# ESSENTIAL SURFACES IN AMALGAMATED 3-MANIFOLDS 

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#### Abstract

Let $M_{\phi}$ denote the 3-manifold obtain by identifying the boundaries of two small hyperbolic 3 -manifolds by the homeomorphism $\phi$. The genus of any essential surface, other than the amalgamating surface, in $M_{\phi}$ is forced to be arbitrarily high by making the map $\phi$ sufficiently complicated.


Keywords: Incompressible Surface

## 1. Introduction

In his thesis K. Hartshorn show that when a map identifying the boundaries of two handlebodies is complicated then the genus of any essential surface in the resulting 3 -manifold must be high [Har02]. Here we consider what happens when gluing two small, hyperbolic 3-manifolds along their boundaries. Of course, we have no control over the genus of the amalgamating surface. But we do show that when the gluing map is complicated then the genus of any other essential surface must be high.

Our first task is to define the complexity of the map used to glue two 3-manifolds together. Following Hempel [Hem01], we will make use of the curve complex to do this. Let $F$ be a 2-manifold. It's curve complex is defined by the following relations:

- vertices $\leftrightarrow$ isotopy classes of essential loops on $F$.
- $n$-simplices $\leftrightarrow$ sets of $n$ non-isotopic essential loops which can be isotoped to be pairwise-disjoint.
Note that the path metric on the 1-skeleton of the curve complex gives a well-defined, integer-valued metric on the 0 -skeleton.

Let $X$ and $Y$ be 3-manifolds with homeomorphic boundaries. Fix triangulations $\Delta_{X}$ and $\Delta_{Y}$ of $X$ and $Y$. Let $C_{X}$ and $C_{Y}$ denote the sets of loops on $\partial X$ and $\partial Y$ which intersect each edge of the respective induced triangulations at most once. Let $\phi: \partial X \rightarrow \partial Y$ be a homeomorphism. We define $d(\phi)$ to be $d\left(\phi\left(C_{X}\right), C_{Y}\right)$, the distance between the sets $\phi\left(C_{X}\right)$ and $C_{Y}$ as measured in the curve complex of $\partial Y$.

Note that this definition depends on the choice of triangulations $\Delta_{X}$ and $\Delta_{Y}$. However, given any such choice the sets $C_{X}$ and $C_{Y}$ are finite. It follows that as the distance of $\phi$ gets large with respect to one choice, it must also become large with respect to any other choice. In this sense results of the form "When the distance of $\phi$ is sufficiently large then ..." do not depend on the choice of triangulations.

The main result of this paper is the following:
Date: December 16, 2003.

Theorem 3.1. Let $X$ and $Y$ be small hyperbolic 3-manifolds and $\phi: \partial X \rightarrow \partial Y$ a homeomorphism. Let $G$ be an incompressible surface in $X \cup_{\phi} Y$, other than the amalgamating surface. There are constants $c(X)$ and $c(Y)$ such that the distance $d(\phi)$ is at most $c(X)+c(Y)-\chi(G)$.

If $t_{X}$ and $t_{Y}$ are the number of tetrahedra in $\Delta_{X}$ and $\Delta_{Y}$, and each is at least 16 , then we will show that $c(X) \leq 15 t_{X}$ and $c(Y) \leq 15 t_{Y}$.
Corollary 1.1. For every pair of small, hyperbolic 3-manifolds $X$ and $Y$, every number $g$, and every psuedo-Anosov homoemorphism $\phi: \partial X \rightarrow \partial Y$ there is an $N$ such that for all $n \geq N$ the only incompressible surface in the 3-manifold $X \cup_{\phi^{n}} Y$ whose genus is possibly less than $g$ is the amalgamating surface.

## 2. Normal Surfaces

In this section we discuss the necessary background material on normal surfaces. A normal curve on the boundary of a tetrahedron is a simple loop which is transverse to the 1 -skeleton, made up of arcs which connect distinct edges of the 1-skeleton. The length of such a curve is the number of times it crosses the 1 -skeleton. A normal disk in a tetrahedron is any embedded disk whose boundary is a normal curve of length three or four, as in Figure 1.


Figure 1
A normal surface in a triangulated 3-manifold is the image of a proper embedding $p$ of some surface $S$ such that $p(S)$ is a union of normal disks. Normal surfaces were first defined by Kneser in [Kne29] and later used extensively by Haken [Hak61].

According to Haken's theory the set of normal surfaces in a particular 3-manifold forms a finite union of finitely generated semigroups. A generator of such a semigroup is referred to as a fundamental surface, and the semigroup operation is called Haken sum. Haken sum always yields a normal surface that can be obtained from the summands by a cut-and-paste operation. In particular, Euler characteristic is additive. That is, if $S_{1}$ and $S_{2}$ are normal surfaces then

$$
\chi\left(S_{1}+S_{2}\right)=\chi\left(S_{1}\right)+\chi\left(S_{2}\right)
$$

There is an important subset of the fundamental surfaces called the vertex surfaces. We recall the following result of Jaco and Oertel [JO84].

Theorem 2.1. If $G$ is an incompressible, boundary-incompressible, normal surface then it is the sum of incompressible, boundary-incompressible vertex surfaces.

Each normal surface corresponds to a vector in $\mathbb{Z}^{7 t}$, where $t$ is the number of tetrahedra. If such a vector represents a normal surface then at most $5 t$ of its entries are non-zero. The sum of the entries tells one precisely how many normal triangles and quadrilaterals are contained in the surface. The following is a result of Hass, Lagarius, and Pippenger [HLP99].

Theorem 2.2. If $S$ is a vertex surface and $\mathbf{v}(S)$ the vector in $\mathbb{Z}^{7 t}$ representing $S$ then for each $i$ the entry $v_{i}(S)$ is at most $2^{7 t-1}$.

From this we deduce the following theorem.
Theorem 2.3. If $S$ is an incompressible, boundary-incompressible, vertex surface then

$$
|\partial S| \leq 15 t 2^{7 t-1}
$$

Proof. By Theorem 2.2 each coordinate in $\mathbf{v}(S)$ is at most $2^{7 t-1}$. As there are at most $5 t$ non-zero coordinates the total number of normal triangles and quadrilaterals in $S$ is at most $5 t 2^{7 t-1}$. Each such normal disk has at most 3 edges on the boundary. (If some quadrilateral had all four of its edges on the boundary then every face of the tetrahedron which contains it would be on the boundary. This is a contradiction, unless the entire manifold was a single tetrahedron.) Hence, the number of boundary edges is at most $15 t 2^{7 t-1}$. But the number of boundary edges equals the length of the boundary, so the result follows.

## 3. Proof of Theorem 3.1

Recall the statement:
Theorem 3.1. Let $X$ and $Y$ be small hyperbolic 3-manifolds and $\phi: \partial X \rightarrow \partial Y$ a homeomorphism. Let $G$ be an incompressible surface in $X \cup_{\phi} Y$, other than the amalgamating surface. There are constants $c(X)$ and $c(Y)$ such that the distance $d(\phi)$ is at most $c(X)+c(Y)-\chi(G)$.

We now begin the proof. Let $F=\partial X=\partial Y$. Isotope $G$ so that $|F \cap G|$ is minimal. Note that this quantity must be non-zero, since $X$ and $Y$ are assumed to be small. It follows from our minimality assumption that $G_{X}=G \cap X$ and $G_{Y}=G \cap Y$ are incompressible in $X$ and $Y$.

We wish to bound the distance between $G \cap F$ and both $C_{X}$ and $C_{Y}$, in the curve complexes of $\partial X$ and $\partial Y$. Our argument will be symmetric, so henceforth we will work completely in $X$.

As $X$ is hyperbolic, it has incompressible boundary. Hence, no component of $G_{X}$ is a disk. It follows that $\chi\left(G_{X}\right)$ is at least $\chi(G)$. Let $\left\{G_{i}\right\}_{i=0}^{n}$ be a sequence of surfaces in $X$ such that $G_{0}=G_{X}, G_{i}$ is obtained from $G_{i-1}$ by a boundary compression, and $G_{n}$ is incompressible and boundary incompressible. Since each such boundary compression increases Euler characteristic by exactly one, and $\chi\left(G_{n}\right) \leq 0$, we have $n \leq \chi\left(G_{n}\right)-\chi\left(G_{X}\right)$.

Note that the distance between the set $\partial G_{i-1}$ and $\partial G_{i}$ is at most one in the curve complex of $\partial X$. Hence, the distance between $\partial G_{X}$ and $\partial G_{n}$ is at most $n$. Putting this together we have

$$
\begin{equation*}
d\left(\partial G_{X}, \partial G_{n}\right) \leq \chi\left(G_{n}\right)-\chi\left(G_{X}\right) \tag{1}
\end{equation*}
$$

As $G_{n}$ is both incompressible and boundary incompressible in $X$ it is isotopic to a normal surface in the triangulation $\Delta_{X}$ [Hak61]. As $X$ is annannular, by Theorem 2.1 we may write $G_{n}$ has a sum of vertex fundamental surfaces with negative Euler characteristic. By Theorem 2.3 the length of the boundary of each such summand is at most $15 t 2^{7 t-1}$, where $t$ is the number of tetrahedra in $\Delta_{X}$. As Euler characteristic is additive when summing normal surfaces, the number of summands of $G_{n}$ is at most $-\chi\left(G_{n}\right)$. We conclude

$$
\begin{equation*}
\left|\partial G_{n}\right| \leq-15 t 2^{7 t-1} \chi\left(G_{n}\right) \tag{2}
\end{equation*}
$$

Let $\gamma$ denote the shortest representative among the isotopy classes of all loops of $\partial G_{n}$. Hence,

$$
\begin{equation*}
|\gamma| \leq\left|\partial G_{n}\right| \tag{3}
\end{equation*}
$$

Suppose $\gamma$ meets some edge $e$ of $\Delta_{X}$ more than once. Then there is a subarc $a \subset e$ such that $\gamma \cap a=\partial a$. Pick some orientation of $\gamma$. If this orientation is opposite at the points of $\partial a$ then we use $a$ to surger $\gamma$ as in Figure 2, left. This results in two curves, $\gamma^{\prime}$ and $\gamma^{\prime \prime}$. Otherwise $\gamma^{\prime}$ and $\gamma^{\prime \prime}$ are obtained by the exchange depicted at the right of Figure 2. In either case, it follows from our assumption that $\gamma$ has minimal length in its isotopy class that both $\gamma^{\prime}$ and $\gamma^{\prime \prime}$ are essential in $\partial X$. Assume that $\left|\gamma^{\prime}\right| \leq \frac{1}{2}|\gamma|$.


Figure 2
Continuing in this way we obtain a sequence of essential curves $\left\{\gamma_{i}\right\}_{i=0}^{l}$ on $\partial X$ such that $\gamma_{0}=\gamma, \gamma_{1}=\gamma^{\prime},\left|\gamma_{i-1}\right| \leq \frac{1}{2}\left|\gamma_{i}\right|$, and $\gamma_{l}$ meets each edge of $\Delta_{X}$ at most once.

Hence,

$$
\begin{equation*}
l \leq \log _{2}|\gamma| \tag{4}
\end{equation*}
$$

Note that for each $i$ the curve $\gamma_{i-1}$ meets $\gamma_{i}$ at most once. If they are disjoint then they are at a distance of one in the curve complex of $\partial X$. If they meet once then they are both disjoint from the boundary of a neighborhood of their union, so they are at a distance of at most two from each other. We conclude that the distance between $\gamma$ and $\gamma_{l}$ as at most $2 l$. As $\gamma$ is isotopic to a loop of $\partial G_{n}$ and $\gamma_{l} \in C_{X}$, we have

$$
\begin{equation*}
d\left(\partial G_{n}, C_{X}\right) \leq 2 l \tag{5}
\end{equation*}
$$

Combining Equations 2 through 5 gives us

$$
\begin{align*}
d\left(\partial G_{n}, C_{X}\right) & \leq 2 l  \tag{6}\\
& \leq 2 \log _{2}|\gamma|  \tag{7}\\
& \leq 2 \log _{2}\left(-15 t 2^{7 t-1} \chi\left(G_{n}\right)\right)  \tag{8}\\
& \leq 2 \log _{2}(15 t)+14 t-2+2 \log _{2}\left(-\chi\left(G_{n}\right)\right)  \tag{9}\\
& \leq 2 \log _{2}(15 t)+14 t+2 \log _{2}\left(-\chi\left(G_{n}\right)\right) \tag{10}
\end{align*}
$$

Combining this last inequality with Equation 1 yields

$$
\begin{align*}
d\left(\partial G_{X}, C_{X}\right) & \leq d\left(G_{X}, G_{n}\right)+d\left(G_{n}, C_{X}\right)  \tag{11}\\
& \leq \chi\left(G_{n}\right)-\chi\left(G_{X}\right)+2 \log _{2}(15 t)+14 t+2 \log _{2}\left(-\chi\left(G_{n}\right)\right)  \tag{12}\\
& \leq 2 \log _{2}(15 t)+14 t-\chi\left(G_{X}\right) \tag{13}
\end{align*}
$$

Let $c(X)=2 \log _{2}(15 t)+14 t$ and define $c(Y)$ similarly. Note that when $t$ is at least $16, c(X) \leq 15 t$. Inequality 13 then becomes

$$
\begin{equation*}
d\left(\partial G_{X}, C_{X}\right) \leq c(X)-\chi\left(G_{X}\right) \tag{14}
\end{equation*}
$$

Finally, this gives us

$$
\begin{align*}
d(\phi) & =d\left(\phi\left(C_{X}\right), C_{Y}\right)  \tag{15}\\
& \leq d\left(\partial G_{X}, C_{X}\right)+d\left(\partial G_{Y}, C_{Y}\right)  \tag{16}\\
& \leq c(X)-\chi\left(G_{X}\right)+c(Y)-\chi\left(G_{Y}\right)  \tag{17}\\
& \leq c(X)+c(Y)-\chi(G) \tag{18}
\end{align*}
$$

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