the edge 3. In the last case, folding along the univalent edge, we get the bad minimal layered-triangulation of $L(3, 1)$ and have all three sets of intersection numbers possible. As above, for $2, 2, 2$, we get the vertex-linking 2–sphere and the vertex-linking 2–sphere with a thin edge-linking tube along the edge 3. For, intersection numbers $2, 1, 1$, we have an edge-linking annulus about the interior edge 3, which gives an embedded normal 2–sphere; and an almost normal octagonal annulus, which gives an almost normal octagonal 2–sphere. Finally, we have intersection numbers $0, 1, 1$ and get a vertex-linking 2–sphere with thin edge-linking tube along the thin edge labeled 1, the vertex-linking 2–sphere with thin edge-linking tubes along the thin edge labeled 1 and along the edge labeled 3 and an almost normal octagonal torus. The last example is *not* 0–efficient. $\square$

Notice that we have almost normal octagonal 2–spheres in the minimal layered-triangulations of S^3, $\mathbb{R}P^3$ and the bad layered-triangulations of $L(3, 1)$. Every triangulation of S^3 must have an almost normal 2–sphere and a one-vertex triangulation must have such an octagonal 2–sphere. The triangulations of $\mathbb{R}P^2$ and the bad minimal layered-triangulation for $L(3, 1)$ are not 0–efficient; thus they have non vertex-linking normal 2–spheres. The almost normal octagonal 2–spheres are formed as the minimax in a sweep-out between these non vertex-linking normal 2–sphere and the vertex-linking 2–sphere. See more on this below.

7.1. **Corollary.** *For a lens space distinct from S^3, $\mathbb{R}P^3$, $L(3, 1)$, and $S^2 \times S^1$, the only embedded, orientable normal surfaces are the vertex-linking 2–spheres with thin edge-linking tubes. There are no almost normal octagonal surfaces.*

8. APPLICATIONS OF LAYERED-TRIANGULATIONS OF THE SOLID TORUS AND LENS SPACES

In this section we use layered-triangulations to capture some classical results about the topology of lens spaces. We obtain earlier results of Bredon and Wood [3] characterizing those lens spaces that admit embedded nonorientable surfaces and classifying the embedded nonorientable surfaces in each such lens space. We apply layered-triangulations to obtain new proofs of the results of Waldhausen [34] and those of Bonahon and Otal [2] to classify Heegaard splittings of S^3 and $S^2 \times S^1$ and all (other) lens spaces. We also use layered-triangulations of the solid torus to construct what we consider canonical triangulations for Dehn-fillings of cusped manifolds (called triangulated Dehn fillings); in particular, we use these triangulations as a tool to study Heegaard splittings of Dehn-fillings of cusped manifolds. Finally, we give sufficient conditions for 0–efficient triangulations of cusped manifolds to extend to 0–efficient triangulated Dehn fillings of these manifolds.

8.1. **Embedded, non orientable surfaces in lens spaces.** A determination of those lens spaces that admit embeddings of non orientable surfaces and the classification of non orientable surfaces embedded in such lens spaces is given in [3]. Here we apply layered-triangulations of lens spaces to obtain these results, including the cases of S^3 and $S^2 \times S^1$.

We shall use U_h to denote the non orientable surface of genus h (connected sum of h copies of $\mathbb{R}P^2$). In an orientable 3–manifold a non orientable surface is obtained by possibly adding handles to an embedded, incompressible non orientable surface

or to an incompressible, non separating, orientable surface. For a lens space, except $S^2 \times S^1$, we only have the first possibility; for $S^2 \times S^1$, we also have the second possibility, interpreting the essential non separating 2–sphere as incompressible.

If the lens space $L(X, Y)$ is distinct from $S^2 \times S^1$ and contains a non orientable surface, then such a surface is isotopic to an embedded, incompressible non orientable surface with possibly some trivial handles attached; it follows that if $L(X, Y)$ contains an embedded, non orientable surface, then for any triangulation, there must be a *normal* (incompressible) non orientable surface and our given surface is isotopic to this normal surface with possibly some trivial handles attached. For $S^2 \times S^1$ a similar statement applies where we have in place of the incompressible nonorientable surface the "incompressible", non separating 2–sphere. Of course, once the handles are attached, there is no reason the surface need be normal or even isotopic to a normal surface. In any case, to understand all embedded non orientable surfaces, it is sufficient to understand the embedded, *normal*, non orientable surfaces in a triangulation of $L(X, Y)$. To do this, we use a minimal layered-triangulation for $L(X, Y)$. As above for orientable surfaces, such a normal surface in a minimal layered-triangulation of a lens space must come from an embedded normal surface in a minimal layered-triangulation of the solid torus with identifications in its boundary determined by the folding of the boundary of the solid torus to get the lens space. Furthermore, and again just as above, such a normal surface in the layered-triangulation of the solid torus must have intersection numbers with the edges e_1, e_2 and e_3 in the boundary of the solid torus one of $1, 1, 0$; $1, 1, 2$; or $2, 2, 2$, respectively, where we are, again, denoting the edge on which we fold by e_3.

A normal surface with intersection number $2, 2, 2$ can not give a non orientable surface upon folding. A normal surface with intersection numbers $1, 1, 2$ indicates that the normal surface meets the folding edge e_3 maximally; but this can only happen if we fold on the univalent edge or the layered-triangulation of the solid torus is the minimal $1/1$–layered-triangulation and we fold on the edge 2. For a minimal layered-triangulation, we only fold on the univalent edge when we have the one-tetrahedron $1/2$–layered-triangulation and then we get S^3; if we fold on the edge 2 in the $1/1$–layered-triangulation, we get $S^2 \times S^1$. There are no normal surfaces in the one-tetrahedron $1/2$-layered-triangulation with boundary slope having intersection numbers $1, 1, 2$. Hence, except for $S^2 \times S^1$, a non orientable normal surface in a minimal layered-triangulation of a lens space must come from a normal surface in a minimal layered-triangulation of a solid torus having boundary parallel to the folding edge. From our classification of normal surfaces in Theorem 5.3 such a surface is either a thin edge-linking annulus, possibly with thin edge-linking tubes, or a non orientable surface, or we have the two-tetrahedra $0/1$ triangulation of the solid torus, the normal surface is the meridional disk, and folding over along the edge 0 gives $\mathbb{R}P^3$. In the first case, the edge-linking annulus identifies to a vertex-linking 2–sphere with thin edge-linking tubes, which is orientable. Thus, except for possibly $\mathbb{R}P^3$ and $S^2 \times S^1$, we only get a non orientable normal surface in the lens space, if we have a non orientable surface in the layered solid torus with boundary slope the slope of the edge upon which we are folding; in particular, the folding edge is even.

Now, we wish to determine the genus of the unique non orientable normal surface in a minimal p/q–layered-triangulation of a solid torus that has its boundary slope

the slope of an edge of the triangulation that is in the boundary; furthermore, we want the genus in terms of the layered-triangulation of the solid torus. Let $e(p, q)$ denoted the number of even order edges in the minimal p/q–layered-triangulation of the solid torus. From Theorem 5.3 we have that the genus of the surface of interest is just $e(p, q)$. We use the Möbius band as the minimal $1/1$–layered-triangulation; hence, we set $e(1, 1) = 1$; since we have one edge of even order.

8.1. Lemma. *For, p, q relatively prime, $q > p \geq 1$, we have*

$$e(p, q) = \begin{cases} e(q - p, p) + 1 & \text{if } p \text{ and } q \text{ are both odd,} \\ e(q - p, p) & \text{if } p \text{ or } q \text{ is even.} \end{cases}$$

Proof. If we consider Euclid's original algorithm (or the difference version of the Euclidean algorithm) for $p/q = (a_0, \ldots, a_n)$ the continued fraction expansion of p/q, we get a sequence of numbers that index the order of the edges in the corresponding minimal p/q–layered-triangulation of the solid torus. Hence, for $\{p, q\}$ we have the sequence:

$$p + q, q, q - p, \ldots, q - a_0 p, p, p - r_1, \ldots, p - a_1 r_1, r_1, \ldots, 2, 1,$$

which contains the sequence for $\{q - p, q, p\}$. So, if $p + q$ is even we have one more even edge in the minimal p/q–layered-triangulation than we have for the minimal layered-triangulation associated with the triple $\{q - p, q, p\}$. $\qquad\square$

EXAMPLES:

(1) $e(2, q) = 1$; q is odd,
(2) $e(2k, 1) = k$,
(3) $e(16, 7) = e(9, 7) = e(2, 7) + 1 = 2$.

In the statement of the following theorem, we exclude the lens spaces $\mathbb{R}P^3$ and $\mathcal{S}^2 \times S^1$, both of which admit embedded non orientable surfaces; the non orientable surfaces in these two lens spaces arise in a different way from those in other lens spaces. We provide the answers for $L(2, 1) = \mathbb{R}P^3$ and $L(0, 1) = S^2 \times S^1$ in the examples following the proof of the theorem.

8.2. Theorem. [3] *Suppose $L(X, Y)$ is a lens space distinct from $\mathbb{R}P^3$ and $\mathcal{S}^2 \times S^1$. Then $L(X, Y)$ admits an embedded non orientable surface iff X is even. Furthermore, there is a unique embedded, <u>incompressible</u> non orientable surface U_h in $L(X, Y)$, where $h = e(X, Y)$.*

Proof. Assume $L(X, Y)$ has a minimal layered-triangulation. If $L(X, Y)$ admits an embedded non orientable surface, then there is an embedded, incompressible, non orientable surface in $L(X, Y)$; hence, a normal such surface. From the above discussion, we have that the minimal layered-triangulation for $L(X, Y)$ is derived by folding over a minimal layered-triangulation of a solid torus where the edge along which we fold has even order and is not univalent. We denote the triangulation of the layered solid torus by $\{2k, q, 2k + q\}$. We fold over on the edge labeled $2k$; therefore, $X = 2k + 2q$ and for $0 < Y \leq X/2$, $Y = q$, whereas, for $X/2 \leq Y < X$, $Y = 2k + q$. We have X is even and

$$\begin{cases} k = X/2 - Y & q = Y, & \text{if } 0 < Y \leq X/2, \\ k = Y - X/2 & q = X - Y, & \text{if } X/2 \leq Y < X. \end{cases}$$

The minimal layered-triangulation of the solid torus therefore extends the triangulation on the boundary of a solid torus where the edges meet the meridional disk in the triples

$$\begin{cases} \{\text{X-2Y,Y,X-Y}\}, & \text{if } 0 < Y < X/2, \text{ or} \\ \{\text{2Y-X,X-Y,Y}\}, & \text{if } X/2 < Y < X. \end{cases}$$

In general, we have the unique non orientable surface having boundary slope that of the edge $X - 2Y$ for $0 < Y < X/2$, and that of the edge $2Y - X$ for $X/2 < Y < X$. Hence, the genera of the embedded, incompressible, non orientable surfaces in $L(X, Y)$ for these two possibilities are $e(X-2Y, Y)+1$, for $0 < Y < X/2$, and $e(2Y - X, X - Y) + 1$, for $X/2 < Y < X$.

But since X is even, we have from Lemma 8.1 that $e(X, Y) = e(X - Y, Y) = e(X - 2Y, Y) + 1$ ($X - Y, Y$ are both odd), $0 < Y < X/2$; and $e(X, Y) = e(X - Y, Y) = e(X - Y, (2Y - X) + (X - Y)) = e(X - Y, 2Y - X) + 1$ ($X - Y, Y$ are both odd), $X/2 < Y < X$. Thus in both cases, we have $h = e(X, Y)$. $\square$

Examples:

(1) The unique incompressible, non orientable surface in $L(30, 7)$ is U_3.

   In this case we have $X = 30, Y = 7$; hence, $Y < X/2$ and e(X-2Y,Y) = e(16,7) = 2 and h = 3.

(2) For $L(30, 23)$, we have $X = 30, Y = 23$ and now $X/2 < Y$. We use $e(2Y - X, X - Y) = e(16, 7)$, and, as expected, $h = 3$.

(3) $L(2, 1) = \mathbb{R}P^3$ is the only lens space in which we can embed $\mathbb{R}P^2$.

   The minimal layered-triangulation of the lens space $\mathbb{R}P^3$ is obtained from the two-tetrahedra extension of the 0/1-triangulation by "folding over" along the edge 0. The 0/1–layered-triangulation of the solid torus admits a Möbius band with boundary having the slope of the interior edge 2 and a once-punctured Klein bottle with boundary having slope of the edge 0 in the boundary. However, we also have the meridional disk with boundary having the slope of the edge 0. Folding along the edge 0, the boundary of the Möbius band has intersection numbers $3, 1, 2$, the boundary of the once-punctured Klein bottle has intersection numbers $1, 1, 0$, and the boundary of the meridional disk also has intersection numbers $1, 1, 0$. From the meridional disk we get an embedded $\mathbb{R}P^2$ and from the the once-punctured Klein bottle we get a genus 3 nonorientable surface, which is the embedded $\mathbb{R}P^2$ with a trivial handle.

   It follows that $\mathbb{R}P^3$ admits an embedded nonorientable surface of every *odd* genus.

(4) Those lens spaces $L(X, Y)$ that admit an embedding of the Klein bottle have the form $L(4n, 2n - 1), n \geq 1$.

   In this case, the minimal layered-triangulation of the solid torus that gives the minimal layered-triangulation of $L(X, Y)$ has $e(p, q) = 1$; hence, the even edge in the boundary, say p, must be 2. It follows that $X = 2 + 2q$, $Y = q \geq 1$ and odd. Set $q = 2n - 1$. In particular, for $n = 1$, we have $L(4, 1)$; for $n = 2$, we have $L(8, 3); \ldots$; for $n = 11$, we have $L(44, 21); \ldots$.

(5) There are no *incompressible* non orientable surfaces embedded in $S^2 \times S^1$; however, there is an embedding of a nonorientable surface of every even genus in $S^2 \times S^1$.

If we add a handle from one side of the non separating 2–sphere in $S^2 \times S^1$ to its other side, then we have an embedded Klein bottle; so, any non orientable surface with even Euler number can be embedded in $S^2 \times S^1$. On the other hand, the minimal two-tetrahedra, layered-triangulation of $S^2 \times S^1$ is obtained by folding the boundary of the two-tetrahedra, layered-triangulation of the solid torus extending $\{2, 1, 1\}$ along the edge labeled 2.

The normal surfaces in this two-tetrahedra triangulation of the solid torus are given in the proof of Theorem 5.3. The only possibilities leading to a nonorientable normal surface in $S^2 \times S^1$ is the Möbius band with boundary having slope that of the boundary edge 2. Upon folding over the edge 2, this surface becomes a Klein bottle. There are no incompressible non orientable surfaces. It is curious that this minimal genus non orientable surface is captured by the minimal layered-triangulation.

8.2. 0– and 1–efficient layered-triangulations of lens spaces.

Layered-triangulations of lens spaces are quite simple and do not exhibit the full richness of 0– and 1–efficient triangulations; however, they do provide examples of triangulations that are not efficient as well as allow a complete characterization of those layered-triangulations that are efficient.

A triangulation of a closed 3–manifold is said to be 0–*efficient* if and only if the only normal 2–spheres are vertex-linking [14]. A layered-triangulation of a lens space is said to be 1–*efficient* if and only if it is 0–efficient and the only normal tori are thin edge-linking. There is a definition of 1–efficient that applies to general triangulations of closed 3–manifolds; however, for layered-triangulations of lens spaces, what we have given is an equivalent definition.

0–*efficient layered-triangulations of lens spaces.* A layered-triangulation of a lens space can be viewed in two ways as a layered-triangulation of a solid torus with its boundary folded over; one view is the reverse of the other. In other words, once we fold over, we can start from either end and we have a layered solid torus with its boundary folded over. If in either of these views we have the meridian slope of the solid torus the slope of an edge in the triangulation, we say "a meridional slope is the slope of an edge". If a meridional slope is the slope of an edge, there is a level in the triangulation where the meridional slope of a solid torus is the slope of an edge in the boundary of that solid torus. It follows that the layered-triangulation of the lens space can be decomposed as the union of two subcomplexes, each a layered-triangulation of a solid torus where one is an extension of the 0/1–triangulation on the boundary of the solid torus. It is possible in such a case that we have the creased 3–cell. We have the following characterization of 0–efficient layered-triangulations of lens spaces.

8.3. Theorem. *A layered triangulation of a lens space, distinct from S^3 and $S^2 \times S^1$, is 0–efficient if and only if no edge has the slope of a meridian.*

Proof. Suppose an edge has the slope of a meridian. Then split the layered-triangulation of the lens space at a level where the meridional slope is the slope of an edge in the boundary of the solid torus; i.e., extends the 0/1–triangulation. One possibility is that we are folding the boundary of a layered-triangulation that extends the 0/1–triangulation; hence, we get S^3 or we are folding along the edge 0

and we get $\mathbb{R}P^3$. Since we are assuming we do not have S^3, we must get $\mathbb{R}P^3$ and the meridional disk identifies to a projective plane and its double is a non vertex-linking 2–sphere. So, the triangulation is not 0–efficient. The other possibility is that we have two solid tori and in one solid torus, the meridional curve has the slope of an edge in the boundary. In the other solid torus there are two possibilities: the edge in the boundary of this solid torus, which is identified to the edge having the slope of the meridian, is either thick or it is thin. If it is thick, then we must have S^3. Again this is ruled out by hypothesis, the edge in this solid torus, identified to the edge having slope that of a meridian, must be thin. Hence, there is a thin edge-linking annulus about the edge. This annulus and two copies of the meridional disk form a non vertex-linking normal 2–sphere. Hence, if we do not have a layered-triangulation of S^3, then the existence of an edge having the slope of a meridian gives a non vertex-linking normal 2–sphere and the triangulation is not 0–efficient.

Now, suppose a layered-triangulation of a lens space is not 0–efficient. Hence, there must be a non vertex-linking, normal 2–sphere, say Σ. The layered-triangulation of a lens space is obtained by folding over a layered-triangulation of a solid torus and Σ meets this solid torus in a planar surface (the 2-sphere Σ is obtained from this planar surface after identifications along its boundary). Again, the possible intersection numbers for such a planar normal surface with the edges e_1, e_2, e_3 in the boundary of the solid torus, folding over edge e_3, are: $2, 2, 2; 1, 1, 2;$ or $1, 1, 0$, respectively.

Now, by Theorem 5.4, the possible planar (genus 0) normal surfaces embedded in a layered-triangulation of a solid torus are (using the labeling from Theorem 5.4)

(1) the vertex-linking disk with boundary intersection $2, 2, 2$, which gives a vertex-linking 2–sphere in the lens space;

(2) the meridional disk either with boundary intersection $1, 1, 2$, which then gives $S^2 \times S^1$ and contradicts our hypothesis, or with boundary intersection $1, 1, 0$ in which case we have a meridional slope the slope of an edge and we have $\mathbb{R}P^3$;

(3) an edge-linking annulus with boundary slope the slope of an edge. If the intersection numbers are $1, 1, 2$, then we are folding along the univalent edge and thus by reversing our layered solid torus, we see that we have a creased 3–cell and an edge with meridional slope; this includes the possibility of a fat edge-linking annulus, where we have two edges with the same slope. If the intersection numbers are $1, 1, 0$, then we get a torus upon identification of the boundary and not a 2–sphere;

(4) a trivial, non vertex-linking disk having boundary with intersection numbers $2, 2, 2$, in which case an edge has the slope of the meridional disk;

(5) a non edge-linking annulus with boundary slope the slope of an edge; so, again, an edge has the slope of a meridional disk.

Hence, if the lens space is distinct from S^3 or $S^2 \times S^1$ and no edge in the layered-triangulation has slope that of a meridian, then the layered-triangulation is 0–efficient.   □

Examples.

(1) There are infinitely many distinct 0–efficient layered triangulations of S^3, each with an edge having the slope of a meridian.

In the previous proof we noted that if a lens space is formed by identifying the boundaries of two layered-triangulations of the solid torus via a simplicial identification with an edge having the slope of the meridian in one layered solid torus being identified to a thick edge in the other layered solid torus, then we get a layered-triangulation of S^3. This is used to construct infinitely many distinct 0–efficient, layered-triangulations of S^3, each with an edge having the slope of the meridian.

Consider the layered-triangulations of S^3 given by folding over along the edge $2n + 1$ in the minimal $(n/n + 1)$–layered-triangulation of the solid torus. We write this as $\{n, n + 1, 2n + 1\} \leftrightarrow \{1, 1, 2\}$, $n \geq 0$. Since we have folded over along the univalent edge, each of these triangulations is the same as attaching the creased 3–cell to the minimal layered-triangulation of the solid torus extending the $1/n$–triangulation; namely,

$$\{n, n + 1, \underline{2n + 1}\} \leftrightarrow \{1, 1, \underline{2}\}$$
$$\{n, n + 1, 1\} \leftrightarrow \{1, 1, 0\}.$$

So, in each triangulation, we have an edge having the slope of a meridian being attached to the thick edge.

Now, to see that these are 0–efficient triangulations, we analyze the possible normal surfaces in these layered-triangulations. A normal surface in such a triangulation of S^3 is determined by a normal surface in the minimal $(n/n+1)$–layered-triangulation of the solid torus, along with some identifications in its boundary. So, we can proceed, as above, and consider those normal surfaces in the minimal $(n/n+1)$–layered-triangulation of the solid torus, where the surface has boundary intersection numbers with the edges labeled n, $n+1$ and $2n+1$ one of: $2, 2, 2$; $1, 1, 2$; or $1, 1, 0$, respectively. The only interesting set of intersection numbers here is $1, 1, 2$. Typically there would be an edge-linking annulus in the solid torus with boundary slope having the intersection numbers $1, 1, 2$; however, in these examples, the intersection numbers $1, 1, 2$ correspond to the slope of the thick edge and as such there is no associated edge-linking annulus. We do, however, have in each an almost normal annulus with boundary slope having the intersection numbers $1, 1, 2$ and from these we get almost normal 2–spheres in each of these triangulations of S^3.

Notice for $n = 0$, we have one of the two, two-tetrahedra 0–efficient layered triangulations of S^3, which is two creased 3–cells identified along their boundaries (*not* doubled); and for $n = 1$, we have the one-tetrahedron, one-vertex minimal triangulation of S^3. For these examples, we fold along the univalent edge. We also can get a two-tetrahedra, layered 0–efficient triangulation of S^3 by folding along the thick edge in the minimal $0/1$–layered-triangulation of the solid torus.

(2) All lens spaces (except $\mathbb{R}P^3$ and $S^2 \times S^1$) admit an infinite family of distinct 0–efficient layered-triangulations; furthermore, by Theorem 8.3, except for S^3, none of these layered-triangulations have an edge with the slope of a meridian.

Recall that a path in the L–graph starting at p/q and ending at $1/1$ determines a layered-triangulation of the solid torus and if the path runs through $0/1$, then we have an edge with the slope of a meridian. To obtain a layered-triangulation of a lens space we fold along an edge in the boundary

of the p/q–layered-triangulation of the solid torus. Now, the reverse layering gives us another path in the L–graph. Does it run through $0/1$? If we have $\{p, q, p+q\} \leftrightarrow \{1, 1, 2\}$ (we fold over along the univalent edge, except in the case $\{1, 1, 2\} \leftrightarrow \{1, 1, 2\}$ where 2 is not univalent and we get $S^2 \times S^1$), then the reverse path has

$$\{p, q, \underline{p+q}\} \leftrightarrow \{1, 1, \underline{2}\}$$
$$\{p, q, \overline{|p-q|}\} \leftrightarrow \{1, 1, 0\},$$

which gives an edge having the slope of a meridian. So, we can not fold along the univalent edge. If we fold along the edge p or q, say q, then the reverse path passes through $1/1$ every time our original path passes through p/q. Hence, the reverse path has

$$\{p, \underline{q}, p+q\} \leftrightarrow \{1, \underline{2}, 1\}$$
$$\{p, \overline{2p+q}, p+q\} \leftrightarrow \{1, 0, 1\},$$

whenever we go from p/q in the L–graph to $p/p+q$ in the L–graph in the original path.

We conclude, with the exception of folding along an edge in the boundary of a $0/1$-layered-triangulation or a $1/1$–layered-triangulation, for any point p/q in the L–graph and any path from p/q to $1/1$ that does not pass through $0/1$ or $p/p+q$ we have a layered-triangulation of the solid torus so that folding along the edge q gives a layered-triangulation of $L(2p+q, p)$ with no edge having slope a meridian; hence, we have a 0–efficient layered-triangulation by Theorem 8.3.

Recall for paths beginning at $0/1$, if we fold, we get S^3 or $\mathbb{R}P^3$; for paths beginning at $1/1$, we get $S^2 \times S^1$ if we fold over the edge 2 and get $L(3, 1)$, otherwise. If we fold over the univalent edge (thin edge) labeled 1 we do not get a 0–efficient triangulation; whereas, if we fold over the thick edge labeled 1 we do get a 0–efficient triangulation. Hence, paths in the L–graph beginning at $1/1$ and never layering along the edge paired with 2 when passing through $1/1$ provide an infinite family of 0–efficient triangulations of $L(3, 1)$.

8.4. Corollary. *For any lens space other than $S^2 \times S^1$ and $\mathbb{R}P^3$, there exists a 0–efficient layered-triangulation having an arbitrarily large number of tetrahedra. Hence, for each such lens space there are infinitely many distinct 0–efficient layered-triangulations.*

1–efficient layered-triangulations of lens spaces. We have the following character-ization of 1–efficient layered-triangulations of lens spaces. Just as in the case of a layered-triangulation of the solid torus, we say a layered-triangulation of a lens space is *nearly-minimal* if no edge has the slope of a meridian and no two *thin* edges have the same slope. Recall that in a layered-triangulation of a lens space there are two thick edges; whereas, only one in a layered-triangulation of a solid torus. For lens spaces the two-tetrahedron layered-triangulation of $\mathbb{R}P^3$ is minimal but is not nearly-minimal and the two-tetrahedron layered triangulation of $L(3, 1)$ obtained by folding over along the univalent edge in the two-tetrahedron layered-triangulation of the solid torus extending the $1/1$–triangulation is minimal but is not nearly-minimal. Otherwise, a minimal triangulation is nearly-minimal.

8.5. Theorem. *A layered-triangulation of a lens space, distinct from S^3 and $S^2 \times S^1$, is 1–efficient if and only if it is nearly-minimal.*

Proof. Suppose the layered-triangulation is obtained by folding along an edge in the boundary of a layered-triangulation of the solid torus $\mathbb{T}$ and it is not nearly-minimal. Then either an edge has the slope of a meridian or two thin edges have the same slope.

If an edge has the slope of a meridian, then by Theorem 8.3 and our assumption that the lens space is not S^3, the triangulation is not 0–efficient and so can not be 1–efficient.

Thus we may assume, no edge has the slope of a meridian; hence, there are two thin edges (in the lens space) having the same slope. It follows that two thin edges, say e and e', in $\mathbb{T}$ have the same slope. Now, by Theorem 5.4, there is either a vertex-linking disk with a non thin edge-linking tube in $\mathbb{T}$ or an edge-linking annulus about both e and e' and having boundary slope the slope of these two edges. In the first case, we get a non thin edge-linking torus in the lens space and the triangulation of the lens space is not 1–efficient. In the second case, we have two possibilities for the intersection numbers for the slope of e and e': either $1, 1, 2$ or $1, 1, 0$. Now, if the slope of e and e' has intersection number $1, 1, 2$, then in the reverse layering there is an edge having the same slope as e and e' (we could actually choose one of these to be such an edge) and having the slope of a meridian. An easy way to see this is that the slope with intersection numbers $1, 1, 2$ bounds the meridional disk in the degenerate layered-triangulation of the solid torus, the one-triangle Möbius band. But we have assumed no edge has the slope of a meridian. If the intersection numbers for the boundary slope are $1, 1, 0$ and since we have that both e and e' are both thin edges, then we have a non thin edge-linking torus in the lens space.

Hence, under the assumption of an edge having the slope of a meridian or two thin edges having the same slope, the layered-triangulation of the lens space is not 1–efficient. It follows that our condition is necessary.

Now, suppose the layered-triangulation of the lens space is *not* 1–efficient. By Theorem 8.3, we may assume it is 0–efficient. Thus we must have a normal torus, say T, that is not thin edge-linking in the lens space. We again consider the layered-triangulation of the lens space as coming from a layered-triangulation of a solid torus $\mathbb{T}$ with its boundary folded over. The torus T meets $\mathbb{T}$ in a genus 0 or a genus 1 normal surface. By Theorem 5.4, the possibilities that can lead to a normal torus are (we use the numbering in the conclusions of Theorem 5.4):

(3) An edge-linking annulus with boundary slope the slope of an edge. The possibilities are: the thin edge-linking annulus whose boundary slope has intersection numbers $1, 1, 0$ and is about the edge along which we are folding; or a thin edge-linking annulus whose boundary slope has intersection numbers $1, 1, 2$ and meets the edge along which we are folding in 2 points. The former gives a thin edge-linking torus in the lens space, which is not T; the latter, as above, can only happen if the reverse layering has an edge having the slope a meridian, which contradicts that the layered-triangulation of the lens space is 0–efficient and the lens space is not S^3.

(4) We have (4B), a non edge-linking annulus with boundary slope the slope of an edge. But we only have such an annulus if there is an edge having the slope of a meridian.

(5) A fat edge-linking annulus. We only have such an annulus if there are two edges with the same slope.

(6) In the case of (6A), a vertex-linking disk with a thin edge-linking tube, we have a thin edge-linking torus and not T. In the case of (6B), an edge-linking annulus with a thin edge-linking tube, we have the possibilities of intersection numbers $1, 1, 0$ and $1, 1, 2$. In the former case, after folding we get a genus 2 surface and not T; in the latter, we would have to have an edge with the slope of a meridian, which we have ruled out.

In the cases of (7), (8) and (9) of Theorem 5.4, we have an edge with the slope of a meridian, two edges with the same slope or both possibilities.

Hence, if the lens space is distinct from S^3 or $S^2 \times S^1$ and the layered-triangulation is nearly-minimal, then the layered-triangulation is 1–efficient. $\qquad\square$

Note there are triangulations of S^3 that are 1–efficient and not nearly-minimal; also, the minimal two-tetrahedra triangulation of $S^2 \times S^1$ is nearly-minimal but is not 1–efficient.

Examples.

(1) There are infinitely many distinct 1–efficient triangulations of S^3. In fact, the triangulations, $\{n, n+1, 2n+1\} \leftrightarrow \{1, 1, 2\}$, given in the above example of infinitely many 0–efficient triangulations for S^3 also are 1–efficient.

(2) An opening at $1/1$ of a nearly-minimal layered-triangulation of a lens space gives a nearly-minimal layered-triangulation; hence, we can generate from a minimal layered-triangulation of a lens space, that is distinct from the two-tetrahedra minimal layered-triangulation for $\mathbb{R}P^3$, three additional nearly-minimal layered-triangulations (two by opening at one end, or the opposite end, and a third by opening at both ends).

(3) An opening at $1/1$ is a special case of *opening at a thick edge*; that is, if we have the minimal $1/k$–layered triangulation, $k \geq 2$, and layer on 1, we get the minimal $(k/k + 1)$–layered-triangulation. Then we can undo this layering by layering on the univalent edge $2k + 1$ to return to a $1/k$–layered-triangulation (not minimal). We still have an edge that meets the meridional disk once but now it is not a thick edge. The resulting $1/k$–layered-triangulation is nearly-minimal. It follows that a minimal $1/n$–layered-triangulation, $n \geq 2$, of the solid torus has $n-1$ such openings, one for each $k, 2 \leq k < n$, giving $n - 1$ distinct nearly-minimal $1/n$–layered-triangulations. See Figure 25.

8.6. **Corollary.** *A lens space distinct from S^3 has only finitely many nearly-minimal (hence, 1–efficient) layered-triangulations.*

Proof. Suppose $\mathcal{T}$ is a layered-triangulation of a solid torus and e is an edge of $\mathcal{T}$ in the boundary. Let $\mathcal{T}' = \mathcal{T} \cup_e \Delta$ be a layering of $\mathcal{T}$ along the edge e. Now, let e' be the univalent edge in $\mathcal{T}'$ and let $\mathcal{T}'' = \mathcal{T}' \cup_{e'} \Delta'$ be a layering of $\mathcal{T}'$ along the univalent edge e' of $\mathcal{T}'$. Finally let e'' be the univalent edge of $\mathcal{T}''$. Then e'' has the same slope as e. Hence, any time we layer on the univalent edge we get two edges with the same slope.

Now, using this notation, if the edge e is a thick edge, then we do not have two thin edges with the same slope; however, this, which we call opening at a thick edge,
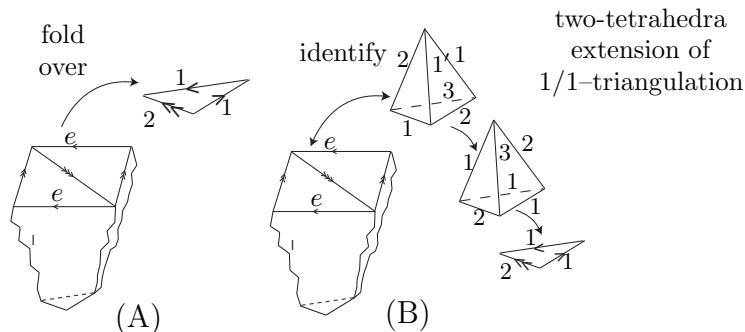
FIGURE 25. Examples of opening a layered-triangulation (A) at a thick edge and (B) at 1/1.

can only happen once. See Example 3 in the previous paragraph; this is precisely what is happening in that example.

When e is not a thick edge, this leads to a layered-triangulation of the solid torus that is not a nearly-minimal layered-triangulation. If such a triangulation of the solid torus is folded over to form a lens space, then the layered-triangulation of the lens space is typically not nearly-minimal. However, it is possible that we fold over the boundary of $\mathcal{T}''$ and e'' becomes a thick edge; i.e., we have *not* folded over along e''.

If this is the case, the two tetrahedra $\Delta \cup_{e'} \Delta'$, become a two-tetrahedra extension of the 1/1-triangulation after folding along an edge in the boundary of Δ'. Thus the layered-triangulation of the lens space is just an opening at 1/1 of the layered-triangulation of the *same* lens space had we folded over the boundary of the layered-triangulation $\mathcal{T}$. In this case, we have the situation given in Example 2 in the paragraphs preceding this Corollary.

Finally, if we fold over along e'', which is the univalent edge, then we have an edge with slope the meridian and the triangulation can not be nearly-minimal. It follows there are only finitely many ways to make the triangulation of a fixed lens space a nearly-minimally layered-triangulation.   □

Note that the typical situation would be at most five 1–efficient triangulations, the minimal one, opening along 1/1 at either end, which typically gives two more, opening at both ends and, finally, opening at the thick edge. There also are those paths in the L–graph that run through a number of vertices of the form $1/k$. For example, layered-triangulations of the lens space $L(n,1), n \geq 4$ are formed from a $(1/n-2)$–layered-triangulation by folding along the edge labeled $n-2$ in the boundary. We saw above in Example 3 that each of these have $n-3$ places where the triangulation can be opened at a thick edge. If we fold the two-tetrahedra 1/1–layered-triangulation along the thick edge, we get $L(3,1)$, which we can then open along the thick edge to get a nearly minimal four-tetrahedron triangulation of $L(3,1)$.

8.3. **Heegaard splittings of lens spaces.** We are able to use layered-triangulations for a new proof of the uniqueness of Heegaard splittings for S^3 and $S^2 \times S^1$, first done by F. Waldhausen [34], and for other lens spaces, first done by F. Bonohan and J.P. Otal [2]. While the combinatorics of this proof make it seem quite elementary,

we have lurking in the background, the observation, first made by Rubinstein [32], that for any triangulation of a closed 3–manifold, a strongly irreducible Heegaard surface is isotopic to an almost normal surface; and the observation by A. Casson and C. McA. Gordon [7] that for non Haken manifolds, an irreducible Heegaard surface is strongly irreducible. We comment here that the methods initiated by Rubinstein and presented in [32] do *not* use the classification of Heegaard splittings of S^3, as do those methods coming from thin position. One of our reasons for considering this alternate proof of uniqueness of Heegaard splittings for lens spaces is a belief that one can use nice triangulations to gain important topological/geometric information about three-manifolds as exhibited in this proof. In particular, these methods may extend to layered-triangulations of higher genera three-manifolds and offer analogous results. We use simple triangulations in [17] to study the Heegaard splittings of small Seifert fibered spaces.

We assume familiarity with basic notions and results regarding Heegaard splittings. We refer the reader to the exposition by M. Scharlemann [33].

We remind the reader that, except for the lens spaces $S^3, \mathbb{R}P^3, S^2 \times S^1$, and $L(3,1)$, the only normal surface in a minimal layered-triangulation of a lens space is a vertex-linking 2–sphere (possibly) with some thin edge-linking tubes. Hence, an almost normal tubed surface must be a vertex-linking 2–sphere (possibly) with some thin edge-linking tubes and an almost normal tube; or two such surfaces with an almost normal tube between them. There are no almost normal octagonal surfaces. We are able to characterize which of these surfaces are Heegaard surfaces.

If we have an almost normal tubed surface in a layered-triangulation of a lens space, we say the *almost normal tube is at the same level as a thin edge-linking tube* if the almost normal tube has at least one end in a quadrilateral of the thin edge-linking tube. See Figure 26.
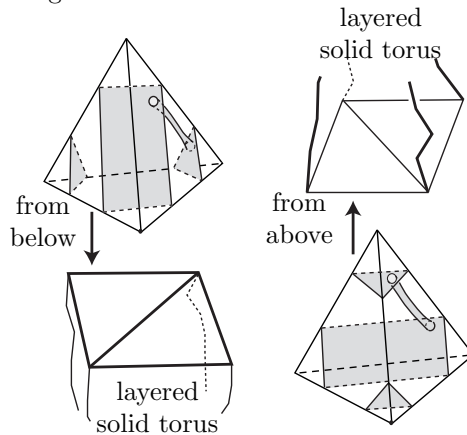
FIGURE 26. An almost normal tube at the same level as a thin edge-linking tube.

The following lemma and its corollary are used in our understanding of Heegaard splittings of lens spaces and also used later when we consider Heegaard splittings of Dehn-fillings. Note the following lemma is not necessarily true if we have an almost normal tubed surface or an almost normal octagonal surface without thin edge-linking tubes; also, in the case of the two-tetrahedron 0/1–layered-triangulation, we have an edge-linking annulus that is compressible.

8.7. **Lemma.** *Suppose $\mathcal{T}$ is a non-degenerate minimal layered-triangulation of the solid torus $\mathbb{T}$ distinct from the $0/1$–layered-triangulation. Furthermore, suppose A is an edge-linking annulus possibly with thin edge-linking tubes or an almost normal octagonal annulus with thin edge-linking tubes. Let B' denote the complementary annulus to A. Then A does not ∂–compress into B'.*

Proof. The embedded tori which correspond to a boundary torus at each level of the layering are called *level tori*. If we have a thin edge-linking tube of A, then in each level torus that meets the tube, we have a companion and complementary annulus. If we denote the component of the complement of A that meets B' by H', then the complementary annuli in the various level-tori that meet a thin edge-linking tube of A determine a family of parallel, properly embedded, annuli in H'.

We order the thin edge-linking tubes in A from top to bottom, $tube_1, tube_2, \ldots,$ $tube_k$. If we split the layered triangulation along a level torus meeting the ith tube of A, then we split off an initial segment of the layered solid torus $\mathbb{T}$, say $\mathbb{T}_i$, and split A into an edge-linking annulus A_i with tubes embedded as a normal surface in $\mathbb{T}_i$. For simplicity, we shall always assume we have done this at the lowest level possible for $tube_i$.

If A has no tubes, then a ∂-compression of A into B' gives that A must be longitudinal. But in a minimal layered-triangulation of a solid torus, there are no longitudinal normal annuli (only almost normal octagonal annuli). But we have assumed that if A is longitudinal, then it does have tubes. Thus, we may suppose A has tubes.

If A ∂–compresses into B', denote the compressing disk by D_0'. Two copies of D_0' along with a disk in B' determined a compressing disk D_0 for A and we may assume that D_0 does not meet any of the complementary annuli in the various level-tori that are parallel in H' to B', including, of course, B'.

We have that A has tubes. Consider the way D_0 meets the collection of complementary annuli to the tubes of A. We may assume, up to isotopy, D_0 has minimal intersection with these complementary annuli.

Suppose first that it may be the case that D_0 does not meet any of the complementary annuli. Then D_0 is a compressing disk in a region between two level tori in the layering and D_0 meets each of these level tori in annuli that correspond to complementary annuli for some edge slope. Thus we must have the edge slope of A the same as the slope of the edge that $tube_1$ links. But two edges can not have the same slope in a minimal layered-triangulation of the torus. So, if D_0 exists, it must meet some of the complementary annuli associated with the thin edge-linking tubes of A.

If we have a simple closed curve component in the intersection, then we have an innermost, on D_0, simple closed curve, say in the intersection with B_i'. If this curve were essential in B_i', this would mean we have an edge-slope the slope of a meridian, which can not happen in a minimal layered-triangulation distinct from the two-tetrahedra $0/1$–layered-triangulation of the solid torus. Hence, such a component of intersection must be trivial in B_i'. But now, using standard techniques, we could modify the intersection to reduce the number of components; this is a contradiction to our having minimal intersection. If we have a component of intersection that is an outermost, on D_0, spanning arc and if this is a component common with, say B_i', then we have a boundary compression of A_i into B_i', or the arc of intersection is inessential in B_i'. The latter would contradict that the intersection of D_0 with the

collection of complementary annuli is minimal; hence, we must have a boundary compression of A_i into B'_i.

We now let A_i and B'_i play the role of A and B' in the above argument and let D'_i denote the outermost disk in D_0 that gives the boundary compression of A_i into B'_i. Again, we have a compressing disk, say D_i, that plays the role of the disk D_0 above. Continuing in this way, we finally have as the only possibility, that for some A_j, we have a compressing disk that meets no complementary annuli. But as above, this means that two edges must have the same slope. This contradiction completes the argument.   □

We are now ready to characterize those normal or almost normal surfaces in a minimal layered-triangulation of a lens space that are Heegaard surfaces. First, we consider lens spaces distinct from $S^3, \mathbb{R}P^3, S^2 \times S^1$, and $L(3,1)$; following this we will consider these special cases.

8.8. Theorem. *Suppose L is a lens space distinct from $S^3, \mathbb{R}P^3, S^2 \times S^1$, and $L(3,1)$. A normal or almost normal surface S in a minimal layered-triangulation of L is a Heegaard surface if and only if S has an almost normal tube along a thick edge or S has an almost normal tube at the same level as a thin edge-linking tube. In particular, there are no normal or almost normal octagonal Heegaard surfaces in a minimal layered-triangulation of L.*

Proof. We begin by showing that either of these conditions is sufficient for an almost normal surface to be a Heegaard surface.

Suppose S is an almost normal surface with an almost normal tube along a thick edge in a minimal layered-triangulation of L. Since we have excluded $S^3, \mathbb{R}P^3, S^2 \times S^1$, and $L(3,1)$, the layered-triangulation of L has no creased 3–cells and no layering along a univalent edge. Hence, the layered-triangulation of L is formed by folding along an edge in the boundary of a minimal (non-degenerate) layered-triangulation of a solid torus. We may assume we have opened the layered-triangulation of the solid torus so that the thick edge along which we have an almost normal tube is in the one-tetrahedron solid torus at the bottom of the layered-triangulation of the solid torus. If S has more than one tube, then the thin edge-linking tubes come in an order in the layered solid torus, ordering from the bottom toward the top, where we have opened the lens space triangulation. Beginning with the first thin edge-linking tube from the bottom, there is a pinched disk embedded in the layered solid torus between the edge the tube is linking and a curve in the boundary of the one-tetrahedron solid torus at the bottom of the solid torus. Thus this tube unwinds about the thick edge and corresponds to a trivial handle attached to S. We can remove this handle to get an almost normal surface S_1 with an almost normal tube along a thick edge. In this way we see that S is a stabilization of the standard genus one Heegaard surface of L and therefore is a Heegaard surface.

Now, suppose S is an almost normal surface with an almost normal tube at the same level as a thin edge-linking tube. Then there is a tetrahedron in our layering as shown in Figure 26. After a small isotopy of the almost normal tube, it and the thin edge-linking tube provide a spine in the torus where they occur at the same level. Hence, if there are any other tubes (necessarily thin edge-linking), they come in an order and thus any one adjacent to the level of the spine would unwind about it and form a trivial handle. It follows that S is obtained by adding trivial handles to a surface S', which is the boundary of a small neighborhood of the spine of a level

torus; so, S' bounds a genus-2-handlebody, say H', on one side. However, the other side of S' is two solid tori attached by a 1-handle determined by the complementary disk of this spine on the level torus and therefore S' is a genus 2 Heegaard surface for L. The surface S is a stabilization of S' and thus S is a Heegaard surface for L.

We have shown that an almost normal surface that has either an almost normal tube along a thick edge or an almost normal tube at the same level as a thin edge-linking tube, in a minimal layered-triangulation of a lens space distinct from $S^3, \mathbb{R}P^3, S^2 \times S^1$, and $L(3, 1)$, is sufficient for it to be a Heegaard surface.

Conversely, suppose we have a normal or almost normal Heegaard surface S splitting L into the handlebodies H and H'. Then S is the vertex-linking 2–sphere with tubes, possibly an almost normal tube, or two vertex-linking 2–spheres, one necessarily with thin edge-linking tubes, that are connected by an almost normal tube. In particular, S can not be an almost normal octagonal surface.

We first consider the case that we have a vertex-linking 2–sphere with tubes. If S has only one tube, then for S to be a Heegaard surface, S must be an almost normal surface with the almost normal tube along a thick edge. So, we may assume S has more than one tube and thus must have a thin edge-linking tube. Since there can be only one almost normal tube, there is a direction in the layering so that the last tube is a thin edge-linking tube. If we split the layering and the thin edge-linking tube at this level, we have two solid tori, each with a layered-triangulation, $\mathbb{T}$ and $\mathbb{T}'$, and the surface S is split into a thin edge-linking annulus with tubes, $A \subset \mathbb{T}$, and a thin edge-linking annulus, $A' \subset \mathbb{T}'$. It is possible that $\mathbb{T}'$ is degenerate (a Möbius band) and A' is also degenerate (a simple closed curve).

Recall that if A has an almost normal tube along an edge and if it does not have the almost normal tube along the thick edge, or at the same level as a thin edge-linking tube, or at the same level as the edge-linking annulus associated with A, then A is isotopic, keeping the boundary of $\mathbb{T}$ fixed, to a *normal* surface that is a thin edge-linking annulus with all tubes thin edge-linking. Thus if A does not have an almost normal tube along the thick edge or does not have an almost normal tube at the same level as a thin edge-linking tube, including the level of the edge-linking annulus associated with A, then we may assume A is normal.

On the other hand, if we let B' denote the complementary annulus to A and have chosen notation so $B' \subset H'$, then B' must compress or ∂–compress in H'. B' can not compress, since no edge has the slope of a meridian. Also, B' can not ∂–compress into $\mathbb{T}'$, since A' can not be a longitudinal. It follows that the only possibility is that B' ∂–compresses into $\mathbb{T}$. But this is the same as A ∂–compressing into $\mathbb{T}$. However, by Lemma 8.8, this is impossible for A normal.

In particular, a normal surface can not be a Heegaard surface.

If S were two vertex-linking 2–spheres connected by an almost normal tube, then we conclude that S can not be a Heegaard surface. For in this situation, if we compress the almost normal tube, then we have two vertex-linking 2–spheres, one necessarily with tubes, say S_1 and S_2, where we may assume S_1 separates S_2 from the vertex (they are nested). If S separates L into the handlebodies H and H', then notation may be chosen so that S_2 bounds a handlebody that is a disk connected summand of H' and S_1 is disjoint from this summand of H'. However, if this were the case, then S_2 would necessarily bound a handlebody on both sides and be, itself, a Heegaard surface. But the earlier argument concludes that a normal surface can not be a Heegaard surface.

We conclude that a Heegaard surface, in a lens space distinct from $S^3, \mathbb{R}P^3, S^2 \times S^1$, and $L(3,1)$, is a vertex-linking 2–sphere with an almost normal tube along a thick edge or an almost normal tube at the same level as a thin edge-linking tube. $\square$

We now consider those cases excluded in Theorem 8.8; i.e., the lens space L is one of $S^3, \mathbb{R}P^3, S^2 \times S^1$, or $L(3,1)$.

For S^3:

By Theorem 7 a normal or almost normal surface in the one-tetrahedron layered-triangulation of S^3 is: if normal, then it is the vertex-linking 2–sphere or the vertex-linking 2–sphere with a thin edge-linking tube about the edge 3 (the trefoil knot); if almost normal, then it is the vertex-linking 2–sphere with an almost normal tube along the thick edge, or the vertex-linking 2–sphere with a thin edge-linking tube along the edge 3 and an almost normal tube along the thick edge, or the almost normal octagonal 2–sphere. There also is the vertex-linking 2–sphere with an almost normal tube along the edge 3 but it is isotopic to the vertex-linking 2–sphere with a thin edge-linking tube along the edge 3. All of these but the vertex-linking 2–sphere with a thin edge-linking tube along the edge 3 is a Heegaard surface. It is curious that the genus 2 example (the vertex-linking 2–sphere with a thin edge-linking tube along the edge 3 and an almost normal tube along the thick edge gives the classical example of an embedding of a genus 2 handlebody in S^3 with one handle the trefoil knot and the other unknotted but linking in such a way that the complement has closure a handlebody. Just as in this example, the edge 3 (the trefoil knot) unwinds about the trivial handle; thus the genus 2 example is stabilized. Finally, the minimal layered-triangulation of S^3 is the only minimal layered-triangulation of a lens space with a normal Heegaard surface (the vertex-linking 2–sphere). In addition, and as expect by the theory of sweep-outs, this triangulation of S^3 *must* have an almost normal octagonal 2–sphere; it is the only octagonal Heegaard surface in a minimal layered-triangulation of a lens space.

For $\mathbb{R}P^3, S^2 \times S^1$ and $L(3,1)$:

In each of these cases, we have two-tetrahedra layered-triangulations. For $L(3,1)$ we use the minimal (two-tetrahedra) $1/1$–layered-triangulation of the solid torus and fold over along the thick edge. Again, the argument is just a matter of listing the possible normal and almost normal surfaces for the lens spaces given in Theorem 7. In each case we have the same characterization as in Theorem 8.8; namely, a normal or almost normal surface is a Heegaard surface if and only if there is an almost normal tube at the same level as a thin edge-linking tube or there is an almost normal tube along a thick edge. There are no normal Heegaard surfaces.

8.9. **Corollary.** *An almost normal Heegaard surface of genus ≥ 2 in a minimal layered-triangulation of a lens space is stabilized.*

Proof. If the genus is at least 2, then by Theorem 8.8 and the preceding paragraphs, there is an almost normal tube along a thick edge or there is an almost normal tube at the same level as a thin edge-linking tube. Furthermore, we showed that an almost normal tube along a thick edge and any other tube leads to a stabilization; and an almost normal tube along a thin edge-linking tube and any other thin edge-linking tube, leads to a stabilization. Hence, we only need to show that in the genus

2 case, when we have an almost normal tube at the same level as a thin edge-linking tube, we get a stabilization.

In this case, we may consider the thin edge-linking tube and the almost normal tube in the same level-torus, sat T; thus we have a spine, say Γ, for T made up of the edge which the thin edge-linking torus links and the core of the almost normal tube, which may also be taken as an edge. Let $N(\Gamma)$ be a small neighborhood of Γ in T, which is a once-punctured torus, and consider the small product neighborhood $N(\Gamma) \times I$. $N(\Gamma) \times I$ is (up to isotopy) one of the handlebodies of the genus 2 Heegaard splitting. The other handlebody is the two solid tori, say $\mathbb{T}_1$ and $\mathbb{T}_0$, complementary to the interior of $T \times I$, which are connected by the solid tube that is the closure of $(T \setminus N(\Gamma)) \times I$. Since Γ is a spine for T, there is a meridional disk D_0 in the solid torus $\mathbb{T}_0$ with boundary in the interior of $N(\Gamma) \times \{0\}$. On the other hand, since $N(\Gamma) \times \{0\}$ is a punctured torus, there is an arc α in $N(\Gamma) \times \{0\}$ that crosses the boundary of D_0 precisely once. Let $D = \alpha \times I$. Then D_0 and D determine a stabilization for the genus 2 Heegaard splitting.

Note that following this stabilization, the genus 1 Heegaard splitting may not be normal or almost normal but after an isotopy, it is a vertex-linking 2–sphere with an almost normal tube along a thick edge. □

8.10. **Theorem.** [34, 2] *Up to isotopy a lens space has a unique genus g Heegaard splitting for every $g \geq 1$.*

Proof. It is sufficient to consider only irreducible Heegaard surfaces and splittings with genus ≥ 1. By [7], for a lens space an irreducible Heegaard surface is strongly irreducible; hence, by Rubinstein [32], for any triangulation such a Heegaard surface is isotopic to an almost normal surface. In particular, this is true for a minimal layered-triangulation of the lens space (for the lens space $L(3, 1)$, we can avoid the bad minimal layered-triangulation). Now, from the above, if the Heegaard surface is irreducible, it must be genus $g \geq 1$ and is the vertex-linking 2–sphere with an almost normal tube along a thick edge. □

8.4. **Dehn fillings of knot and link manifolds.** Our encounter with layered-triangulations began with a problem on Dehn filling a knot-manifold (a compact, orientable, irreducible 3–manifold, with a single incompressible torus boundary). In particular, we had constructed a one-vertex triangulation, $\mathcal{T}$, of the knot-manifold, X, along with a designated slope α on ∂X. We wanted a one-vertex triangulation of the Dehn filling of X along α that retained (extended) the triangulation $\mathcal{T}$ of X. In addition, we needed to understand how the normal and almost normal surfaces in the triangulation of the Dehn filling could meet the solid torus of the filling.

So, suppose X is a knot-manifold and $\mathcal{T}$ is a one-vertex triangulation of X. Let α be a slope on ∂X. The isotopy class of α in ∂X determines a unique triple of intersection numbers with the three edges in the one-vertex triangulation of the torus, ∂X, say $\{p, q, p + q\}, 0 \leq p < q$. Let $\mathbb{T}$ denote a solid torus with the minimal layered-triangulation extending the p/q-triangulation on its boundary. We can attach $\mathbb{T}$ to X via a simplicial isomorphism from $\partial \mathbb{T} \to \partial X$ taking the edges with intersection numbers $\{p, q, p + q\}$ in $\partial \mathbb{T}$ to those with corresponding intersection numbers $\{p, q, p+q\}$ in ∂X. Hence, we get the closed 3–manifold $X(\alpha)$ along with a unique triangulation extending the triangulation $\mathcal{T}$ of X. We denote this triangulation by $\mathcal{T}(\alpha)$ and say we have a *triangulated Dehn filling* (of X with respect to $\mathcal{T}$). Without changing the triangulation $\mathcal{T}$, we get unique triangulated

Dehn fillings $\mathcal{T}(\alpha)$ of $X(\alpha)$ for all slopes α. In case the triple is either $\{0,1,1\}$ or $\{1,1,2\}$, we can use the creased 3–cell or the one-triangle Möbius band to achieve a triangulated Dehn filling. In these cases, there will be identifications on the faces of $\mathcal{T}$ that are in the boundary torus. One last remark. Typically, when working with triangulated Dehn fillings, it is convenient to have that the frontier of a small regular neighborhood of the layered solid torus is a normal torus. This is equivalent to the frontier of a small neighborhood of the boundary of X being normal in $\mathcal{T}$. If this is the case, we say ∂X *is normal in* $\mathcal{T}$. It is not necessary that the boundary of a manifold be normal in a triangulation of the manifold. However, we do show in [16] that any compact 3–manifold with nonempty boundary, no component of which is a 2–sphere, admits a triangulation with normal boundary; furthermore, we have such triangulations with all vertices in the boundary and just one vertex in each boundary component. Hence, normal boundary and minimal vertex triangulations are not necessarily restrictions. Throughout our work we assume that any triangulation of X, used in triangulated Dehn filling, has normal boundary.

Heegaard splittings of Dehn fillings. The study of Heegaard splittings of Dehn fillings has attracted much interest. A key to success in these efforts is an understanding of how a strongly irreducible Heegaard surface can meet a solid torus. This was done in [25], followed by new proofs in [33], and has been studied extensively by Y. Reick and E. Sedgwick [29, 30, 31]. A reader familiar with these earlier results can easily see that layered-triangulations of the solid torus provide an ideal combinatorial environment for an investigation of how a normal or almost normal, strongly irreducible, Heegaard surface can meet the solid torus in a triangulated Dehn filling.

We give a combinatorial proof of a theorem that is in the spirit of the main result in both the work of Moriah-Rubinstein [25] and Scharlemann [33]. We follow the format of the statement in [33]; however, here we give a modest generalization and we note that possibility $3(c)$ in the conclusion of Theorem 8.12 seems to have been missed in the earlier versions. Following this result, we are able to give a completely general analysis of how a strongly irreducible Heegaard surface can meet a solid torus with a minimal layered-triangulation. We extend our investigation and give examples in both [12] and [16].

Suppose S is a normal or almost normal surface in a triangulated 3–manifold meeting a subcomplex $\mathbb{T}$ that is a solid torus with a minimal layered-triangulation. Furthermore, suppose S does not meet $\partial\mathbb{T}$ in any trivial simple closed curves. From our classification of normal and almost normal surfaces in a minimal layered triangulation of a solid torus, it follows that S can meet $\mathbb{T}$ in at most one of:

(1) a collection of meridional disks, possibly with an almost normal tube, or
(2) an almost normal octagonal annulus (possibly) with thin edge-linking tubes, or
(3) a collection of normal edge-linking annuli (possibly) with tubes, including the possibility of an almost normal tube.

In case (3) the components of $S \cap \mathbb{T}$ nest in the sense that the annuli must all be parallel (normally isotopic) edge-linking annuli, each separates $\mathbb{T}$ into two components, one containing the vertex and, if A_i and A_j are components of $S \cap \mathbb{T}$ and A_i separates A_j from the vertex, then any thin edge-linking tube for A_i must

run through a thin edge-linking tube for A_j. Similarly, if there is an almost normal tube between two normal surfaces and if both have thin edge-linking tubes, then we also have the thin edge-linking tubes of one running through the thin edge-linking tubes of the other. In either of these situations we say we have *nested tubes* in $S \cap \mathbb{T}$.

8.11. **Lemma.** *If in the above situation, S is a strongly irreducible Heegaard surface, then there are no nested tubes in $S \cap \mathbb{T}$.*

Proof. If there were nested tubes, then there is an edge e and thin edge-linking tubes t and t' about e so that the meridian disk d' for t' contains the meridian disk d for t and only meets S in ∂d (see Figure 27). Let C denote the annulus in the disk d' complementary to the interior of d.
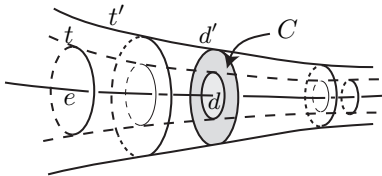

FIGURE 27. There are no nested tubes.

The annulus C is contained in one of the handlebodies bounded by S and has both of its boundary components contained in S; so, it must compress or ∂-compress. If C compresses, then using the disk determined by the compression and the disk d, we have a reduction of S, which is impossible. If C ∂-compresses, then since C does not separate, C is not parallel into the boundary of S and so, we get a disk upon boundary compression that along with the disk d gives a weak reduction of S, which also is impossible. $\square$

8.12. **Theorem.** *Suppose X is a knot-manifold with a one-vertex triangulation $\mathcal{T}$, α is a slope on ∂X, and $\mathcal{T}(\alpha)$ is the triangulated Dehn filling of X with respect to $\mathcal{T}$. Let $\mathbb{T}$ denote the solid torus of the Dehn filling. If S is a strongly irreducible Heegaard surface in $X(\alpha)$ that is normal or almost normal in $\mathcal{T}(\alpha)$ and $S \cap \partial\mathbb{T}$ has no components that are vertex-linking simple closed curves, then S meets $\mathbb{T}$ in one of the following:*

    (1) *a collection of meridional disks along with possibly one other component that is two meridional disks with an almost normal tube between them along an edge in $\partial\mathbb{T}$, or*

    (2) *a longitudinal, almost normal, octagonal annulus, or*

    (3) *a collection of edge-linking normal annuli and possibly one other component that is obtained either*

        (a) *by adding an almost normal tube between two of the annuli, along an edge in $\partial\mathbb{T}$, or*

        (b) *by adding an almost normal tube to the outermost annulus, along an edge in the complementary annulus, or*

        (c) *by adding an almost normal tube to the outermost annulus along the thick edge.*

Proof. First consider the situation where α does not have intersection numbers $\{0, 1, 1\}$ or $\{1, 1, 2\}$.

Suppose $M = H \cup_S H'$ is a strongly irreducible Heegaard splitting of $X(\alpha)$. We have the possibilities $(1), (2)$ and (3) from above, where in case (3) we have no nesting. We analyze each of these possibilities.

Case (1) $S \cap \mathbb{T}$ is a collection of meridional disks, possibly with an almost normal tube.

Note that any almost normal tube from a meridional disk to itself or between two meridional disks can be isotoped to an almost normal tube along an edge in $\partial \mathbb{T}$. If such a tube were on a single meridional disk, then we would have a weak reduction, which is a contradiction. Hence, we are left only with possibility (1) in the statement of the theorem.

Case (2) $S \cap \mathbb{T}$ is an almost normal octagonal annulus (possibly) with thin edge-linking tubes.

Let B' be the complementary annulus to A and suppose $B' \subset H'$. B' must compress or ∂–compress into H'. Since B' has the slope of an edge, it is not trivial in $\partial \mathbb{T}$ nor can it have the slope of a meridian. It follows that B' can not compress. We have that B' must ∂–compress.

If B' ∂–compresses to the outside of $\mathbb{T}$, then we can use this ∂–compression to construct a properly embedded disk D in H'. If D is not essential, then $S = A \cup A'$, where A' is an annulus parallel into $\mathbb{T}$. But then we would have X in H and ∂X incompressible in H; this is impossible. So, if B' ∂–compresses outside of $\mathbb{T}$, the disk D must be essential. Note, however, by S being strongly irreducible, then we can have no thin edge-linking tubes for A, for otherwise, we would get a weak reduction of S. It follows in this case that A is an almost normal octagonal annulus without thin edge-linking tubes. Hence, A is either longitudinal or is isotopic to a normal edge-linking annulus. The former gives us conclusion (2) of the theorem and the latter gives us conclusion (3).

Now, suppose B' does not compress and does not ∂–compress to the outside of $\mathbb{T}$. Thus B' must ∂–compress into $\mathbb{T}$. This is equivalent to saying that A ∂–compresses into B'. But by Lemma 8.7, this is impossible unless A is a longitudinal almost normal octagonal annulus with no tubes. Again, we have conclusion (2) of the theorem.

Case (3) $S \cap \mathbb{T}$ is a collection of normal edge-linking annuli (possibly) with tubes, including the possibility of an almost normal tube.

Hence, the only possibility for $S \cap \mathbb{T}$ is a number of copies of the same annulus with the outer most one (possibly) having thin edge-linking tubes and (possibly) an almost normal tube either between two of these annuli or attached to the outermost one. Notice that any almost normal tube between any two of these annuli is isotopic to an almost normal tube between them and along an edge in $\partial \mathbb{T}$.

Let A denote the outermost annulus (possibly) with tubes. While the argument here is similar to that above in Case (2), we may have an almost normal tube on A. However, if A does has an almost normal tube, it may be isotoped to form a new thin edge-linking tube on A or it is at the same level as a thin edge-linking tube, or along the thick edge, or, possibly, at the same level as the thin edge-linking annulus associated with A.

So, we may assume we either have no almost normal tube or we have an almost normal tube at the same level as a thin edge-linking tube, or along the thick edge, or, possibly, at the same level as the thin edge-linking annulus associated with A.

We shall consider, first, the case where we must have an almost normal tube. If we have the almost normal tube along the thick edge, then other thin edge-linking tubes could be unwound about the tube along the thick edge and thus would give us a reduction. If we have the almost normal tube at the level of a thin edge-linking tube or at the level of the annulus associated with A, Then we have a spine for a level torus at that level in A and thus any other thin edge-linking tubes could be unwound about this spine and thus would give us a reduction. We conclude that A is precisely one of:

(a) an edge-linking annulus with a single thin edge-linking tube and an almost normal tube at the same level as this thin edge-linking tube, or

(b) an edge-linking annulus with an almost normal tube along an edge in the complementary annulus, or

(c) an edge-linking annulus with an almost normal tube along the thick edge.

But as in the proof of Corollary 8.9, if we have an almost normal tube at the same level as a thin edge-linking tube, then we have a reduction. We conclude we can not have situation (a). Situation (b) gives conclusion $3(b)$ and situation (c) gives conclusion $3(c)$ of the theorem.

Finally, we consider the situation where we do not have an almost normal tube; that is, the outermost edge-linking annulus A does not have an almost normal tube. Let B' be the complementary annulus to A and suppose notation has been chosen so that $B' \subset H'$. Then B' must compress or ∂–compress in H'. As above, it can not compress; it therefore, must ∂–compress.

If it ∂–compresses to the outside, we get a properly embedded disk D in H'. If D is essential, then A has no tubes or we would have a weak reduction of S; if it is inessential, then $S = A' \cup A$, where A' is an annulus parallel into $\mathbb{T}$. Again, as we saw above, this would place $\partial \mathbb{T}$ as an incompressible torus in H, a contradiction. Hence, in this situation, we conclude that A has no tubes. This gives conclusion (3) of the theorem where the collection has no component with tubes.

Since A is normal, by Lemma 8.7, B' can not compress or ∂–compress into $\mathbb{T}$. $\square$

While Theorem 8.12 is in a form analogous to those in [25, 33], our version does not exclude curves in the intersection between the Heegaard surface and the boundary of the solid torus that have meridional slope. In fact, we can extend our analysis to the general case, allowing all possible curves of intersection between the Heegaard surface and the boundary of the solid torus. This is a telling difference in using normal and almost normal surfaces and layered-triangulations; we can address situations that were totally intractable without these tools. We now state the result in the general case; the earlier version provides a rather direct comparison between our methods and the earlier methods. Furthermore, by allowing inessential curves of intersection, the proofs are not more difficult but there are more possibilities for almost normal tubes, which makes the conclusions a bit unwieldy.

8.13. **Theorem.** *Suppose X is a knot-manifold with a one-vertex triangulation $\mathcal{T}$, α is a slope on ∂X, and $\mathcal{T}(\alpha)$ is the triangulated Dehn filling of X with respect to $\mathcal{T}$. Let $\mathbb{T}$ denote the solid torus of the Dehn filling. If S is a strongly irreducible*

Heegaard surface in $X(\alpha)$ that is normal or almost normal in $\mathcal{T}(\alpha)$, then S meets $\mathbb{T}$ in (possibly) a collection of vertex-linking disks and one of the following:

(1) *a collection of meridional disks along with possibly one other component that is formed by adding an almost normal tube along an edge in $\partial\mathbb{T}$ between two meridional disks, or between two vertex-linking disks, or between a meridional disk and a vertex-linking disk, or*

(2) *a longitudinal, almost normal, octagonal annulus, or*

(3) *a collection of edge-linking normal annuli and possibly one other component that is obtained either*

   (a) *by adding an almost normal tube along an edge in $\partial\mathbb{T}$ between two of the annuli, or between two vertex-linking disks, or between a vertex-linking disk and the innermost edge-linking annulus, or*

   (b) *by adding an almost normal tube to the outermost annulus, along an edge in the complementary annulus, or*

   (c) *by adding an almost normal tube to the outermost annulus along the thick edge, or*

(4) *a vertex-linking disk with an almost normal tube along the thick edge.*

Proof. The really new part of the theorem is conclusion (4). We have this possibility since we now have that the intersection between S and $\mathbb{T}$ might be a collection of vertex-linking disks, possibly, with thin edge-linking tubes along with a collection of edge-linking linking annuli, possibly, with thin edge-linking tubes. However, we note that the nesting of tubes in the case of just edge-linking annuli with tubes, extends to having vertex-linking disks with tubes and edge-linking annuli with tubes. But as before, there can be no nesting of tubes. Thus if we have any edge-linking annuli, then the vertex-linking disk will not have any thin edge-linking tubes and we are really back in the case of considering edge-linking annuli.

So, we may assume the intersection between S and $\mathbb{T}$ is a collection of vertex-linking disks with, possibly, the outermost one having tubes and, possibly one other component that is formed by adding an almost normal tube between two vertex-linking disks or between a vertex-linking disk and the outermost vertex-linking disk whit thin edge-linking tubes.

An almost normal tube on the outermost vertex-linking disk can be isotoped to a thin edge-linking tube, which we also do should the almost normal tube be along a thin edge in $\partial\mathbb{T}$, or it is along the thick edge, or at the same level as a thin edge-linking tube. If we have an almost normal tube along a thick edge, then there can be no other tubes or we would have a reduction. This gives us conclusion (4). If there is an almost normal tube at the same level as a thin edge-linking tube, then in any case there must be a reduction, which is a contradiction. So, we have conclusion (4) or we may assume that all the tubes from the outermost vertex-linking disk are thin edge-linking.

We can order the thin edge-linking tubes from the top to the bottom. Having done this, we can split the layered solid torus at the level of the first tube. Using an initial segment of the layered-triangulation of the solid torus, we have the Heegaard surface S meeting this solid torus in a collection of vertex-linking disks and a thin edge-linking annulus, possibly with thin edge-linking tubes and, possibly one other component formed by adding an almost normal tube between two vertex-linking disks or between this edge-linking annulus and a vertex-linking disk. Now, as in earlier arguments the complementary annulus to the annulus we have must compress

or ∂–compress into one side of the Heegaard splitting. However, the slope of the boundary of this annulus can not be meridional nor longitudinal; hence, the only possibility is that it ∂–compress into $\mathbb{T}$. But by Lemma 8.7, this is impossible.

We find that in this situation we have conclusion (4) or we have a collection of vertex-linking disks and, possibly, one other component that is two vertex-linking disks with an almost normal tube between them and along an edge in $\partial\mathbb{T}$. $\square$

We end this section with a summary which can be drawn from the conclusions of the previous theorem.

  (1) The intersection contains a meridional disk. There are only finitely many slopes [15] for which this can happen. It is typical to exclude these slopes; however, in some examples, the normal surfaces in the knot space can be analyzed and often we can make conclusions about such exceptional slopes.

  (2) The intersection contains a longitudinal, almost normal, octagonal annulus. The knot (core of the solid torus) is in the Heegaard surface.

  (3) The intersection includes any of the possibilities with an edge-linking annulus. Then the Heegaard splitting is a Heegaard splitting of the knot manifold X. In some of these possibilities, we use the so-called Daisy Lemma of Rieck and Sedgwick [30].

  (4) The intersection contains the vertex-linking disk with an almost normal tube along the thick edge. Then the Heegaard splitting is a Heegaard splitting of the knot manifold X.

Efficient triangulations of Dehn fillings. The following result on efficiency of triangulated Dehn fillings indicates some of the techniques for studying interesting surfaces in Dehn fillings; however, as in the study of Heegaard splittings of Dehn fillings, it requires knowledge of normal surfaces in the knot-manifold that is being filled. We, again, point out that triangulated Dehn fillings fix a triangulation of the knot-manifold that remains for all Dehn fillings; so, in particular, the collection of normal and almost normal surfaces and, hence, the slopes of these surfaces in the knot-manifold are the same for all fillings.

We consider when a 0–efficient triangulation of a knot-manifold carries over to a 0–efficient triangulation of a Dehn filling of the manifold. Recall from [14] that a triangulation of a closed 3–manifold is said to be 0–*efficient* if and only if the only normal 2–spheres are vertex-linking; for a compact 3–manifold with boundary, a triangulation is 0–*efficient* if and only if the only normal disks are vertex-linking. The latter condition guarantees that there are no normal 2–spheres and the manifold has incompressible boundary. In our consideration of Dehn fillings, we shall assume we have a minimal triangulation of the knot-manifold and consider triangulated Dehn fillings, extending this triangulation. It follows from [14] that a minimal triangulation of a compact, irreducible, ∂–irreducible 3–manifold is 0–efficient; so, a minimal triangulation of a knot-manifold is 0–efficient and has just one vertex.

8.14. **Theorem.** *Suppose X is a knot-manifold and $\mathcal{T}$ is a minimal triangulation of X. Then for all but finitely many slopes α the triangulated Dehn filling $\mathcal{T}(\alpha)$ is 0–efficient.*

Proof. Let $\mathbb{T}(\alpha)$ denote the solid torus in the triangulated Dehn filling $\mathcal{T}(\alpha)$. $\mathbb{T}(\alpha)$ has a minimal layered-triangulation determined by the slope α and possibly is degenerate (a Möbius band or creased 3–cell).

Suppose Σ is a normal 2–sphere embedded in $\mathcal{T}(\alpha)$. Since we have assumed $\mathcal{T}$ is minimal (0–efficient by [14]), $\Sigma \cap \mathbb{T}(\alpha) \neq \emptyset$. The possibilities for the components of $\Sigma \cap \mathbb{T}(\alpha)$ are:

(1) a collection of vertex-linking disks, or
(2) a collection of meridional disks, possibly, with a collection of vertex-linking disks, or
(3) a collection of edge-linking annuli, possibly, with a collection of vertex-linking disks.

Since the triangulation $\mathcal{T}$ for X is fixed, there are only finitely many slopes that are boundary slopes for normal (or almost normal) surfaces in $\mathcal{T}$ [15]; hence, if we avoid these slopes, then we may assume possibility (2) does not occur. In our particular situation, the only slopes that need to be avoided are slopes for normal planar surfaces in $\mathcal{T}$. These slopes can be algorithmically determined from planar surfaces at the vertices of the projective solution space for $\mathcal{T}$; and, in the case of essential planar surfaces we need to avoid at most three slopes [9]. On the other hand, note that excluding surgery slopes that are slopes of embedded normal surfaces in the knot-manifold does not exclude possibility (3), as boundary slopes of normal annuli in the solid torus come in several varieties and *a priori* can match with slopes of normal surfaces embedded in the knot-manifold, even if excluded as surgery slopes. However, we see below that, almost magically, possibility (3) can not occur.

Assuming that (2) does not occur, we consider disk components (innermost disks) in $\Sigma \setminus (\Sigma \cap \partial \mathbb{T}(\alpha))$. If such an innermost disk were in X, then its boundary must be a vertex-linking curve ($\mathcal{T}$ is 0–efficient). But a component of $\Sigma \cap X$ with a trivial boundary can only match up with a vertex-linking disk in $\mathbb{T}$, leaving only the possibility that we are in (1) and Σ is the vertex-linking 2–sphere. Thus all disk components in $\Sigma \setminus (\Sigma \cap \partial \mathbb{T}(\alpha))$ must be in $\mathbb{T}(\alpha)$ and, therefore, vertex-linking.

If there is any curve in $\Sigma \cap \partial \mathbb{T}(\alpha)$ that does not bound a disk component of $\Sigma \setminus (\Sigma \cap \partial \mathbb{T}(\alpha))$, then there is a component, say P, of $\Sigma \setminus (\Sigma \cap \partial \mathbb{T}(\alpha)) \subset X$, where all curves in its boundary, except one, say δ, bound innermost disks in Σ; we call δ an outermost boundary for P. Each boundary curve of an innermost disk in Σ is a vertex-linking curve and thus P can be capped off with disks in ∂X leading to δ bounding an embedded disk in X. But ∂X is incompressible, it follows that δ must be vertex-linking in ∂X as well. But then it too would bound an innermost disk in $\mathbb{T}(\alpha)$. We conclude that Σ meets X in a planar surface all of whose boundary components bound vertex-linking disks in $\mathbb{T}(\alpha)$. It follows that possibility (3) can not occur. We are left with only possibility (1).

We note that in particular, our methods show that except for possibly the finite number of excluded slopes, $X(\alpha)$ is irreducible.

If we have (1), then we need to call upon deeper results from [14]; namely, we show that if Σ is not vertex-linking and we are in (1), then we can crush Σ and contradict the minimality of $\mathcal{T}$. There are a number of conditions to verify before we can crush Σ. We refer the reader to [14], Section 4, and in particular, Theorem 4.1.

First, we must have Σ bounding a 3–cell that contains the vertex. Since X is irreducible and Σ is isotopic into X, Σ bounds a 3–cell. If the vertex is not in the 3–cell bounded by Σ, then we can engulf the vertex, getting a new normal 2–sphere bounding a 3–cell, B, that contains the vertex. (It is possible in the engulfing

process we arrive at a punctured 3–sphere containing the vertex and having all boundary components normal 2–spheres; but as noted, $X(\alpha)$ is irreducible and from above, normal 2–spheres can not meet the filling torus in other than vertex-linking disks. So we get the desired conclusion.) We shall continue to call this 2–sphere Σ and note that since we have excluded Dehn-fillings along slopes of planar normal surfaces in X, then this new 2–sphere meets $\mathbb{T}(\alpha)$ only in vertex-linking disks, and is not, itself, vertex-linking.

Let $Y = X(\alpha)\backslash \overset{\circ}{B}$. Then Y has a nice cell-decomposition consisting of truncated-tetrahedra, truncated-prisms, and triangular and quadrilateral product blocks. We denote this cell-decomposition of Y by $\mathcal{C}$.

We need to observe that there are not too many product blocks; i.e., $\mathcal{P}(\mathcal{C}) \neq Y$. However, since Y does not meet $\mathbb{T}(\alpha)$ in only product blocks, we have that $\mathcal{P}(\mathcal{C}) \neq Y$. Next we need to assure that we can fill product blocks in Y to get trivial product blocks, denoted $\mathcal{P}(Y)$. However, each region that needs to be filled has a 2–sphere boundary and is contained in X; since X is irreducible, we can fill these blocks missing $\mathbb{T}(\alpha)$. So, $\mathcal{P}(Y) \neq Y$.

Now, we must consider having too many truncated prisms; i.e., possible cycles of truncated-prisms that are not in $\mathcal{P}(Y)$. If we have a cycle of truncated-prisms about a single edge, then we have a compression and obtain two new normal 2–spheres by adding a 2–handle to Σ; neither of these 2–spheres is vertex-linking. Furthermore, one of these 2–spheres bounds a 3–cell containing the 3–cell B and the vertex and meets $\mathbb{T}(\alpha)$ in a subcollection of the vertex-linking disks from Σ. By Kneser's Finiteness Theorem [18], this can only happen a finite number of times. We shall continue to call the normal, non vertex-linking 2–sphere Σ. So, now we consider having a cycle of truncated-prisms about different edges. In this situation, we have a solid torus formed by the cycle of truncated-prisms and we have either three disjoint annular bands on Σ meeting the boundary of this torus, each a longitude of the torus, or just one annular band on Σ meeting the torus in a curve that intersects the meridian of the torus three times. In the first situation, the cycle of truncated-prisms must be in the product region $\mathcal{P}(Y)$. In the second, by using a disk on Σ complementary to the annular band, we have that the cycle of truncated-prisms along with this disk determine a copy of $L(3,1)$. However, this cycle of truncated-prisms does not meet $\mathbb{T}(\alpha)$, thus such a copy of $L(3,1)$ can be isotoped into$X$ which contradicts X being irreducible. We conclude there are no cycles of truncated-prisms in $\mathcal{C}$ that are not in $\mathcal{P}(Y)$.

So, by [14], we can crush Y along Σ. Since Σ meets $\mathbb{T}(\alpha)$ only in vertex-linking disks, $\mathbb{T}(\alpha)$ remains after crushing and thus the complement of $\mathbb{T}(\alpha)$ after crushing has closure X. Since Σ is not vertex-linking it meets a tetrahedron of $\mathcal{T}$ in a quadrilateral; such a tetrahedron disappears after crushing. Hence, the new triangulation of X would have fewer tetrahedra than $\mathcal{T}$. But this contradicts $\mathcal{T}$ a minimal triangulation of X. This completes the proof. $\qquad\square$

The only algorithm we know to decide if a given triangulation of a 3–manifold is minimal uses a solution to the homeomorphism problem; whereas, there is a straight forward algorithm to decide if a triangulation is 0–efficient [14]. So, it seems a bit unpleasant that we have to assume the triangulation is minimal. But, note from the proof of Theorem 8.14 that the problem we encountered is the existence of a normal planar surface in $\mathcal{T}$ with all its boundary components vertex-linking but it is not a vertex-linking disk (it has more than one boundary component). The

method of proof in this situation, which comes from [14], shows that starting with X irreducible and ∂–irreducible, then we can alter the triangulation $\mathcal{T}$ of X to eliminate such planar surfaces. Having done this, then the above proof works to show that all but finitely many triangulated Dehn fillings of such a triangulation are 0–efficient. We do not need to know the given triangulation $\mathcal{T}$ is minimal; we can deform it so that it is not only 0–efficient but rid it of unwanted planar normal surfaces with all boundary vertex-linking curves and make it "super" 0–efficient.

We have a similar result as to when we can obtain 1–efficient triangulated Dehn fillings. In Section 8, we defined a 1–efficient triangulation in the special case of layered-triangulations of lens spaces. However, in working with triangulated Dehn fillings, we need the full definition of a 1–efficient triangulation. In addition, as we needed methods from [14] for the proof of Theorem 8.14, we need methods from [13] to establish this analogous result for 1–efficient triangulations. Thus, we have chosen to leave the 1–efficient case and include it in [13].

9. Layered-triangulations of Handlebodies

Having layered-triangulations of the solid torus and genus one manifolds, it is natural to question if there are analogous triangulations for higher genera handle-bodies and higher genera manifolds. The answer is yes; there are direct generalizations. We provide the definitions and a brief introduction here. While the generalizations are straight forward, the level of complexity is not. We have not given much time to the study of layered-triangulations of handlebodies or layered-triangulations of 3–manifolds; however, we believe it reveals a very interesting direction of study and we hope to follow-up on these ideas.

A solid torus has a minimal triangulation with just one tetrahedron. In generalizing to handlebodies, we first determine the minimal number of tetrahedra needed in a triangulation of a genus g handlebody.

If we have a 3–manifold with connected, genus g boundary, then for any triangulation, we have from Euler characteristic,

$$t = 3g + 2 + e_i - v_i,$$

where t is the number of tetrahedra, e_i is the number of interior edges, and v_i is the number of interior vertices. Since, $e_i - v_i$ must be non negative, we have $3g - 2$ as a lower bound for the number of tetrahedra needed to triangulate a 3–manifold with connected boundary having genus g (Euler characteristic $1 - g$). Notice if we assume the manifold is irreducible and we have $e_i = 0$ (and hence, $v_i = 0$), then we must have a handlebody.

For $g = 1$, we layer a single tetrahedron onto the center (only interior) edge of the one-triangle Möbius band and we get a one-tetrahedron solid torus. For higher genera, we note that a one-vertex triangulation of a compact surface with one boundary component and Euler characteristic $1 - g$ has $3g - 2$ interior edges. Thus we are lead to layering $3g - 2$ tetrahedra onto the interior edges of such a surface. Specifically, let S be a compact, surface with one boundary component and Euler characteristic $1 - g$. Let $\mathcal{S}$ be a one-vertex triangulation of S and let $e_1, \ldots, e_{3g-2}$ denote the $3g - 2$ interior edges of $\mathcal{S}$. Let $\boldsymbol{\Delta} = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_{3g-2}\}$ be a pairwise disjoint collection of oriented tetrahedra. Let $K_1 = S \cup_{e_1} \Delta_1$ denote the complex determined by layering $\tilde{\Delta}_1$ onto S along e_1 via an orientation reversing attaching map, where Δ_1 is the image of $\tilde{\Delta}_1$ after attaching (here the attaching

map can take two faces of $\tilde{\Delta}_1$ to the same face of S; so, the orientation reversing condition may apply in this initial step, as well as later in the attaching). Having defined K_{i-1}, $1 < i < 3g - 2$, let $K_i = K_{i-1} \cup_{e_i} \Delta_i$ be formed by layering $\tilde{\Delta}_i$ onto K_{i-1} along e_i via an orientation reversing attaching map.

Notice that by layering a tetrahedron on each interior edge, we have a 3–manifold at each point, except possibly at the vertex (there is only one vertex). Since we have layered on all interior edges, the vertex-linking triangles form a 2–manifold and we can determine its Euler characteristic.

To see this let D denote the vertex-linking surface; let χ_D be its Euler characteristic, f_D, e_D and v_D be the number of its faces, edges and vertices, respectively. $f_D = 4(3g-2)$, $e_D = (3f_D - e_{\partial D})/2 + e_{\partial D}$, where $e_{\partial D}$ is the number of edges in the boundary of D. However, the number of edges in ∂D is three times the number of faces in a genus g surface with a one-vertex triangulation, which is $12g - 6$. Hence, $e_D = 6(3g - 2) + 6g - 3$. Finally, v_D is twice the number of edges in the triangulation of the handlebody. In this case it is easy to see that the number of edges in the triangulation, which are the edges in the once-punctured surface with Euler characteristic $1 - g$, is $3g - 1$, and then we get a new edge for each tetrahedron we add; hence, we have $(3g - 1) + (3g - 2) = 6g - 3$ edges and therefore $v_D = 12g - 6$. (In general, if we have a compact 3–manifold with boundary and a one-vertex triangulation, having Euler characteristic $1 - g$, then $1 - g = -t + f - e + 1$ and since the number of faces in the boundary must be $4g - 2$, the number of edges is $e = t + 3g - 1$.) We conclude that

$$\chi_D = f_D - e_D + v_D = 12g - 8 - (24g - 15) + 12g - 6 = 1.$$

So, D is a disk and the underlying point set is a 3–manifold. We implicitly used that the layering is via orientation reversing attaching maps to guarantee that the resulting identification space is a manifold at the midpoints of the edges.

There are clearly no closed normal surfaces in such a triangulation; so we have an irreducible 3–manifold. Since the triangulation collapses onto the once-punctured surface S, we have a handlebody having Euler characteristic $1 - g$ and thus has genus g. The surface S along with a fixed triangulation is called a *spine* or a *g–spine*. Just as we considered the one-triangle Möbius band a degenerate layered-triangulation of the solid torus, we shall consider a g–spine a degenerate layered-triangulation of the genus-g-handlebody. We summarize in the following proposition.

9.1. **Proposition.** *A genus-g-handlebody has a one-vertex triangulation; furthermore, a minimal triangulation is a one-vertex triangulation having $3g - 2$ tetrahedra.*

See Figure 28 for an explicit example of a minimal triangulation of a genus 2 handlebody. In Figure 28, we have used the number j for the edge e_j and the number k with a tilde, $\tilde{k}$, for the edge $\tilde{e}_k$, which eliminates the figure from being overwhelmed with notation.

A triangulation of the genus-g-handlebody obtained by layering $3g - 2$ tetrahedra onto the interior edges of a g-spine will be referred to as a *minimal layered-triangulation*. These are analogous to the one-tetrahedron solid torus obtained by layering a tetrahedron onto the interior edge of the one-triangle Möbius band. There is only one choice for layering in the genus one case. However, for genus $g > 1$, there are many choices. For an even genus handlebody, we have the choice of layering on a nonorientable or an orientable surface. Having made our choice of surface, we have the many choices of a one-vertex triangulation of the surface; and
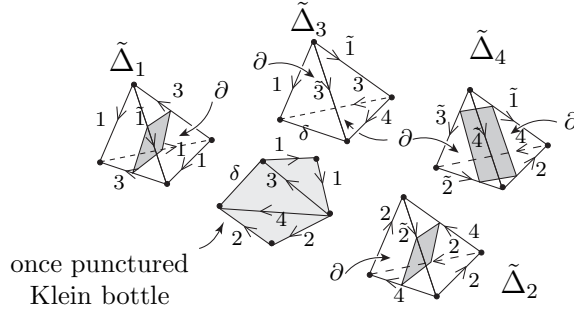
FIGURE 28. A minimal layered-triangulation of the genus 2 handlebody, having 4 tetrahedra and one vertex. The four tetrahedra are "layered" on a once-punctured Klein bottle. The six faces in the triangulation of the genus 2 boundary are indicated by the symbol ∂. Normal quadrilaterals $\{2,2\}, \{3,3\}$ and $\{\tilde{4},4\}$ are shown.

we have the various choices for the order in which we layer the $3g-2$ tetrahedra along the interior edges of the surface to get a minimal layered-triangulation of the genus-g-handlebody. One is tempted to place restrictions on the various choices but for the moment we have no good basis for determining restrictions. In these higher genera cases, there also are homotopy genus-g-handlebodies analogous to the creased 3–cell; and as the creased 3–cell, they are not manifolds. We do not know the role they might play here; so, at this time we shall ignore these possibilities. Finally, we point out two curious questions. The first should be easy to answer. Is a minimal triangulation of a genus-g-handlebody a minimal layered-triangulation? We suspect it is. The second is much more formidable. How many distinct, up to isomorphism, minimal layered-triangulations of the genus-g-handlebody are there? Recall (see [26]) that, except for small genus, we do not know the number of distinct triangulations, up to homeomorphism, of the once-punctured surface of genus g. In our first case, genus 2, there is, up to isomorphism, 1 one-vertex triangulation of the once-punctured torus and 4 distinct one-vertex-triangulations of the once-punctured Klein bottle. REGINA [4] has determined that there are 196 non isomorphic minimal triangulations (four tetrahedra) for the genus-2-handlebody. This may seem horribly large. However, for the once-punctured torus, alone, we have 384 ways to layer four tetrahedra; and all give a genus-2-handlebody. We have not determined if all of the 196 that REGINA found are layered. We discuss these examples more below.

9.1. **Layered-triangulations of a genus-g-handlebody.** As in the case of layered-triangulations of the solid torus, we inductively define layered-triangulations of a handlebody. A triangulation $\mathcal{T}_t$ is said to be a *layered-triangulation of the genus-g-handlebody*, with t–layers, if

   (1) $\mathcal{T}_{-1}$ is a g-spine,
   (2) $\mathcal{T}_0$ is a minimal layered-triangulation of the genus-g-handlebody (obtained by layering $3g-2$ tetrahedra onto the interior edges of a g-spine),
   (3) $\mathcal{T}_t = \mathcal{T}_{t-1} \cup_e \tilde{\Delta}_t$ is a layering along the edge e of a layered-triangulation $\mathcal{T}_{t-1}$ having $t-1$ layers, $t \geq 1$. See Figure 29.
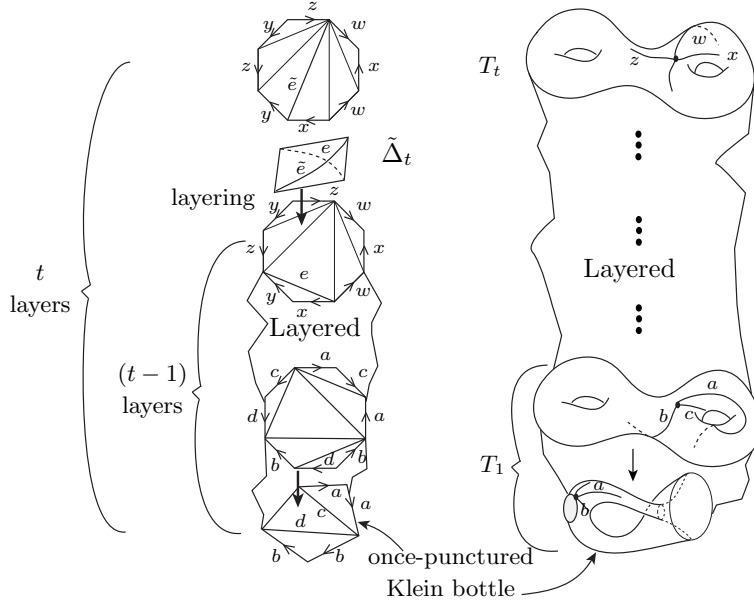
FIGURE 29. Layered-triangulation of a handlebody. Shown for genus 2.

If we have a genus-g-handlebody, there are infinitely many ways we can place a one-vertex (minimal) triangulation of the genus g surface on its boundary. If τ and τ' are one-vertex triangulations on the boundary of a genus-g-handlebody, we shall say that τ and τ' are *equivalent* if and only if there is a homeomorphism of the handlebody taking τ onto τ' via a simplicial isomorphism. This is an equivalence relation on one-vertex triangulations on the boundary of a genus-g-handlebody. We denote the equivalence class of τ by $[\tau]$. For genus one, we have that the equivalence classes of one-vertex-triangulations on the boundary of the solid torus are parameterized by reduced fractions $p/q, 0 < p < q$, along with $0/1$ and $1/1$. We do not have an analogous parametrization for higher genera. However, we do have that these triangulations on the boundary of the genus-g-handlebody can be extended to layered-triangulations of the handlebody.

First, we give the following application of Lemma 3.4, which is very useful to the theory of one-vertex triangulations of 3–manifolds.

9.2. **Lemma.** *Suppose $\mathcal{T}$ is a triangulation of the compact 3–manifold M and S is a component of ∂M. Furthermore, suppose $\mathcal{P}$ is the triangulation induced by $\mathcal{T}$ on S and $\mathcal{P}$ has one vertex. Then for any one-vertex triangulation $\mathcal{P}'$ of S, there is a sequence $\mathcal{T} = \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n = \mathcal{T}'$ of triangulations of M and a sequence $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1 \ldots, \mathcal{P}_n = \mathcal{P}'$ of triangulations of S, where $\mathcal{P}_i$ is the triangulation of S induced by the triangulation $\mathcal{T}_i$ of M and the triangulation $\mathcal{T}_{i+1}$ is isomorphic to the triangulation obtained by layering on $\mathcal{T}_i$ along an edge of $\mathcal{P}_i$. Hence, $\mathcal{P}_{i+1}$ is isomorphic to the triangulation obtained from $\mathcal{P}_i$ by a diagonal flip along an edge of $\mathcal{P}_i, 0 \le i \le n$. If S' is a component of ∂M distinct from S, then the triangulation induced by $\mathcal{T}'$ on S' is identical to that induced by $\mathcal{T}$ on S'.*

Proof. This is now a familiar argument in our work. The proof is by induction on the length of the sequence $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1 \ldots, \mathcal{P}_n = \mathcal{P}'$. We have organized this result as a sequence of triangulations on M rather than triangulations on homeomorphic copies of M, each obtained from the preceding by layering a tetrahedron along an edge in the induced triangulation on the boundary. $\square$

We note in Lemma 9.2 that in going from the triangulation $\mathcal{P}$ on S to the triangulation $\mathcal{P}'$ on S, we have embedded in the triangulation $\mathcal{T}'$ of M a sequence $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_{n-1}$ of subcomplexes of $\mathcal{T}'$, each is homeomorphic to S; furthermore, the triangulation of S_i is isomorphic to $\mathcal{P}_i$. We call such an S_i a *simplicial pleated surface* and think of this sequence of pleated surfaces as *modifying the combinatorial structure on M*, in a neighborhood of S, changing the induced triangulation on S from $\mathcal{P}$ to $\mathcal{P}'$.

Recall that if $\mathcal{T}_\partial$ is a triangulation on the boundary of a 3–manifold M, we say the triangulation $\mathcal{T}$ of M extends $\mathcal{T}_\partial$ if $\mathcal{T}$ restricted to ∂M is $\mathcal{T}_\partial$ and the vertices of $\mathcal{T}$ are precisely the vertices of $\mathcal{T}_\partial$ (we do not add vertices).

9.3. Lemma. *Suppose $\mathcal{T}_\partial$ is a triangulation on the boundary of a compact 3–manifold M and $\mathcal{T}_\partial$ has just one vertex in each component of ∂M. Then $\mathcal{T}_\partial$ can be extended to a triangulation of M.*

Proof. For this, we first get a triangulation of M, ignoring $\mathcal{T}_\partial$ for the moment, that has all its vertices in the boundary and has just one vertex in each boundary component. Such a triangulation can be obtained starting with the observation of R.H.Bing [1] that any compact 3–manifold with nonempty boundary admits a triangulation with all of its vertices in the boundary. Having this, one can use the operation of "closing-the-book" (see [14]) to get a triangulation of M that not only has all vertices in the boundary but has just one vertex in each boundary component (no component of the boundary is a 2–sphere or projective plane, since by hypothesis each component of the boundary admits a triangulation with just one vertex). This last step may require adding tetrahedra by layering along edges in the boundary but even if this is the case, at most two such tetrahedra are required for each boundary component. Now, we have a triangulation, say $\mathcal{T}$, of M with all vertices in the boundary of M and precisely one vertex in each boundary component of M. The triangulation $\mathcal{T}$ induces a one-vertex triangulation $\mathcal{P}$ on ∂M and, up to isotopy, we may assume that $\mathcal{P}$ and $\mathcal{T}_\partial$ have the same vertex in each boundary component.

So, the problem is the triangulation $\mathcal{T}$ does not induce $\mathcal{T}_\partial$; hence, we must modify the PL structure on M induced by $\mathcal{T}$ in a neighborhood of each boundary component of M from $\mathcal{P}$ to $\mathcal{T}_\partial$. By Lemma 9.2, we can do this, one boundary component at a time; having modified the triangulation in a neighborhood of a component S' of ∂M, we do not change the triangulation induced on S' when we move on to a boundary component S distinct from S' and modify the triangulation in a neighborhood of S. Thus, Lemma 9.2 shows that we can, through a sequence of steps, modify the combinatorial structure in a neighborhood of each boundary component of M so as to eventually include all boundary components. $\square$

While this result gives us immediately that any one-vertex triangulation on the boundary of a genus-g-handlebody can be extended to a triangulation of the handlebody, we want to sharpen the conclusion so that not only have we extended the triangulation but we have extended it to a layered-triangulation. This is easily done

by substituting a minimal layered-triangulation of the genus-g-handlebody at the point in the preceding proof where we used the triangulation coming from Bing's observation and closing-the-book. The following corollary provides, yet, another way to extend a one-vertex triangulation on the boundary of a solid torus to a layered triangulation of the solid torus.

9.4. Corollary. *A one-vertex triangulation on the boundary of a genus-g-handlebody can be extended to a layered-triangulation of the handlebody.*

Proof. Let M be a genus-g-handlebody and suppose $\mathcal{P}'$ is a one-vertex triangulation on ∂M. Now, M can be triangulated by any one of the minimal layered-triangulations, having $3g-2$ tetrahedra. Let such a triangulation be denoted $\mathcal{T}$ and let $\mathcal{P}$ denote the triangulation on ∂M induced by $\mathcal{T}$. Then we can modify $\mathcal{T}$ in a neighborhood of ∂M via a sequence of pleated surfaces to obtain a triangulation $\mathcal{T}'$ of M that induces $\mathcal{P}'$ on ∂M. The triangulation $\mathcal{T}'$ of M is a layered-triangulation because we started with $\mathcal{T}$ layered and each subsequent step in pleating the PL structure is a layering along an edge of the previous step. $\square$

What we provide here are existence proofs. One can, however, construct a layered-triangulation extending a given one-vertex-triangulation on the boundary of the genus-g-handlebody. That is, given a one-vertex triangulation $\mathcal{P}$ of a closed surface S and a complete system of closed curves for the surface, then we can construct a layered-triangulation of a genus-g-handlebody and a homeomorphism of the surface S to the boundary of the handlebody taking the given system of curves to meridians for the handlebody.

From Corollary 9.4, if τ is a one-vertex triangulation on the boundary of a genus-g-handlebody, τ can be extended to a layered-triangulation of the handlebody. We call such an extension a $[\tau]$-*layered-triangulation* of the genus-g-handlebody. If in addition, such a $[\tau]$-layered-triangulation of the genus-g-handlebody is minimal among all $[\tau]$-layered-triangulations, we say it is a *minimal* $[\tau]$-*layered-triangulation*. This should *not* be confused with a minimal layered-triangulation of a genus-g-handlebody, which has $3g - 2$ tetrahedra. In the genus one case, the minimal layered-triangulation (the one-tetrahedron torus) is also the minimal 1/2-layered-triangulation. In the higher genera cases, we do not know what triangulations appear on the boundaries of the minimal layered-triangulations. This is, itself, an interesting question. Below, we give examples of three different triangulations of the genus 2 surface that appear on the boundary of minimal layered-triangulations of the genus-2-handlebody. These were discovered by doing three examples by hand. There are nine distinct (up to orientation preserving homeomorphism) triangulations of the genus 2 surface; and any one of these can be placed infinitely many ways on the boundary of the handlebody. Here the issue is which of these classes have a representative that can be extended to a minimal layered-triangulation of the genus-2-handlebody. Ben Burton is adding code to REGINA [4] so such questions can be aided by a computer search.

We have the following conjecture analogous to our conjecture in the case of layered-triangulations of the solid torus.

9.5. Conjecture. *The minimal triangulation extending the one-vertex triangulation τ on the boundary of a genus-g-handlebody is a minimal $[\tau]$-layered-triangulation.*

The L_g graph: Classification of layered-triangulations of the genus-g–handlebody.
We construct a graph to organize our study of one-vertex triangulations on the
boundary of a genus-g-handlebody and their extensions to layered-triangulations of
the handlebody. This is a preliminary attempt for a genus g analog of the L–graph
used above in the study of one-vertex triangulations on the boundary of the solid
torus and their extensions to layered-triangulations of the solid torus.

For genus g, we shall denote this graph by L_g. The 0-cells of L_g are in one-
one correspondence with equivalence classes of one-vertex triangulations on the
boundary of a genus-g-handlebody, where equivalence is up to homeomorphism of
the handlebody (we allow orientation reversing as well as preserving homeomor-
phisms). The 1–cells are in one-one correspondence with diagonal flips on edges of
these triangulations. Specifically, there is a 1–cell between the 0–cells corresponding
to the equivalence classes $[\tau]$ and $[\tau']$ if and only if some representative of $[\tau']$ can
be obtained from τ via a diagonal flip along an edge of τ (hence, a representative
of $[\tau]$ can be obtained from τ' by a diagonal flip). We can perform diagonal flips
on any edge ($6g - 3$ edges) in one of these triangulations, since an edge must be
adjacent to two distinct triangles. Note, the L_1–graph is isomorphic to the L–graph
above.

It follows from Proposition 3.4 that L_g is connected. Also, by changing the order
of independent diagonal flips, it is easy to see that L_g, $g > 1$, has cycles. We have
the following relationship between edge paths in L_g and layered-triangulations of
the genus-g-handlebody.

9.6. **Proposition.** *The $[\tau]$-layered-triangulations of a genus-g-handlebody are in
one-one correspondence with the paths in L_g that begin at the vertex $[\tau]$ and end at
some g-spine.*

A minimal $[\tau]$–layered-triangulation corresponds to a minimal edge path from
the vertex $[\tau]$ to a g–spine. There are various possibilities for such a path; hence,
unlike the genus one case, there probably are a number of distinct minimal $[\tau]$–
layered-triangulations. The answers to such questions will come from the study
and understanding of the properties of the L_g graph and possibly its enlargement
to a complex much like the complex used by J.Harer [10] and others (see L. Mosher
[26]) to study the mapping class group.

Normal surfaces in layered-triangulations of handlebodies. One of our goals is to
undertake an analysis of layered-triangulations of higher genera handlebodies anal-
ogous to that for the solid torus. A major step toward understanding and using
layered-triangulations of the solid torus is the classification of normal and almost
normal surfaces and their boundary slopes in minimal p/q–layered-triangulations
of the solid torus. While we have minimal $[\tau]$–layered-triangulations in the higher
genera case, we do not expect to get a sharp classification like we had for the solid
torus. However, we do think there should be a useful organization of the theory
of normal surfaces in a layered-triangulation of a genus-g-handlebody, as well as
an understanding of their boundary slopes. For example, the layering serves as a
height function and the normal and almost normal surfaces are in a good position
relative to this height function. The local minima occur in the g-spine; if critical
values are introduced, they occur in a tetrahedron upon layering and correspond to
a saddle singularity formed by a "banding quadrilateral"; and the maxima are in
the boundary. As a step toward what we might expect, we provide a small example

by classifying the normal surfaces in one of the minimal layered-triangulations of the genus-2-handlebody. Recall that while there are infinitely many incompressible distinct annuli and non orientable surfaces embedded in a solid torus, a layered-triangulation sees only finitely many of these as normal surfaces. For higher genera handlebodies there are two constructions of interesting families of embedded incompressible surfaces that are consistent with the generation of normal surfaces. Namely "stair step surfaces" formed by adding several copies of an incompressible normal surface to an incompressible normal annulus and "spun surfaces" formed by adding several copies of a incompressible normal annulus to an incompressible normal surface (Dehn twisting about the annulus). In normal surface parlance, surfaces of the form $kF + A$ and $F + kA$, where F is an incompressible normal surface and A is an incompressible normal annulus, both satisfying the same quadrilateral conditions. In fact, we see in the simple example below, a minimal layered-triangulation of a genus-2-handlebody, a disk and annulus that generate an infinite family of "spun" disks by Dehn twisting a disk about the annulus.

Example: Consider the minimal layered-triangulation of the genus-2-handlebody given in Figure 28. To determine the normal surfaces in this layered-triangulation, we use quadrilateral coordinates. There are 12 variables (the 12 normal quadrilaterals in the four tetrahedra) and no equations. (For quadrilateral coordinates, there is one equation for each *interior* edge; in this example, there are no interior edges.) To simplify notation, if e and f are opposite edges in a tetrahedron, we shall denote the quadrilateral separating these edges by $\{e, f\}$ and we shall use j for the edge e_j and $\tilde{k}$ for the edge $\tilde{e}_k$. In the example, using this notation, the three quadrilaterals in the tetrahedron $\tilde{\Delta}_1$ are: $\{1, 1\}, \{3, 3\}$, and $\{\tilde{1}, 1\}$; in the tetrahedron $\tilde{\Delta}_2$, the quadrilaterals are: $\{2, 2\}, \{4, 4\}$ and $\{\tilde{2}, 2\}$; in the tetrahedron $\tilde{\Delta}_3$, the quadrilaterals are: $\{1, 4\}, \ldots$, and so on. We show the quadrilaterals $\{2, 2\}, \{3, 3\}$ and $\{\tilde{4}, 4\}$ in Figure 28. Each quadrilateral is a quad solution and the 12 quads give a complete set of fundamental quadrilateral solutions; in fact, they are vertex solutions. A quadrilateral solution determines a unique normal solution, up to some multiple of the vertex-linking disk. The fundamental surfaces in the example are:

$\{1, 1\} \Rightarrow$ meridional disk $\{3, 3\} \Rightarrow$ Möbius band $\{\tilde{1}, 1\} \Rightarrow$ annulus

$\{2, 2\} \Rightarrow$ meridional disk $\{4, 4\} \Rightarrow$ Möbius band $\{\tilde{2}, 2\} \Rightarrow$ annulus

$\{1, 4\} \Rightarrow$ meridional disk $\{\delta, \tilde{1}\} \Rightarrow$ annulus $\{\tilde{3}, 3\} \Rightarrow$ meridional disk

$\{2, \tilde{3}\} \Rightarrow$ meridional disk $\{\tilde{1}, \tilde{2}\} \Rightarrow$ meridional disk $\{\tilde{4}, 4\} \Rightarrow$ annulus.

Hence, any normal surface in this minimal layered-triangulation of the genus-2-handlebody is a disk, annulus, or Möbius band. This is as expected; however, we do get some complexity with just these four tetrahedra that we did not see in layered-triangulations of the solid torus. We have infinitely many distinct normal surfaces. For example, each sum $\{3, 3\} + k\{\tilde{4}, 4\}, k = 0, 1, 2, \ldots$ is a meridional disk. These additions are the same as Dehn twisting the disk determined by $\{3, 3\}$ around the annulus determined by $\{\tilde{4}, 4\}$; the latter is a thin edge-linking annulus about the edge $e_{\tilde{4}}$ and the former is a meridional disk orthogonal to this edge. The disks formed in this way are isotopic but not normally isotopic. We say more about this in the next section. There is another interesting example here; the sum of the two quad solutions $\{1, 1\} + \{1, 4\}$ determines a meridional disk; the normal solution for the disk is a fundamental solution (actually a vertex solution) in normal solution

space but its image in quad space is not even fundamental. The sum in normal
solution space of the disk determined by $\{1,1\}$ and the disk determined by $\{1,4\}$
have the vertex-linking disk as a component.

10. Layered-triangulations of 3–manifolds

Layered-triangulations for general 3–manifolds are defined analogously to those
for lens spaces using genus g Heegaard splittings and layered-triangulations of
genus-g-handlebodies.

We shall use compact surfaces having one boundary component and Euler char-
acteristic $1-g$ as degenerate genus-g-handlebodies, just as we used the Möbius band
as a degenerate solid torus. Furthermore, a minimal triangulation (one-vertex) of
these surfaces will be considered a (degenerate) layered-triangulation. Unlike the
one-triangle Möbius band, there are many minimal triangulations of these sur-
faces; as pointed out above, there are four distinct minimal triangulations of the
once-punctured Klein bottle. In the genus 1 case, we allowed the creased 3–cell
as a degenerate solid torus and its one-tetrahedron triangulation as a degenerate
layered-triangulation. An analogous situation occurs for higher genera. However,
we have not investigated the role these degenerate triangulations play; we suspect
the role might be as for genus 1, where the creased 3-cell was involved in minimal
layered-triangulations (Heegaard splittings) of small examples.

2–symmetric triangulations of closed orientable surfaces. Recall that we defined
a simplicial attachment between the boundaries of two solid tori with layered-
triangulations, incorporating an attachment in the cases where one of the factors
was degenerate. A simplicial attachment of the two-triangle torus to the one-
triangle Möbius band was a folding along any one of the three edges of the two-
triangle torus. Thus the torus is a two-sheeted branched covering of the Möbius
band, where the branching locus is the boundary of the Möbius band. The one-
triangle triangulation of the Möbius band lifts to the two-triangle triangulation of
the torus inducing a simplicial involution on the torus fixed along an edge. There
is such a symmetry in the two-triangle torus about each edge.

There is an analogous situation for higher genera surfaces. If we have a compact
surface B_g with one boundary component and Euler characteristic $1 - g$, then the
closed orientable surface, S_g, with Euler characteristic $2 - 2g$ is a branched double
cover over B_g with branching locus the boundary of B_g. Hence, if we have a mini-
mal (one-vertex) triangulation of B_g it lifts to a minimal (one-vertex) triangulation
of S_g; furthermore, such a triangulation of S_g has a simplicial involution fixed along
an edge, the edge over the branching locus (∂B_g). We call such a triangulation of
S_g *2–symmetric*; if the fixed edge, under the simplicial involution, is separating, we
say we have a *separating 2–symmetry*; and if it is non separating, we say we have
a *non separating 2–symmetry*. The quotient manifold in the case of a separating
2–symmetry is orientable and in the case of a non separating 2–symmetry, is non
orientable. The 2–symmetric triangulation of S_g induces a minimal triangulation of
B_g so that the projection map is simplicial; and conversely, a minimal triangulation
of B_g lifts to a 2–symmetric triangulation of S_g so the projection map is simplicial.
In Figure 30, we give the 2–symmetric triangulations for a genus 2 closed, orientable
surface. Chord diagrams, such as those given in [26], are very helpful in recognizing
2–symmetric triangulations; the chord diagrams of a 2–symmetric triangulation are

symmetric about a chord. For the examples in Figure 30, we also use the notation for the distinct minimal triangulations of the closed, orientable surface of genus 2 given in [26]. In the case of genus 2, where there are 9 distinct minimal triangulations, there are five 2–symmetric triangulations: T_1, T_2, T_3, T_5 and T_8. Since the 2–symmetric triangulations of a genus g surface are in one-one correspondence with the minimal triangulations of a compact surface with one boundary component and Euler characteristic $1 - g$, the ratio of 2–symmetric triangulations to the totality of minimal triangulations of a closed, orientable surface of genus g approaches zero as the genus increases. More about this later.

separating 2–symmetry
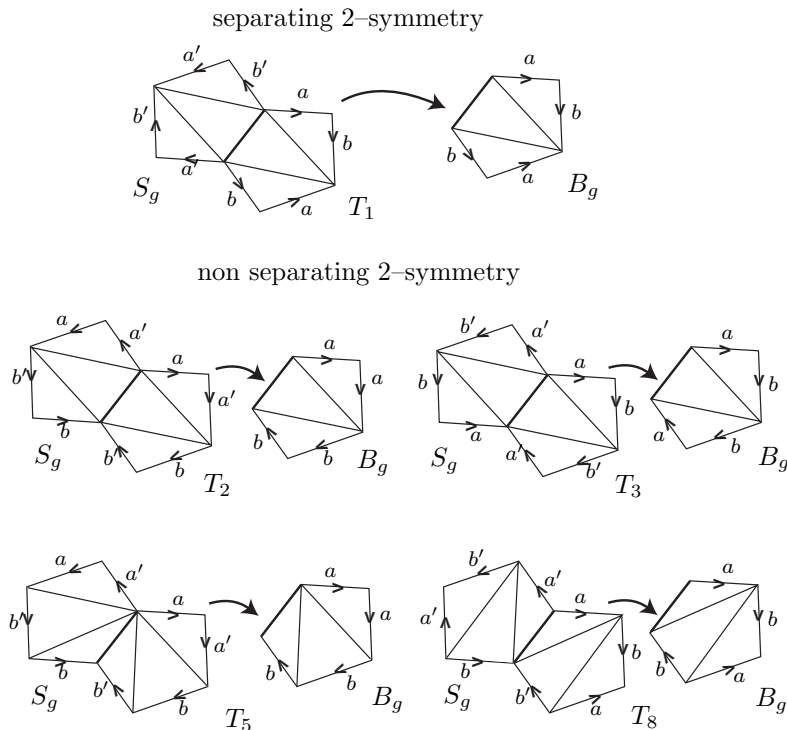

non separating 2–symmetry

FIGURE 30. The five 2–symmetric triangulations of a closed, orientable surface of genus 2 with orbit surfaces and quotient triangulations.

Definition. Suppose M is a closed 3–manifold and $M = H \cup_{\partial H \to \partial H'} H'$ is a genus g Heegaard splitting of M. If H and H' can be endowed with a layered-triangulation so that the attaching map $\partial H \to \partial H'$ is simplicial, then the layered-triangulations on H and H' give a triangulation of M. In this case, we call the induced triangulation of M a *layered-triangulation* and more specifically a *genus g layered-triangulation*. (Note that the four tetrahedra triangulation of $S^2 \times S^1$ shown in Figure 32 indicates that we should distinguish the genus of a layered-triangulation; we typically think of a layered-triangulation of this manifold being genus 1 but clearly any manifold with a genus g layered-triangulation has a genus g' layered-triangulation for all $g' > g$.) If one of the handlebodies is degenerate, then

only one of H or H' can be degenerate (say H') and in this situation the induced triangulation on ∂H must be 2–symmetric and the attaching map from $\partial H \to \partial H'$ is the projection of the 2–symmetric triangulation on ∂H to the g–spine H'. If e is the fixed edge under the 2–symmetry of ∂H, we call the attaching map a *folding (of the triangulation of ∂H) along the edge e*. While the situation is more complex than in the genus one case, we still have that a fixed layered-triangulation of a 3–manifold can be viewed in a number of different ways as a union of two handlebodies, each with a layered triangulation.

10.1. Theorem. *Every closed, orientable 3–manifold admits a layered-triangulation.*

Proof. Suppose M is a closed, orientable 3–manifold. Then M admits a Heegaard splitting, $M = H \cup_S H'$. Place any one-vertex triangulation $\mathcal{P}$ on S. Then by Theorem 4.2, $\mathcal{P}$ may be extended to a layered-triangulation of H and to a layered-triangulation of H'. This gives a layered-triangulation of M. $\square$

In [14], we provided a constructive proof that every closed, orientable and irreducible 3–manifold admits a one-vertex triangulation. However, in [14], we were interested in 0–efficient triangulations and placed additional conditions on the triangulation; the methods there do not necessarily extend to reducible 3–manifolds. On the other hand, there are many ways to arrive at one-vertex triangulations for 3–manifolds. The previous theorem provides one such method as a layered-triangulation has one vertex. Thus as a corollary, we have:

10.2. Corollary. *Every closed, orientable 3–manifold admits a one-vertex triangulation.*

There is a more constructive way to give a closed orientable 3–manifold a layered-triangulation, when it is given by a Heegaard splitting. Specifically, suppose $M = H \cup_{f:\partial H \to \partial H'} H'$ is a Heegaard splitting of M. Select any minimal layered-triangulations $\mathcal{T}$ for H and $\mathcal{T}'$ for H'. If g is the genus of the Heegaard splitting, then $\mathcal{T}$ and $\mathcal{T}'$ each have $3g - 2$ tetrahedra and, in fact, may be chosen to be isomorphic. Let $\mathcal{P}$ and $\mathcal{P}'$ be the induced one-vertex triangulations on ∂H and $\partial H'$ respectively, and let $\mathcal{P}'' = f^{-1}(P')$ be the one-vertex triangulation on ∂H given by pulling the triangulation $\mathcal{P}$ on $\partial H'$ back to ∂H. Then by Theorem 9.2 there is a sequence $\mathcal{T} = \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n = \mathcal{T}''$ of triangulations of H and a sequence $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1 \ldots, \mathcal{P}_n = \mathcal{P}''$ of triangulations of ∂H, where $\mathcal{P}_i$ is the triangulation of S induced by the triangulation $\mathcal{T}_i$ of M and the triangulation $\mathcal{T}_{i+1}$ is isomorphic to the triangulation obtained by layering on $\mathcal{T}_i$ along an edge of $\mathcal{P}_i$. Furthermore, in going from the triangulation $\mathcal{P}$ on ∂H to the triangulation $\mathcal{P}''$ on ∂H, we have embedded in the triangulation $\mathcal{T}''$ of M a sequence $S_0, S_1, \ldots, S_{n-1}$ of subcomplexes of $\mathcal{T}''$, each is homeomorphic to ∂H. The triangulation of S_i is isomorphic to $\mathcal{P}_i$. Hence, we can go through a sequence of simplicial pleated surfaces building a *new* layered-triangulation for H that modifies the combinatorial structure on ∂H so the attaching map f is a simplicial attachment. We provide a picture of this in Figure 31.

Triangulated Heegaard splittings. We now give what seems to be the most interesting method of representing a 3–manifold via Heegaard splittings and layered-triangulations. Let H be a handlebody with a layered-triangulation where the induced triangulation on the boundary is 2–symmetric along the edge e. If M is
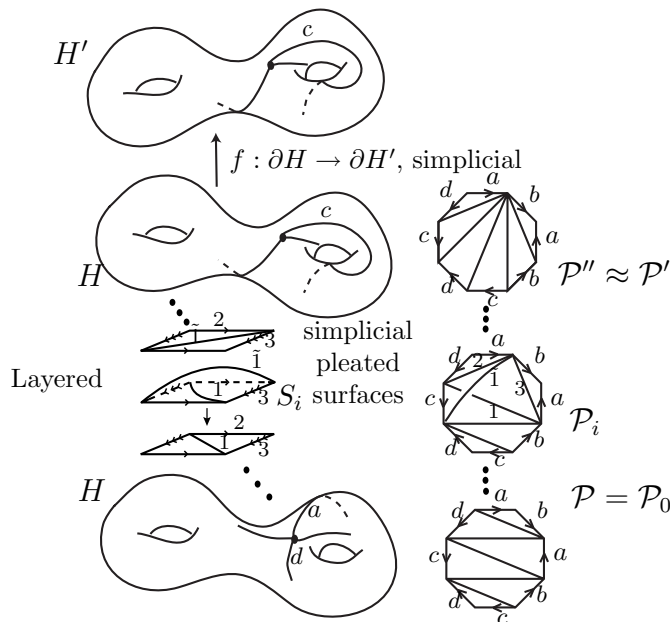
FIGURE 31. Modification of combinatorial structure through a sequence of simplicial pleated surfaces making the attaching map a simplicial isomorphism.

the 3–manifold obtained by folding the boundary along the edge e, then M has a layered-triangulation and we say M is presented by a *triangulated Heegaard splitting*.

10.3. Theorem. *Every closed, orientable 3–manifold can be presented by a triangulated Heegaard splitting.*

Proof. Consider any layered-triangulation of M. This gives $M = H \cup H'$, where H and H' both have layered-triangulations and the attaching map is simplicial. If we open either H or H' where they are layered onto a g-spine, then we have M presented by a triangulated Heegaard splitting. $\square$

A triangulated Heegaard splitting of a lens space, where the layered-triangulation of the solid torus is a minimal p/q-layered-triangulation, is the method we used to present lens spaces in the applications of layered-triangulations given in Section 8. Presenting a 3–manifold by a triangulated Heegaard splitting, where the layered-triangulation of the handlebody is a minimal $[\tau]$–layered-triangulation, gives a compelling direction for study of higher genera manifolds. There are numerous questions that these presentations raise. For example, in the generation of 3–manifolds via triangulations, a major problem is that the probability of getting a 3–manifold by face pairings of tetrahedra goes to 0 as the number of tetrahedra gets large. So, one might think that triangulated Heegaard splittings offer a nice alternative to face pairings for the generation of triangulated 3–manifolds. However, the probability of getting 2–symmetric triangulations on the boundary of layered-triangulations of a genus-g-handlebody goes to zero as the genus gets large. On the other hand, each

equivalence class of minimal triangulations of the genus g surface appears at infinitely many vertices of the L_g-graph. An interesting question is the distribution of 2–symmetric triangulations in the L_g-graph. Each gives rise to a closed, orientable 3–manifold and by Theorem 10.3, every closed, orientable 3–manifold must appear.
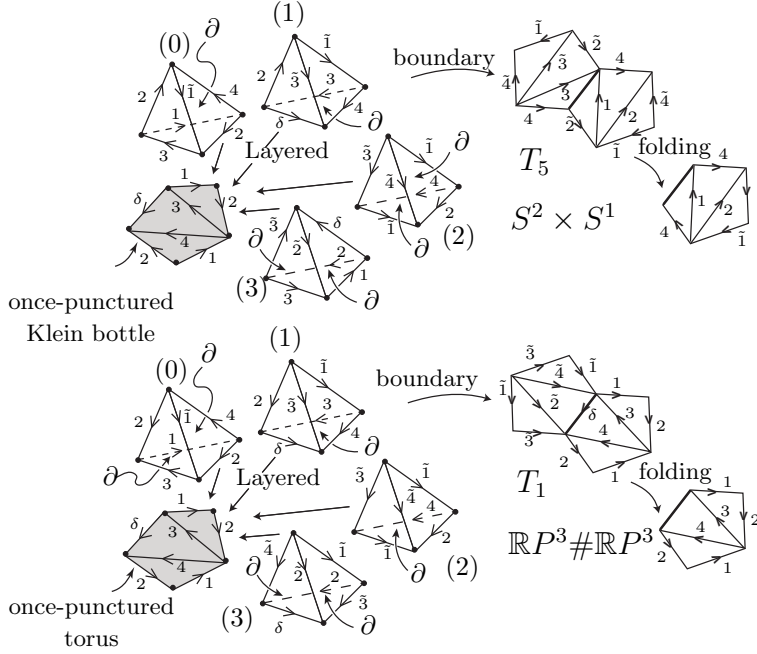

FIGURE 32. Two examples of triangulated Heegaard splittings of genus 2. The first gives a four-tetrahedron, layered-triangulation of $S^2 \times S^1$; and the second gives a four-tetrahedron, (minimal) layered-triangulation of $\mathbb{R}P^3 \# \mathbb{R}P^3$. For $S^2 \times S^1$, the Heegaard splitting is stabilized and for $\mathbb{R}P^3 \# \mathbb{R}P^3$, the Heegaard splitting is minimal genus but is a reducible splitting.

For example, in the case of genus 2, there are 196 distinct minimal triangulations of the genus-2-handlebody. We do not know if all of these are layered-triangulations. However, of those that are layered-triangulations (and we suspect all are), we can determine which of these have 2–symmetric triangulations on their boundaries; and for those that have a 2-symmetric triangulation on the boundary, we can determine the 3–manifold presented in this way. Ben Burton is adding code to REGINA to assist with the this exercise. We have done some examples beyond the one given in Figure 28. The example in Figure 28 does not have a 2–symmetric triangulation on its boundary. One can easily check that the triangulation induced on the boundary in this example is T_6, using the notation in [26]. Having discovered this triangulation is not 2–symmetric, we constructed three additional examples by hand, two of these had 2-symmetric triangulations induced on their boundaries. We give these examples in Figure 32. The first is a triangulated Heegaard splitting of $S^2 \times S^1$, of genus 2 with four tetrahedra; the second is a triangulated Heegaard splitting of $\mathbb{R}P^3 \# \mathbb{R}P^3$, also of genus 2 with four tetrahedra. The first splitting is stabilized. However, the second is not stabilized but must be reducible as the manifold is

reducible; it is a minimal triangulation of $\mathbb{R}P^3 \# \mathbb{R}P^3$. There are 15 manifolds having minimal triangulations with 4 tetrahedra and few with more than one such minimal triangulation. Only 5 of the 15 manifolds requiring 4 tetrahedra are manifolds having genus 2. Many of the 196 minimal triangulations of the genus-2-handlebody probably do not have 2–symmetric triangulations on their boundaries and many of those that do have 2–symmetric triangulations on their boundaries probably do not give minimal triangulations. REGINA will give us this answer.

Finally, suppose M is a closed 3–manifold with a triangulated Heegaard splitting. Then the number of tetrahedra gives a measure of complexity of the Heegaard splitting. There are many questions related to the number of tetrahedra in such a triangulated Heegaard splitting relative to the genus of the splitting and distance of the Heegaard splitting as defined in [11]. Also, if we consider all possible triangulated Heegaard splittings of M, then the number of tetrahedra in a minimal triangulated Heegaard splitting of M is an invariant of M. It may be the minimal number of tetrahedra needed to triangulate M. Indeed, we have a question similar to the conjectures given above.

10.4. **Question.** *With possibly a few exceptions, is a minimal triangulation of a 3–manifold a layered-triangulation?*

Hopefully with the aid of REGINA, we can explore these questions and gain a better understanding toward the use of layered-triangulations and triangulated Heegaard splittings in the study and understanding of 3–manifolds of genus larger than one.

Layered-triangulations and foliations. There is an interesting connection between layered triangulations and foliations, following Lackenby's view of how to construct taut ideal triangulations [19]. We sketch this procedure in the closed case and will discuss it further in a later paper. Given a tetrahedron Δ, choose a pair of opposite edges e, e'. Now we can build a "compressed" foliation on Δ with leaves being a family of quadrilateral disks which all have common boundary being the 4 edges of Δ other than e, e'. Note that if these four edges are removed from Δ, we see a genuine foliation. In Lackenby's construction, a taut foliation can be constructed by gluing together such foliations on all the ideal simplices and then slightly uncompressing the leaves along the edges. There is a simple compatibility condition required to make this work. Note that in our construction of layered triangulations of handlebodies and closed manifolds like lens spaces, similar compressed foliations can be built on all the simplices. The opposite edges e, e' chosen always to include the one being layered on. It is easy to see that this foliation can be uncompressed to be the standard product foliation, when the layered manifold is homeomorphic to a surface $\times I$. When a closed manifold is obtained from a triangulated Heegaard splitting by folding a 2–symmetric triangulation of the boundary of a handlebody along an edge, then a spine of the surface which is the g-spine can be chosen as the singular limit of the leaves of the foliation (for example, the center line of a Möbius band in case of a solid torus).

An interesting point about this is that normal and almost normal surfaces behave nicely relative to this singular foliation, when we use layered triangulations. So a basic idea is that normal surfaces can have saddle singularities and death singularities relative to this foliation, but birth singularities can only occur at the limit leaf. (A birth is the appearance of an inessential simple closed curve and a death is

the disappearance of one. Also, here we are following normal surface intersections with the leaves in from the boundary). This is what gives the strong control of normal and almost normal surface theory in layered solid tori and partial control in layered-triangulations of more general handlebodies.

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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078
E-mail address: `jaco@math.okstate.edu`

Department of Mathematics and Statistics, University of Melbourne, Parkville, VIC 3052, Australia
E-mail address: `rubin@maths.unimelb.edu.au`