# Laminations and flows transverse to finite lepth foliations 

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December 16, 1996
Proliminary vcrsion, part I only
${ }^{1}$ Research at, MSRI is supported in part by NSF grant DMS-9022140. The anthor was supported by an additional NSF grant.

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## Introduction

The main purpose of this monograph is to present a theorem of D . Gabai which produces essential laminations on a great many 3 -manifolds. We shall produce these laminations via a construction of pseudo-Anosov flows on closed 3-manifolds, and of "pA flows" on compact 3 -manifolds $M$ which are torally bounded, meaning that each component of $\partial M$ is a torus. The construction works whenever:

- $M$ is oriented and irreducible.
- The rank of $H_{2}(M, \partial M ; \mathbf{Z})$ is positive (automatically true if $\left.\partial M \neq \emptyset\right)$.
- $M$ is not a Seifert fibered space.
- $M$ is atoroidal, which means that every incompressible torus $T$ in $M$ is isotopic to a boundary component; in other words, $T$ is peripheral.

See theorem A below for a precise statement. The laminations produced by the construction are "very full" laminations, which means that each complementary piece of the lamination is obtained from a solid torus or a (torus) $\times I$ by a simple "paring" operation. See below for a discussion of pseudo-Anosov and pA flows, and very full laminations.

When $\partial M \neq \emptyset$, very full laminations and pA flows are nicely situated for Dehn filling. Given a very full lamination $\Lambda$ in $M$, for each component $T$ of $\partial M$ there are coordinates ( $m, l$ ) for $H_{1}(T ; \mathbf{Z})$, depending on $\Lambda$, such that as long as the surgery coefficients $l=0, \pm 1$ are avoided the lamination $\Lambda$ remains very full in the filled manifold. A similar statement is true for pA flows. See theorem B below.

Along the way, we develop the theory of pseudo-Anosov and pA flows:

- On a closed manifold there is a close relation between pseudo-Anosov flows and pA flows given by the "double DA" operation.
- Associated to a pA flow there is a transverse pair of very full laminations $\Lambda^{s}, \Lambda^{u}$, the "stable" and "unstable" laminations of the flow (theorem 4.10.3).

We also produce combinatorial tools which aid in the construction of pseudo-Anosov and pA flows. By melding Markov partitions with the dynamic branched surfaces of J. Christy, we produce the concept of a dynamic pair of branched surfaces, and prove:

- Every dynamic pair of branched surfaces carries a pA flow (theorem 3.3.2). On a closed manifold every dynamic pair carries a pseudo-Anosov flow (theorem 3.4.1). A converse is also true: every pA flow is carried by some dynamic pair of branched surfaces (theorem 3.3.2).

See chapter 2 for the theory of dynamic pairs, and chapter 3 for the relation to flows. The elements of this theory are extended to the setting of sutured manifolds in chapter 4.

In the rest of this introduction, we give a more leisurely overview of the monograph.

## Laminations and flows

Essential laminations are a simultaneous generalization of Reebless foliations and incompressible surfaces in 3-manifolds [GO89]. An essential lamination $\Lambda$ carries topological information about the 3 -manifold $M$ in which it lives. For example, if a 3-manifold $M$ has an essential lamination then $M$ is irreducible and has infinite fundamental group, and if $M$ is closed then the universal cover of $M$ is homeomorphic to $\mathbf{R}^{3}$. Many workers have labored at constructing essential laminations in many different contexts; see Gabai's survey [Gab95].

There is a well known connection between foliations and Anosov flows on 3-manifolds. If $\Phi$ is an Anosov flow on a closed 3 -manifold $M$, the stable manifold theory of Hirsch, Pugh, and Shub [HPS77] produces a transverse pair of 2-dimensional foliations, the weak stable and unstable foliations of $\Phi$, and these are Reebless foliations of $M$.

There is similar well known connection between very full laminations and suspension flows of pseudo-Anosov flows. Suppose $f: S \rightarrow S$ is a homeomorphism of a closed surface, with mapping torus $M_{f}=S \times \mathbf{R} /(x, s+1) \sim(f(x), s)$, and with suspension flow $\operatorname{susp}(f)$ on $M_{f}$, the quotient of the flow on $S \times \mathbf{R}$ given by $(x, s) \cdot t=(x, s+t)$. If $f$ is a pseudoAnosov homeomorphism, then (by definition) there is associated to $f$ a transverse pair of singular 1-dimensional foliations of $S$, the stable and unstable foliations of $f$ [ $\left.\mathrm{FLP}^{+} 79\right]$. The flow $\operatorname{susp}(f)$ is an example of a pseudo-Anosov flow. On the surface $S$, the singular stable and unstable foliations may be "split" along singular leaves to produce a transverse pair of laminations on $S$, each filling the surface $S$; one can then suspend these laminations to obtain a transverse pair of very full laminations $\Lambda^{s}, \Lambda^{u}$ on $M$.

The general concept of a pseudo-Anosov flow on a 3-manifold arises by simultaneously generalizing Anosov flows and suspension flows of pseudo-Anosov surface homeomorphisms (see $\S 3.1$ ). We shall offer two definitions, a smooth definition and a topological definition, and we shall formulate some conjectures about how these definitions are related. Mostly we
will stick to the topological definition, in part because that is what comes most naturally out of our combinatorial constructions (such as theorem 3.4.1 mentioned above). A pseudoAnosov flow has 2-dimensional weak stable and unstable foliations, which are singular along a finite number of periodic pseudohyperbolic orbits. These foliations may be split open along their singular leaves, producing a transverse pair of very full laminations in $M$. Thus if $M$ supports a pseudo-Anosov flow, then $M$ has an essential lamination, with all the topological consequences that entails.

If $\Phi$ is a pseudo-Anosov flow on $M$, the splitting operation performed above on the singular stable and unstable foliations of $\Phi$ may be performed dynamically on $\Phi$ itself, using a variant of Smale's DA operation. The result is what we call a pA flow; see $\S 3$ for the definition. Loosely speaking, a pA flow is one which satisfies axiom A except for certain basic sets which are pseudohyperbolic periodic orbits; the definition also places strict conditions on the connections between the pseudohyprbolic orbits and the other basic sets. The letters "pA" can be read as "pseudo axiom A", or as "derived from pseudo-Anosov", or something like that (the letters "DA" stand for "derived from Anosov"-acronyms are in general a bad idea, but I can't think of what else to call a $p A$ flow). The advantage of $p A$ flows over pseudo-Anosov flows is that the definition can be formulated on any compact, oriented, torally bounded 3 -manifold $M$. pA flows enjoy all the topological advantages of pseudo-Anosov flows: they have 2-dimensional weak stable and unstable laminations which are very full, essential laminations in $M$. Also, pA flows are technically easier to work with than pseudo-Anosov flows, because the tools of smooth hyperbolic dynamics may be applied directly to pA flows.

To see how pA flows arise in nature, consider the problem of defining a pseudo-Anosov homeomorphism $f: S \rightarrow S$ when $S$ is a compact, oriented surface with nonempty boundary. Because $\partial S$ is a union of circles invariant under $f$, some aspect of the definition for closed surfaces will have to be discarded. In [FLP $\left.{ }^{+} 79\right]$ exposé 11 , a definition is offered which retains the property of topological transitivity (some orbit is dense) at the expense of uniform hyperbolicity. Inspired by axiom A diffeomorphisms, we propose a different definition, discarding topological transitivity but keeping uniform hyperbolicity on the chain recurrent set, except on a finite collection of basic sets each of which is a pseudohyperbolic orbit. Thurston's classification of surface mapping classes will still hold using this definition: on any compact oriented surface $S$, any orientation preserving homeomorphism $f: S \rightarrow S$ can be isotoped so that: $f$ is finite order; or $f$ preserves some system of nonperipheral, nontrivial simple closed curves; or $f$ is a pA homeomorphism. By applying the suspension construction, we obtain a proof of theorem A for the mapping torus $M_{f}$.

In general, given a pA flow $\Phi$ on $M$, we may use the stable manifold theory to associate to $\Phi$ a transverse pair of laminations called the stable and unstable laminations of $\Phi$. These laminations are of a special type called "very full" laminations, defined as follows. Let $M$ be a compact, oriented, torally bounded 3 -manifold. Let $\Lambda$ be a compact lamination contained in $\operatorname{int}(M)$. Consider the manifold with boundary $M_{\Lambda}$ obtained from $M-\Lambda$ by adding on
boundary leaves, that is, leaves of $\Lambda$ which are adjacent to $M-\Lambda$. We say that $\Lambda$ is very full if there are no sphere leaves or Reeb components, and if each component $C$ of $M_{\Lambda}$ falls into one of the following two types:

Pared solid torus $C \approx H-K$ where $H=D^{2} \times S^{1}$ and $K \subset \partial H$ is a nonempty family of nontrivial simple closed curves which is essential in the sense that the minimal geometric intersection number between $K$ and the meridian curve $\partial D^{2} \times$ (point) is at least 2 .

Pared torus shell $C \approx T^{2} \times[0,1]-K \times 1$ where $T^{2}$ is the torus, $K \subset T^{2}$ is any nonempty family of nontrivial simple closed curves, and $T^{2} \times 0$ corresponds to a boundary component of $M$; in this case, the curve family $K \times 0$ lives in a boundary component of $M$, and is called the degeneracy locus of that boundary component.

These objects are jointly called pared torus pieces. Very full laminations are evidently essential, and they are a special case of the "full laminations" studied by Hatcher and Oertel [HO96].

We can now state a simplified version of our main theorem:
Theorem A. Let $M$ be a compact, oriented, irreducible, torally bounded 3-manifold such that the rank of $H_{2}(M, \partial M ; \mathbf{Z})$ is positive. One of the following is true:

- $M$ is Seifert fibered.
- $M$ has a nonperipheral, incompressible torus.
- $M$ has a $p A$ flow, and if $M$ is closed then it has a pseudo-Anosov flow. In particular, $M$ has a very full lamination.

Remark. As a special case, the complement of every nontorus, nonsatellite knot in $S^{3}$ has a pA flow. This leads to a completely general existence theorem for "knot holders" in the sense of Birman and Williams [BW83], or "templates" as they are known in more recent literature [GHS96]).

Theorem A may be regarded as a 3-dimensional generalization of Thurston's classification of surface diffeomorphisms mentioned above. That classification has an equivalent reformulation which is often useful: every compact surface homeomorphism $f: S \rightarrow S$ may be isotoped so that $f$ preserves some family $T$ of nontrivial, nonperipheral, pairwise disjoint, simple closed curves, and for each component $C$ of $S$ cut along $T$, the first return map of $f$ to $C$ is isotopic to either a finite order homeomorphism or a pA homeomorphism. There is a similar reformulation of theorem A:

Theorem $\mathbf{A}^{\prime}$. Let $M$ be a compact, oriented, irreducible, torally bounded 3-manifold such that the rank of $H_{2}(M, \partial M ; \mathbf{Z})$ is positive. There exists a family $\mathcal{T}$ of incompressible,
nonperipheral, pairwise disjoint, embedded tori, such that for each component $C$ of $M$ cut along $\mathcal{T}$, one of the following is true:

- $C$ is a Seifert fibered; or
- $C$ has a pA flow, and if $\mathcal{T}=\partial M=\emptyset$ then $C=M$ has a pseudo-Anosov flow. In particular, $C$ has a very full lamination.


## Application to Dehn filling

pA flows and very full laminations are nicely situated for performing Dehn filling-for "almost all" Dehn fillings on each boundary torus, the lamination remains essential in the filled manifold, and the flow remains pA. "Almost all" means that in the appropriate coordinates on Dehn filling space, the only bad filling coefficients ( $m, l$ ) are those with $|l| \leq 1$.

The following theorem is due to D. Gabai:
Theorem B: Dehn Filling Theorem. Let $M$ be a compact, oriented, irreducible, torally bounded 3-manifold, and suppose that $\partial M \neq \emptyset$. Suppose also that $M$ is not Seifert fibered. For each component $T_{i} \subset \partial M$ there exist Dehn filling coordinates $\left(m_{i}, l_{i}\right): H_{1}\left(T_{i}\right) \rightarrow \mathbf{Z} \oplus \mathbf{Z}$ with the following property. Let $M_{\gamma}$ be obtained from $M$ by filling some components $T_{i}$ of $\partial M$ along curves $\gamma_{i} \subset T_{i}$, so that $\left|l_{i}\left(\gamma_{i}\right)\right| \geq 2$ for each $T_{i}$ that is filled. Then $M_{\gamma}$ has an essential lamination, and so $M_{\gamma}$ is irreducible and has infinite fundamental group, and if $\partial M_{\gamma}$ is closed then the universal cover of $M_{\gamma}$ is homeomorphic to $\mathbf{R}^{3}$.

Remark. In a knot complement, the coordinates given by the theorem may not agree with the usual meridian-longitude coordinates; an example is given in ???.

Proof. Since $\partial M \neq \emptyset$ it follows that the rank of $H_{2}(M, \partial M ; \mathbf{Z})$ is positive. Applying theorem $\mathrm{A}^{\prime}$ let $\mathcal{T}$ be a family of incompressible, nonperipheral, pairwise disjoint, embedded tori in $M$ such that each component of $M$ cut along $\mathcal{T}$ either is Seifert fibered or has a very full lamination. For each component $T_{i}$ of $\partial M$, let $C$ be the component of $M$ cut along $T$ which contains $T_{i}$, and choose the coordinates ( $m_{i}, l_{i}$ ) so that:

- If $C$ is Seifert fibered then each Seifert fiber on $T_{i}$ has coordinates $\left(m_{i}, l_{i}\right)=(1,0)$.
- If $C$ has a very full lamination then each component of the degeneracy locus on $T_{i}$ has coordinates $\left(m_{i}, l_{i}\right)=(1,0)$.

Now consider $M_{\gamma}$ as in the statement of the theorem. For each component $C$ of $M$ cut along $\mathcal{T}$, let $C_{\gamma} \subset M_{\gamma}$ be the filling of $C$. If $C$ has a very full lamination $\Lambda$ we shall prove that $\Lambda$ remains very full in $C_{\gamma}$. If $C$ has a Seifert fibration we shall prove that this extends
to a Seifert fibration on $C_{\gamma}$, and $C_{\gamma}$ is irreducible and boundary incompressible. It follows that if $\mathcal{T} \neq \emptyset$ then $\mathcal{T}$ is itself an essential lamination in $M_{\gamma}$, whereas if $\mathcal{T}=\emptyset$ then the very full lamination in $M$ is also a very full lamination in $M_{\gamma}$.

Case 1: $C$ has a very full lamination $\Lambda$. Consider a component $T_{i}$ of $C \cap \partial M$, which is filled along the curve $\gamma_{i} \subset T_{i}$. Let $K_{i}$ be the degeneracy locus of $\Lambda$ on $T_{i}$. Since $\left(m_{i}\left(K_{i}\right), l_{i}\left(K_{1}\right)\right)=(1,0)$ and $\left|l_{i}\left(\gamma_{i}\right)\right| \geq 2$, the pared torus shell containing $T_{i}$ is filled in to form a pared solid torus in $C_{\gamma}$, which is essential because of the fact that $\left\langle\gamma, K_{i}\right\rangle \geq 2$. It follows that $\Lambda$ is very full in $C_{\gamma}$, and so $C_{\gamma}$ is irreducible and boundary incompressible.

Case 2: $C$ is Seifert fibered Let $Q$ be the base orbifold of $C$. For any component $T_{i}$ of $C \cap \partial M$ which is filled, let $\gamma_{i} \subset T_{i}$ be the filling curve, let $K_{i}$ be a Seifert fiber on $T_{i}$, and let $c_{i}$ be the component of $\partial Q$ to which $T_{i}$ projects. By hypotheses we have $\left\langle\gamma, K_{i}\right\rangle \neq 0$ and so $C_{\gamma}$ is Seifert fibered.

From the hypothesis that $M$ is not Seifert fibered it follows that $C$ contains at least one boundary component which arises from cutting along $\mathcal{T}$, and so $\partial C_{\gamma} \neq \emptyset$. It follows that $C_{\gamma}$ is irreducible-the only reducible Seifert fibered spaces are those with empty boundary and with an $S^{2} \times \mathbf{R}^{1}$ geometric structure.

Let $Q_{\gamma}$ be the base orbifold of $C_{\gamma}$. Note that $Q_{\gamma}$ is obtained from $Q$ by capping off $c_{i}$ with a disc containing a cone point of order $\left|l_{i}\left(\gamma_{i}\right)\right| \geq 2$, for each component $T_{i}$ of $C \cap \partial M$ which is filled.

Suppose that $C_{\gamma}$ has compressible boundary. It follows that $Q_{\gamma}$ is a disc with at most one cone point. The number of cone points in $Q_{\gamma}$ is equal to the number of cone points in $Q$ plus the number of filled components of $C \cap \partial M$, and so at most one component of $C \cap \partial M$ is filled. If no components are filled then $Q_{\gamma}=Q$, and so $\partial C$ contains a component of $\mathcal{T}$ which is compressible in $M$, a contradiction. If one component is filled then $Q$ is an annulus with no cone points and $C$ is a (torus) $\times I$; it follows that $\partial C$ contains a component of $\mathcal{T}$ which is peripheral in $M$, also a contradiction.

Remark. In many special cases this theorem can be strengthened by weakening the hypothesis to say that $\left|l_{i}\left(\gamma_{i}\right)\right| \geq 1$.

For example, let $C$ be a component of $M$ cut along $\mathcal{T}$, and suppose that $C$ has a very full lamination. If $T_{i}$ is a component of $\partial M \cap C$, and if the degeneracy locus on $T_{i}$ has two or more components, then the weaker hypothesis $\left|l_{i}\left(\gamma_{i}\right)\right| \geq 1$ still implies $\left\langle\gamma_{i}, K_{i}\right\rangle \geq 2$ which is enough to insure that the pared torus shell containing $T_{i}$ gets filled in to an essential pared solid torus.

For another example, suppose that $C$ is Seifert fibered with base orbifold $Q$, and let $C_{\gamma}, Q_{\gamma}$ be as in the proof. In almost all cases, for each component $T_{i}$ of $\partial M \cap C$ we need only the weaker restriction that $\left|l_{i}\left(\gamma_{i}\right)\right| \geq 1$. This restriction is still enough to conclude that $C_{\gamma}$ is Seifert fibered; if $\left|l_{i}\left(\gamma_{i}\right)\right|=1$ then $c_{i}$ is capped off by a disc with no cone points.

Also, under the hypotheses that $M$ is not Seifert fibered it still follows that $\partial C_{\gamma} \neq \emptyset$ and so $C_{\gamma}$ is still irreducible. The only possible problem is if $C_{\gamma}$ has a compressible boundary component, and the only way that fillings with $\left|l_{i}\left(\gamma_{i}\right)\right|=1$ could cause this problem is if: $Q$ is a planar surface with $\leq 1$ cone point; all but one component of $\partial C$ is filled; and for every filled component $T_{i}$ we have $\left|l_{i}\left(\gamma_{i}\right)\right|=1$. These are precisely the assumptions which lead to $Q_{\gamma}$ being a disc with $\leq 1$ cone point.
Remark. Because of the previous remark, it is desirable to have very full laminations whose degeneracy locus has two or more components on each torus boundary. There is, however, a certain intuition saying that generically the degeneracy locus has only one component. For example, the generic element of the mapping class group of a punctured surface is pseudoAnosov with 1-pronged singularities at the puncture, and under suspension this leads to a very full lamination whose degeneracy locus has one component on each boundary torus. On the other hand, as suggested to me by W. P. Thurston, this intuition about "generic" properties of the degeneracy locus may be suspicious in special situations, for example knot complements in the 3 -sphere.
Remark. When $M$ is an oriented Seifert fibered space, for example the complement of a torus link, the hypothesis $\left|l_{i}\left(\gamma_{i}\right)\right| \geq 1$ is sufficient for the Seifert fibration to extend over the filled manifold $M_{\gamma}$. As mentioned above, the base orbifold of $M_{\gamma}$ is easily computed by capping off boundary components of the base orbifold of $M$. It is well known how to use the Seifert fibration to decide basic topological properties of $M_{\gamma}$ such as irreducibility, incompressible boundary, and infinite fundamental group. In many cases the hypothesis $\left|l_{i}\left(\gamma_{i}\right)\right| \geq 1$ is not enough to establish these properties; for example if $M$ is a torus link and if every boundary torus is filled so that $\left|l_{i}\left(\gamma_{i}\right)\right|=1$ then $M_{\gamma}$ is a lens space and so it has finite fundamental group.

## Finite depth foliations

In [Gab83] Gabai proves that if $M$ is a compact, irreducible, oriented torally bounded 3-manifold, and if the rank of $H_{2}(M)$ is positive, then $M$ has a transversely oriented, Reebless finite depth foliation $\mathcal{F}$ transverse to $\partial M$. Recall that $\mathcal{F}$ is finite depth if there is a nested sequence of sublaminations $\mathcal{F}_{0} \subset \cdots \subset \mathcal{F}_{n}=\mathcal{F}$ such that each leaf of $\mathcal{F}_{0}$ is compact, and for each $i=1, \ldots, n$, each leaf $L$ of $\mathcal{F}_{i}-\mathcal{F}_{i-1}$, and each sequence of points $x_{k} \in L$ leaving every compact subset of $L$, all limit points in $M$ of the sequence ( $x_{k}$ ) are contained in $\mathcal{F}_{i-1}$. Recall that a Reeb component of $\mathcal{F}$ is a solid torus whose boundary is a leaf and whose interior is foliated by planes that accumulate on the boundary. A foliation is Reebless if it has no Reeb components.

For example, if the depth is zero then $\mathcal{F}$ is a fibration over the circle with fiber $F$, and in this case theorem A follows from Thurston's classification of surface mapping classes $\left[\mathrm{FLP}^{+} 79\right]$ as remarked above.

In general, theorem A is an immediate consequence of [Gab83] and the following:
Theorem C. Let $M$ be a compact, oriented irreducible torally bounded 3-manifold, and let $\mathcal{F}$ be a transversely oriented, Reebless, finite depth foliation transverse to $\partial M$. One of the following is true:

1. $M$ is Seifert fibered.
2. $M$ has a nonperipheral, incompressible torus $T$ transverse to $\mathcal{F}$ and not isotopic to a leaf of $\mathcal{F}$.
3. $M$ has a pA flow whose stable and unstable laminations are transverse to $\mathcal{F}$. If $M$ is closed then $M$ has a pseudo-Anosov flow which is almost transverse to $\mathcal{F}$.

Remark. "Almost transversality" of a pseudo-Anosov flow and a closed surface was introduced in [Mos90]. The needed generalization, replacing the surface by a foliation, is given in $\S 3.5$; here is a rough sketch of the definition. If $\gamma$ is a pseudohyperbolic orbit of a flow $\Phi$, there are several ways to "dynamically blow up" $\gamma$; this means that the stable and unstable manifolds of $\gamma$ are pulled apart, and invariant annuli are inserted, creating a new flow $\Phi^{\#}$. There is a semi-conjugacy from $\Phi^{\#}$ to $\Phi$, homotopic to the identity, that collapses the invariant annuli back to $\gamma$. A pseudo-Anosov flow $\Phi$ is said to be almost transverse to a foliation $\mathcal{F}$ if there is a way to dynamically blow up the singular orbits of $\Phi$ so that the blown up flow $\Phi^{\#}$ is transverse to $\mathcal{F}$. On a hyperbolic manifold $M$, a pseudo-Anosov flow $\Phi$ which is almost transverse to some finite depth foliation has strong geometric properties: $\Phi$ is a "quasigeodesic" flow on $M$ [FM95].

Note that clause 3 of theorem C does not say that $M$ has a pA flow which is almost transverse to $\mathcal{F}$. While it is possible to formulate a true statement of this sort, to do so requires a rather technical and perhaps not very useful definition of "almost transversality" between pA flows and finite depth foliations.
Remark. In the case that $M$ is Seifert fibered, more information is available from work of Brittenham [Bri93], which shows that $\mathcal{F}$ has a sublamination $\Lambda$ such that $\Lambda$ is parallel to the Seifert fibration and $\mathcal{F}-\Lambda$ is transverse to the Seifert fibration. As the reader will see, the Seifert fibrations popping out of the proof of the main theorem are easily seen to have this property.

There is an equivalent reformulation of theorem C , modelled on theorem $\mathrm{A}^{\prime}$.
Theorem $\mathbf{C}^{\prime}$. Let $M$ be a compact, oriented, irreducible, torally bounded 3-manifold, and let $\mathcal{F}$ be a transversely oriented, Reebless, finite depth foliation transverse to $\partial M$. There exists a finite family $\mathcal{T}$ of pairwise disjoint, incompressible, nonperipheral tori transverse to $\mathcal{F}$, none of which are isotopic to leaves of $\mathcal{F}$, such that for each component $C$ of $M$ cut open along $\mathcal{T}$, either $C$ is Seifert fibered, or the following statements are true:

1. $C$ has a $p A$ flow whose stable and unstable laminations are transverse to $\mathcal{F} \mid C$.
2. If $\mathcal{T}=\partial M=\emptyset$ then the closed manifold $C=M$ has a pseudo-Anosov flow which is almost transverse to $\mathcal{F}$.

Remark. This theorem can be used to produce pseudo-Anosov flows on some 3-manifolds which have nonperipheral, incompressible tori, as long as each such torus is isotopic to a leaf of $\mathcal{F}$. Examples of this sort are given by Bonatti and Langevin [BL94], where the flows produced are Anosov-our construction applied to the Bonatti-Langevin example produces the same flows as their construction.

## Application to the Thurston norm

Theorem C has consequences for the structure of the Thurston's seminorm $\boldsymbol{x}$ on $H_{2}(M ; \mathbf{R})$. This seminorm measures the least complexity of a surface representing a class: if $\alpha \subset$ $H_{2}(M ; \mathbf{Z})$ then $x(\alpha)$ is the minimum of $\left|\chi_{-}(A)\right|$ where $A$ is any surface representing $\alpha$, and $\chi_{-}(A)$ is the Euler characteristic of $A-$ (sphere components). When $M$ is irreducible and atoroidal, e.g. when $M$ is hyperbolic, it follows that $x$ is a norm. The unit ball $B_{x}$ is a finite-sided polyhedron in $H_{2}(M ; \mathbf{R})$ whose faces are defined by level sets of linear functionals defined over $\mathbf{Z}$.

In [Mos92b] a connection is drawn between pseudo-Anosov flows and the Thurston norm. Let $\Phi$ be a pseudo-Anosov flow on a closed, hyperbolic 3-manifold $M$. Let $C_{\Phi} \subset H_{1}(M ; \mathbf{R})$ be the smallest closed cone containing the homology class of every periodic orbit, a finite cone that can be calculated from a Markov partition for $\Phi$. Let $D_{\Phi} \subset H_{2}(M ; \mathbf{R})$ be the dual cone of $C_{\Phi}$ with respect to the intersection pairing. Let $\chi_{\Phi} \in H^{2}(M ; \mathbf{R})$ be the Euler class of the normal plane bundle of $\Phi$, regarded as a linear functional on $H_{2}(M ; \mathbf{R})$. The class $\chi_{\Phi}$ may be computed from the pseudo-hyperbolic orbits of $\Phi$. The main result of [Mos 92 b ] says that if $\Phi$ is quasigeodesic then $\Sigma_{\Phi}=\chi_{\Phi}^{-1}(1) \cap D_{\Phi}$ is a subpolyhedron of some face of $B_{x}$.

In [FM95] it is proved that if $M$ is a closed, hyperbolic 3-manifold, and if a pseudoAnosov $\Phi$ on $M$ is almost tranverse to a finite depth foliation on $M$, then $\Phi$ is indeed quasigeodesic, and so the results of [Mos92b] apply.

According to [Gab83], for any surface $A \subset M$ which realizes the Thurston norm in its homology class, there is a finite depth foliation $\mathcal{F}$ containing $A$ as a leaf. By theorem A , there is a pseudo-Anosov flow $\Phi$ that is almost transverse to $\mathcal{F}$. By [FM95] the flow $\Phi$ is quasigeodesic. Applying the results of [Mos92b] as described above, we have proved:

Corollary D. If $M$ is a hyperbolic 3-manifold, then the boundary of $B_{x} \subset H_{2}(M ; \mathbf{R})$ is covered by the polyhedra $\Sigma_{\Phi}$ associated to quasigeodesic, pseudo-Anosov flows on $M$.

Remark. This corollary raises an important question: can $\partial B_{x}$ be covered by finitely many of the polyhedra $\Sigma_{\Phi}$ ? Conjecture: Yes. See the questions section below for some discussion of this conjecture.

There should be a computational process which takes as input some finite depth foliation and produces as output a complete description of $B_{x}$, without invoking normal surface theory as is done in [Oer85] or [TW96]. This process would go as follows: construct a pseudo-Anosov flow $\Phi$ almost transverse to the given foliation, and compute $\Sigma_{\Phi}$. Now invent a process for moving around $\partial B_{x}$, constructing pseudo-Anosov flows whose corresponding $\Sigma$ 's piece together to give all of $\partial B_{x}$.

## A sketch of the proof of Theorem C

As remarked above, when $\mathcal{F}$ is depth 0 theorem C follows from an application of Thurston's classification of surface mapping classes $\left[\mathrm{FLP}^{+} 79\right]$. For higher depth foliations the technical details are quite different.

The major elements of the construction are already present in the special case of a depth 1 foliation $\mathcal{F}$ on $M$, which we now describe. We may suppose that $\mathcal{F}$ has finitely many compact leaves, and no complementary component of the compact leaves is foliated as a product. The compact leaves form a compact surface $S$. Let $N$ be obtained by cutting $M$ open along $S$. The "scars" of $S$ form disjoint subsurfaces $\mathcal{R}_{-} N, \mathcal{R}_{+} N \subset \partial N$, and $M$ is obtained from $N$ by gluing $\mathcal{R}_{+} N$ to $\mathcal{R}_{-} N$ via a homeomorphism $g: \mathcal{R}_{+} N \rightarrow \mathcal{R}_{-} N$. The surfaces $\mathcal{R}_{-} N, \mathcal{R}_{+} N$ give $N$ the structure of a sutured manifold [Gab83]. The suture set $\gamma N=\operatorname{cl}\left(\partial N-\left(\mathcal{R}_{-} N \cup \mathcal{R}_{+} N\right)\right)$ is a union of tori and annuli, each annulus with one boundary circle in $R_{-} N$ and the other in $R_{+} N$. The restriction of $\mathcal{F}$ to $N-\mathcal{R}_{ \pm} N$ is a fibration over the circle. The fiber is a surface $F$ with finitely many ends, and the monodromy map $f: F \rightarrow F$ is end periodic, i.e. $f$ acts as a semi-covering transformation on a neighborhood of each end, either attracting towards the end or repelling away from the end. Each end of $F$ spirals into some component of $S$. The dynamics of end periodic maps have been analyzed by Handel and Miller, with results similar to Thurston's analysis of compact surface dynamics (but with significantly different proofs). An account of the Handel-Miller theory is given in [Fen96a]. The result of this theory is that $f$ may be isotoped so that one of three alternatives happens:

1. $f$ permutes a finite, simple family of non-peripheral essential closed curves on $F$.
2. $f$ is a covering transformation over a compact surface.
3. $f$ is a pA surface homeomorphism (defined appropriately in the category of end periodic maps), with invariant 1-dimensional stable and unstable laminations. This case encompasses the possibility that $f$ respects a finite, simple family of proper lines
which cut off components on which $f$ acts as a covering transformation over a compact surface.

Case (1) leads to an incompressible, nonperipheral torus transverse to $\mathcal{F}$. Case (2) implies that $N$ is a (surface) $\times I$ and so $M$ is fibered over the circle.

Case (3) is the most interesting. Suspending $f$ and then accelerating the time parameter near $\partial N$ gives a pA flow $\Phi_{N}$ defined on $N$. The repelling ends of $F$ spiral into $\mathcal{R}_{-} N$, and $\Phi_{N}$ enters $N$ along $\mathcal{R}_{-} N$. The attracting ends spiral into $\mathcal{R}_{+} N$, where $\Phi_{N}$ leaves $N$. The unstable lamination $\Lambda_{N}^{u}$ has boundary $\lambda^{u} \subset \mathcal{R}_{+} N$, and the stable lamination $\Lambda_{N}^{s}$ has boundary $\lambda^{s} \subset \mathcal{R}_{-} N$. There is a maximal compact totally invariant subset $C \subset \operatorname{int}(N)$ of $\Phi_{N}$, containing the intersection $\Lambda_{N}^{u} \cap \Lambda_{N}^{s}$.

Now let $\mathcal{R}_{-} N$ and $\mathcal{R}_{+} N$ be identified by $g: \mathcal{R}_{+} N \rightarrow \mathcal{R}_{-} N$ to form $S \subset M$. Isotope the identification map $g$ so that $\lambda^{s}$ and $\lambda^{u}$ intersect efficiently in $S$. There are special cases to handle in which this isotopy does not exist, but these cases typically lead to incompressible, nonperipheral tori. If we make the extra assumption that closed leaves of $\lambda^{s}$ are not isotopic to closed leaves of $\lambda^{u}$, then the isotopy exists.

Note that $\Lambda_{N}^{u}$ is not a lamination in $M$, because it has boundary $\lambda^{u}$ lying on the surface $S$ which is not part of $\partial M$. To correct this problem, extend $\Lambda_{N}^{u}$ by flowing up past $\lambda^{u} \subset S$ and back into $N$. The points where $\lambda^{u}$ intersects $\lambda^{s}$ extend to flow lines spiralling into $C$, and the rest of the flow lines escape past $C$ and back out to $S$, augmenting $\Lambda_{N}^{u}=\Lambda_{0}^{u}$ to a larger lamination $\Lambda_{1}^{u} \supset \Lambda_{0}^{u}$, with boundary $\partial \Lambda_{1}^{u}=\lambda_{1}^{u} \supset \lambda_{0}^{u}=\lambda^{u}$. Now continue the process inductively: isotope the identification map so that $\lambda_{1}^{u}$ intersects $\lambda^{s}$ efficiently in $S$, flow up past $S$, etc. This generates an increasing sequence of laminations $\Lambda_{0}^{u} \subset \Lambda_{1}^{u} \subset \Lambda_{2}^{u} \subset \cdots$. Passing carefully to a limit, we obtain a true, boundaryless lamination $\Lambda^{u}=\operatorname{cl}\left(\Lambda_{0}^{u} \cup \Lambda_{1}^{u} \cup \cdots\right)$ of $M$. A similar argument with $\Lambda_{N}^{s}$, flowing down past $\lambda^{s} \subset S$, produces a true lamination $\Lambda^{s}$ of $M$. Then one proves that one of the following occurs:

- $\Lambda^{s}, \Lambda^{u}$ are very full in $M$, and are the stable and unstable laminations of a pA flow $\Phi$ on $M$ that is transverse to $\mathcal{F}$; or
- $M$ has a nonperipheral, incompressible torus.

Although we made some special assumptions about closed leaves of $\lambda^{s}$ and $\lambda^{u}$, these assumptions may be drastically weakened at the expense of losing strict transversality of $\Phi$ and $\mathcal{F}$, as long as one is willing to accept almost transversality.

## A combinatorial formulation: Dynamic pairs of branched surfaces

Rather than trying to formalize the above proof directly, we take an indirect approach, via the combinatorial tools of dynamical systems: branched surfaces and Markov partitions. My whole approach to this subject is highly combinatorial. This is due in part to
availability of good combinatorial tools. But to be honest, the actual reason for thinking combinatorially is because one's mind is bent that way (apologies to M. Gromov-[Gro93], p. 7). As a child I loved playing with Tinker Toys and Legos, producing gothic constructions from a few simple brightly colored pieces whose forms I admired. This love has colored almost all of my mathematics.

Branched surfaces are an important tool in dynamics. Williams first used branched surfaces to study 2-dimensional hyperbolic attractors of diffeomorphisms [Wil73]. Birman and Williams used a special class of branched surfaces with boundary, which they called "knot holders", to study 1-dimensional hyperbolic invariant sets of flows on the 3 -sphere [BW83]. Nowadays knot holders are known as "templates", and a rich literature of templates has grown up. See, for example, the recent notable work of Ghrist [Ghr96], [Ghr95], who proves that if $K$ is the figure-eight knot in $S^{3}$, and if $\Phi$ is any flow transverse to the fibration of $S^{3}-K$ over $S^{1}$, then $\Phi$ contains periodic orbits of every knot type in $S^{3}$. The paper [Ghr96] also has a bibliography on the subject of templates.

Closer to our present topic, Christy applied branched surfaces to the study of 2-dimensional hyperbolic attractors [Chr93]. Following Christy, we define an unstable dynamic branched surface in a 3-manifold $M$ to be a branched surface $B \subset M$ together with a nonzero vector field $V$ tangent to $B$ that always flows "into" the branch locus. In most cases of interest $V$ generates a forward semiflow on $B$. When this semiflow is expansive, Christy's main result is that there a flow $\Phi$ defined in a neighborhood $N(B)$, and a 2-dimensional lamination $\Lambda$ carried by $B$, such that $\Lambda$ is a hyperbolic attractor. One can also reverse the direction of the vector field on $B$ to obtain a stable dynamic branched surface, which carries a hyperbolic repeller of some flow defined on $N(B)$.

The advantage of dynamic branched surfaces over templates, for our purposes, is that a dynamic branched surface may be regarded as an essential branched surface in the sense of Gabai and Oertel [GO89]. In other words, dynamic branched surfaces are more closely associated to the global topology of the ambient manifold. Nonetheless, template theory plays a crucial role in our development of dynamic branched surfaces and dynamic pairs (see §3.3).

Another classical tool of dynamical systems is "symbolic dynamics", in which directed graphs are used to encode the dynamics of hyperbolic invariant sets of flows, with Markov partitions as an intermediary. According to Bowen [Bow73], the idea of symbolic dynamics first arose with Hadamard and later Marston Morse. The construction of Markov partitions for hyperbolic invariant sets of flows is due to Bowen [Bow78], [Bow73]. Fried, simultaneously generalizing hyperbolic invariant sets of homeomorphisms and pseudo-Anosov homeomorphisms, introduced "finitely presented" homeomorphisms, and constructed Markov partitions for them [Fri87]. (Although it has not yet been done, there should be a theory of finitely presented flows, simultaneously generalizing hyperbolic invariant sets of flows and pseudo-Anosov flows, and including examples such as the geodesic flow of a word hyperbolic group [Gro93]).

Given a pA flow $\Phi$ on a torally bounded 3 -manifold $M$, the stable and unstable laminations $\Lambda^{s}, \Lambda^{u}$ are carried, respectively, by stable and unstable branched surfaces $B^{s}, B^{u}$. We put these into a single package called a dynamic pair of branched surfaces, by requiring the vector fields on $B^{s}$ and $B^{u}$ to be compatible. We also require the pair $B^{s}, B^{u}$ to "fill up" $M$, in a sense which is made precise by describing the allowed types of components of $M-\left(B^{s} \cup B^{u}\right)$. The intersection $B^{s} \cap B^{u}$ is an oriented train track $\tau$ which, regarded as a directed graph, encodes a Markov partition and symbolic dynamics for the flow $\Phi$.

Chapter 2 contains the theory of dynamic pairs. The main results in this chapter are about the structure of dynamic pairs, and methods for constructing them. In particular, proposition 2.6.2 describes how, starting from an unstable dynamic branched surface, one may construct a dynamic pair.

Chapter 3 describes the relations between dynamic pairs, pA flows, and pseudo-Anosov flows. Theorem 3.3.2 describes precisely how to pass back and forth between pA flows and dynamic pairs. Theorem 3.4.1 describes how to pass from a dynamic pair on a closed manifold to a pseudo-Anosov flow.

Chapter 3 also contains transversality constructions, designed to establish transversality statements in the conclusion of theorems C and $\mathrm{C}^{\prime}$. Given a transversely oriented, Reebless foliation $\mathcal{F}$ carried by a branched surface $\beta$, and given a dynamic pair $B^{s}, B^{u}$, we define the property that $B^{s}, B^{u}$ is "vertical" with respect to $\beta$. Theorem 3.5.4 says that if $B^{s}, B^{u}$ is vertical with respect to $\beta$, and if the manifold $M$ is closed, then a pseudo-Anosov flow carried by $B^{s}, B^{u}$ is almost transverse to $\mathcal{F}$.

Almost transversality is a delicate property, and theorem 3.5.4 is hard to prove. We also offer a very simple result, proposition 3.5.3, which says that if the dynamic pair $B^{s}, B^{u}$ is vertical with respect to $\beta$, then the stable and unstable laminations of a pA flow carried by $B^{s}, B^{u}$ are transverse to $\mathcal{F}$. This simpler result is probably adequate for most applications, and it does not require the ambient manifold to be closed.

All of the theory from chapters 2 and 3 can be carried out in the setting of sutured manifolds, and this is the subject of chapter 4 . Formulating the correct definition of a dynamic pair is the hardest part, and there are many new cases and extra details needed to supply complete proofs, but the basic structure of the sutured manifold theory is the same as the torally bounded theory.

The proof of theorems C and $\mathrm{C}^{\prime}$ are contained in chapters 5 and 6 (NOT INCLUDED IN THIS VERSION). Here is a sketch of the proof, for the above example of a depth 1 foliation $\mathcal{F}$ on $M$. First use the Handel-Miller process, appropriately combinatorialized, to construct a dynamic pair of branched surfaces $B_{N}^{s}, B_{N}^{u}$ in $N$, the sutured manifold obtained by cutting $M$ open along the compact leaves of $\mathcal{F}$. These branched surfaces have boundary train tracks $\tau^{s}=\partial B_{N}^{s} \subset \mathcal{R}_{-} N, \tau^{u}=\partial B_{N}^{u} \subset R_{+}$. We may not regard $B_{N}^{s}, B_{N}^{u}$ as a dynamic pair in the torally bounded manifold $M$, because these branched surfaces have boundary on the surface $S$. To correct this problem, let $B_{N}^{u}=B_{0}^{u}$ "flow up" past $S$, generating a new branched surface $B_{1}^{u} \supset B_{0}^{u}$ with boundary train track $\tau_{1}^{u} \supset \tau_{0}^{u}$. Continue this process,
defining a sequence $B_{0}^{u} \subset B_{1}^{u} \subset B_{2}^{u} \subset \cdots$ with boundary train tracks $\tau_{0}^{u} \subset \tau_{1}^{u} \subset \tau_{2}^{u} \subset \cdots$. This train track sequence fills up more and more of $S$, and so must eventually stabilize at some finite stage: $\tau_{I}^{u}=\tau_{I+1}^{u}=\cdots$. Once stabilization is acheived, the branched surfaces also stabilize: $B_{I+1}^{u}=B_{I+2}^{u}=\cdots$, and this becomes the unstable branched surface $B^{u}$ in $M$. By using the tools of chapters $2-4$, in particular proposition 2.6.2 and its ultimate generalization 4.11.2, we produce automatically the stable branched surface $B^{s}$. One shows that either $B^{s}, B^{u}$ fill up $M$, producing the desired dynamic pair, or they do not fill up and $M$ has an incompressible, nonperipheral torus.

In general, given a sutured manifold hierarchy on a torally bounded 3-manifold $M$ [Gab83], $M=M_{0} \leadsto M_{1} \leadsto \cdots \leadsto M_{n}$, we inductively construct dynamic pairs starting with $M_{n}$ and working down through the hierarchy, or we eventually produce a nonperipheral, incompressible torus. The basis step of the induction, an adaptation of the methods of Handel and Miller, is proved in chapter 5. The inductive step is proved in chapter 6.

## Questions

Here are some questions that are raised by these results.

Tori and flows Can you have both an incompressible torus and a pseudo-Anosov flow? The answer is an easy yes, in fact you can have an incompressible torus and an Anosov flow, e.g. the geodesic flow of a hyperbolic surface. For a deeper understanding, one needs a geometric description of the relationship between a pseudo-Anosov flow and the torus decomposition of the manifold. In principle, one should be able to take a description of the flow and decide if there is an incompressible torus (given the geometrization conjecture, this would help decide if the manifold is hyperbolic). The work of Fenley should be useful here [Fen96b].

A closely related question is: For which finite depth foliations $\mathcal{F}$ is there an almost transverse pseudo-Anosov flow? In other words, even if there exists an incompressible, nonperipheral torus transverse to $\mathcal{F}$, not isotopic to a leaf of $\mathcal{F}$, can one still find a pA or pseudo-Anosov flow?

Existence of flows Which manifolds have pseudo-Anosov flows? At this writing, the only closed, oriented, irreducible, atoroidal 3-manifolds with infinite fundamental group which are known not to have pseudo-Anosov flows are certain small Seifert fiber spaces which were proved, by Brittenham, not to have essential laminations [Bri93].

For an intriguing special case of this question, let $M$ be a compact, oriented 3-manifold with $\partial M$ a torus, suppose $\operatorname{int}(M)$ has a finite volume hyperbolic structure, and suppose $M$ has a pA flow $\Phi$ whose degeneracy locus on $\partial M$ has one component. Let $M_{\gamma}$ be obtained by filling $M$ along a curve $\gamma$ which intersects the degeneracy locus exactly once. By choosing
the filling coefficients $(m(\gamma), l(\gamma))$ sufficiently close to $\infty$, the manifold $M_{\gamma}$ has a hyperbolic structure (see [Thu]). Does $M_{\gamma}$ have a pseudo-Anosov flow? Note that if $M$ fibers over the circle and $\Phi$ is transverse to the fibration, and if $\gamma$ is parallel to the boundary of the fiber, then the answer is yes, because $M_{\gamma}$ also fibers over the circle, and the monodromy map must be isotopic to a pseudo-Anosov homeomorphism.

Geometrization As a special case of the geometrization conjecture one can ask: if $M$ has a pseudo-Anosov flow and no incompressible torus, is $M$ hyperbolic? Is $\pi_{1}(M)$ a word hyperbolic group?

Thurston's norm We alluded earlier to the following conjecture:
Conjecture (Norm and flow finiteness conjecture). If $B_{x}$ is the Thurston norm unit ball in $H_{2}(M ; \mathbf{R})$, the boundary of $B_{x}$ may be covered by the finitely many polygons $\Sigma_{\Phi}$ associated to pseudo-Anosov flows $\Phi$ on $M$ that are almost transverse to finite depth foliations.

It is easy to see that if $\mathcal{F}$ is a fibration of $M$ over $S^{1}$, and if $M$ is atorioidal, then there is a unique pseudo-Anosov flow transverse to $\mathcal{F}$, up to reparameterization and isotopic preserving $\mathcal{F}$; this follows from the uniqueness of pseudo-Anosov surface homeomoprhisms in their isotopy classes [ $\left.\mathrm{FLP}^{+} 79\right]$.

Here is series of successively stronger conjectures which would resolve the finiteness conjecture for the Thurston unit ball. The first conjecture should be reasonably easy to attack, by induction down through the depths:

Conjecture (Transverse finiteness conjecture). For any finite depth foliation $\mathcal{F}$ on an atoroidal 3 -manifold $M$, there are only finitely many pseudo-Anosov flows that are almost tranverse to $\mathcal{F}$.

Remark. The construction used in the proof of theorem C produces at most a finite collection of pseudo-Anosov flows that are almost transverse to $\mathcal{F}$, as long as there are no incompressible, nonperipheral tori. A good understanding of this proof should provide a clue to the transverse finiteness conjecture.

The following would imply the finiteness conjecture for the Thurston norm:
Conjecture (pseudo-Anosov finiteness conjecture). Given an atoroidal 3-manifold $M$, there exists only finitely many pseudo-Anosov flows $\Phi$ on $M$, up to isotopy and reparameterization, such that $\Phi$ is almost tranverse to some finite depth foliation on $M$.

Here is one possibility for attacking the pseudo-Anosov finiteness conjecture. Let $\mathcal{F}$, $\mathcal{F}^{\prime}$ be two Reebless finite depth foliations whose tangential Euler classes are equal. By the transverse finiteness conjecture, there exist finitely many pseudo-Anosov flows that are
almost transverse to $\mathcal{F}$. We conjecture that one of these flows is almost transverse to $\mathcal{F}^{\prime}$. As a starting point for this conjecture, one wants to prove that for each norm-minimizing surface $S \subset M$ such that $|\chi(S)|=\left|\chi_{\mathcal{F}}(S)\right|$, there exists a pseudo-Anosov flow which is almost transverse to $\mathcal{F}$ and to $S$.

In general it is not true that if $\Phi$ is pseudo-Anosov and $\mathcal{F}$ is finite depth, and if the tangential Euler class of $\mathcal{F}$ is equal to the normal Euler class of $\Phi$, then $\Phi$ is almost transverse to $\mathcal{F}$; a counterexample is given in [Mos92b].

Introduction

## Part I

## Branched surfaces and dynamics

## Chapter 1

## Preliminaries

In this section we review material from the theory of laminations and branched surfaces ( $\S 1.1-1.4$ ) and we begin the study of dynamic branched surfaces ( $\S 1.5-1.6$ ).

### 1.1 Laminations

In this section we review from [GO89] the definition of an essential lamination on a 3-manifold, and we define "very full" essential laminations, a special case of the full laminations of [HO96].

A familiar operation in topology is that of "cutting open" a manifold along a submanifold. For instance, if a closed surface is cut along an embedded circle the result is a compact surface with one or two boundary components, depending on whether or not the circle is orientation reversing. In order to make use of this cutting operation in many different contexts, we formalize it as follows.

Let $X$ be a smooth compact manifold, or a smooth compact subcomplex of a smooth manifold. Let $Z$ be an open subset of $X$. Choosing a Riemannian metric on $X$, there is an induced Riemannian metric on $Z$. Define a topological metric on $Z$ where $d(x, y)$ is the infimum of path lengths from $x$ to $y$. Let $\mathfrak{C} Z$ denote the completion of $Z$. Since any two Riemannian metrics on $X$ are bilipschitz equivalent, as a topological space $\mathfrak{C} Z$ is well-defined independent of the metric on $X$. The inclusion map $Z \hookrightarrow X$ extends uniquely to a continuous map $\mathfrak{C} Z \rightarrow X$ called the overlay map. Abusing terminology we sometimes refer to the overlay map as the inclusion map. If $Y$ is a subset of $X$, the remains of $Y$ in $\mathfrak{C} Z$ is the inverse image of $Y$ under the overlay map $\mathfrak{C} Z \rightarrow X$. Sometimes we abuse notation and write $\mathfrak{C}(Z) \cap Y$ for the remains of $Y$ in $\mathfrak{C} Z$.
Example. If $M$ is a manifold and $S$ is a codimension-1 submanifold, then $\mathfrak{C}(M-S)$ is what we usually mean when we talk about " $M$ cut open along $S$ ". Each component of the remains of $S$ is called a scar of $S$. If $\partial M=\emptyset$ then the remains of $S$ is $\partial \mathfrak{C}(M-S)$.

Example. If $U$ is an open cell in a smooth cell complex $X$, the overlay map takes $\mathfrak{C} U$ homeomorphically to the associated closed cell $\operatorname{cl}(U)$-the overlay map "lays $\mathfrak{C} U$ over" the set $\operatorname{cl}(U)$. There are many similar situations where the completion operator " $\mathfrak{C}$ " and the closure operator "cl" give the same result, but when cutting along submanifolds or subcomplexes it is usually safer to use the completion operator.
Example. If $U$ is a complementary component of a codimension-1 lamination with smooth leaves, $\mathfrak{C} U$ is the union of $U$ with boundary leaves incident to $U$, as described below.

Given an $n$-manifold $M$, a $k$-dimensional lamination (without boundary) is a closed subset $\Lambda$ of $M$ contained in $\operatorname{int}(M)$ which is decomposed into $k$-dimensional manifolds called leaves, so that $\Lambda$ is covered by open charts of the form $U \approx D^{k} \times D^{n-k}$, where for each leaf $L$, each component of $U \cap L$ has the form $D^{k} \times t$ for some $t \in D^{n-k}$. A $k$-dimensional lamination with boundary $(\Lambda, \partial \Lambda) \subset(M, \partial M)$ is similarly defined, with the additional requirement that $\partial \Lambda$ is a $(k-1)$-dimensional lamination in $\partial M$, and $\partial M$ has a collar neighborhood $U \approx \partial M \times[0,1)$ such that $\Lambda \cap U \approx \partial \Lambda \times[0,1)$.

Given a manifold $M$ and a lamination $\Lambda$ in $M$, a homeomorhism $f: M \rightarrow M$ is said to preserve $\Lambda$ if $f$ takes each leaf of $\Lambda$ to itself, and $f$ is said to respect $\Lambda$ is $f$ takes each leaf of $\Lambda$ to some other leaf. Dennis Sullivan told me that this terminology is due to Michael Gromov.

Let $M$ be a compact 3-manifold and $\Lambda$ a 2 -dimensional lamination without boundary in $M$. A transversely oriented leaf $L$ of $\Lambda$ is called a boundary leaf if $L$ has a one-sided collar neighborhood $L \times[0,1) \hookrightarrow M$ on the positive side of $L$, which is $1-1$ immersed in $M$, and whose interior $L \times(0,1)$ is disjoint from $\Lambda$. Let $M_{\Lambda}=\mathfrak{C}(M-\Lambda)$, a 3-manifold with $\operatorname{int}\left(M_{\Lambda}\right)=M-\Lambda$ and $\partial M_{\Lambda}$ identified with the union of boundary leaves of $\Lambda$. For each boundary leaf $L$, the one-sided collar neighborhood $L \times[0,1)$ described above may be regarded as a collar neighborhood of $L$ in $M_{\Lambda}$. A lamination $\Lambda$ is essential if the following conditions hold:

- $\Lambda$ has no sphere leaves.
- $\Lambda$ has no Reeb components.
- $M_{\Lambda}$ is irreducible.
- $\partial M_{\Lambda}$ is incompressible in $M_{\Lambda}$.
- $\partial M_{\Lambda}$ is end incompressible in $M_{\Lambda}$.

We refer the reader to [GO89] for a detailed discussion of this definition, with the following brief reminders.

A Reeb component of $\Lambda$ is a solid torus $H \approx D^{2} \times S^{1}$ embedded in $M$ such that $\partial H$ is a leaf of $\Lambda$, each leaf $L \subset \operatorname{int}(H)$ is a topological plane, $L$ is symmetric with respect to
rotation of the $D^{2}$ coordinate of $H$, and $\bar{L}-L \subset \partial H$. The leaf $L$ is like an infinite snake eating its tail.

End incompressibility is defined as follows. Let $E$ be a closed disc with one boundary point removed, $E=\left\{(x, y) \in \mathbf{R}^{2} \mid\|(x, y)\| \leq 1,(x, y) \neq(1,0)\right\}$. Given a subsurface $F \subset \partial M_{\Lambda}$, an end compression of $F$ is a proper embedding $f:(E, \partial E, \operatorname{int}(E)) \hookrightarrow$ $\left(M_{\Lambda}, F, \operatorname{int}\left(M_{\Lambda}\right)\right.$ ), where proper means that the inverse image of a compact subset of $M_{\Lambda}$ is compact. The subsurface $F$ is end incompressible if for every end compression $f$, the restricted map $\partial f: \partial E \hookrightarrow F$ extends to a proper embedding $f^{\prime}: E \hookrightarrow F$.

There is an equivalent formulation of end incompressibility which makes use of extra structure on $M_{\Lambda}$, which it has by virtue of being the completion of a component of $M-\Lambda$. The manifold $M_{\Lambda}$ has a compact submanifold-with-corners $N$, a "core", having the following properties:

- $\partial_{h} N=\partial N \cap \partial M_{\Lambda}$ is a union of faces.
- $\partial_{v} N=\operatorname{cl}\left(\partial N-\partial_{h} N\right)$ is also a union of faces.
- Each component of $\operatorname{cl}\left(M_{\Lambda}-N\right)$ is noncompact.
- $\operatorname{cl}\left(M_{\Lambda}-N\right)$ is an $I$-bundle over a noncompact surface $F$, such that $\partial_{v} N$ is the restriction of the $I$-bundle to $\partial F$.

The existence of the "core" is a standard result in lamination theory. We say that an end compression $f: E \rightarrow M_{\Lambda}$ is vertical near the end if $f(E) \cap \operatorname{cl}\left(M_{\Lambda}-N\right)$ is a union of $I$-fibers. It follows easily that a subsurface $F \subset \partial M_{\Lambda}$ is end incompressible if and only if there is no end compression $f: E \rightarrow M_{\Lambda}$ with $f(\partial E) \subset F$ which is vertical near the ends.

In the definition of essentiality for $\Lambda$ we refer only to end incompressibility of $\partial M_{\Lambda}$. However, when we study laminations on sutured manifolds in $\S 4.10$ we will use more general subsurfaces of $\partial M_{\Lambda}$.

A lamination $\Lambda$ is very full if it has neither sphere leaves nor Reeb components, and each component of $M_{\Lambda}$ is an essential pared torus piece as defined in the introduction. In particular, $M_{\Lambda}$ has a core whose closed complementary components are products of the form $F \times I$ where the surface $F$ is a half-open annulus. Obviously pared torus pieces satisfy properties (3-5) in the definition of an essential lamination, and hence a very full lamination is essential; the essentiality condition in the definition of a pared solid torus guarantees that there are no end compressions which are vertical near the ends. Note that for each pared solid torus $H-K$, there exist integers ( $m, k$ ) with $m \geq 2$ and $0 \leq k<m$, such that $m, k$ are the geometric intersection number of $K$ with, respectively, a meridian and a longitude of $H$. The pair $(m, k)$ is called the type of the pared solid torus. Also, given a pared torus shell $T^{2} \times[0,1]-K \times 1$, the number $n \geq 1$ of components of $K$ is called the type .

### 1.2 Branched surfaces

In this section we review branched surfaces and laminations carried by branched surfaces.
Let $M$ be a smooth, compact $n$-manifold. A branched $k$-manifold (without boundary) in $M$ is a smooth, compact, $k$-dimensional subcomplex $B \subset \operatorname{int}(M)$ with the property that each $x \in B$ has a neighborhood in $B$ which is a union of smooth $k$-discs embedded in $M$, all tangent at $x$, thereby defining a unique tangent $k$-plane $T_{x} B \subset T_{x} M$. A branched $k$-manifold (with boundary) is a smooth $k$-complex in $M$ with the same property at points of $B \cap \operatorname{int}(M)$, with the extra requirement that $\partial B=B \cap \partial M$ is a branched $(k-1)$ manifold without boundary, and for some collar neighborhood $U \approx \partial M \times[0,1)$ of $\partial M$ we have $B \cap U \approx \partial B \times[0,1)$. A branched 2-manifold is called a branched surface, and a branched 1-manifold is called a train track. Given a branched $k$-manifold $B$, a point $x \in B$, and a smooth $k$-manifold $S \subset B$ with $\partial S=S \cap \partial B$ such that $x \in S$, the germ of $S$ at $x$ is called a sheet of $B$ at $x$. By local finiteness of $B$ it follows that each point of $B$ has finitely many sheets. Nonmanifold points of $B$, i.e. points where $B$ is not locally a manifold with boundary, are characterized by having two or more sheets.

If $B$ is a branched surface, the set of nonmanifold points of $B$ is a smooth 1-complex denoted $\Upsilon B$, the branch locus of $B$. A nonmanifold point of $\Upsilon B$, i.e. a point where $\Upsilon B$ is not a 1-manifold with boundary, is called a crossing point of $B$. A completed component of $B-\Upsilon B$ is called a sector of $B$. If $\tau$ is a train track, nonmanifold points of $\tau$ are called switches, and completed components of $\tau-$ (switches) are called branches.

Henceforth, unless specified otherwise, we shall assume that every train track and every branched surface has generic branch locus. For a train track $\tau$, generic branch locus means that every switch $s \in \tau$ is trivalent, with exactly two sheets. On one half of the tangent line $T_{s} \tau$, called the one-sheeted side, the two sheets coincide. On the other half of $T_{s} \tau$, the two-sheeted side, the two sheets are distinct. For a branched surface $B$, generic branch locus means first of all that $B$ is trivalent at each noncrossing point $x \in \Upsilon B$, that is, there is a neighborhood of $x$ in $B$ of the form $Y \times(0,1)$ where $Y$ is a neighborhood of a generic switch in a train track. It follows that $B$ has two sheets at $x$, and $\Upsilon B$ divides $T_{x} B$ into two halves, the one-sheeted side on which the two sheets coincide, and the two-sheeted side on which the two sheets are distinct. For a crossing point $x \in \Upsilon B$, generic branching means that there are four ends of $\Upsilon B-\{x\}$, defining four distinct directions in $T_{x} B$ arranged in cyclic order. These four directions divide $T_{x} B$ into four quadrants: a one sheeted quadrant; two two sheeted quadrants adjacent to the one-sheeted quadrant; and a three sheeted quadrant opposite the one sheeted quadrant (see figure 1.1). There are three sheets at $x$, arranged from top to bottom as $S_{1}, S_{2}, S_{3}$; all three coincide in the one-sheeted quadrant; all three are distinct in the three-sheeted quadrant; in one two-sheeted quadrant $S_{1}$ and $S_{2}$ coincide and are distinct from $S_{3}$; in the other two-sheeted quadrant $S_{2}$ and $S_{3}$ coincide and are distinct from $S_{1}$. Note that $\Upsilon B$ is the image of a piecewise smooth immersion of a 1-manifold which passes from one branching direction at $x$ to the opposite direction, whenever it passes over


Figure 1.1: Generic branching of a branched surface
a crossing point $x$. The domain of this immersion is a 1-manifold called the maw, and its components are called maw curves. A maw curve is therefore either an immersion of a circle into $B$ with image contained in $\Upsilon B=\partial B$, or an immersion of an arc into $B$ with image contained in $\Upsilon B$ and endpoints contained in $\partial B$.

In figure 1.1 the maw appears smooth at each crossing point, but this is neither necessary nor convenient-it is useful to allow the maw to be piecewise smooth, especially when defining branched surface hierarchies (see figure 1.5).

A branched surface $B \subset M$, possibly with boundary, has a regular neighborhood $N(B)$ and a decomposition of $N(B)$ into interval fibers or $I$-fibers, each $I$-fiber intersecting $B$ transversely. The local model for $N(B)$ near $\Upsilon B$ and near $\partial B$ is shown in figure 1.2. Define the frontier Fr $N(B)=N(B) \cap \operatorname{cl}(M-N(B))$. There is a decomposition Fr $N(B)=$ $\mathrm{Fr}_{h} N(B) \cup \mathrm{Fr}_{v} N(B)$ as follows. The horizontal frontier $\mathrm{Fr}_{h} N(B)$ is the set of endpoints of the $I$-fibers, and the vertical frontier $\operatorname{Fr}_{v} N(B)$ is $\mathrm{cl}\left(\operatorname{Fr} N(B)-\operatorname{Fr}_{h} N(B)\right)$. The restriction of the $I$-fibration to $\mathrm{Fr}_{v} N(B)$ defines a fibration over the maw of $B$; we may embed the maw as a section of $\operatorname{Fr}_{v} N(B)$ transverse to the $I$-fibers. Note that a circular maw curve embeds as the core of an annulus component of $\operatorname{Fr}_{v} N(B)$, and an interval maw curve embeds as the core of a rectangle component of $\mathrm{Fr}_{v} N(B)$.

There is a map $q: M \rightarrow M$, homotopic to the identity, that maps $N(B)$ onto $B$ so that the inverse image of a point $x \in B$ is an $I$-fiber in $N(B)$, and so that $q \mid M-N(B)$ is a homeomorphism onto $M-B$; note that an $I$-fiber is not necessarily taken to a point where that fiber intersects $B$. We call $q$ an $I$-collapsing map for $B$. Restricting $q$ to the core of $\mathrm{Fr}_{v} N(B)$ we obtain the maw curve parameterization of $\Upsilon B$.

Note: we follow the tradition in branched surface theory of using the word "fiber" somewhat loosely. The map $q: N(B) \rightarrow B$ is not a fibration in the sense of homotopy theory [Spa81], for it need not satisfy the path lifting property-given a path in $B$, a chosen lift of the starting point to $N(B)$ need not extend to a lift of the whole path.


Figure 1.2: An $I$-fibered neighborhood of a branched surface $B$, showing the $I$-fibers of the vertical frontier. In (a) we show $N(B)$ near a noncrossing point of $\Upsilon B$, with $B$ itself shaded. In (b) we show $N(B)$ near a crossing point of $B$, with the horizontal frontier of $N(B)$ shaded on the outside of $N(B)$; the branched surface itself is suppressed in (b).

Denote $P(B)=\operatorname{cl}(M-N(B)), \operatorname{Fr} P(B)=\operatorname{Fr} N(B), \operatorname{Fr}_{v} P(B)=\operatorname{Fr}_{v} N(B)$, and $\operatorname{Fr}_{h} P(B)=\operatorname{Fr}_{h} N(B)$. Note that the manifold boundary of $P(B)$ is $\partial P(B)=\operatorname{Fr} P(B) \cup$ $(P(B) \cap \partial M)$. We also denote $\gamma P(B)=\operatorname{Fr}_{v} P(B) \cup(P(B) \cap \partial M)$, a disjoint union of annuli and tori in $\partial P(B)$. Choosing a core curve for each annulus component of $\gamma P(B)$, the union of the core curves is denoted $\sigma P(B)$. Each annulus component of $\gamma P(B)$ lies either in $\partial M$, or in $\operatorname{Fr}_{v} P(B)$, or is a union of rectangles in $\partial M$ and rectangles in $\mathrm{Fr}_{v} P(B)$. Note that $\sigma P(B) \cap \operatorname{Fr}_{v} P(B)$ is equal to the maw curves.

We may regard $N(B)$ and $P(B)$ as manifolds-with-corners: in $P(B)$, an edge where $\mathrm{Fr}_{v} P(B)$ meets $\mathrm{Fr}_{h} P(B)$ is a corner, with interior angle strictly between $0^{\circ}$ and $180^{\circ}$; whereas in $N(B)$, an edge where $\operatorname{Fr}_{v} N(B)$ meets $\operatorname{Fr}_{h} N(B)$ is a reflex corner with interior angle strictly between $180^{\circ}$ and $360^{\circ}$.

The manifold-with-corners structure on $P(B)$ and $N(B)$ described in the last paragraph and depicted in figure 1.2 will be called the corner model for these manifolds. We shall use two other models for different purposes (figure 1.3).

First is the cusp model, where each $I$-fiber of $\mathrm{Fr}_{v} P(B)$ and of $\mathrm{Fr}_{v} N(B)$ is collapsed to a point. The collapsed image of $\operatorname{Fr}_{v} P(B)$ is a cusp edge, locally modelled on the set $\left\{(x, y, z) \in \mathbf{R}^{3}|x \geq 0,|z| \leq f(x)\}\right.$, where $f:[0, \infty) \rightarrow[0, \infty)$ is a cusp function, a $C^{\infty}$ function whose value and derivatives all vanish at 0 , and which is positive on $(0, \infty)$. Similarly, the collapsed image of $\operatorname{Fr}_{v} N(B)$ is a culvert edge, whose local model is the closure of the complement of the local model of a cusp edge, namely $\left\{(x, y, z) \in \mathbf{R}^{3} \mid x \leq\right.$ 0 or $|z| \leq f(x)\}$. In the cusp model of $N(B)$, the $I$-fiber over a non-crossing point of $\Upsilon B$ touches the culvert at a single point, and the $I$-fiber over a crossing point touches the culvert at two points.

We also need the smooth model for $N(B)$, where $\operatorname{Fr} N(B)$ is a smooth surface in $M$;


Figure 1.3: Three models of the $I$-fibered neighborhood of a branched surface.
the $I$-fiber over a non-crossing point of $\Upsilon B$ has one tangential intersection with Fr $N(B)$; and the $I$-fiber over a crossing point has two tangential interections with Fr $N(B)$.

A lamination $\Lambda \subset M$ is carried by a branched surface $B$ if $\Lambda \subset N(B)$ and $\Lambda$ is transverse to the $I$-fibers. If $\Lambda$ intersects each $I$-fiber then we say that $B$ fully carries $\Lambda$.

Next we define splitting of a branched surface $B$. Choose an $I$-bundle neighborhood $p: N(B) \rightarrow B$ in the corner model. Let $F$ be a compact surface embedded in $N(B)$ transverse to the $I$-fibers, and suppose that $\partial F=\partial_{v} F \cup \partial_{i} F$, where $\partial_{v} F$ and $\partial_{i} F$ are compact submanifolds with disjoint interior, and $\partial_{v} F=F \cap \partial_{v} N(B)$. Suppose moreover that $p \mid \operatorname{int}\left(\partial_{i} F\right)$ is in general position with respect to $\Upsilon B$. Then we call $F$ a splitting surface, and we define a branched surface $B_{F}$ obtained by splitting $B$ along $F$, as follows. Let $N(F)$ be an $I$-bundle over $F$ each of whose fibers is embedded in an $I$-fiber of $N(B)$. Let $N\left(B_{F}\right)=\operatorname{cl}(N(B)-N(F))$. The branched surface $B_{F}$ is defined abstractly as the quotient of $N\left(B_{F}\right)$ obtained by crushing each $I$-fiber of $N\left(B_{F}\right)$ to a point. To get a concrete embedding of $B_{F}$ in $M$, perturb the map $p \mid N\left(B_{F}\right)$ so that it crushes distinct $I$-fibers to distinct points. Clearly $B_{F}$ is embedded in $N(B)$ transverse to the $I$-fibers, so the map $p \mid B_{F}$ is defined and is a submersion called the carrying map from $B_{F}$ to $B$. Notice that if $F$ is altered by an isotopy of $N(B)$ preserving $I$-fibers, then $B_{F}$ is unchanged. If $p \mid F: F \rightarrow B$ is an embedding, and if $E=p(F)$, then we sometimes abuse terminology by saying that $B_{F}$ is obtained by splitting $B$ along $E$; this abuse of terminology always assumes that $E$ lifts to some splitting surface $F$ unique up to an $I$-fiber preserving isotopy of $N(B)$.

### 1.3 Finite depth foliations

Let $M$ be a manifold with torus boundaries. A lamination covering all of $M$ and transverse to $\partial M$ is called a foliation of $M$. A transversely oriented foliation $\mathcal{F}$ is called taut if for every $x \in M$ there exists an immersion $S^{1} \rightarrow M$ transverse to $\mathcal{F}$ passing through $x$. If $\mathcal{F}$ is taut then $\mathcal{F}$ is Reebless.

Let $\mathcal{F}$ be a transversely oriented foliation of $M$. Suppose that every leaf $L$ of $\mathcal{F}$ is proper, meaning that the leaf topology coincides with the subspace topology on $L$, that is, $L$ is
covered by foliation charts each intersecting $L$ in a disc. Define $\mathcal{F}_{0}$ to be the sublamination of $\mathcal{F}$ consisting of the compact leaves, and for each $n \geq 1$ define $\mathcal{F}_{n}$ by induction to be the sublamination consisting of $\mathcal{F}_{n-1}$ union all leaves $L$ such that $\operatorname{cl}(L)-L \subset \mathcal{F}_{n-1}$. We say that $\mathcal{F}$ is a finite depth foliation if $\mathcal{F}_{n}=M$ for some integer $n \geq 0$. The depth of a leaf $L$ is the minimal $k$ such that $L \subset \mathcal{F}_{k}$.

Recall the construction of Gabai [Gab83, Gab87], which shows that taut, finite depth foliations are ubiquitous in 3 -manifolds. Given a compact 3 -manifold $M$, the "Thurston norm" on $H_{2}(M, \partial M ; \mathbf{R})$ is a semi-norm which describes the minimal complexity of a surface representing a given integer homology class. More precisely, given a compact surface $S$ let

$$
x(S)=\sum_{\chi\left(S_{0}\right) \leq 0}-\chi\left(S_{0}\right)
$$

where the sum is taken over components $S_{0}$ of $S$. Given $\sigma \in H_{2}(M, \partial M ; \mathbf{Z})$, define $x(\sigma)$ to be the infimum of $x(S)$ over all properly embedded oriented surfaces $(S, \partial S) \subset(M, \partial M)$ representing $\sigma$. Then $x$ extends to a pseudo-norm on $H_{2}(M, \partial M ; \mathbf{R})$ called the Thurston norm. An oriented surface $(S, \partial S) \subset(M, \partial M)$ is norm-minimizing if each component of $S$ is incompressible, and $S$ realizes the minimum of $x$ in the homology class of $S$.

Theorem 1.3.1 ([Gab83]). Let $M$ be a compact, irreducible 3-manifold with torus boundaries. If $S$ is a norm-minimizing surface, then there exists a taut, finite depth foliation $\mathcal{F}$ with $S$ as a compact leaf.

### 1.4 Hierarchies

In this section, we recall the combinatorial methods used by Gabai to construct finite depth foliations. Gabai's original construction in [Gab83] was couched in terms of "sutured manifold hierarchies", and in [Gab87] constructions 4.16 and 4.17, he reworded the construction in terms of "branched surface hierarchies". Both points of view will be useful to us: when viewing the hierarchy as a whole we usually think in terms of branched surfaces; but when looking at a particular level in the hierarchy we think in terms of sutured manifolds.

### 1.4.1 Sutured manifolds and their decompositions

Recall that if $M$ is an oriented manifold and $S \subset M$ is a submanifold, each orientation on $S$ induces a transverse orientation on $S$, and vice versa, giving a 1-1 correspondence; we use this correspondence without comment in what follows. The orientation on $\partial M$ corresponding to the outward transverse orientation is called the boundary orientation.

A sutured manifold is a compact, oriented 3 -manifold $P$, equipped with a decomposition $\partial P=\gamma P \cup \mathcal{R}_{-} P \cup \mathcal{R}_{+} P$ into submanifolds with disjoint interior, so that the following properties are satisfied:

1. $\gamma P$ is a disjoint union of tori and annuli in $\partial P$.
2. $\mathcal{R}_{-} P \cap \mathcal{R}_{+} P=\emptyset$.
3. $\partial R_{-} P \cup \partial R_{+} P=\partial \gamma P$.
4. For each annulus component $A \subset \gamma P$, one component of $\partial A$ is in $\mathcal{R}_{-} P$ and the other is in $\mathcal{R}_{+} P$.

The union of annulus components is denoted $\gamma_{A} P$, the union of torus components is $\gamma_{T} P$, and we denote $\mathcal{R} P=\mathcal{R}_{-} P \cup \mathcal{R}_{+} P$. The core of an annulus component is called a suture, and the union of sutures is denoted $\sigma P$. We shall sometimes regard a sutured manifold as a smooth manifold-with-corners, so that each component of $\partial \gamma_{A} P$ is a corner with $90^{\circ}$ interior angle; this is called the corner model of $P$.

We sometimes put a different manifold-with-corners structure on a sutured manifold $P$, called the cusp model (compare figure 1.3), which is smooth except for a cusp along the suture $\sigma P$, which is locally modelled on the set $\left\{(x, y, z) \in \mathbf{R}^{3} \mid x \geq 0,-f(x) \leq\right.$ $z \leq f(x)\}$ where $f:[0, \infty) \rightarrow[0, \infty)$ is a cusp function. In the cusp model we have $\sigma P=\mathcal{R}_{-} P \cap \mathcal{R}_{+} P=\partial \mathcal{R}_{-} P=\partial \mathcal{R}_{+} P$.
Example. Suppose $M$ is a manifold with torus boundaries, and $B \subset M$ is a transversely oriented branched surface with boundary. Suppose also that $B$ is groomed which means that for each annulus component $A$ of $\mathfrak{C}(\partial M-\partial B)$, the transverse orientation points into $A$ along one boundary circle and out of $A$ along the other. Then $P(B)$ has the natural structure of a sutured manifold in the corner model where $\mathcal{R} P(B)=\partial_{h} P(B)$, and the transverse orientation points inward along $\mathcal{R}_{-} P(B)$ and outward along $\mathcal{R}_{+} P(B)$; the transverse orientation on $\mathcal{R} P(B)$ is defined by pulling back the transverse orientation on $B$ under the $I$-fiber collapsing map $P(B) \rightarrow B$. Also, $\gamma P(B)=\operatorname{Fr}_{v} P(B) \cup(P(B) \cap \partial M)$, as defined earlier for any branched surface.

Example. A sutured manifold $P$ is a product if $(P, \gamma P) \approx(F \times I, \partial F \times I)$ for some compact surface $F$.

Now we recall Gabai's operation of sutured manifold decomposition. Consider a sutured manifold $P$ in the cusp model. A properly embedded, transversely oriented surface $S \subset P$ is a decomposing surface if the following are satisfied:

1. $S$ is transverse to $\sigma P$.
2. $S$ is groomed, that is, for each component $T$ of $\gamma_{T} P$, the components of $\partial S \cap T$, with transverse orientation induced from $S$, are isotopic as transversely oriented circles.
3. No component of $S$ is a disc with boundary in $\mathcal{R}_{-} P$ or $\mathcal{R}_{+} P$.
4. No component of $\partial S$ is the boundary of a disc in $\mathcal{R}_{-} P$ or $\mathcal{R}_{+} P$.


Figure 1.4: Sutured manifold decomposition. The transverse orientation, on $S$ as well as on $\mathcal{R}$, is the orientation pointing out of the page.

We define a sutured manifold decomposition $P \stackrel{S}{\leadsto} P^{\prime}$ as follows. As a topological manifold we have $P^{\prime}=\mathfrak{C}(P-S)$, with scars $S^{-}, S^{+}$, where the notation is chosen so that the pullback of the transverse orientation on $S$ defines transverse orientations on $S^{-}$and $S^{+}$which point into $P^{\prime}$ along $S^{-}$and out of $P^{\prime}$ along $S^{+}$. Let the remains in $P^{\prime}$ of the surfaces $\mathcal{R}_{-} P$ and $\mathcal{R}_{+} P$ be denoted $\mathcal{R}_{-}^{\prime}$ and $\mathcal{R}_{+}^{\prime}$. Define $\mathcal{R}_{-} P^{\prime}=S^{-} \cup \mathcal{R}_{-}^{\prime}$ and $\mathcal{R}_{+} P^{\prime}=S^{+} \cup \mathcal{R}_{+}^{\prime}$. This gives $P^{\prime}$ the structure of a sutured manifold in a mixture of the cusp and corner models. To convert to a pure cusp model, collapse each annulus component of $\gamma_{T} P=S$ to a cusp. See figure 1.4 to see how the suture structure is affected near a point of $\partial S \cap \sigma P$. The overlay map $P^{\prime} \rightarrow P$ takes each of $S^{-}, S^{+}$homeomorphically to $S$, thereby inducing a gluing map $g: S^{+} \rightarrow S^{-}$. The overlay map $P^{\prime} \rightarrow P$ is the quotient map obtained by identifying each $x \in S^{+}$with $g(x) \in S^{-}$.
Remark. Gabai's original definition in [Gab83] works in the corner model. In place of the condition that $S$ intersects $\sigma P$ transversely, the requirement is that for each component $A$ of $\gamma_{A} P$, every component of $S \cap A$ is either an arc connecting opposite boundary circles of $A$, or a "groomed circle", a circle which, when equipped with a transverse orientation inherited from $S$, is oriented isotopic to the boundary circles of $A$, equipped with their transverse orientations inherited from $\mathcal{R} P$. The difference between these two definitions is not very strong, because circle components of $S \cap \gamma_{A} P$ can always be pushed out of $\gamma_{A} P$ by isotopy of $S$, and the grooming condition implies that the decomposed manifold is unaffected by this isotopy.

### 1.4.2 Branched surface hierarchies

Let $M$ be a compact, oriented, torally bounded 3-manifold. A branched surface hierarchy in $M$ is a sequence of transversely oriented branched surfaces $B_{0} \subset B_{1} \subset \cdots \subset B_{K}$ in $M$ such that:

- $B_{0}$ is a surface.
- Each component of $S_{k}^{\prime}=\mathfrak{C}\left(B_{k}-B_{k-1}\right)$ is a sector of $B_{k}$.


Figure 1.5: Given branched surfaces $B_{0} \subset B_{1}$ such that $S=S_{1}^{\prime}=\mathfrak{C}\left(B_{1}-B_{0}\right)$ is a sector of $B_{1}$, this figure shows how $S$ attaches to $B_{0}$ at a point where $\partial S$ crosses $\Upsilon B_{0}$, producing a crossing point $x$ of $B_{1}$. In (a) the point $x$ is a non-smooth manifold point of $\Upsilon B_{0}$. In (b) the sector $S$ has been attached, with boundary crossing $\Upsilon B_{0}$ at $x$. In (c) we show the view inside the cusp model for $P\left(B_{0}\right)$, looking down the maw. Compare with the sutured manifold decomposition depicted in figure 1.4.

- No component of $S_{k}^{\prime}$ is a disc whose boundary is a maw circle of $B_{k}$.
- There is no smoothly embedded disc $D \subset B_{k-1}$ whose boundary is a maw circle of $B_{k}$.
- Each annulus component $A$ of $\mathfrak{C}\left(\partial M-\partial B_{i}\right)$ is groomed, and so $P\left(B_{i}\right)=\mathfrak{C}\left(N-N\left(B_{i}\right)\right)$ is a sutured manifold.
- $P\left(B_{K}\right)$ is a product sutured manifold.

Suppose that we convert $P\left(B_{k-1}\right)$ to the cusp model by collapsing each component of $\gamma_{A} P\left(B_{k-1}\right)$ to a cusp. Do this collapsing so that each component of $S_{k}^{\prime} \cap \gamma_{A} P\left(B_{k-1}\right)$, an arc connecting opposite boundary circles, is collapsed to a point. Let $S_{k}$ be the image of $S_{k}^{\prime}$ under this collapsing. It follows that $S_{k}$ is a decomposing surface in $P\left(B_{k-1}\right)$, and moreover we have a sutured manifold decomposition $P\left(B_{k-1}\right) \stackrel{S_{k}}{\rightsquigarrow} P\left(B_{k}\right)$. Therefore, associated to each branched surface hierarchy $B_{0} \subset B_{1} \subset \cdots \subset B_{K}$ in $M$ there is a sutured manifold hierarchy

$$
M \stackrel{B_{0}}{\rightsquigarrow} P\left(B_{0}\right) \stackrel{S_{1}}{\rightsquigarrow} P\left(B_{1}\right) \leadsto \ldots \stackrel{S_{K}}{\rightsquigarrow} P\left(B_{K}\right)
$$

in the sense of [Gab83]. Figure 1.5 shows how $S_{k}^{\prime}$ attaches to $B_{k-1}$ at a point where $\partial S_{k}^{\prime}$ crosses $\Upsilon B_{k-1}$.

When the whole sequence $B_{0} \subset B_{1} \subset \cdots \subset B_{K}$ is understood, we will sometimes abuse notation and refer to $B=B_{K}$ as a branched surface hierarchy.

Now we turn to the relation between branched surfaces and finite depth foliations. Let $\mathcal{F}$ be a finite depth foliation and let $B_{0} \subset \cdots \subset B_{N-1}$ be a branched surface hierarchy.

We have previously defined what it means for a lamination to be carried by a branched surface. We say that the foliation $\mathcal{F}$ is carried by the hierarchy $B$ if there are sublaminations $\mathcal{F}_{0} \subset \cdots \subset \mathcal{F}_{N-1} \subset \mathcal{F}_{N}=\mathcal{F}$ such that $\mathcal{F}_{n}$ is fully carried by $B_{n}$ for $0 \leq n<N$, and if $N\left(B_{N}\right), P\left(B_{N}\right)$ are the $I$-fibered neighborhood and complementary sutured manifold of $B_{N}$ in the corner model, then $\mathcal{F} \mid N\left(B_{N}\right)$ is transverse to $I$-fibers and $\mathcal{F} \mid P\left(B_{N}\right)$ is transverse to the $I$-fibration of the product structure $P\left(B_{N}\right) \approx($ surface $) \times I$. The definition does not require that $\mathcal{F}_{n}$ consist exclusively of leaves of depth $\leq n$ in $\mathcal{F}$.

The construction of finite depth foliations given by Gabai in [Gab87] starts by constructing certain branched surface hierarchies (ones which are "groomed" in a stronger sense than we have described), and then produces foliations carried by such hierarchies. We shall need the following straightforward fact, whose proof is given in [FM95] for the special case of a foliation with isolated levels, meaning that each leaf has a saturated neighborhood in which all other leaves have lower depth; the general case is similar.

Proposition 1.4.1. Every finite depth foliation is carried by some branched surface hierarchy.

### 1.4.3 Sliding a hierarchy

Part of our strategy for proving the main theorem will be to construct a lamination or flow transverse to a given branched surface hierarchy, and apply the previous proposition. But there is a hitch: our construction requires operations which change the hierarchy, although the changes are quite mild and do not alter the finite depth foliations that are carried by the hierarchy. Here is a description of the operations needed.

Consider a branched surface hierarchy $B_{0} \subset \cdots \subset B_{N}=B$. If we hold $B_{n}$ fixed and isotope the attaching maps of $\mathfrak{C}\left(B-B_{n}\right)$, we obtain a new hierarchy $B_{0} \subset \cdots \subset B_{n} \subset$ $B_{n+1}^{\prime} \subset \cdots \subset B_{N}^{\prime}=B^{\prime}$, which we say is obtained from $B$ by sliding, or more specifically by sliding along level $n$. To be more precise, let $N\left(B_{n}\right), P\left(B_{n}\right)$ be the $I$-fibered neighborhood and complementary sutured manifold in the corner model, and let $q: M \rightarrow M$ be an $I$-fiber collapsing map for $B_{n}$, taking $N\left(B_{n}\right)$ onto $B_{n}$. Let $B_{n}^{c}=\operatorname{cl}\left(q^{-1}\left(B-B_{n}\right)\right)$, a branched surface with boundary in $P\left(B_{n}\right)$, which may be identified with $B \cap P\left(B_{n}\right)$; think of $B_{n}^{c}$ as a "complement" of $B_{n}$. Now consider a map $h: P\left(B_{n}\right) \rightarrow P\left(B_{n}\right)$ which is a sutured manifold homeomorphism isotopic to the identity relative to $\gamma P\left(B_{n}\right)$. Let $B^{\prime}=B_{n} \cup q\left(h\left(B_{n}^{c}\right)\right)$. Note that there is a branched surface hierarchy

$$
B_{0} \subset \cdots \subset B_{n} \subset B_{n+1}^{\prime} \subset \cdots \subset B_{N}^{\prime}=B^{\prime}
$$

where

$$
B_{i}^{\prime}=B_{n} \cup q\left[h\left(\mathfrak{C}\left(q^{-1}\left(B_{i}-B_{n}\right)\right)\right)\right] \quad \text { for } n+1 \leq i \leq N .
$$

The relation of sliding generates an equivalence relation among branched surface hierarchies in $M$ : two branched surface hierarchies are equivalent if you can get from one to the other by a sequence of slidings along levels. We have the following easy fact:

Proposition 1.4.2. If $B, B^{\prime}$ are equivalent branched branched surface hierarchies, and if $B$ carries a finite depth foliation $\mathcal{F}$, then $B^{\prime}$ also carries $\mathcal{F}$.

It is convenient to translate equivalence of branched surface hierarchies into sutured manifold terms, as follow.

Given a sutured manifold decomposition $P \stackrel{S}{\leftrightarrows} \Pi$, the homeomorphism $g: S^{+} \rightarrow S^{-}$that is consistent with the quotient map $\Pi \rightarrow P$ is called the gluing map for the decomposition. Suppose that $h_{+}: \mathcal{R}_{+} \Pi \rightarrow \mathcal{R}_{+} \Pi$ and $h_{-}: \mathcal{R}_{-} \Pi \rightarrow \mathcal{R}_{-} \Pi$ are homeomorphisms isotopic to the identity rel boundary. Then we regard the homeomorphism $h_{-} \circ g \circ h_{+}^{-1}: h_{+}\left(S^{+}\right) \rightarrow$ $h_{-}\left(S^{-}\right)$as a new gluing map, obtained by sliding the old gluing map $g$. If $\hat{P}$ is the manifold obtained from $\Pi$ by using the new gluing map, then there is an obvious induced suture structure on $\hat{P}$, and there is an obvious induced isotopy class of sutured manifold homeomorphisms $\hat{P} \leftrightarrow P$.

Proposition 1.4.3. Let $B_{0} \subset \cdots \subset B_{N}$ be a branched surface hierarchy in $M$, and for each $n$ let $P\left(B_{n-1}\right) \stackrel{S_{n}}{\rightsquigarrow} P\left(B_{n}\right)$ be the induced sutured manifold decomposition. If we slide the gluing map $g: S_{n}^{+} \rightarrow S_{n}^{-}$to produce a sutured manifold $\hat{P}\left(B_{n-1}\right)$, and if we choose a sutured manifold homeomorphism $\hat{P}\left(B_{n-1}\right) \rightarrow P\left(B_{n-1}\right)$ in the correct isotopy class, then there is an induced slide in level $n$ on the branched surface hierarchy.

Proof. Sliding $g$ means precomposing by $h_{+}: \mathcal{R}_{+} P\left(B_{n}\right) \rightarrow \mathcal{R}_{+} P\left(B_{n}\right)$ and postcomposing by $h_{-}: \mathcal{R}_{-} P\left(B_{n}\right) \rightarrow \mathcal{R}_{-} P\left(B_{n}\right)$, where $h_{-}$and $h_{+}$are homeomorphisms isotopic to the identity rel boundary. The maps $h_{-}, h_{+}$extend to a sutured manifold homeomorphism $h: P\left(B_{n}\right) \rightarrow P\left(B_{n}\right)$ isotopic to the identity rel $\gamma P\left(B_{n}\right)$, and the effect of $g$ on the hierarchy is to slide it along level $n$ using the map $h$.

### 1.5 Dynamic branched surfaces

Branched manifolds are often used to study the structure of hyperbolic attractors. Williams [Wil73] put this idea on solid mathematical ground, for expanding attractors of diffeomorphisms. Christy [Chr93] later extended this to expanding attractors of flows. In this section we shall describe several concepts due to Christy.

Recall that a semiflow on a space $X$ is a continuous map $\phi$ from a subset $D \subset X \times \mathbf{R}$ to $X$ such that

- For all $x \in X$, the set $I_{x}=\{t \in \mathbf{R} \mid(x, t) \in D\}$ is a closed connected subset of $\mathbf{R}$ containing 0 .
- $\phi(x, 0)=x$.
- For all $x \in X, s \in I_{x}$, and $t \in I_{\phi(x, s)}$ we have $s+t \in I_{x}$ and

$$
\phi(x, s+t)=\phi(\phi(x, s), t)
$$

A forward semiflow is one whose domain contains $X \times[0, \infty)$, and the domain of a backward semiflow contains $X \times(-\infty, 0]$. The trajectory through a point $x$ is the map $I_{x} \rightarrow X$ given by $t \rightarrow \phi(x, t)$. If $X$ is a smooth subcomplex of a manifold then we can speak about smooth or piecewise smooth semiflows on $X$. Also, a vector field $V$ on $X$ generates the semiflow $\phi$ if $V_{x}$ is the tangent vector to the trajectory through $x$, for each $x \in X$. If $\phi$ is understood, we often write $x \cdot t$ as a shorthand for $\phi(x, t)$. Similarly given $A \subset X$ and $J \subset \mathbf{R}$ such that $J \subset I_{x}$ for each $x \in A$, we write $A \cdot J=\{x \cdot t \mid x \in A, t \in J\}$.

A sink of a forward semiflow on $X$ is a closed subset $S \subset X$ such that for all $x \in X$ there exists $t \geq 0$ such that $x \cdot t \in S$, and if $x \in S$ then $x \cdot[0, \infty) \subset S$. A source of a backward semiflow is similarly defined.

Consider a branched surface $B \subset M$ and a nowhere zero vector field $V$ in $M$ tangent to $B$. We say that $V$ points forward along $\Upsilon B$ if:

- For each noncrossing point $x \in \Upsilon B$, the vector $V_{x}$ points from the two-sheeted side to the one-sheeted side.
- For each crossing point $x \in B$, the vector $V_{x}$ points from the three-sheeted quadrant to the one-sheeted quadrant.

In the top view of figure 1.1, a vector field tangent to the branched surface and pointing towards the northwest is a forward vector field. We say that $V$ points backward along $\Upsilon B$ if $-V$ points forward.

Remark. Note that if $\Upsilon B \neq \emptyset$, and if $V$ is tangent to $B$ and forward along $\Upsilon B$, then $V$ cannot be smooth or even Lipschitz on $M$, because $V$ has at least two local trajectories through each point of $\Upsilon B$, violating uniqueness of trajectories for Lipschitz vector fields.

An unstable dynamic branched surface in a compact 3-manifold $M$ is a pair ( $B, V$ ) where $B \subset M$ is a branched surface without boundary and:

- $V$ is a nowhere zero, $C^{0}$ vector field on $M$.
- $V$ is tangent to $B$.
- $V$ points forward along $\Upsilon B$.

Despite the above remark, we say that $V$ is smooth if it is smooth in the ordinary sense on $M-\Upsilon B$, and $V$ has a unique smooth forward trajectory starting at each point of $M$,
which depends continuously on the initial point. In other words, $V$ generates a unique forward semiflow on $M$ with smooth trajectories. It is easy to construct local models for vector fields near $\Upsilon B$ which satisfy these properties. Notice that $B$ is invariant under the forward semiflow generated by a smooth $V$, and so the restriction $V \mid B$ generates a forward semiflow on $B$. Note that the definition of an unstable, dynamic branched surface ( $B, V$ ) does not require smoothness of $V$.

A stable dynamic branched surface $(B, V)$ is similarly defined by requiring $V$ to point backward along $\Upsilon B$. We say that $V$ is smooth if it generates a unique backward semiflow with smooth trajectories.

When $(B, V)$ is a (stable or unstable) dynamic branched surface, we often abuse terminology and say that $B$ is a (stable or unstable) dynamic branched surface with dynamic vector field $V$.

Sectors of a dynamic branched surface $B$ are described as follows. Given a sector $\Sigma$, a point $p \in \partial \Sigma$ is called an external tangency if the overlay map $\Sigma \rightarrow B$ takes $p$ to a crossing point $s$ and takes a neighborhood of $p$ to one of the two-sheeted quadrants at $s$. Equivalently, $p$ has a neighborhood in $\Sigma$ locally modelled on the subset $\left\{(x, y) \in \mathbf{R}^{2} \mid\right.$ $x \geq 0$ and $-|x| \leq y \leq|x|\}$, where $p$ corresponds to $(0,0)$, and the vector field near $p$ corresponds to $\partial / \partial y$. Note that if $p \in \partial \Sigma$ is not an external tangency, then the vector field either points into $\Sigma$ or out of $\Sigma$ at $p$. An application of the Euler-Poincaré index theorem, using that the vector field is nowhere zero, shows:

Proposition 1.5.1. Each sector $\Sigma$ of a dynamic branched surface has one of the following types: a torus or Klein bottle, an annulus or Möbius band with no external tangencies, or a disc with two external tangencies.

The latter type will be called a bigon sector.
Remark. Given any branched surface $B \subset M$, the existence of a dynamic vector field $V$ is a purely combinatorial property of $B$ and of the inclusion $B \hookrightarrow M$. To see why, first note that one can always construct $V \mid \Upsilon B$ to point forward. Then, for each sector $\sigma$, one can extend $V$ over $\sigma$ if and only if $\sigma$ has zero index in the sense of proposition 1.5.1. Finally, for each component $C$ of $\mathfrak{C}(M-B)$, one can extend $V$ over $C$ if and only if $C$ has zero index in a certain sense.

Thus, in some sense the concept of a dynamic vector field is purely combinatorial, and we often regard the dynamic vector field $V$ as a purely combinatorial object associated to the branched surface $B$.

Remark. Christy [Chr93] requires that the forward semiflow on $B$ generated by $V$ be expansive, which means that there exists $\epsilon, R>0$, such that for any two trajectories $\alpha, \beta: \mathbf{R} \rightarrow B$, if $d(\alpha(t), \beta(t))<\epsilon$ for all $t \in \mathbf{R}$, then there exists $r<|R|$ such that $\alpha(t+r)=\beta(t)$ for all $t \in \mathbf{R}$.

For our purposes, we do not want to worry about the global dynamics of the forward semiflow, which leaves us free to choose any vector field satisfying the defining properties. This allows us to regard $V$ as a purely combinatorial object.

Nevertheless, in $\S 2.6$ the idea of expansivity is re-introduced in the combinatorial disguise of a "Markov section" for $V$ (see proposition 2.6.3).

Let $(B, V)$ be an unstable dynamic branched surface. We define "dynamic splitting" of $B$ as follows. Let $F \subset N(B)$ be a splitting surface (or an embedded surface in $B$ ). The map from $F$ to $B$ is a submersion, so we may pull $V$ back to a smooth vector field $V^{\prime}$ on $F$. Suppose that $V^{\prime}$ points into $F$ along $\partial_{v} F$ and out of $F$ along $\partial_{i} F$. Then we say that $F$ is a dynamic splitting surface, and $B_{F}$ is a dynamic splitting of $B$ along $F$. There is a vector field $V_{F}$ on $M$ whose restriction to $B_{F}$ is the pullback of $V$ under the submersion $p \mid B_{F}$. Since $F$ is a dynamic splitting surface, it is easily checked that $V_{F}$ points toward the one-sheeted side of each noncrossing point in $\Upsilon B_{F}$, and $V_{F}$ points toward the one-sheeted quadrant of each crossing point.

Dynamic splitting is similarly defined for stable dynamic branched surfaces, except that the vector field on the splitting surface $F$ points outward along $\partial_{v} F$ and inward along $\partial_{i} F$. We have:

Lemma 1.5.2. If $\left(B_{F}, V_{F}\right)$ is obtained by dynamic splitting from an unstable (resp. stable) dynamic branched surface $(B, V)$, then $\left(B_{F}, V_{F}\right)$ is an unstable (resp. stable) dynamic branched surface.

### 1.6 The taffy pulling example

Some examples of dynamic branched surfaces are given in [Chr93]. Here we shall describe how to produce examples on mapping tori of pseudo-Anosov homeomorphisms. These examples have previously been described in the language of "affine branched surfaces" by Oertel [Oer96].

Let $S$ be a compact, connected, oriented surface of genus $g$ and with $n$ boundary components, such that if $g=0$ then $n \geq 4$. By work of Thurston, such a surface always has a pseudo-Anosov homeomorphism $f: S \rightarrow S$ (see [FLP $\left.{ }^{+} 79\right]$ ). Moreover the map $f$ has an invariant train track $\tau$, which means that $f(\tau)$ is isotopic to a train track in $N(\tau)$ transverse to the $I$-fibers of $N(\tau)$. An invariant train track can be found concretely by results of Bestvina and Handel [BH95]. Using an invariant train track, one can construct an unstable dynamic branched surface $B$ in the mapping torus $M_{f}=S \times I /(x, 1) \sim(f(x), 0)$. We shall illustrate this construction with a single example.

Let $S$ be the four holed sphere, depicted in figure 1.6 as a disc with three holes. Let $t_{1}: S \rightarrow S$ be the "half Dehn twist" under which the left and middle holes are rotated halfway through a circle in the counterclockwise direction, and let $t_{2}$ be the half Dehn twist
for the middle and right holes. Let $f=t_{1}^{-1} \circ t_{2}: S \rightarrow S$. This is the "taffy-pulling example", one of the simplest examples of pseudo-Anosov phenomena.

The topmost diagram in figure 1.6 shows an invariant train track $\tau$ for $f$, the middle diagram shows $t_{2}(\tau)$, and the bottom diagram shows $t_{1}^{-1} \circ t_{2}(\tau)=f(\tau)$. Also shown are eight points marked (a-h) which partition $\tau$ into ten segments, each segment identified by its endpoints, e.g. $[a, b],[b, c],[b, d], \ldots$. These segments form a "Markov partition" of $\tau$, as in [BH92] (see $\S 2.6$ for Markov partitions in the context of unstable dynamic branched surfaces). The vertex and edge maps are:

$$
\begin{aligned}
& f, h \mapsto a \quad \overline{a b} \mapsto \overline{b d} \\
& a \mapsto b \quad \overline{b c} \mapsto \overline{d e} \\
& d \mapsto c \quad \overline{c f} \mapsto \overline{e h g f c b a} \\
& b, e \mapsto d \quad \overline{b d} \mapsto \overline{\text { dehgfc }} \\
& c \mapsto e \quad \overline{d e} \mapsto \overline{\text { cbafghed }} \\
& g \mapsto g \quad \overline{e h} \mapsto \overline{d b a} \\
& \overline{f g} \mapsto \overline{a f g} \\
& \overline{g h} \mapsto \overline{g h a} \\
& \overline{f a} \mapsto \overline{a b} \\
& h a \mapsto \overline{a b}
\end{aligned}
$$

To construct the unstable dynamic branched surface in $M$, first we construct a branched surface $B^{\prime} \subset S \times[0,1]$, with $B^{\prime} \cap(S \times 1)=\tau \times 1$ and $B^{\prime} \cap(S \times 0)=f(\tau) \times 0$. The branched surface $B^{\prime}$ "interpolates" between $f(\tau)$ and $\tau$, realizing the folding map $f(\tau) \mapsto \tau$. To describe the folding map, we regard $f(\tau)$ as embedded in an $I$-fibered neighborhood $N(\tau)$. If $G$ is the grey area in the lower diagram of figure 1.6 , for each component $C$ of $\mathfrak{C}(G-f(\tau))$, the $I$-fibration of $N(\tau)$ induces an $I$-fibration of $C$ which may be parameterized as $\alpha^{C}:[0,1] \times I \rightarrow C$, where $\alpha_{t}^{C}=\alpha^{C}(t \times I)$ is an $I$-fiber, $\alpha_{0}^{C}$ is the "degenerate" fiber mapped to the cusp of $C$, and $\alpha_{1}^{C} \subset \partial C$. We require that these parameterizations are "generic" in the sense that for each $t \in[0,1]$, there are at most two $I$-fibers of the form $\alpha_{t}^{C}, \alpha_{t}^{C^{\prime}}$ which share an endpoint. There is a 1-parameter family of train tracks $\tau_{t}$, where $\tau_{1}=\tau, \tau_{0}=f(\tau)$, and $\tau_{t}$ is obtained from $f(\tau)$ by collapsing the fibers $\alpha_{s}^{C}$ for each $C$ and each $s \in[0, t]$. The branched surface $B^{\prime}$ is defined by $B^{\prime} \cap(S \times t)=\tau_{t} \times t$. A crossing point of $B^{\prime}$ occurs on a level surface $S \times t$ if there exist fibers $\alpha_{t}^{C}, \alpha_{t}^{C^{\prime}}$ which share an endpoint. Since the $I$-fiber parameterizations are generic, the branched surface $B^{\prime}$ has generic branching. There are two kinds of crossing points, depicted in figure 1.7. The branched surface $B^{\prime} \subset S \times[0,1]$ glues up, under the map $(x, 1) \mapsto(f(x), 0)$, to give the desired unstable dynamic branched surface $B \subset M$. The dynamic vector field on $B$ is induced by a vector field on $S \times[0,1]$ which points tranverse upward on each $S \times t$ and is tangent to $B^{\prime}$. The method of folding guarantees that the vector field is forward along $\Upsilon B^{\prime}$ (see figure 1.7 for the vector field near a crossing point).


The example of this section can be generalized to any compact, oriented, connected surface $S$ and any map $f: S \rightarrow S$ with pseudo-Anosov mapping class. Apply the results of [BH95] to get an invariant train track $\tau$ with a Markov partition. By twiddling their construction, we may assume that $\tau$ is trivalent, and that every switch is folded. Then follow the exact same method described in this section to produce an unstable dynamic branched surface in the mapping torus $M_{f}=(S \times[0,1]) /(x, 1) \sim(f(x), 0)$. Using $f^{-1}$ one can also produce a stable dynamic branched surface.

## Chapter 2

## Dynamic pairs

In this section we define and study dynamic pairs of branched surfaces on oriented 3manifolds with torus boundaries. One starts with a pair of branched surfaces $B^{s}, B^{u} \subset M$ in general position, from which it follows that $B^{s}$ and $B^{u}$ are transverse to each other and to each other's branch locus, and their branch loci are disjoint. Then one takes a $C^{0}$ vector field $V$ on $M$ such that $\left(B^{s}, V\right)$ is a stable dynamic branched surface and $\left(B^{u}, V\right)$ is unstable. The manifold $Q=\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ inherits, by pullback from $M$, the structure of a smooth manifold-with-corners. Also, $Q$ has a vector field obtained by pulling back $V$. Certain faces of $Q$ that come from $B^{s}$ are labelled with the symbol " s ", faces coming from $B^{u}$ are labelled " $\mathbf{u}$ ", and faces coming from $\partial M$ are labelled " $\mathbf{b}$ " for "bare". The manifold-with-corners $Q$, equipped with its vector field and labelling, is an example of a "dynamic manifold". We will require that each component of $Q$ is topologically simple and has "simple dynamics", which says roughly that trajectories of $Q$ are either intervals or circles. The main work in defining dynamic pairs is to formulate the precise requirements on $Q$.

In $\S 2.2$ we define manifolds with corners, and in $\S 2.3$ we define dynamic manifolds. The definition of dynamic pairs is given in §2.4. In the remaining sections, we explore some properties of dynamic pairs and develop tools for constructing dynamic pairs. In $\S 2.5$ we study "dynamic train tracks", which occur for example as the intersection of the branched surfaces in a dynamic pair. We use dynamic train tracks to investigate the branched surfaces which make up a dynamic pair. In $\S 2.6$ we study Markov branched surfaces, a concept due to Joe Christy, and we use them to give a method for constructing a dynamic pair starting from just an unstable dynamic branched surface. In $\S 2.7$ we show by example how to contruct a dynamic pair on any pseudo-Anosov mapping torus.

The results in this section are combinatorial in nature, and yet they are motivated by dynamical considerations. The motivations may not, however, become clear until section $\S 3$; on the other hand many of the results in $\S 3$ depend on technical results from this section.

This has led to some difficulties in ordering the presentation. The reader may want to shift back and forth between the present section and $\S 3$, as needed in order to understand the material.

### 2.1 Motivation: dynamic pairs in pseudo-Anosov mapping tori

The definition of a dynamic pair will be conceptually simple, but the formal definition requires some unfamilar combinatorial machinery, and it may be helpful to visualize dynamic pairs in a familiar situation before launching into the formal definition. To simplify the discussion we will stick to the boundaryless case, making a few comments afterward to explain the case of nonempty boundary.

Let $f: S \rightarrow S$ be a pseudo-Anosov homeomorphism. Let $M_{f}=S \times I /(x, 1) \sim(f(x), 0)$ be the mapping torus of $f$. Recall the construction of an unstable dynamic branched surface $B^{u} \subset M$ in §1.6. Start from an invariant train track $\tau^{u}$ for $f$. Choose a 1-parameter sequence of foldings $\tau_{t}^{u}$ where $\tau_{0}^{u}=f\left(\tau^{u}\right)$ and $\tau_{1}^{u}=\tau^{u}$, so folding occurs as $t$ increases. Construct $B^{u}$ so that it intersects $S \times t / \sim$ in $\tau_{t}^{u} \times t / \sim$. Because folding occurs as $t$ increases, there is an upward pointing vector field $V$ tangent to $B^{u}$ such that $\left(B^{u}, V\right)$ is an unstable dynamic vector field.

Let's examine the components of $\mathfrak{C}\left(M-B^{u}\right)$. Recall that a pseudo-Anosov homeomorphism has finitely many "pseudohyperbolic" periodic orbits. At each point $x$ in such an orbit, the stable and unstable foliations each have $n$-prongs for some $n \geq 3$, and the first return map of $x$ induces a $k$-fold cyclic rotation on these prongs for some $k=0, \ldots, n-1$. Associated to $x$ there is a component of $\mathfrak{C}\left(S-\tau^{u}\right)$ which is an $n$-cusped disc. As $t$ increases, one traces out an $n$-cusped disc component of $\mathfrak{C}\left(S-\tau_{t}^{u}\right)$, and the $n$-cusped disc for $x$ at level $t=1$ is glued to the $n$-cusped disc for $f(x)$ at level $t=0$. Continuing around the orbit of $x$, the $n$-cusped disc associated to $x$ at level $t=0$ eventually returns to itself, with the cusps undergoing a $k$-fold cyclic rotation. Thus, associated to the orbit of $x$ there is a component of $\mathfrak{C}\left(M-B^{u}\right)$ which has the structure of a solid torus with cusps on its boundary, and these cusps trace out an ( $n, k$ ) torus knot on the boundary of the solid torus; this will be called a u-cusped solid torus in the next section.

One may similarly construct a stable dynamic branched surface $B^{s} \subset M$ : choose an invariant train track $\tau^{s}$ for $f$; choose a 1-parameter sequence of foldings $\tau_{t}^{s}$ where $\tau_{0}^{s}=\tau^{s}$ and $\tau_{1}^{s}=f^{-1}\left(\tau^{s}\right)$, so folding occurs as $t$ decreases; and construct $B^{s}$ so that it intersects $S \times t / \sim$ in $\tau_{t}^{s} \times t / \sim$. The components of $\mathfrak{C}\left(M-B^{u}\right)$ are $\mathbf{u}$-cusped solid tori.

The key observation is that the train tracks $\tau_{t}^{u}$ and $\tau_{t}^{s}$ can be chosen to intersect "efficiently", after possibly replacing $\tau_{t}^{u}$ by $f^{n}\left(\tau_{t}^{u}\right)$ for some sufficiently large $n$. After this is done, the same vector field $V$ on $M$ will suffice as a dynamic vector field for both $B^{s}$ and $B^{u}$. Moreover, we may also assume that for each $t \in[0,1]$ and for each pseudohyperbolic


Figure 2.1: A 3 -cusped disc of $\mathfrak{C}\left(S-\tau_{t}^{u}\right)$, intersected by $\tau_{t}^{s}$.
point $x$ of $f$, if $P^{s}$ is the $n$-pronged disc of $\mathfrak{C}\left(S-\tau_{t}^{s}\right)$ associated to $x$, and if $P^{u}$ is the $n$-pronged disc of $\mathfrak{C}\left(S-\tau_{t}^{u}\right)$ associated to $x$, then $P^{s} \cap P^{u}$ has a component which is a $2 n$-sided polygon $P$, with sides of $P$ alternating between arcs in $\tau_{t}^{s}$ and arcs in $\tau_{t}^{u}$.

With the above assumption, we now describe how components of $\mathfrak{C}\left(M-B^{s}\right)$ interact with components of $\mathfrak{C}\left(M-B^{u}\right)$. First, if $T^{s}, T^{u}$ are the cusped solid torus components of $\mathfrak{C}\left(M-B^{s}\right), \mathfrak{C}\left(M-B^{u}\right)$ associated to the same periodic orbit of $f$, then $T^{s} \cap T^{u}$ contains a component of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ which is a solid torus $T$ with corners on its boundary, and these corners trace out a $(2 n, 2 k)$ torus knot on $\partial T$. In the next section we shall refer to $T$ as a "dynamic solid torus".

Now we describe the remaining components of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$. If $D_{t}$ is an $n$-cusped disc component of $\mathfrak{C}\left(S-\tau_{t}^{u}\right)$, figure 2.1 shows how $D_{t}$ might intersect $\tau_{t}^{s}$. One component of $\mathfrak{C}\left(D_{t}-\tau_{t}^{s}\right)$ is a disc with $2 n$ corners, leading to a dynamic solid torus as described above. Each remaining component is either a rectangle, a disc with 4 corners, or a onecusped triangle, a disc with 2 corners and one cusp. As $t$ increases, the cusps of $D_{t}$ are folded, and meanwhile the cusps of $\tau_{t}^{s}$ are being split, creating and destroying rectangles and one-cusped triangles, or converting one into the other.

In figure 2.2, we examine the creation and destruction of a certain component of $\mathfrak{C}(S-$ $\left.\left(\tau_{t}^{s} \cup \tau_{t}^{u}\right)\right)$ as $t$ increases, yielding the component of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ shown in figure 2.3. In the next section objects of this type are referred to as "pinched tetrahedra" (see figure 2.6).

In conclusion, the components of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ are either dynamic solid tori or pinched tetrahedra. This statement is the main clause in the definition of a dynamic pair in $M$. Other clauses in the definition describe how components fit together in $M$.

In the following sections we give detailed descriptions of objects encountered when studying $\mathfrak{C}\left(M-B^{s}\right), \mathfrak{C}\left(M-B^{u}\right)$ and $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$; in addition to dynamic solid tori and pinched tetrahedra, we will also need "dynamic torus shells" which arise when $M$ has torus boundaries.


Figure 2.2: At time $t_{0}$ a cusp of $\tau_{t_{0}}^{s}$, while being split, pierces through a branch of $\tau_{t_{0}}^{u}$, giving birth to a 1 -cusped triangle at greater times $t_{1}$. At time $t_{2}$ this cusp has passed through to an adjacent branch, converting the 1-cusped triangle into a rectangle at greater times $t_{3}$. At time $t_{4}$ a cusp of $\tau_{t_{4}}^{u}$, while being folded, converts this rectangle into a 1-cusped triangle for greater times $t_{5}$. At time $t_{6}$, this cusp continues to fold, leading to the death of the 1 -cusped triangle.


Figure 2.3: If the shaded regions in figure 2.2 are stacked one atop the other, the corresponding component of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ is a "pinched tetrahedron" (figure 2.6).

### 2.2 Manifolds with corners

Let $N$ be a compact topological $n$-manifold, possibly with boundary. A stratification of $N$ is a filtration $N^{(0)} \subset N^{(1)} \subset \cdots \subset N^{(n)}=N$ such that $N^{(0)}$ is a finite set (i.e. a compact 0 -manifold), and for each $n>0$ the set $N^{(n)}-N^{(n-1)}$ is an $n$-manifold without boundary, the completion of which is a compact $n$-manifold whose boundary immerses topologically in $N^{(n-1)}$. A component of $N^{(n)}-N^{(n-1)}$ (or its completion) is called an $n$-stratum. 0 -strata are called vertices, 1 -strata are called edges, and 2 -strata are called faces. Note that a stratum may be compact and boundaryless, and it may also be empty.

A surface with corners is a compact 2 -manifold $F$ equipped with a covering by charts that are locally modelled on certain closed subsets of $\mathbf{E}^{2}$, with $C^{\infty}$ overlap maps. The closed subsets, called standard local models at the origin $\mathcal{O}$, are as follows:

- (Interior point) $\mathbf{E}^{2}$
- (Boundary point) $y \geq 0$
- (Corner) $x \geq 0$ and $y \geq 0$
- (Cusp) $x \geq 0$ and $-f(x) \leq y \leq f(x)$, where $f:[0, \infty) \rightarrow[0, \infty)$ is a cusp function.

Formally, a chart for a surface with corners $F$ at $p \in F$ is an open set $U$ containing $p$ and a homeomorphism $(U, p) \approx\left(D^{2} \cap C, \mathcal{O}\right)$, where $D^{2}$ is the open unit disc in $\mathbf{E}^{2}$, and $C$ is one of the four sets above, the names of which define the type of the point $p$. A surface with corners $F$ has a natural stratification of the form $\partial^{0} F \subset \partial^{1} F=\partial F \subset F$, where the vertex set $\partial^{0} F$ is the set of corners and cusps.

A 3-manifold with corners is similarly defined, using the following standard local models at the origin in $\mathbf{E}^{3}=\{(x, y, z)\}$ (with $f$ a cusp function):

- (Interior point) $\mathbf{E}^{3}$
- (Boundary point) $z \geq 0$
- (Corner edge) $x \geq 0$ and $y \geq 0$
- (Cusp edge) $z \leq 0$ and $-f(z) \leq x \leq f(z)$.
- (Apex) $x \geq 0$ and $y \geq 0$ and $z \leq 0$
- (Gable) $z \leq 0$ and $y \geq 0$ and $-f(z) \leq x \leq f(z)$

A 3-manifold with corners $Q$ has a natural stratification of the form $\partial^{0} Q \subset \partial^{1} Q \subset \partial^{2} Q=$ $\partial Q \subset Q$, where the vertex set $\partial^{0} Q$ is the set of apexes and gables, and $\partial^{1} Q$ is the closure of the union of all corner and cusp edges. Note that each face of a 3 -manifold with corners is a surface with corners.

### 2.3 Dynamic manifolds

A dynamic manifold consists of a 3-manifold with corners $Q$, a $C^{0}$ vector field $V$ on $Q$, and a labelling of each face of $Q$ with one of the symbols $\mathbf{b}, \mathbf{u}, \mathbf{s}, \mathbf{p}, \mathbf{m}$, (for bare, unstable, stable, plus, minus) such that the following axioms hold (on first reading it may be easier to ignore all axioms involving $\mathbf{p}$ and $\mathbf{m}$ labels, since these occur only in the context of sutured manifolds):

1. For each face $F$ of $Q$ :
(a) If $F$ is an $\mathbf{m}$-face then $V$ is transverse to $F$, pointing out of $Q$ along $\operatorname{int}(F)$.
(b) If $F$ is a p-face then $V$ is transverse to $F$, pointing into $Q$ along $\operatorname{int}(F)$.
(c) If $F$ is a $\mathbf{b}, \mathbf{u}$, or $\mathbf{s}$-face then $V$ is tangent to $F$.
2. Labelling each edge of $Q$ with the pair of symbols labelling the faces on either side of the edge, we have:
(a) Each uu, ss, and pm-edge is a cusp edge.
(b) There are no $\mathbf{p p}, \mathbf{m m}, \mathbf{b b}, \mathbf{b u}$, or bs-edges.
(c) All other edges are corners.
(d) If $E$ is a uu-edge, then $V$ is transverse to $E$, pointing out of $Q$ along $E$ (see figure 2.4 for a concrete local model); in particular, if $F$ is a u-face incident to $E$ then $V$ points out of $F$ along $E$.
(e) If $E$ is an ss-edge and $F$ is an s-face incident to $E$ then $V$ points into $F$ along $E$ (reverse the direction of the vector field in item 2 d ).
(f) If $E$ is an su-edge then $V$ is tangent to $E$ at each point; more precisely in the standard local model for a corner edge, $V(x, y, z)=(0,0,1)$. (This property follows from (c)).

The union of $\mathbf{b}$-faces of $Q$ is denoted $\partial_{\mathbf{b}} Q$, and similarly for $\partial_{\mathbf{u}} Q, \partial_{\mathbf{s}} Q, \partial_{\mathbf{p}} Q$, and $\partial_{\mathbf{m}} Q$.
Formally a dynamic manifold is a triple $(Q, V, \ell)$ where: $V$ is a $C^{0}$ vector field on $Q ; \ell$ is a function from the set of faces of $Q$ to the set $\{\mathbf{b}, \mathbf{u}, \mathbf{s}, \mathbf{p}, \mathbf{m}\}$; and the above conditions are satisfied. We shall also say that $V$ is a dynamic vector field on $Q$. Notice in the local model for an ss or uu-edge, the formula for $V$ given in figure 2.4 forces $V$ to be $C^{\infty}$ except at a point $p$ on the edge itself; $V$ obviously does not have a unique integral curve at $p$, therefore $V$ is not even Lipschitz at $p$, by the uniqueness theorem for solutions of ordinary differential equations. Nevertheless we shall say that $V$ is smooth if it is $C^{\infty}$ at each point not on an ss or uu-edge; but it should be emphasized that we do not always require smoothness.

Despite the formal nature of the definition of a dynamic manifold, there is some geometric meaning to the labels $\mathbf{b}, \mathbf{u}, \mathbf{s}, \mathbf{p}$, and $\mathbf{m}$, which hopefully will be clarified as properties and examples of dynamic manifolds are presented.

cross section of a uu-edge

cross section of a pm-edge

Figure 2.4: The vector field near a uu-edge and a pm-edge, in cross-section, using the standard local model $z \leq 0,-f(z) \leq x \leq f(z)$ for cusp edges. Near a uu-cusp edge the vector field is tangent to each curve of the form $(x, y, z)=(a \cdot f(t), b, t)$, where $t$ parameterizes the curve, $a \in[-1,+1]$ is a constant, and $b \in \mathbf{R}$ is a constant. Near a pmcusp edge in the standard local model, the vector field points in the positive $\boldsymbol{x}$-direction.

From the defining axioms of a dynamic manifold, other properties are deduced as follows. If $E$ is a pm-edge then $V$ is not tangent to $E$ at any point; more precisely, in the standard local model for a cusp edge we can write $V(x, y, z)=(1,0,0)$ (see figure 2.4). The types of corner edges are: $\mathbf{s u}, \mathbf{p b}, \mathbf{p s}, \mathbf{p u}, \mathbf{m b}, \mathbf{m s}, \mathbf{m u}$. If $E$ is a $\mathbf{p b}, \mathbf{p s}$, or $\mathbf{p u}$-edge and if $F$ is the $\mathbf{b}, \mathbf{s}$, or $\mathbf{u}$-face incident to $E$ then $V$ points out of $F$ along $E$; take $V(x, y, z)=(0,0,1)$ in the standard local model for a corner edge; similar descriptions hold for mb, ms, and mu-edges. Labelling each vertex with the triple of symbols associated to the three faces incident to the vertex, the types of gables are: uus, uup, uum; ssu, ssp, ssm; pmb, pmu, pms; and the types of apexes are $\mathbf{p s u}$ or $\mathbf{~ m s u ( a p e x e s ~ w i l l ~ r a r e l y ~ o c c u r ) . ~ A t ~ u u s , ~ u u p , ~ a n d ~}$ uum-gables $V$ is described as follows (see figure 2.5). At a uus-gable, in the standard local model for gables, we can use the same formula for $V(x, y, z)$ as given above for uu-edges. At a uup-gable, we may use the same formula for $V(x, y, z)$ but a nonstandard local model for the gable, namely $z \leq 0, y \leq z,-f(z) \leq x \leq f(z)$. At a uum-gable, again we may use the same formula for $V(x, y, z)$, but we use a nonstandard local model for the gable, namely $z \leq 0, y \geq-z,-f(z) \leq x \leq f(z)$. At an ssu, ssp, or ssm-gable $V$ is similarly described. At a pmb, pmu, or pms-gable with the standard local model for gables, we may take $V(x, y, z)=(1,0,0)$ as in figure 2.4 for a pm-edge, intersected with $y \geq 0$. At a corner edge with the standard local model, take $V=(0,0,1)$.

The distinction between $\mathbf{b}, \mathbf{u}$, and $\mathbf{s}$-faces-all of which are tangent to the vector fieldis clarified by using the Euler-Poincaré formula together with the restrictions on edge and vertex labels to list the possible types of faces, a tedious but finite task. Rather than give an exhaustive list of the possible types, we point out that only an s-face can have a cusp where the vector field leaves the face, as in the uus-gable in figure 2.5 ; for example one possible type of s-face is a one-cusped triangle with two us-edges meeting at a uus-gable, and the third edge being an ss-cusp edge (see figure 2.6). Similarly, only a u-face can have a cusp where the vector field enters the face. The idea is that on a "stable" face flow lines converge in forward time, while on an "unstable" face flow lines converge in backward time.


Figure 2.5: The vector field near gables incident to a uu-edge.

If $V$ is a dynamic vector field on $Q$, a trajectory of $V$ is a differentiable path $\alpha: J \rightarrow Q$, where $J$ is a connected subset of $\mathbf{R}$, such that for each $t_{0} \in J$ we have $d \alpha / d t\left(t_{0}\right)=V\left(\alpha\left(t_{0}\right)\right)$. The trajectory is complete if it is not the restriction of a trajectory with larger domain; in this case each point of $\alpha(\partial J)$ lies in an $\mathbf{m}$-face, $\mathbf{p}$-face, ss-edge, or uu-edge. We also allow the degenerate case of a complete trajectory which is a single point lying on a pm-edge, or a single point on an muu-gable or a pss-gable. When $V$ is smooth, each point of $Q$ lies on some trajectory, and the trajectory passing through each point not on an ss or uu-edge is unique; this follows from the existence and uniqueness theorem for solutions of ordinary differential equations. Notice that each us-edge is a trajectory; the orientation on a us-edge inherited from $V$ is called the dynamic orientation on that edge. On any us-edge $E$ which is not a circle, the dynamic orientation on $E$ is determined by the labelling structure: the negative endpoint of $E$ is either a uss-gable or msu-corner, and the positive endpoint is either an suu-gable or a psu-corner.

If $V$ is smooth, we say that $(Q, V)$ has interval dynamics if each trajectory is a closed interval, and circle dynamics if each trajectory is a circle. Roughly speaking, "simple dynamics" means either interval dynamics or circle dynamics (but see the "maw pieces" below).

Here are some examples of dynamic manifolds.
Example. If $M$ is a torally bounded 3 -manifold and $V$ is a vector field on $M$ tangent to the boundary, then $(M, V)$ is a dynamic manifold where $\partial M$ is labelled $\mathbf{b}$.

Example. Suppose $M$ is a manifold with torus boundaries, $V$ is a vector field on $M$, and $\left(B^{s}, V\right)$ is a stable dynamic branched surface. Let $Q=\mathfrak{C}\left(M-B^{s}\right)$. The pullback of $V$ to $Q$ defines a dynamic manifold with only $\mathbf{b}$ and s-faces, and only ss-edges. A similar construction works with unstable dynamic branched surfaces.
Example. A dynamic manifold $Q$ is an s-cusped solid torus if $Q$ is a solid torus, all edges are ss-circles, all faces are s-annuli, and the ss-circles form a family of homotopically nontrivial, simple closed curves, which intersects any meridian curve of $Q$ in at least two points. There is an ordered pair ( $n, k$ ) called the type of $Q$, where $n \geq 2$ is the minimal intersection number
of the ss-circle family with a meridian curve, and $Q$ is the mapping torus of a rotation on an $n$-cusped disc through an angle $2 \pi k / n$. A u-cusped solid torus is similarly defined.

The only requirement on the dynamic vector field $V$ on $Q$ is that it be tangent to faces, exit along a uu-cusp edge and enter along an ss-cusp edge. If $V$ is smooth, one can use this property to prove that there exist bi-infinite trajectories of $V$, using the Conley index [Con78]. We say that $V$ is circular on $Q$ if there exists a homotopy equivalence $Q \rightarrow S^{1}$ such that each trajectory of $V$, when mapped to $S^{1}$, has positive derivative with respect to the standard orientation on $S^{1}$. Pulling back the standard generator of $H_{1}\left(S^{1}\right) \approx \mathbf{Z}$ we obtain a generator of $H_{1}(Q)$ called the positive generator.
Example. A dynamic manifold $Q$ is an s-cusped torus shell if $Q \approx T^{2} \times[0,1], T^{2} \times 0$ is a b-face, all edges on $T^{2} \times 1$ are ss-circles, and all faces on $T^{2} \times 1$ are s-annuli. The number $n$ of ss-circles is called the type of $Q$. A u-cusped torus shell is similarly defined. Circularity of $V$ on cusped torus shells is defined as for cusped solid tori.

If the symbol $\mathbf{s}$ or $\mathbf{u}$ is understood, we will drop it from the terminology for a cusped solid torus or torus shell. Cusped solid tori and torus shells are known collectively as cusped torus pieces. Note, for example, that if $M_{f}$ is the mapping torus of a pseudo-Anosov homeomorphism $f: S \rightarrow S$, and if $B \subset M_{f}$ is the unstable dynamic branched surface constructed by the method of $\S 1.6$, then each component of $\mathfrak{C}\left(M_{f}-B\right)$ is a $\mathbf{u}$-cusped torus piece: there is one $\mathbf{u}$-cusped solid torus for each orbit of singular points of $f$, and there is one $\mathbf{u}$-cusped torus shell for each orbit of boundary components of $f$.
Example. Let $P$ be a sutured manifold in the corner model, and let $V$ any smooth vector field on $P$ which points inward along $\mathcal{R}_{-} P$, outward along $\mathcal{R}_{+} P$, and is tangent along $\gamma P$. Labelling $\partial P$ so that $\partial_{\mathbf{m}} P=\mathcal{R}_{-} P, \partial_{\mathbf{p}} P=\mathcal{R}_{+} P$, and $\partial_{\mathbf{b}} P=\gamma P$, the pair $(P, V)$ is called a dynamic sutured manifold. We can always alter $V$ by a homotopy supported in a neighborhood of $\gamma_{A} P$ so that $V$ restricted to $\gamma_{A} P$ has interval dynamics, each trajectory connecting opposite boundary components. Assuming $V$ is so altered, if we collapse the each trajectory of $V$ on $\gamma_{A} P$ to a point, the result is a dynamic manifold which is the cusp model for $Q$, and each suture becomes a pm-cusp.

A dynamic sutured manifold is also called an isolating block in the terminology of [CE71]. Isolating blocks are useful in studying Conley's homotopy index for isolated invariant sets of flows. Indeed, in our present context the complementary sutured manifolds of a branched surface hierarchy will be isolating blocks for the pseudo-Anosov flow that we eventually construct.
Example. If $B^{s}, B^{u}$ is a transverse pair of branched surfaces, and if $V$ is a vector field on $M$ such that ( $B^{s}, V$ ) is a stable dynamic branched surface and ( $B^{u}, V$ ) is an unstable dynamic branched surface, then $Q=\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ equipped with the pullback of $V$ is a dynamic manifold. These examples have $\mathbf{b}, \mathbf{s}$, and $\mathbf{u}$-faces, as well as ss, uu, and su-edges.

As in the example just considered, the examples to follow will have only $\mathbf{b}$, $\mathbf{s}$, and $\mathbf{u}$-faces, and only ss, uu, and su-edges. The vector field $V$ will be implicitly determined


Figure 2.6: A pinched tetrahedron
up to homotopy on edges of $Q$ by the labelling structure on $\partial Q$, except that the dynamic orientation on us-circles must be given explicitly. In each example with "simple" dynamics it is easy to construct $V$ explicitly, and we usually leave the construction to the reader.

Example. A pinched tetrahedron, shown in figure 2.6, is a topological 3-ball with two u-faces and two s-faces fitting together in a tetrahedral pattern, with one uu-edge, one ss-edge, four us-edges, two suu-gables and two uss-gables. Note that a smooth dynamic vector field may be chosen to have interval dynamics. The trajectories make the tetrahedron into the join of the ss-edge and the uu-edge.

Example. A dynamic solid torus is a solid torus $Q$ whose edges form a nonempty family of oriented isotopic, nontrivial, nonmeridinal us-circles. Each face is an sor u-annulus, and there are no vertices. There exist integers $m \geq 1$ and $0 \leq k<m$ such that $Q$ is the mapping torus of a rotation on a $2 m$-gon through an angle $2 \pi k / m$; the number of us-circles, and the number of annulus faces, is $2 \cdot \operatorname{gcf}(m, k)$. The pair $(m, k)$ is called the type of $Q$. We say that $Q$ is essential if $m \geq 2$.

The only restriction on the dynamic vector field $V$ is that the us-circles all be oriented isotopic in $Q$. There are many vector fields satisfying this condition. One reasonably canonical choice is a vector field tangent to a Seifert fibration of $Q$; when $k \neq 0$ there is one singular fiber at the core of the solid torus; and when $k=0$ there are no singular fibers. Another less canonical but still reasonable property is that $V$ be circular, which means
that there is a fibration of $Q$ over $S^{1}$ such that each trajectory of $V$, when mapped to $S^{1}$, have positive derivative with respect to the standard orientation on $S^{1}$.

Example. A dynamic torus shell is a torus shell $Q \approx T \times[0,1]$ such that $T \times 0$ is a b-face, and the edges on $T \times 1$ form a nonempty family of oriented isotopic us-circles. The dynamic vector field may be homotoped to have circle dynamics, making $T$ a product circle bundle over an annulus. The faces on $T \times 1$ are all $\mathbf{s}$ and $\mathbf{u}$-annuli, and there are no vertices. The number of annulus faces is $2 n$ for some integer $n \geq 1$ called the type of the dynamic torus shell.

We refer to dynamic solid tori and dynamic torus shells collectively as dynamic torus pieces. When the context is clear the adjective "dynamic" may be dropped.

Example. Our final example (for now) of a dynamic manifold is a maw piece (see figure 2.7). Topologically, a maw piece is a solid torus. As a manifold with corners, a maw piece is the cartesian product of a circle with a one-cusped triangle, a triangle with one cusp and two corners. There are two types of maw pieces: a uss and an suu-maw piece. A uss-maw piece has one $\mathbf{u}$-annulus and two $\mathbf{s}$-annuli as faces. The edges consist of one ss-circle and two oriented isotopic us-circles, each homotopic to the core of the solid torus. The vector field on a uss-maw piece can be homotoped so that the $\mathbf{u}$-annulus is foliated by circular trajectories, and so that every other trajectory is the $1-1$ immersed image of $[0, \infty)$, starting at a point on the ss-circle and spiraling asymptotically into a circular trajectory on the $\mathbf{u}$-annulus (as with torus pieces, later we will use other models for $V$ ). An suu-maw piece is defined similarly. Note that maw pieces have neither interval dynamics nor circle dynamics, but some kind of hybrid; we still consider maw pieces to have "simple dynamics".

Maw pieces will not occur in the definition of a dynamic pair, but they will appear in later results which describe the structure of a dynamic pair. For now, we observe that if $Q$ is a dynamic torus piece with $\mathbf{u}$-faces $F_{1}, \ldots, F_{n}$, and if $\mu_{1}, \ldots, \mu_{n}$ are uss-maw pieces, then we may attach $\mu_{1}, \ldots, \mu_{n}$ to $Q$ by identifying $F_{i}$ with the $\mathbf{u}$-face of $\mu_{i}$; the result of these identifications is an s-cusped torus piece of the same type as $Q$. Similarly, attaching suu-maw pieces to the s-faces of $Q$ results in a $\mathbf{u}$-cusped torus piece of the same type as $Q$.

### 2.4 Definition of dynamic pairs

A dynamic pair of branched surfaces, on a compact, oriented, torally bounded 3-manifold $M$, is a pair of branched surfaces $B^{s}, B^{u} \subset M$ in general position, disjoint from $\partial M$, together with a $C^{0}$ vector field $V$ on $M$, so that the following are satisfied.

1. $(M, V)$ is a dynamic manifold, in other words $V$ is tangent to $\partial M$.


Figure 2.7: A uss-maw piece (glue the top to the bottom).
2. $\left(B^{s}, V\right)$ and $\left(B^{u}, V\right)$ are stable and unstable dynamic branched surfaces. Let $Q=$ $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$, a dynamic manifold with dynamic vector field obtained by pulling back $V$ under the overlay map $Q \rightarrow M$.
3. The vector field $V$ is smooth on $M$, except along $\Upsilon B^{s}$ where backward trajectories are locally unique, and along $\Upsilon B^{u}$ where forward trajectories are locally unique.
4. $Q$ has simple dynamics. Each component of $Q$ is either a pinched tetrahedron, an essential dynamic solid torus, or a dynamic torus shell. In a dynamic torus piece, $V$ is circular.
5. Transience of forward trajectories. For each component $K$ of $\mathfrak{C}\left(B^{u}-B^{s}\right)$, there exists a u-face $F$ of some torus piece such that $F \subset K$ and $F$ is a sink of the forward semiflow on $K$.
6. Transience of backward trajectories. For each component $K$ of $\mathfrak{C}\left(B^{s}-B^{u}\right)$, there exists an s-face $F$ of some torus piece such that $F \subset K$ and $F$ is a source of the backward semiflow on $K$.
7. Separation of torus pieces. Let $Q_{T}$ be the union of torus piece components of $Q$. The overlay $\operatorname{map} Q_{T} \mapsto M$ has no face gluings, where a face gluing is a factorization $Q_{T} \xrightarrow{f} X \mapsto M$ such that the quotient map $f: Q_{T} \rightarrow X$ either identifies two faces homeomorphically or identifies one face to itself by a double covering map over a Möbius band.

Here are some remarks to clarify various points.
Remark. To clarify the axiom 5 Transience of forward trajectories, note that by axiom 3 each point of $B^{u}-B^{s}$ has a unique forward trajectory defined for all future time and similarly for axiom 6. Also note that the vector field on $Q$ is smooth, as a consequence of axiom 3.

Remark. Given a dynamic pair $B^{s}, B^{u}$ on $M$, the intersection train track $\tau=B^{s} \cap B^{u}$ has an orientation induced by $V$, and $\tau$ is called the dynamic train track. The intuition behind a dynamic pair is that the "interesting" trajectories are the ones contained in $\tau$ : the set of trajectories staying in $\tau$ may be regarded as a Markov flow, with stable direction tangent to $B^{s}$ and unstable direction tangent to $B^{u}$. On the other hand, trajectories not contained in $\tau$ have "boring" behavior. For example, trajectories in a dynamic solid torus just wind around and around and around. Now a circular flow on a solid torus can still have pretty interesting dynamics, but the point is that the trajectories are boring from a homotopic perspective-for example, all periodic orbits in the solid torus are homotopic to an iterate of the core.

Trajectories that stay entirely within $\tau$ are called orbits of $\tau$. An orbit $\mathbf{R} \rightarrow \tau$ is said to be periodic if it factors through a map $\mathbf{R} \rightarrow S^{1} \rightarrow \tau$, where $\mathbf{R} \rightarrow S^{1}$ is a universal covering map; we also say that the map $S^{1} \rightarrow \tau$ is a periodic orbit. Given a periodic orbit $f: S^{1} \rightarrow \tau$, there is a plane bundle $f^{*}\left(T B^{u}\right)$ defined over $S^{1}$, where $T B^{u}$ is the tangent plane bundle of $B^{u}$. If the total space of $f^{*}\left(T B^{u}\right)$ is an annulus we say that $f$ is an untwisted periodic orbit; otherwise, the total space is a Möbius band, and we say that $f$ is a twisted periodic orbit. Note that twistedness may be defined equivalently using $f^{*}\left(T B^{s}\right)$.

Remark. Axioms 5-7 may seem technical and mysterious at this stage, but they are very important for getting good dynamical and topological behavior. For example, they will be crucial in the proof of proposition 2.5 .1 which says in part that each component $C$ of $\mathfrak{C}\left(M-B^{u}\right)$ is a u-cusped torus piece. This is a key component in the proof that the unstable manifold of a dynamic pair carries a very full lamination (theorems 3.3.1 and 3.3.2).
Remark. Axiom 7 is independent of the others-here is an example of a pair $B^{s}, B^{u}$ satisfying axioms 1-6 but not axiom 7. Let $p: M \rightarrow S$ be a Seifert fibration of $M$ over some compact, oriented 2-orbifold $Q$. Let $C^{s}, C^{u}$ be closed 1-manifolds in $S$ which are transverse to each other and disjoint from the cone points of $S$, such that each component of $\mathfrak{C}\left(S-\left(C^{s} \cup C^{u}\right)\right)$ is either an even-sided polygon with at most one cone point, or an annulus without cone points and with one boundary circle in $\partial S$ and the other boundary circle an even sided polygonal curve. Then $B^{s}=p^{-1}\left(C^{s}\right)$ and $B^{u}=p^{-1}\left(C^{u}\right)$ give the desired example. Note that the overlay map $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right) \rightarrow M$ identifies faces of torus pieces in pairs (see also the next remark).

In the presence of axioms 1-6, the nonexistence of face gluings may be reformulated as follows:

Proposition 2.4.1 (Equivalence of face and corner gluings). Suppose $M$ is a compact, oriented, torally bounded 3-manifold, and suppose $B^{s}, B^{u}, V$ satisfy axioms 1-6. Let $Q_{T}$ be the union of torus piece components of $Q=\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$. Then $Q_{T}$ has face gluings if and only if it has "corner gluings". That is, axiom 7 Separation of torus pieces is satisfied if and only if $Q_{T}$ has no corner gluings, where a corner gluing is a factorization $Q_{T} \xrightarrow{g} Y \rightarrow M$ where $g: Q_{T} \rightarrow Y$ either identifies two corner circles homeomorphically or identifies one corner circle to itself by a double covering map over a circle.

Proof. If there is a face gluing then by looking at boundary circles of the glued faces we obtain corner gluings.

Suppose there is a corner gluing. Let $\gamma$ be the periodic orbit in $\tau$ resulting from the gluing. Consider first the case that $\gamma$ is untwisted, and suppose for the moment that $\gamma$ is embedded in $\tau$. Let $N(\gamma)$ be a regular neighborhood of $\gamma$ in $M$, and let $A^{s} \subset N(\gamma) \cap$ $B^{s}, A^{u} \subset N(\gamma) \cap B^{u}$ be smooth, properly embedded annuli in $N(\gamma)$, dividing the solid torus $N(\gamma)$ into four quadrants $q_{1}, q_{2}, q_{3}, q_{4}$ numbered in circular order. By assumption at least two of these quadrants lie in dynamic torus pieces. If two adjacent quadrants lie in dynamic torus pieces then there is a face gluing, a contradiction. Suppose that two opposite quadrants lie in dynamic torus pieces, say $q_{1}$ and $q_{3}$. Choose the notation so that $A^{s}$ separates $q_{1} \cup q_{2}$ from $q_{3} \cup q_{4}$, and $A^{u}$ separates $q_{2} \cup q_{3}$ from $q_{4} \cup q_{1}$. Since $B^{u} \cap q_{1}=\emptyset$ it follows that $B^{u} \cap q_{2}=\emptyset$. Since $B^{s} \cap q_{3}=\emptyset$ it follows that $B^{s} \cap q_{2}=\emptyset$. Therefore, $q_{2}$ lies in a dynamic torus piece (and similarly $q_{4}$ lies in a dynamic torus piece). Therefore the two adjacent quadrants $q_{1}, q_{2}$ both lie in dynamic torus piecees, and so there is a face gluing, a contradiction.

This argument did not really depend on $\gamma$ being embedded: if $\gamma$ is not embedded replace $N(\gamma)$ by an immersed solid torus. And if $\gamma$ is twisted, there is an analogous argument where $A^{s}, A^{u}$ are Möbius bands.

Remark. Another possible alternative to axiom 7 Separation of torus pieces is Torus piece disjointness, which says that the union of torus pieces embeds in $M$ under the overlay map. However, Torus piece disjointness is strictly stronger than Separation of torus pieces-in section 2.7 we give an example of a dynamic pair which violates Torus piece disjointness by having two corner circles of torus pieces intersect nontrivially under the overlay map.

This is a somewhat unfortunate state of affairs-several technical details would be simplified if one had Torus piece disjointness. On the other hand, from a constructive point of view, Torus piece disjointness is more difficult to verify, being stronger than Separation of torus pieces. The reader is encouraged, over the next several sections, to imagine how the theory might be changed by requiring Torus piece disjointness.

### 2.5 Dynamic train tracks

The main results of this section describe the structure of the two branched surfaces that make up a dynamic pair, and the train track which occurs as their intersection.

Suppose that ( $B, V$ ) is an unstable dynamic branched surface in a compact, oriented 3 -manifold $M$ with torus boundaries. Note that $\mathfrak{C}(M-B)$ is a dynamic 3 -manifold (with respect to the pullback of $V$ ), all faces of which are labelled $\mathbf{u}$ or $\mathbf{b}$. We say that $B$ is very full in $M$ if each component of $\mathfrak{C}(M-B)$ is a u-cusped torus piece. Very full is similarly defined for a stable dynamic branched surface.

Proposition 2.5.1. Let $M$ be a compact, oriented 3-manifold with torus boundaries. Suppose that $B^{s}, B^{u}$ is a dynamic pair in $M$. Let $Q=\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$, $P^{s}=\mathfrak{C}\left(M-B^{s}\right)$, $P^{u}=\mathfrak{C}\left(M-B^{u}\right)$. Then:

1. $B^{s}$ is very full in $M$.
2. Inclusion induces a type preserving, 1-1 correspondence between dynamic torus pieces of $Q$ and components of $P^{s}$. If $T \subset Q$ and $C \subset P^{s}$ are corresponding components, then (abusing notation) each component of $\mathfrak{C}(C-T)$ is a uss-maw piece $\mu$, and $\mu$ is attached to $T$ by identifying the $\mathbf{u}$-face of $\mu$ with some $\mathbf{u}$-face of $T$.
3. The dynamic vector field on each component of $P^{s}$ is circular.
4. $B^{s}$ does not carry a closed surface.
5. No sector of $B^{s}$ contains a periodic trajectory of the dynamic vector field.

Similar statements hold for $B^{u}$ and $P^{u}$.
Remark. With $T, C$ as above, the proof will show that for each maw piece component $\mu$ of $\mathfrak{C}(C-T)$, the branched surface $B^{u} \cap \mu$ consists of "tongues" dividing $\mu$ into pinched tetrahedra, as shown for example in figure 2.8.

Remark. This proposition shows that the axiom Separation of torus pieces of a dynamic pair implies a seemingly stronger property, namely that the interiors of faces of torus pieces map disjointly under the overlay map.
Remark. From proposition 1.5.1 and the fact that $B^{s}$ carries no closed surface it follows that each sector is either a bigon, an annulus, or a Möbius band. If we knew that $B^{s}$ contained no annulus or Möbius band sectors, then property 5 of proposition 2.5.1 would follow easily, because if a bigon sector $\sigma$ contained a periodic trajectory of $V$ then the Euler-Poincaré formula would imply that $\sigma$ contains a zero of $V$, but $V$ has no zeroes. We would therefore have a stronger theorem if we could prove that $B^{s}$ has no annulus or Möbius band sectors. I suspect that this is true, at least in the "transitive" case when $\tau$ is strongly connected, but I am not certain.


Figure 2.8: A uss-maw piece divided into six pinched tetrahedra by tongues. The model vector field depicted in figure 2.7 must be homotoped, to make it tangent to the tongues. After homotopy, the vector field is not uniquely integrable along the unstable branch locus, but it does generate a forward semiflow.

The rest of this section is devoted to the proof of proposition 2.5.1. The proof will use some tools whose importance will grow throughout the rest of this paper, so we will take a leisurely path through the proof, taking care to develop the tools in some generality.

The main tool used in the proof is a "dynamic train track" on a dynamic branched surface, a concept motivated by the train track $\tau=B^{s} \cap B^{u}$ associated to a dynamic pair $B^{s}, B^{u}$.

Let $\tau$ be a train track with generic branching, let $s$ be a switch, and suppose $\tau$ is oriented in a neighborhood of $s$. We say that $s$ is a converging switch if the orientation points from the two-sheeted side to the one-sheeted side, and $s$ is a diverging switch if the orientation points the other way.

A dynamic train track in an unstable dynamic branched surface $(B, V)$ is an oriented train track $\tau$ embedded in $B$ such that for some dynamic vector field $V^{\prime}$ on $B$ we have:

1. $V^{\prime}$ is tangent to $\tau$, and $V^{\prime}$ is smooth on $B$, except at diverging switches of $\tau$. It follows that each $\boldsymbol{x} \in \mathfrak{C}(B-\tau)$ which is not a cusp of $\mathfrak{C}(B-\tau)$ has a unique forward trajectory.
2. $\tau$ is disjoint from the crossing points of $B$, and the set of converging switches of $\tau$ is $\tau \cap \Upsilon B$.
3. Transience of forward trajectories. For each component $K$ of $\mathfrak{C}(B-\tau)$, there exists a smooth, compact, connected surface $A \subset K$ such that $\partial A \subset \partial K$, and $A$ is a sink of the forward semiflow on $K$ generated by $V^{\prime}$.

As a consequence of item 2 we have:
4. Each diverging switch of $\tau$ lies in $B-\Upsilon B$, i.e. in the interior of some sector of $B$.

Given a component $K$ of $\mathfrak{C}(B-\tau)$, if $A$ is the sink of $K$, then $V^{\prime}$ is tangent to $A$ and to $\partial A$, and so $A$ is a torus, Klein bottle, annulus, or Möbius band. We say that $\tau$ fills up $B$ if the sink of each component of $\mathfrak{C}(B-\tau)$ is an annulus or Möbius band.

The definition of a dynamic train track $\tau$ in a stable branched surface $B$, and the definition of filling up, are obtained by obvious analogy with the word "sink" replaced by "source".

Lemma 2.5.2. Given a dynamic pair $B^{s}, B^{u}$ in $M$, the train track $\tau=B^{s} \cap B^{u}$ is a dynamic train track filling up each of the dynamic branched surfaces $B^{s}, B^{u}$.

In order to further understand dynamic train tracks, we study the structure of $\mathfrak{C}(B-\tau)$ in more detail.

A cusped branched surface is an object $K$ satisfying the definition of a branched surface with boundary, except that a boundary point may be locally modelled on a cusp point, as in the definition of a surface with corners. Given a cusped branched surface $K$, the boundary
$\partial K$ is a train track, with nongeneric behavior at the cusp points: each cusp point of $K$ is a switch of $\partial K$ whose one-sheeted side is empty. Each cusp point of $K$ is a topological 1-manifold point of $\partial K$, so we may smooth $\partial K$ at each cusp point to obtain a train track with generic switching. Now consider a vector field $V_{K}$ on $K$ such that $V_{K}$ is tangent to $\partial K, V_{K}$ points inward along each cusp, and $V_{K}$ points forward along $\Upsilon K$; we call $K$ a cusped unstable dynamic branched surface. As an immediate consequence of items 2-1 in the definition of a dynamic train track, we have:

Lemma 2.5.3. If $B \subset M$ is an unstable dynamic branched surface and $\tau \subset B$ is a dynamic train track, then $\mathfrak{C}(B-\tau)$ is a cusped unstable dynamic branched surface. Moreover, pulling $V$ back via the overlay map we obtain a vector field making $\mathfrak{C}(B-\tau)$ into a cusped unstable dynamic branched surface.

The structure of the train track $\partial \mathfrak{C}(B-\tau)$ is described in the following definition.
A singular orientation on a train track $\beta$ is an orientation defined on the complement of a finite set of manifold points of $\beta$ called singularities, such that the orientations point in opposite directions on the two sides of a singularity. Given a singularity $s$, if the orientations point away from $s$ then we say that $s$ is a source, or more specifically an orientation source; if the orientations point away from $s$ then $s$ is a sink or an orientation sink. A trivalent train track $\beta$ equipped with a singular orientation is called unstable if each singularity is a source, and each switch is converging. Similarly, $\beta$ is stable if each singularity is a sink and each switch is diverging. As a further consequence of the definitions we have:

Lemma 2.5.4. Continuing the notation from the above lemma, the restriction of the vector field to $\partial \mathfrak{C}(B-\tau)$ makes it into an unstable train track with one orientation source for each cusp. Similar statements hold for a dynamic train track in a stable dynamic branched surface.

The train track $\partial \mathfrak{C}(B-\tau)$ might be called the "tangentially peripheral train track" associated to $\tau$. We can also associate a "transversely peripheral train track", as follows:

Lemma 2.5.5. Continuing the above notation, consider $\mathfrak{C}(M-B)$, a dynamic manifold with $\mathbf{u}$ and $\mathbf{b}$-faces, and uu-cusps. Let $\tau^{\prime}$ be the remains of $\tau$ in $\mathfrak{C}(M-B)$. The restriction of the vector field on $\mathfrak{C}(M-B)$ to $\tau^{\prime}$ determines an orientation which is singular at the cusps, making $\tau^{\prime}$ into a stable train track. Similar statements hold for a dynamic train track in a stable dynamic branched surface.

Further structure of stable and unstable train tracks is described as follows.
Given a train track $\beta$ and a smoothly embedded circle $\gamma \subset \beta$, a spiralling orientation on $\gamma$ is an orientation for which, if extended continuously to a neighborhood of $\gamma$ in $\beta$, each switch of $\beta$ on the curve $\gamma$ is a converging switch. In other words, all train paths arriving at $\gamma$ agree with the spiralling orientation on $\gamma$. The train track $\beta$ is said to be rational if:

- There are only finitely many smoothly immersed circles in $\beta$, called cycles.
- Each cycle is embedded, and the cycles are pairwise disjoint.
- Each cycle has a spiralling orientation.

Note that any infinite train path in a rational train track $\beta$ eventually spirals around a cycle.

The notions of stable and unstable train tracks are related to rationality by the following easily proved fact:

Lemma 2.5.6. Suppose $\beta$ is a connected train track not homeomorphic to a circle. If $\beta$ is unstable then:

- $\beta$ is rational.
- For each cycle $\gamma \subset \beta$, there are no singularities on $\gamma$, and the orientation of $\beta$ restricted to $\gamma$ is a spiralling orientation. We say that $\gamma$ is a circular $\operatorname{sink}$ of $\beta$.
- A branch of $\beta$ contains a source if and only if both ends of the branch are on the two-sheeted side of a switch.
- No branch of $\beta$ has both ends on the one-sheeted side of a switch.
- Given a train path $p:[0,1] \rightarrow \beta$, there is at most one point $t \in[0,1]$ such that $p(t)$ is a source of $\beta$.

If $\beta$ is stable, the same statements hold replacing the word "agrees" with "disagrees" and "circular sink" with "circular source".

When a dynamic train track $\tau$ fills up an unstable dynamic branched surface $B$, proposition 2.5 .7 will give us more detail about the structure of each component $K$ of $\mathfrak{C}(B-\tau)$. In particular, if $A$ is the sink of $K$-so $A$ is an annulus or Möbius band, also called a ringthen $K$ can be built up from $A$ by inductively attaching certain sectors called "tongues" and so we will call $K$ a "ring with tongues" (figure 2.9 shows a tongue, and figure 2.8 shows tongues attached to a ring). It will follow that the unstable train track $\partial K$ has either one or two circular sinks, depending on whether $A$ is a Möbius band or annulus. We turn to the description of a "ring with tongues".

Let $T$ be a disc with one cusp $c$ and at least two corners (see figure 2.9). Let $\alpha, \beta$ be the two edges adjacent to the cusp, and let $a, b$ be the corners of $T$ which are at the ends of $\alpha, \beta$ opposite $c$. Let $\gamma=\operatorname{cl}(\partial T-(\alpha \cup \beta))$, an arc connecting $a$ and $b$ consisting of one or more edges of $T$ meeting at corners. Let $V_{T}$ be a vector field on $T$ which is tangent to $\alpha$ and $\beta$, pointing inwards at $c$, and transverse outwards at each point of $\gamma$. Then $\left(T, V_{T}\right)$ is called a tongue, or more specifically an unstable tongue, and $\gamma$ is the attaching arc of $T$. If


Figure 2.9: An unstable tongue with cusp $c$ and attaching arc $\gamma$.
$V_{T}$ points outward at the cusp and inward along $\gamma$ then $\left(T, V_{T}\right)$ is a stable tongue. When $V_{T}$ is understood we may drop it from the notation and say simply that $T$ is a tongue.

Let $K^{\prime}$ be a cusped unstable branched surface. Let $\gamma^{\prime}$ be an embedded arc in $K^{\prime}$ with $\partial \gamma^{\prime} \subset \partial K^{\prime}$ such that $\gamma^{\prime}$ is transverse to the vector field on $K^{\prime}$. Form a new cusped unstable branched surface $K$ by gluing an unstable tongue $T$ to $K^{\prime}$, identifying the base $\gamma$ of $T$ with the arc $\gamma^{\prime}$, so that the gluing map is smooth, the tangent planes match up along the gluing locus, and the vector fields agree along the attaching arc. We say that $K$ is obtained from $K^{\prime}$ by attaching a tongue.

A cusped unstable dynamic branched surface $K$ is called a ring with tongues if $K$ is built up from a ring-an annulus or Möbius band-by inductively attaching tongues. That is, there exists a sequence $K_{0} \subset K_{1} \subset \cdots \subset K_{n}=K$, each a cusped unstable dynamic branched surface with respect to the vector field obtained by restriction from $K$, such that:

- $\partial K_{i} \subset \partial K$.
- $K_{0}$ is an annulus or Möbius band smoothly embedded in $K$.
- $K_{i}$ is obtained from $K_{i-1}$ by attaching a tongue $T_{i} \subset K$.

Note that $\Upsilon K$ is the union of the attaching arcs for the tongues.
Proposition 2.5.7. Let $B$ be an unstable dynamic branched surface, and let $\tau \subset B$ be $a$ dynamic train track. Then $\tau$ fills up $B$ if and only if each component of $\mathfrak{C}(B-\tau)$ is a ring with tongues. A similar statement holds for a dynamic train track in a stable dynamic branched surface.

Proof. One direction is an immediate consequence of the observation that in a ring with tongues, the ring is a sink.

To prove the converse, consider a component $K$ of $\mathfrak{C}(B-\tau)$, and let $R$ be the sink. Let $c$ be a cusp of $K$. Let $\gamma^{\prime}$ be the boundary of a regular neighborhood of $c$ in $K$, so $\gamma^{\prime}$ is a properly embedded arc in $K$, which we may choose to be transverse to $V$.

We claim that for each $x \in \gamma^{\prime}$ the forward trajectory $x \cdot[0, \infty)$ intersects $\Upsilon K$. To see why, we know that $x \cdot[0, \infty)$ eventually lands in $R$. Note that $R \cap \gamma^{\prime}=\emptyset$, because each backward trajectory from $\gamma^{\prime}$ ends at $c$, but each point in $R$ has at least one infinite backward trajectory. Thus, $x \cdot[0, \infty)$ has a first intersection point with $R$, and this point must be in $\Upsilon K$, proving the claim.

Let $y(x)$ be the first point of $\Upsilon K$ hit by $x \in \gamma^{\prime}$, and let $\gamma=\left\{y(x) \mid x \in \gamma^{\prime}\right\}$. Clearly $y: \gamma^{\prime} \rightarrow \gamma$ is a homeomorphism, and $\gamma$ is the attaching curve of a tongue $T$ with cusp $c$. Removing $T-\gamma$ from $K$ produces a connected, cusped branched surface with one fewer cusp which still has $R$ as a sink. Continuing inductively, eventually we obtain a connected sub-branched surface $R^{\prime} \subset K$ which has no cusps. Note that $R^{\prime}$ is a sink of the forward semiflow on $K$. Note also that $R \subset R^{\prime}$, because each point of $R$ has at least one infinite backward trajectory, but no point of $K-R^{\prime}$ does.

It remains to show that $R=R^{\prime}$. Arguing by contradiction, suppose $R \neq R^{\prime}$ and let $x \in R^{\prime}-R$. Although $x$ does not have a uniquely defined backward trajectory, nonetheless we claim that there exists an infinite backward trajectory. To define it, start flowing backward from $x$, and whenever the trajectory hits $\Upsilon R^{\prime}$, choose an arbitrary sheet to continue backward along. This process may be continued as long as the trajectory never hits a cusp, but $R^{\prime}$ has no cusps. There exists, therefore, a backward trajectory $\rho=x \cdot(-\infty, 0]$ in $R^{\prime}$. Since $R$ is invariant under the forward semiflow, and since $x \notin R$, it follows that $\rho \cap R=\emptyset$. Choose an accumulation point $y$ for $\rho$, i.e. a limit point of $x \cdot t_{i}$ for some sequence $t_{i} \rightarrow-\infty$. Since $\rho$ is transverse to $\Upsilon R^{\prime}$, and since $\rho$ is disjoint from the sub-branched surface $R \subset R^{\prime}$, it follows that $y \notin R$. We claim that the forward trajectory through $y$ is disjoint from $R$, for if $y \cdot t \in R$ for some $t>0$ then by taking $t_{i}$ sufficiently close to $-\infty$ so that $\rho\left(t_{i}\right)$ is close to $y$ it follows that $\rho\left(x, t_{i}+t\right) \in R$, contradicting the fact that $\rho \cap R=\emptyset$. The forward trajectory through $y$ is therefore disjoint from $R$, contradicting the fact that $R$ is a sink of $K$.

This shows that $R=R^{\prime}$, and therefore $K$ is obtained from the ring $R$ by inductively attaching tongues.

Now we turn to:
Proof of proposition 2.5.1. We start by simultaneously proving items 1 and 2. Let $C$ be a component of $P^{s}$. First we claim there exists a torus piece $T$ of $Q$ such that $T \subset C$. Choose $x \in \operatorname{int}(C)$. In the first case, $x \in T$ for some torus piece $T$, and it follows that $T \subset C$. In the second case $x \in B^{u}-B^{s}$, and by applying axiom 5 in the definition of a dynamic pair, Transience of forward trajectories, it follows that the forward trajectory of $x$ is disjoint from $B^{s}$ and eventually lies in a $\mathbf{u}$-face of some torus piece $T$, from which
it follows that $T \subset C$. The only remaining case is that $x \in \operatorname{int}(t)$ for some pinched tetrahedron $t$, and then the forward trajectory of $x$ eventually hits the uu-edge of $t$ at a point of $\operatorname{int}(C) \cap\left(B^{u}-B^{s}\right)$, reducing to the previous case. This establishes the claim.

Let $A$ be (the image in $C$ of) a $\mathbf{u}$-face of $T$. Let $K$ be the component of $\mathfrak{C}\left(B^{u}-B^{s}\right)$ containing $A$. By proposition 2.5.7, $A$ is an annulus or Möbius band, and $K$ is obtained from $A$ by attaching a possibly empty set of tongues. We claim that in fact $A$ is an annulus, and there is at least one tongue: if $A$ were a Möbius band then $A$ would be double covered by a $\mathbf{u}$-face of $T$, violating axiom 7 ; and if there were no tongues then $K=A$ would be equal to (the image of) another face of a torus piece, also violating axiom 7 .

Let $\mu$ be the component of $\mathfrak{C}(C-T)$ containing $A$ as a $\mathbf{u}$-face. Then $K \subset \mu$, from which it follows that $\mu$ has an ss-cusp circle $\gamma$ intersecting $K$ in one or more cusp points of $K$. The points of $K \cap \gamma$ yield a circular subdivision of $\gamma$ into arcs $\gamma_{1} * \cdots * \gamma_{n}$. Each $\gamma_{i}$ is the ss-cusp edge of a pinched tetrahedron component $T_{i}$ of $Q$. Orient $\gamma$ so that for each $i \in \mathbf{Z} / n$ we have $\operatorname{Head}\left(\gamma_{i}\right)=\operatorname{Tail}\left(\gamma_{i+1}\right)$, and denote this point by $x_{i}$.

Following the proof of 2.5.7, define by induction $K=K_{0} \supset K_{1} \supset \cdots \supset K_{n}=A$, where $K_{i-1}$ is obtained from $K_{i}$ by attaching a tongue $\tau_{i}$ with cusp $x_{i}$. Note that the tongue $\tau_{1}$ is a subset of a $\mathbf{u}$-face of $T_{1}$, and also of a $\mathbf{u}$-face of $T_{2}$; we may glue $T_{1}$ and $T_{2}$ along the tongue $\tau_{1}$ to obtain a pinched tetrahedron denoted $T_{2}^{\prime}$. Continuing inductively, we may glue $T_{i}^{\prime}$ and $T_{i+1}$ along $\tau_{i}$ to obtain a pinched tetrahedron denoted $T_{i+1}^{\prime}$. Consider the pinched tetrahedron $T_{n}^{\prime}$. The tongue $\tau_{n}$ is the entirety of one $\mathbf{u}$-face of $T_{n}^{\prime}$, and a subset of the other $\mathbf{u}$-face; gluing these two $\mathbf{u}$-faces together along $\tau_{n}$ one obtains a uss-maw piece with $\mathbf{u}$-face $A$, and clearly this maw piece is identified with $\mu$.

This shows that the components of $\mathfrak{C}(C-T)$ are maw pieces, one attached to each $\mathbf{u}$-face of $T$, and hence $C$ is a $\mathbf{u}$-cusped torus piece of the same type as $T$. This proves statements 1 and 2 of proposition 2.5.1. Statement 3 follows easily from the dynamical properties of maw pieces combined with the circularity of the vector field on dynamic torus pieces. The analogues of statements $1-3$ for $B^{u}$ follow by similar arguments.

Next we prove statement 4 , switching our point of view to the branched surface $B^{u}$ : we show that $B^{u}$ carries no closed surface. Assuming that $B^{u}$ carries the closed surface $F$, we derive a contradiction. Since $V$ points forward along $\Upsilon B^{u}$ it follows that any trajectory of $V$ that starts in $F$ stays in $F$. If $B^{u} \cap \tau$ were empty it would follow that $F$ is contained in the sink of a component of $\mathfrak{C}\left(B^{u}-\tau\right)$, contradicting the fact that all sinks are annuli and Möbius bands. Therefore $\tau_{F}=\tau \cap F \neq \emptyset$. The train track $\tau_{F}$ is oriented and has no converging switches, and therefore $\tau_{F}$ is a stable train track with no orientation sinks. A simple combinatorial exercise shows that a stable train track with no orientation sinks can have no diverging switches, and therefore each component of $\tau_{F}$ is a circle. Let $\gamma$ be a component of $\tau_{F}$. If $\tau$ had a diverging switch $s$ lying on $\gamma$, then $s$ would also be a diverging switch of $\tau_{F}$, a contradiction. It follows that every switch of $\tau$ lying on $\gamma$ is a converging switch, and so $\gamma$ is a "circular sink" of $\tau$. We have therefore shown that if $B^{u}$ carries a closed surface then $\tau$ has a circular sink.

Now we show that if $\gamma$ is a circular sink of $\tau$ then $\gamma$ is a corner circle of some torus piece of $Q$. Let $R$ be a smoothly embedded annulus or Möbius band in $B^{u}$ with core $\gamma$. Let $C$ be a component of $\mathfrak{C}(R-\gamma)$, and let $\gamma_{C}$ be the component of $\partial C$ mapped to $\gamma$ under the overlay map $C \mapsto R$. There exists a component $K$ of $\mathfrak{C}\left(B^{u}-\tau\right)$ such that $C \subset K$ and $\gamma_{C} \subset \partial K$. Applying proposition 2.5.7, $K$ is obtained from an annulus $A$ by attaching tongues. Clearly $C \subset A$ and $\gamma_{C} \subset \partial A$. There exists a torus piece component $T$ of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ having $A$ as a $\mathbf{u}$-face, and it follows that $\gamma$ is a corner circle of $T$.

Finally, we show that a corner circle $\gamma$ of a torus piece $T$ cannot be a circular $\operatorname{sink}$ of $\tau$, and so $\tau$ has no circular sinks, providing the contradiction that proves statement 4 . Let $A^{\prime}$ be the s-face of $T$ incident to $\gamma$, let $\mu^{\prime}$ be the suu-maw piece attached to $T$ along $A^{\prime}$, and let $F$ be the $\mathbf{u}$-face of $\mu^{\prime}$ incident to $\gamma$. We know that $\mu^{\prime} \cap B^{s}$ is obtained from the annulus $A^{\prime}$ by attaching at least one stable tongue. The intersection of $F$ with the boundary of the first stable tongue contains a branch of $\tau$ that intersects $\gamma$ at a diverging switch, and so $\gamma$ is not a circular sink.

In proving statement 4 we have proved slightly more, namely that $\tau$ has no circular sinks. Repeating the argument for $B^{s}$ it also follows that $\tau$ has no circular sources.

Next we prove statement 5 for $B^{u}$, that no sector $\sigma$ of $B^{u}$ contains a periodic trajectory of $V$. Arguing by contradiction, let $\gamma \subset \sigma$ be a periodic trajectory of $V$. There are two cases, depending on whether $\gamma \cap \tau=\emptyset$.

If $\gamma \cap \tau \neq \emptyset$ then $\gamma \subset \tau$, because $V$ is smooth on $B^{u}$ except at diverging switches, and hence backwards trajectories starting in $\tau \cap \sigma$ stay in $\tau$ as long as they stay in $\sigma$. Also, the only switches of $\tau$ on $\gamma$ are diverging switches, and hence $\gamma$ is a circular source of $\tau$. But we have just proved that $\tau$ has no circular sources.

If $\gamma \cap \tau=\emptyset$, then there exists a component $K$ of $\mathfrak{C}\left(B^{u}-\tau\right)$, with annulus sink $R$, such that $\gamma \subset R$. Note that $\gamma$ is isotopic to a core curve of the annulus $R$, for otherwise $\gamma$ is homotopically trivial and bounds a disc, whose interior contains a zero of $V$ by the EulerPoincaré formula, a contradiction. By the Separation of torus pieces axiom of dynamic pairs, together with proposition $2.5 .7, K$ has at least one tongue attached to $R$ along an arc $\alpha \subset \Upsilon K$ such that $\alpha$ connects opposite components of $\partial R$. It follows that $\gamma$, like any homotopically nontrivial curve in $R$, has nonempty intersection with $\alpha \subset \Upsilon B^{u}$, contradicting that $\gamma$ is contained in a sector of $B^{u}$.

Remark. A converse to proposition 2.5 .1 is also true: if $B^{s}, B^{u}$ is a pair satisfying axioms (1-4) of a dynamic pair, and if each component of $P^{s}$ and $P^{u}$ is obtained from a torus piece and some divided maw pieces by identifying annulus faces as in the conclusion of proposition 2.5.1, then $B^{s}, B^{u}$ satisfy axioms $5-7$ and hence $B^{s}, B^{u}$ is a dynamic pair. To prove axiom 7 Separation of torus pieces, each $\mathbf{u}$-face of each torus piece embeds properly in $P^{s}$, and the interior of $P^{s}$ embeds in $M$, and so different $\mathbf{u}$-faces of torus pieces have interiors mapping disjointly to $M$. The proof of axioms 5, 6 Transience of forward and backward trajectories follows from the behavior of the semiflow in a divided maw piece.

The definition of a dynamic pair can therefore be formulated in two equivalent ways: with axioms $5^{-7}$; or with the description of $P^{s}$ and $P^{u}$ in proposition 2.5.1. In practice axioms 5-7 seem easiest to verify and so are more appropriate for a definition.

### 2.6 Unstable Markov branched surfaces yield dynamic pairs

In proposition 2.6 .2 we give a construction which provides a partial converse to proposition 2.5.1: starting from a dynamic branched surface $B$ which satisfies the conclusions of 2.5.1, and assuming that the dynamics on $B$ are "Markov" in a certain sense, we show how to construct a dynamic pair. We begin with a discussion of Markov branched surfaces.

Let $(B, V)$ be an unstable dynamic branched surface with smooth $V$ generating a forward semiflow $\phi$. A Markov section of $\phi$ is a collection $\mathcal{I}$ of closed intervals smoothly embedded in $B$ satisfying the following properties:

1. Each $I \in \mathcal{I}$ is transverse to $\phi$.
2. For each $I \in \mathcal{I}$, either $\operatorname{int}(I) \cap \Upsilon B=\emptyset$ or $I \subset \Upsilon B$.
3. For each $I \neq I^{\prime} \in \mathcal{I}$ we have $I \cap I^{\prime} \subset \partial I \cap \partial I^{\prime}$.
4. $\mathcal{I}$ is a cross section. For each $x \in B$ there exists $t>0$ such that $x \cdot t \in \bigcup \mathcal{I}$. The smallest such value of $t$, called the first return time of $x$, is denoted $t_{x}$, and the function $x \mapsto t_{x}$ is a bounded function on $B$. The map $f(x)=\phi\left(x, t_{x}\right)$ is called the first return map.
5. The Markov property. For any $I \in \mathcal{I}$ and $x \in \partial I$, there exists $I^{\prime} \in \mathcal{I}$ such that $f(x) \in \partial I^{\prime}$.

If $\phi$ has a Markov section $\mathcal{I}$ then we say that $(B, V, \mathcal{I})$, or more informally $B$, is a Markov branched surface.

An important property of a Markov branched surface is that for every $p \in B$, every backward trajectory starting from $p$ intersects $\bigcup \mathcal{I}$ after a bounded time; this is true despite the nonuniqueness of backward trajectories. To prove this, suppose that there were an infinite backward trajectory that was disjoint from $\bigcup \mathcal{I}$. Let $q$ be an accumulation point of that trajectory. Then the forward trajectory from $q$ would be disjoint from $\bigcup \mathcal{I}$, contradicting that $\mathcal{I}$ is a cross section.

The following proposition says that the Markov property is necessary, in order for a dynamic branched surface to be part of a dynamic pair:

Proposition 2.6.1. Given a dynamic pair $B^{s}, B^{u}$ in a compact, oriented, torally bounded 3-manifold $M$, there exists a dynamic vector field $V$ for $B^{u}$ such that $\left(B^{u}, V, \mathcal{I}\right)$ is a Markov branched surface for some $\mathcal{I}$.

Sketch of proof. If one is given a Markov section $\mathcal{I}$, the set

$$
\beta=\bigcup_{x \in \partial I, I \in \mathcal{I}} x \cdot[0, \infty)
$$

is a finite 1 -complex. Our approach is to construct the appropriate 1 -complex $\beta$ and then construct the Markov section.

Consider a component $K$ of $\mathfrak{C}\left(B^{u}-B^{s}\right)$. By proposition 2.5.1 $K$ is obtained from an annulus $A$ by attaching tongues. For each cusp $z$ of $K$ choose a point $x_{z} \in K-\partial K$ near the cusp, so that the backward trajectory from $x_{z}$ hits $z$ after a short time. Let $X_{K}$ be the collection of all the points $x_{z}$ for cusps $z \in K$, together with all branch points of $B$ lying in $K$. For each $x \in X_{K}$, the forward trajectory $x \cdot[0, \infty)$ eventually lands in $A$. By homotoping the vector field, we may assume that $x \cdot[0, \infty)$ is eventually periodic, once it hits $A$. In other words, there exists a periodic orbit $\gamma_{x} \subset A$ such that the image of $x \cdot[0, \infty)$ intersected with $A$ is $\gamma_{x}$. We can, moreover, homotope so that this property is true for all $x \in X=\bigcup_{K} X_{K}$. Having done this, define $\beta=\bigcup_{x \in X} x \cdot[0, \infty)$, a finite 1-complex parallel to the dynamic vector field, consisting of finitely many periodic orbits plus finitely many finite trajectories each ending on one of these periodic orbits.

The Markov section $\mathcal{I}$ may now be constructed by taking sufficiently many intervals transverse to $V$ whose endpoints lie on $\beta$. To construct a typical element of $\mathcal{I}$, start at a point of $\beta$ and trace out an arc transverse to $V$, avoiding $\Upsilon B^{u}$ and any previously constructed elements of $\mathcal{I}$, and passing right through $\tau=B^{s} \cap B^{u}$, stopping the first time you hit $\beta$. A cross-section $\mathcal{I}$ may be constructed in this manner, by taking the starting points to be an $\epsilon$-dense subset of $\beta$ for some $\epsilon>0$. By construction the Markov property is satisfied, and the remaining properties of a Markov section are easily verified.

Combining the above proposition with proposition 2.5 .1 we obtain a list of necessary condition for an unstable branched surface $B$ to be part of a dynamic pair. The following proposition says that these conditions are also sufficient, after some minor adjustments on $B$ :

Proposition 2.6.2. Let $M$ be a compact, oriented, torally bounded 3-manifold. Suppose that $(B, V, \mathcal{I})$ is an unstable Markov branched surface. Suppose moreover that:

- $B$ is very full in $M$.
- The dynamic vector field on each component of $\mathfrak{C}(M-B)$ is circular.
- B carries no closed surfaces.
- No sector of $B$ contains a periodic trajectory of $V$.

Then we can construct a dynamic pair $B^{s}, B^{u}$ in $M$ such that $B^{u}$ is obtained from $B$ by dynamic splitting.

Remark. The proof will show that $B$ is split along a dynamic splitting surface consisting of a disjoint union of annuli and Möbius bands, thereby creating some new cusped solid tori of type $(2,0)$ or $(2,1)$.

Proof. The proof breaks into three major steps. Step 1 uses the Markov section to construct a dynamic train track $\tau \subset B$ which fills up $B$. Step 2 consists of a sequence of alterations on $\tau$ and $B$, including possibly some dynamic splitting of $B$. Step 3 constructs a stable branched surface $B^{s}$ so that $\tau=B^{s} \cap B$, and shows that $B^{s}, B^{u}=B$ is a dynamic pair in $M$. In each step the dynamic vector field $V$ will be altered, but we make sure to do this so that $V$ is still a dynamic vector field on $B$, and after step 3 we check that $V$ is a dynamic vector field for the pair $B^{s}, B^{u}$.

Step 1: From Markov section to dual dynamic train track. Our immediate goal is to homotop $V$ through dynamic vector fields on $B$ to a new dynamic vector field $V^{\prime}$, and construct a dynamic train track $\tau$ tangent to $V^{\prime}$, so that $\tau$ fills up $B$. This $\tau$ will be called the dual dynamic train track to $\mathcal{I}$. In some sense the construction of $\tau$ is the inverse of the construction in proposition 2.6.1.

Let $\mathcal{I}=\left\{I_{1}, \ldots, I_{M}\right\}$. From the definition of Markov section it follows that each $I_{i}$ has a unique partition into subintervals as $I_{i}=I_{i 1} * I_{i 2} * \cdots * I_{i n}$, so that the first return map $f$ is continuous on $\operatorname{int}\left(I_{i j}\right)$ for each $j=1, \ldots, n$, and $f \mid \operatorname{int}\left(I_{i j}\right)$ extends continuously to a homeomorphism $I_{i j} \rightarrow I^{\prime}$ for some $I^{\prime} \in \mathcal{I}$; by abuse of terminology this extension is called the first return map of the subinterval $I_{i j}$. Define the transition matrix of the Markov section $\mathcal{I}$ to be the $M \times M$ matrix $\mu$ where $\mu(i, j)$ is the number of subintervals of $I_{i}$ mapping homeomorphically onto $I_{j}$ under the first return map.

Choose a base point $x_{i} \in \operatorname{int}\left(I_{i}\right)$ for each $I_{i} \in \mathcal{I}$. For each $i$ we construct a piece $\tau_{i}^{+}$of $\tau$ starting from $x_{i}$ (figure 2.10a) and another piece $\tau_{i}^{-}$ending at $x_{i}$ (figure 2.10b).

To construct $\tau_{i}^{+}$, consider the partition $I_{i}=I_{i 1} * I_{i 2} * \cdots * I_{i K}$ defined above; the integer $K=K_{i}$ depends only on $i$, and is the sum of the entries in row $i$ of $\mu$. For each $k=1, \ldots, K$ let $I_{j_{k}} \in \mathcal{I}$ be the element of the Markov section such that the first return map takes $I_{i k}$ to $I_{j_{k}}$. There is a unique point $\xi_{i k} \in \operatorname{int}\left(I_{i k}\right)$ taken to $x_{j_{k}}$ under the first return map $I_{i k} \rightarrow I_{j_{k}}$. Let $I_{i}^{\prime}$ be a compact subinterval of $\operatorname{int}\left(I_{i}\right)$ containing $\xi_{i 1}, \ldots, \xi_{i K}$; we obtain $I_{i}^{\prime}$ from $I_{i}$ by removing a tiny neighborhood of each endpoint of $I_{i}$. Choose a number $t>0$ so small that $I_{i}^{\prime} \cdot[0, t] \cap(\Upsilon B \cup \bigcup \mathcal{I})=I_{i}^{\prime}$. Let $y_{i k}=\xi_{i k} \cdot t$. Construct a piece of oriented train track $\tau_{i}^{+} \subset I_{i}^{\prime} \cdot[0, t]$ with one backwards endpoint at $x_{i}$ and $K$ forward endpoints at $y_{i 1}, \ldots, y_{i K}$, and with $K-1$ diverging switches, as shown in figure 2.10a.

Next, let $I_{i}^{1}, \ldots, I_{i}^{L}$ be the collection of subintervals of elements of $\mathcal{I}$ whose first return maps take them homeomorphically to $I_{i}$; now $L=L_{i}$ is the sum of column $i$ of $\mu$. Let $y_{i}^{l}=y_{j k}$ where $j, k$ are chosen so that $I_{i}^{l}=I_{j k}$. Recalling that $y_{j k}$ is on the trajectory from $\xi_{j k}$ to $x_{i}$, we may construct a piece of oriented train track $\tau_{i}^{-}$consisting of the trajectories from each $y_{i}^{1}, \ldots, y_{i}^{L}$ to $x_{i}$, as shown in figure 2.10 b . Notice that the intersection of any two

(a) Construction of $\tau_{i}^{+}$

(b) Construction of $\tau_{i}^{-}$

Figure 2.10: Constructing a dynamic train track from a Markov section.
of these trajectories contains $x_{i}$, and the first point of intersection is a converging switch of $\tau_{i}^{-}$which is transverse to $B$; it follows that $\tau_{i}^{-}$has exactly $L-1$ converging switches.

Now define the dual train track $\tau$ of $\mathcal{I}$ to be

$$
\tau=\bigcup_{i=1, \ldots, M}\left(\tau_{i}^{+} \cup \tau_{i}^{-}\right)
$$

Note that $\tau$ has exactly $\mu(i, j)$ oriented paths going from $x_{i}$ to $x_{j}$ whose interiors are disjoint from $\bigcup \mathcal{I}$. We must prove that $\tau$ is a dynamic train track in $B$.

We first prove that $\tau \cap \Upsilon B$ consists of the converging switches of $\tau$. By construction every converging switch of $\tau$ lies in $\Upsilon B$. We must show that for each $i=1, \ldots, M$, each point $p \in \tau_{i}^{-} \cap \Upsilon B$ is a converging switch. To see why this is true, note first that for some $l=1, \ldots, L_{i}$ the point $p$ lies on the trajectory from $y_{i}^{l}$ to $x_{i}$. Going backward from $p$, this trajectory follows one of the two sheets on the two-sheeted side of $p$. Now go backward from $p$ along a trajectory that follows the other sheet, and after a bounded time one must intersect $\bigcup \mathcal{I}$. The first such intersection point must be $x_{i}^{k}$ for some $k=1, \ldots, L_{i}$, and so $p$ also lies on the trajectory from $y_{i}^{k}$ to $x_{i}$, proving that $p$ is a converging switch of $\tau$.

We must homotop $V$ through dynamic vector fields to make $V$ tangent to $\tau$. Fix $i=1, \ldots, M$ and consider $\tau_{i}^{+}$; see figure 2.10a. Note that $\tau_{i}^{+}$is contained in $I_{i}^{\prime} \cdot[0, t]$ which lies in some sector $\sigma$ of $B$. In figure 2.10a, the vector field $V$ points straight upward. We may alter $V$ by a homotopy supported in $I_{i}^{\prime} \cdot[0, t]$ so that $V$ is tangent to $\tau_{i}^{+}$, retaining a positive upward coordinate in figure 2.10a. Doing this for all $i=1, \ldots, M$, and noticing that $V$ is automatically tangent to $\tau_{i}^{-}$, it follows that $V$ is now tangent to $\tau$.

It remains to verify the property Transience of forward trajectories.

Let $K$ be a component of $\mathfrak{C}(B-\tau)$, a cusped unstable dynamic branched surface. From the construction we know that $\partial K \neq \emptyset$. Following the proof of proposition 2.5.7, we inductively remove tongues to obtain a sub-branched surface $R \subset K$, also a cusped unstable dynamic branched surface, such that $\partial R \subset \partial K, K$ is obtained from $R$ by inductively attaching tongues, and $R$ has no cusps. Removal of a tongue does not affect connectivity, and so $R$ is connected. The only diverging switches of the oriented train track $\partial R$ occur at cusps of $R$, but $R$ has no cusps and so $\partial R$ has no diverging switches. It follows that $\partial R$ has no converging switches, and therefore $\partial R$ is a union of circles. Removal of tongues preserves the property that the boundary is nonempty, and so $\partial R \neq \emptyset$.

We shall show that $\Upsilon R=\emptyset$, which together with the previous paragraph implies that $R$ is an annulus or Möbius band, and so $K$ is a ring with tongues.

Each $I \in \mathcal{I}$ is subdivided into two arcs at the point $x_{I}=I \cap \tau$; the components are called half-intervals of $\mathcal{I}$. Let $\mathcal{I}_{K}$ be the collection of half-intervals of $\mathcal{I}$ contained in $K$. For each $\alpha \in \mathcal{I}_{K}$, if $\operatorname{int}(\alpha) \cap R \neq \emptyset$ then $\alpha \subset R$, because $\operatorname{int}(\alpha) \cap \Upsilon K=\emptyset$. Let $\mathcal{I}_{R}$ be the set of half-arcs contained in $R$. Each $\alpha \in \mathcal{I}_{R}$ has one boundary point $x_{\alpha}$ on $\partial R$ and the other boundary point $y_{\alpha} \operatorname{in} \operatorname{int}(R)$. For each $y_{\alpha}$, the first return of $y_{\alpha}$ to $\mathcal{I}_{R}$ is another point $y_{\alpha^{\prime}}$; let $\gamma_{\alpha \alpha^{\prime}}$ be the flow segment from $y_{\alpha}$ to $y_{\alpha^{\prime}}$. Let $Y \subset R$ be the directed graph with vertices $y_{\alpha}$ and directed edges $\gamma_{\alpha \alpha^{\prime}}$. Note that each vertex of $Y$ has exactly one outgoing edge.

We claim that every vertex of $Y$ has at least one incoming edge. To prove this statement, suppose there is a vertex $y_{\alpha}$ with no incoming edges. There exists an infinite backwards trajectory $y \cdot(-\infty, 0]$ staying entirely in $R$, for if one follows any trajectory backwards from $y_{\alpha}$, making arbitrary choices whenever $\Upsilon R$ is hit, the only obstruction to continuing backward forever occurs at a cusp of $\partial R$, but there are no cusps. Now it is evident that every infinite backward trajectory eventually hits $\bigcup \mathcal{I}$, and so we can choose a point $z=y \cdot t \in \bigcup \mathcal{I}$ with $|t|>0$ minimal. By the hypothesis that $y_{\alpha}$ has no incoming edges, it follows that $z \in \operatorname{int}(I)$ for some $I \in \mathcal{I}$. However, from the construction of $\tau$ it follows that the flow segment from $z$ to $y_{\alpha}$ must intersect $\tau$, contradicting that this flow segment is contained in the interior of $R \subset \mathfrak{C}(B-\tau)$, thereby proving the claim.

Each vertex of $Y$ therefore has exactly one outgoing edge and at least one incoming edge, from which it follows that each vertex has exactly one incoming edge, and therefore $Y$ is a disjoint union of circles.

We claim also that $Y \cap \Upsilon R=\emptyset$. If not, there would be a vertex $y_{\alpha}$, a backward trajectory $y_{\alpha} \cdot(-\infty, 0]$, and a number $T<0$ such that $y_{\alpha} \cdot T \in Y$ but $y_{\alpha} \cdot(-\infty, 0]$ diverges from $Y$ for $t<T$. But then arguing as above there would be a first value $t<T$ such that $y_{\alpha} \cdot t \in \bigcup \mathcal{I}$ and $y_{\alpha} \cdot t$ is a boundary point, from which it follows that $y_{\alpha} \cdot t$ is a vertex of $Y$. But then the next vertex of $Y$ going forward from $y_{\alpha} \cdot t$ would have two incoming vertices, a contradiction.

Consider now the branched surface $\mathfrak{C}(R-Y)$. Each element of $\mathcal{I}_{R}$ pulls back to a properly embedded arc in $\mathfrak{C}(R-Y)$, and so we may regard $\bigcup \mathcal{I}_{R}$ as a subset of $\mathfrak{C}(R-Y)$. The first return map on $\bigcup \mathcal{I}_{R}$ is a local homeomorphism which restricts to a bijection on
$\partial \bigcup \mathcal{I}_{R}$, from which it follows that the first return map is a homeomorphism on $\bigcup \mathcal{I}_{R}$. Thus, $\mathfrak{C}(R-Y)$ is the mapping torus of a homeomorphism on a union of closed intervals, and so $\mathfrak{C}(R-Y)$ is a union of annuli and Möbius bands. It follows that $R$ is a union of annuli and Möbius bands, and so by connectivity it is a single annulus or Möbius band, completing the proof that $K$ is a ring with tongues.

We have constructed a dynamic train track $\tau$ that fills up $B$, with dynamic vector field $V^{\prime}$ tangent to $\tau$.

It is not hard to check that $V^{\prime}$ is still circular in each component $T$ of $\mathfrak{C}(M-B)$.

Step 2 Ideally we would now like to construct a branched surface $B^{s}$ so that the pair $B^{s}, B^{u}=B$ is a dynamic pair with $\tau=B^{s} \cap B$. However, the intersection train track of a dynamic pair satisfies several properties that $\tau$ may not satisfy. In a sequence of substeps, we shall describe how to alter $B$ and $\tau$ so as to establish each needed property.

The first property is true without any alterations:

Step 2a: Each component $K$ of $\mathfrak{C}(B-\tau)$ has at least one tongue. If this is not so, then $K$ is an annulus or Möbius band. There exists a sector $\sigma$ of $B$ such that $K \subset \sigma$. Each boundary circle $c$ of $K$ is a periodic trajectory of $V^{\prime}$ contained in $\sigma$. We now show that $V$ also has a periodic trajectory in $\sigma$, contradicting the hypothesis of proposition 2.6.2 and therby proving the claim.

The circle $c$ is a periodic trajectory in $\tau$ contained in $\sigma$. From the construction of $\tau$, corresponding to $c$ is a cycle of elements in $\mathcal{I}$, namely $I_{i_{0}}, I_{i_{1}}, \ldots, I_{i_{K}}=I_{i_{0}}$, such that $\mu\left(i_{k-1}, i_{k}\right) \neq 0$ for $k=1, \ldots, K$, and each $I_{i_{k}}$ is contained in $\sigma$. Moreover, there exist subintervals $J_{i_{k}} \subset I_{i_{k}}$ such that the first return map of $\phi$ takes $J_{i_{k-1}}$ homeomorphically onto $J_{i_{k}}$ for $k=1, \ldots, K$, the trajectories from $J_{i_{k-1}}$ to $J_{i_{k}}$ all lie in $\sigma$, and $J_{i_{K}}=I_{i_{K}}=I_{i_{0}}$. The $K$-fold iterate of the first return map therefore takes $J_{i_{0}} \subset I_{i_{0}}$ homeomorphically onto $I_{i_{0}}$, and so there is a periodic trajectory of $\phi$ entirely contained in $\sigma$, i.e. there is a periodic trajectory of $V$ contained in $\sigma$, a contradiction.

For the remainder of the proof let $K=\mathfrak{C}(B-\tau)$. Let $\mathcal{R}$ be the union of sinks of components of $K$.

Step 2b: Eliminating circular sinks of $\tau$. A circular sink of $\tau$ is a smoothly embedded circle $\gamma \subset \tau$ such that each switch of $\tau$ on $\gamma$ is a converging switch. Recall from the proof of proposition 2.5 . 1 that if $\tau$ were the intersection train track of a dynamic pair then $\tau$ would have no circular sinks.

We describe how to alter $\tau$ so as to eliminate a circular sink $\gamma$. Let $b \subset \tau$ be the "basin of attraction" of $\gamma$, the set of all $y \in \tau$ such that the directed path in $\tau$ going forward from $y$ never encounters any diverging switches and eventually lands in $\gamma$. Alter $\tau$ by deleting $b$,
to produce a train track $\tau^{\prime}$. It is obvious that $\tau^{\prime}$ is a dynamic train track, except perhaps for the property Transience of forward orbits, which we now verify.

The inverse image of $\gamma$ under the overlay map $K \mapsto B$ is a subset $\gamma^{\prime} \subset \partial \mathcal{R}$ and the map $\gamma^{\prime} \mapsto \gamma$ is double covering. Similarly, the inverse image of $c l(b)$ is a subset $b^{\prime} \subset \partial K$, and the map $b^{\prime} \rightarrow \operatorname{cl}(b)$ is a double covering. If $\gamma$ is an orientation preserving curve in $B$, then each of the covering maps $\gamma^{\prime} \mapsto \gamma, b^{\prime} \mapsto \mathrm{cl}(b)$ is a disconnected double covering; whereas if $\gamma$ is orientation reversing then the covering maps are connected.

Regarding $b$ as a subset of $\mathfrak{C}\left(B-\tau^{\prime}\right)$, note that $K$ is obtained from $\mathfrak{C}\left(B-\tau^{\prime}\right)$ by cutting open along $\operatorname{cl}(b)$, and $b^{\prime}$ is the remains of $\operatorname{cl}(b)$ in $K$. Let $C^{\prime}$ be the component of $\mathfrak{C}\left(B-\tau^{\prime}\right)$ containing $\mathrm{cl}(b)$, and let $C \subset K$ be the inverse image of $C^{\prime}$. Clearly $C$ is a union of one or two components of $K$, and $\gamma^{\prime}$ is contained in the boundary of the $\operatorname{sink}(C)(=$ the union of sinks of components of $C$ ). It is now easy to see that $C^{\prime}$ has a sink obtained from $\operatorname{sink}(C)$ by gluing $\gamma^{\prime}$ to itself via the covering transformation of the double covering $\gamma^{\prime} \rightarrow \gamma$, proving Transience of forward orbits for the train track $\tau^{\prime}$ and thereby showing that $\tau^{\prime}$ is a dynamic train track in $B$. Also, $\tau^{\prime}$ fills $B$, for if $\operatorname{sink}\left(C^{\prime}\right)$ were a torus or Klein bottle then this surface would be carried by $B$ contrary to the hypothesis.

We have described how to eliminate one circular sink of $\tau$, replacing $\tau$ by a dynamic train track $\tau^{\prime}$ that fills up $B$. Clearly there is a finite number of circular sinks, and the number in $\tau^{\prime}$ is one fewer than in $\tau$. Also, $\tau^{\prime}$ still has the property that each component of $\mathfrak{C}\left(B-\tau^{\prime}\right)$ has at least one tongue. By repeating the process we may therefore assume that $\tau$ has no circular sinks.

Step 2c: Splitting rings having tongues on both sides. Given a component $K$ of $\mathfrak{C}(B-\tau)$ with $\operatorname{sink} R$, the surface $R$ is either a two-sided annulus or one-sided Möbius band. We may ask, for each side of $R$, whether there is a tongue attached to that side. Since there is at least one tongue attached to $R$, there are three cases:
(a) $R$ is a Möbius band, with at least one tongue attached (to its unique side).
(b) $R$ is an annulus, with at least one tongue attached to each side.
(c) $R$ is an annulus, with tongues attached to only one side.

We remark that if $\tau$ is the intersection train track of a dynamic pair, only case (c) can occur.

If any ring $R$ is of type (a) or (b), we shall split it as follows. Let $R^{\prime}$ be a smoothly embedded ring in $B$ such that $R \subset \operatorname{int}\left(R^{\prime}\right) \subset R^{\prime} \subset N(R)$ where $N(R)$ is a small neighborhood of $R$. After perturbing the dynamic vector field on $B$, we may assume that $R^{\prime}$ is a splitting surface, i.e. the vector field points outward along $R^{\prime}$. Now split $B$ along $R^{\prime}$, resulting in a branched surface $B^{\prime}$ with oriented train track $\tau^{\prime}$. It is easy to verify that $B^{\prime}, \tau^{\prime}$ still satisfy the hypotheses, as well as the earlier properties that have already been
established for $B, \tau$. Also, the dynamic vector field on $B$ tangent to $\tau$ can be perturbed to become a dynamic vector field on $B^{\prime}$ tangent to $\tau^{\prime}$. Note that $\mathfrak{C}\left(M-B^{\prime}\right)$ has one more component than $\mathfrak{C}(M-B)$ : a cusped solid torus, of type $(2,0)$ when $R$ is an annulus, or of type $(2,1)$ when $R$ is a Möbius band. Observe also that $\mathfrak{C}\left(B^{\prime}-\tau^{\prime}\right)$ has one fewer sink of types (a), (b).

After a finite number of such alterations, we may therefore assume that each sink of a component of $\mathfrak{C}(B-\tau)$ is of type (c), and therefore is contained in a face of a component of $\mathfrak{C}(M-B)$.

Step 2d: Splitting lonely face orbits of $\tau$. Let $T$ be a component of $\mathfrak{C}(M-B)$, let $A$ be a face of $T$, and let $\tau_{A}=\tau \cap A$. Since $\tau$ fills up $B$ it follows that $\tau_{A} \neq \emptyset$. Also, $\partial \tau \subset \partial A$. Since $V$ is tangent to $\tau$ and points forward on $\Upsilon B$ it follows that $\tau_{A}$ points out of $A$ at each point of $\partial \tau$. Finally, each switch of $\tau_{A}$ is a diverging switch in $\operatorname{int}(A)$. From these properties it follows that $\tau_{A}$ has at least one periodic orbit, every periodic orbit is embedded, and distinct periodic orbits in $\tau_{A}$ are disjoint. There cannot be three or more of these orbits, because the ones in the middle would be circular sinks of $\tau$, which have been eliminated in an earlier step. Thus, $\tau_{A}$ contains either one or two orbits.

Note that if $\tau$ were the intersection train track of a dynamic pair, then $\tau_{A}$ would contain two orbits.

Suppose, then, that $\tau_{A}$ contains a unique periodic orbit $\gamma$. We alter $\tau$ as follows. There is a regular neighborhood $K(\gamma)$ of $\gamma$ in $B$, consisting of a ring $A$ with tongues attached disjointly, one tongue for each point where $\Upsilon B$ crosses $\gamma$ (see figure 2.11). The intersection of $\tau$ with each tongue is a short arc from the cusp of the tongue to a converging switch of $\tau$. Let $\sigma$ be the union of $\gamma$ with slightly shorter arcs, one for each converging switch of $\tau$ on $\gamma$. Now split $\tau$ along the 1 -complex $\sigma$, creating a new train track $\tau^{\prime}$. Perturb the dynamic vector field on $B$ to be tangent to $\tau^{\prime}$. The pair $B, \tau^{\prime}$ still satisfies the hypotheses of the proposition, as well as all the previously established properties. Note that $\mathfrak{C}\left(B-\tau^{\prime}\right)$ has one more component than $\mathfrak{C}(B-\tau)$; this component is an annulus with tongues.

We have now reduced to the case where the inclusion of sinks of components of $\mathfrak{C}(B-\tau)$ into faces of $\mathfrak{C}(M-B)$ induces a $1-1$ correpondence between components of $\mathfrak{C}(B-\tau)$ and faces of components of $\mathfrak{C}(M-B)$.

Step 3: Constructing the stable branched surface. Now take $B^{u}=B$. We construct $B^{s}$ so that $\tau=B^{s} \cap B^{u}$ and prove that $B^{s}, B^{u}$ is a dynamic pair.

Consider a component $T$ of $\mathfrak{C}(M-B)$; we construct $B_{T}^{s}$, the remains of $B^{s}$ in $T$. Let $\tau_{T}$ be the remains of $\tau$ in $\partial T$, a stable train track by lemma 2.5.5. Let $F_{1}, \ldots, F_{N}$ be the faces of $T$, and let $R_{i} \subset F_{i}$ be the unique annulus whose boundary lies in $\tau_{T}$. For each $n \in \mathbf{Z} / N$ there is a uu-cusp circle $c_{n}=\partial F_{n-1} \cap \partial F_{n}$. Note that $c_{n} \cap \tau_{T} \neq \emptyset$ for each $n=1, \ldots, N$, because no component of $\mathfrak{C}(B-\tau)$ contains a maw circle. Since $\tau_{T}$ is


Figure 2.11: If the orbit $\gamma$ is the only orbit of $\tau$ contained in a certain face of a cusped torus piece, split $\tau$ along a 1-complex $\sigma$ obtained from $\gamma$ by adding a short arc adjacent to each convering switch on $\gamma$.
stable, given $x \in c_{n} \cap \tau_{T}$, the backward trajectory in $\tau_{T}$ that starts from $x$ and stays in the face $F_{n-1}$ must eventually hit a circular source of $\tau_{T}$, and that circular source must be a component $d_{n 1}$ of $\partial R_{n-1}$; similarly, going backward from $x$ in $F_{n} \cap \tau_{T}$ you must eventually hit the component $d_{n 2}$ of $\partial R_{n}$. It follows that $\tau_{T}$ has a component $\tau_{T n}$ containing the circular sources $d_{n 1}, d_{n 2}$ and containing each point of $\tau_{T} \cap c_{n}$. By hypothesis, the flow on $T$ is circular, and so the circular sources $d_{n 1}, d_{n 2}$ are oriented isotopic in $T$.

Now we construct a stable ring with tongues which will be inserted into $T$, with boundary $\tau_{T n}$. Let $A_{n}^{\prime}$ be the subannulus of $\partial T$ containing $c_{n}$ with $\partial A_{n}^{\prime}=d_{n 1} \cup d_{n 2}$. Let $A_{n}$ be a properly embedded annulus in $T$ with boundary $d_{n 1} \cup d_{n 2}$, obtained by perturbing the inclusion map $A_{n}^{\prime} \hookrightarrow T$. Note that there is an suu-maw piece $\mu_{n} \subset T$ with cusp circle $c_{n}$ and boundary $A_{n}^{\prime} \cup A_{n}$.

Since $d_{n 1}, d_{n 2}$ are oriented isotopic in $T$, we may homotop the dynamic vector field on $\operatorname{int}(T)$ to be tangent to $A_{n}$, without altering the fact that the vector field is circular on $T$. We may also homotop so that all forward trajectories in $\mu_{n}-A_{n}$ eventually hit the cusp circle $c_{n}$, and all backward trajectories limit on $A_{n}$.

Now we attach tongues to $A_{n}$. Enumerate the points of $c_{n} \cap \tau_{T}$ in circular order around $c_{n}$ as $x_{1}, \ldots, x_{K}$ of the points of $c_{n} \cap \tau_{T}$. Attach one tongue for each of the points $x_{i}$, as follows.

Consider first $x_{1}$. Let $\gamma_{11}, \gamma_{12}$ be the paths in $\tau_{T}$ connecting $x_{1}$ to $d_{n 1}, d_{n 2}$ respectively. We wish to attach a stable tongue $t_{1}$ to $A_{n}$, tangent to $V$, with edges $\gamma_{11}, \gamma_{12}$ adjacent to
$x_{1}$, and with base curve $\gamma_{13} \subset A_{n}$. To do this, first let $t_{1}^{\prime} \subset \mu_{n}$ be a small 1-cusped triangle tangent to $V$, with cusp at $x_{1}$, and with two edges on short subsegments of $\gamma_{11}, \gamma_{12}$ incident to $x_{1}$. The third edge of $t_{1}^{\prime}$, denoted $\gamma_{13}^{\prime}$, is a properly embedded arc in $\mu_{n}$. Now let $\gamma_{13}^{\prime}$ flow backward in $\mu_{n}$ to an arc $\gamma_{13}^{\prime \prime}$ which is contained in a very small neighborhood of $A_{n}$. There is a 1 -cusped triangle $t_{1}^{\prime \prime}$ which is the union of $t_{1}^{\prime}$ with the trajectories from $\gamma_{13}^{\prime}$ to $\gamma_{13}^{\prime \prime}$. By a perturbation of $V$ and $t_{1}^{\prime \prime}$ supported near $A_{n}$, we obtain the desired stable tongue $t_{1}$.

Next consider $x_{2}$. Let $\gamma_{21}, \gamma_{22}$ be the paths in $\tau_{T}$ going from $x_{2}$ to $d_{n 1} \cup \gamma_{11}, d_{n 2} \cup \gamma_{12}$ respectively. We wish to attach a stable tongue $t_{2}$ to $A_{n} \cup t_{1}$, tangent to $V$, with edges $\gamma_{21}$, $\gamma_{22}$ incident to $x_{2}$, and with base curve $\gamma_{23} \subset A_{n} \cup t_{1}$. Again start with a small 1-cusped triangle $t_{2}^{\prime}$ near $x_{2}$ and flow backward to obtain another 1-cusped triangle $t_{2}^{\prime \prime}$ whose base edge is contained in a very small neighborhood of $A_{n} \cup t_{1}$. By a perturbation of $V$ and $t_{1}^{\prime \prime}$ supported near $A_{n} \cup t_{1}$ we obtain $t_{2}$.

Continuing in this manner, we construct an annulus with tongues $B_{T n}^{s}=A_{n} \cup t_{1} \cup \cdots \cup t_{K}$. Let $B_{T}^{s}=\bigcup_{n=1}^{N} B_{T n}^{s}$.

Now let $B^{s}$ be the (overlay image of the) union of the $B_{T}^{s}$, over all components $T$ of $\mathfrak{C}(M-B)$. From the construction it is easy to check all the axioms of a dynamic pair for $B^{s}, B^{u}$, except possibly for axiom 7 , to which we now turn.

Suppose that there is an s-face gluing of dynamic torus pieces of $B^{s}, B^{u}$. This s-face is contained in some sector $\sigma$ of $B^{s}$, and any boundary component of $\sigma$ is a periodic trajectory in $\sigma$, violating the fact that no sector contains a periodic orbit. A $\mathbf{u}$-face gluing is similarly ruled out.

This finishes the proof of proposition 2.6.2.
Recall that when unstable dynamic branched surfaces were defined, we did not require that the dynamic vector field generate an expansive forward semiflow, contrary to the definition adopted by Christy [Chr93]. It is interesting to note that the existence of a Markov section is closely related to expansivity:

Proposition 2.6.3. If $(B, V)$ is an unstable dynamic branched surface, the following are equivalent:

1. The dynamic vector field $V$ can be chosen so that it generates an expansive forward semiflow.
2. $B$ does not carry a torus or Klein bottle, and the dynamic vector field $V$ can be chosen so that it has a Markov section.

Proof. We only sketch the proof, since this proposition is not needed elsewhere.
The space of dynamic vector fields on $B$ is path connected, that is, any two dynamic vector fields on $B$ are homotopic through dynamic vector fields. We are therefore free to replace $V$ by any other dynamic vector field.

To prove that 2 implies 1 , carry out the above proof up to step $2 \mathbf{b}$, using the nonexistence of tori and Klein bottles. Show that the transition matrix has no non-negative eigenvectors of eigenvalue 1. Use that to construct an expansive first return map to the Markov section, and suspend to get an expansive semiflow.

The proof that 1 implies 2 follows standard methods for construction of Markov partitions, as in [Bow73]. If $B$ carried a torus or Klein bottle $S$, the restriction of $V$ to $S$ would generate an expansive flow on $S$; but a torus or Klein bottle does not support an expansive flow.

### 2.7 The taffy-pulling example revisited

We may now use theorem 2.6 .2 to give the first rigorous example of a dynamic pair. Let $B \subset M$ be the unstable dynamic branched surface in the mapping torus of the taffy pulling map on the four-holed sphere, as described in section 1.6 and figure 1.6. The reader may easily check that the set of arcs

$$
\mathcal{I}=\{\overline{a b}, \overline{b c}, \overline{c f}, \overline{b d}, \overline{d e}, \overline{e h}, \overline{f g}, \overline{g h}, \overline{f a}, \overline{h a}\}
$$

is a Markov section for $B$. Applying the proof of proposition 2.6 .2 we obtain a dynamic train track $\tau \subset B$. It is easy to check directly that $\tau$ fills $B$.

The branched surface $B$ does not carry a torus or Klein bottle; indeed $B$ carries no closed surface at all. To check this, apply the following lemma, whose proof is easily extracted from the proof of statement 4 of proposition 2.5.1.

Lemma 2.7.1. Let $\tau$ be a dynamic train track filling an unstable dynamic branched surface B. Suppose that $\tau$ has no circular sinks. Then $B$ carries no closed surface.

In the taffy pulling example, it is easy to check that the train track constructed in proposition 2.6.2 has no circular sinks, and so $B$ carries no closed surface. Proposition 2.6.2 now applies, and so $B$ may be split to form $B^{u}$, and $B^{s}$ may be constructed, so that ( $B^{s}, B^{u}$ ) is a dynamic pair.

Indeed, if one traces through the proof of proposition 2.6 .2 with this example, it is easily seen that the dynamic train track $\tau$ already satisfies all the needed properties so that there is a dynamic pair ( $B^{s}, B^{u}$ ) with $B=B^{u}$ and $\tau=B^{s} \cap B$; none of the alterations needed for the general proof are necessary for this example.

For a general pseudo-Anosov map $f$, the construction of an invariant train track and Markov partition given by [BH95] can be combined with the methods of 1.6 to produce a Markov unstable dynamic branched surface $B$ in the mapping torus $M_{f}$, whose completed complementary components are cusped torus pieces with a circular flow. It is not hard to show that $\tau$ is strongly connected, using the fact that $f$ is transitive, and so $\tau$ has no circular sinks and $B$ carries no closed surface. It is easily checked that $B$ has no annulus
or Möbius band sectors, and no periodic trajectory can lie in a bigon sector. Proposition 2.6.2 therefore applies, yielding a construction for a dynamic pair in $M_{f}$.

We remark that the above example does not satisfy "torus piece disjointness"-the union of torus pieces of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ does not embed in $M$ under the overlay map. To see why, refer back to figure 1.6, and note that there are two dynamic torus shells, each 1pronged. One torus shell arises from the outer boundary component of the disc. The other torus shell arises from the three inner boundary components, which are cyclically permuted by $f$. We claim that this "inner" torus shell does not embed in $M$. To see why, it suffices to identify the corner circles of this torus shell, as periodic cycles in $\tau$, and to show that one of these circles is not embedded. One can check that all the entries in the transition matrix $\mu$ are zeroes and ones, and so a periodic cycle of $\tau$ is determined by a periodic sequence $I_{n_{0}}, I_{n_{1}}, \ldots, I_{n_{K}}=I_{n_{0}}$ of elements in the Markov section such that $\mu\left(n_{k-1}, n_{k}\right)=1$ for all $k=1, \ldots, K$. The periodic cycle in $\tau$ corresponding to such a sequence is embedded in $\tau$ if and only if the sequence is $1-1$. The two periodic cycles yielding the corner orbits for the inner dynamic torus shell are $(e d, d e, c b)$ and $(e h, d b, c f)$. We write each element of the Markov section as an oriented edge, so that the reader can trace out the cycle in figure 1.6. The oriented edges $e d$ and $d e$ represent the same element of the Markov section, and so the first corner orbit is not embedded.

## Chapter 3

## Flows

In this section we consider pseudo-Anosov flows and pA flows. Because pA flows are technically easier to work with-they are more closely related to dynamic pairs and to very full laminations, and they may be analyzed by direct application of classical tools rather than by reconfiguring those tools-our main focus will be pA flows, and certain proofs involving pseudo-Anosov flows will be sketchy or even conjectural. In order to smooth the exposition we will start with the more familiar territory of pseudo-Anosov flows, and we will show how the double DA operation leads naturally to the definition of pA flows.

The reader who is interested only in essential laminations can safely skip the latter subsections $3.4-3.5$, which deal solely with pseudo-Anosov flows and constitute about half of chapter 3. Also, section 3.1 need only be skimmed to review concepts of hyperbolic dynamics and to learn about pseudohyperbolic orbits.

### 3.1 Pseudo-Anosov flows

If $\Phi$ is a flow on a Riemannian 3-manifold $M$, and if $I \subset M$ is a $\Phi$ invariant set, a stable bundle for $\Phi$ on $I$ is a $\Phi$ invariant subbundle $E^{s}$ of $T M \mid I$ such that for some $\theta>0, a>1$, if $v \in E^{s}$ then $\left|D \Phi_{t}(v)\right| \leq \theta a^{-t}|v|$ for $t>0$. An unstable bundle $E^{u}$ is similarly defined by requiring $\left|D \Phi_{t}(v)\right| \leq \theta a^{t}|v|$ for $t<0$. We say that the exponential expansion rate is at least $a$.

Recall that a smooth flow $\Phi$ on a closed 3-manifold $M$ is Anosov if for some (and hence any) Riemannian metric on $M$ there is a continuous $\Phi$-invariant splitting of the tangent bundle $T M=T \Phi \oplus E^{s} \oplus E^{u}$ into 1 -dimensional subbundles, such that $T \Phi$ is tangent to $\Phi$, $E^{s}$ is a stable bundle for $\Phi$, and $E^{u}$ is an unstable bundle. An important feature of an Anosov flow is the Stable manifold theorem [HPS77], which says that the plane bundle $T \Phi \oplus E^{s}$ is integrable, defining a continuous 2-dimensional foliation $\mathcal{F}^{u}$ called the weak unstable foliation of $\Phi$. The bundle $T \Phi \oplus E^{u}$ is also integrable, defining the weak stable
foliation $\mathcal{F}^{s}$ of $\Phi$. Other important features are the existence of a Markov partition [Rat73], and symbolic dynamics [Bow73].

In [Mos92a] a topological definition of pseudo-Anosov flows is offered which avoids smoothness issues, by requiring the flow to have (singular) weak stable and unstable foliations with appropriate expansion properties. In [FM95] a smooth definition is offered, which mimics the above definition of Anosov flows. Here we present the smooth definition and an improved topological definition, and we give a conjectural explanation for how the two definitions are related.

We begin at the point of departure from the land of hyperbolic flows, with the concept of a "pseudohyperbolic" orbit.

Given $n \geq 2$, consider the quadratic differential $z^{n-2} d z^{2}$ on the complex plane $\mathbf{C}$. Away from the origin $\mathcal{O}$, there are holomorphic coordinate charts that take $z^{n-2} d z^{2}$ to $d z^{2}$; these charts are well defined up to Euclidean translation and $180^{\circ}$ rotation in the range. Pulling back horizontal lines with transverse measure $|d y|$ under such a chart, we obtain the horizontal singular foliation $f^{u}$ with transverse measure $\mu^{u}$. Pulling back vertical lines and $|d x|$, we obtain the vertical singular foliation $f^{s}$ with transverse measure $\mu^{s}$. In polar coordinates, for each $k=0, \ldots, n-1$ there is a horizontal leaf $\theta=2 k \pi / n$ and a vertical leaf $\theta=(2 k+1) \pi / n$; these are called prongs. The singularities of $f^{s}, f^{u}$ at $\mathcal{O}$ are called $n$-pronged singularities. Away from $\mathcal{O}$ the foliations are regular, and transverse to each other. The Euclidean metric $d x^{2}+d y^{2}$ pulls back to a well-defined Riemannian metric $\mu_{u}^{2}+\mu_{s}^{2}$ on $\mathbf{C}-\mathcal{O}$. The topological metric defined by this formula may be completed to a metric on $\mathbf{C}$ denoted $d_{n}$.

Given $\lambda_{s}, \lambda_{u}>0$, let $\psi_{n}: \mathbf{C} \rightarrow \mathbf{C}$ be the unique map which respects $f^{s}$ and $f^{u}$, preserves each prong of $f^{s}$ and $f^{u}$, compresses leaves of $f^{s}$ by the factor $\lambda_{s}$, and stretches leaves of $f^{u}$ by the factor $\lambda_{u}$. Let $R_{\theta}: \mathbf{C} \rightarrow \mathbf{C}$ be rotation about $\mathcal{O}$ through angle $\theta$, that is $R_{\theta}(z)=e^{2 \pi \theta} z$. If $0 \leq k<n$ the map $R_{k / n}$ commutes with $\psi_{n}$ and respects $f^{s}, f^{u}$, inducing a cyclic permutation of the prongs. The map $\psi_{n k}=R_{k / n} \circ \psi_{n}$ defines the local model for a pseudohyperbolic fixed point with stretching $\lambda_{u}$, compression $\lambda_{s}, n$ prongs, and rotation $k$. We also say that $\psi_{n k}$ has type $(n, k)$. Note that

$$
\psi_{n k *}\left(\mu_{u}^{2}+\mu_{s}^{2}\right)=\lambda_{s}^{-2} \mu_{s}^{2}+\lambda_{u}^{2} \mu_{u}^{2}
$$

Now take the suspension flow of $\psi_{n k}$, a flow $\Psi_{n k}$ defined on the mapping torus $N_{n k}=$ $\mathbf{C} \times \mathbf{R} /(z, r+1) \sim\left(\psi_{n k}(z), r\right)$, where $\Psi_{n k}$ is induced by the flow $(z, s) \cdot t=(z, s+t)$ on $\mathbf{C} \times \mathbf{R}$. The suspension of the origin defines a periodic orbit $\gamma_{n k} \subset N_{n k}$, and we say that ( $N_{n k}, \gamma_{n k}$ ) is the local model for a pseudohyperbolic periodic orbit of a flow, with compression $\lambda_{s}$, expansion $\lambda_{u}$, and type ( $n, k$ ). The Riemannian metric

$$
\lambda_{u}^{-2 t} \mu_{u}^{2}+\lambda_{s}^{2 t} \mu_{s}^{2}+d t^{2}
$$

on $(\mathbf{C}-\mathcal{O}) \times \mathbf{R}$ is preserved by the covering transformation $(z, r+1) \mapsto\left(\psi_{n k}(z), r\right)$, and so it descends to a Riemannian metric on $N_{n k}-\gamma_{n k}$, which completes to a geodesic metric
on $N_{n k}$ denoted $d s_{n k}$; note that although all definitions depend on the compression $\lambda_{s}$ and the expansion $\lambda_{u}$, these numbers are suppressed in the notation. Note that the suspension of the foliations $f^{u}, f^{s}$ define 2-dimensional foliations on $N_{n k}$, singular along $\gamma_{n k}$, called the local weak stable and unstable foliations of $\Psi_{n k}$.

Let $\Phi$ be a flow without stationary points on a closed, oriented smooth 3-manifold $M$. We say that $\Phi$ is a smooth pseudo-Anosov flow if there exists a geodesic metric $d_{M}$ on $M$ such that the following are satisfied.

- There is a finite set $\Gamma$ of periodic orbits, called singular orbits, such that when restricted to $M-\bigcup \Gamma$, the flow $\Phi$ is smooth and $d_{M}$ is a smooth Riemannian metric.
- Each $\gamma \in \Gamma$ is pseudohyperbolic, defined as follows. For some $\lambda_{s}, \lambda_{u}>1, n \geq 3$, and $k=0, \ldots, n-1$, there exists a neighborhood $U$ of $\gamma$, and an embedding $f: U \hookrightarrow N_{n k}$ taking $\gamma$ to $\gamma_{n k}$, such that:
$-f$ respects orbits.
- $f$ is smooth on $U-\gamma$.
- $f$ is bilipschitz with respect to the metrics $d_{M}$ and $d s_{n k}$.
- On $M-\Gamma$, there is a continuous splitting of the tangent bundle into three 1-dimensional $\Phi$-invariant line bundles $T \Phi \oplus E^{s} \oplus E^{u}$, such that $T \Phi$ is tangent to trajectories of $\Phi$, $E^{s}$ is a stable bundle and $E^{u}$ is an unstable bundle for $\Phi$, with respect to the metric $d_{M}$.
- Near a pseudohyperbolic orbit $\gamma$, the bundles $E^{s}, E^{u}$ are tangent to the local weak stable and unstable foliations near $\gamma$, respectively.

Remark. Given a pseudohyperbolic orbit $\gamma$, the condition that the local conjugacy $f$ be bilipschitz implies that on the complement of $\gamma$ the norm of $D f$ is bounded, where the norm is computed with respect to the metrics $d_{M}$ and $d_{n k}$. Note that the local conjugacy need not respect the parameterization of the flow; but it is easily checked that the parameter ratio is a smooth function bounded away from zero and infinity.

Remark. Given a pseudohyperbolic periodic orbit $\gamma$ of type ( $n, k$ ), compression $\lambda_{s}$ and expansion $\lambda_{u}$, the numbers $n, k, \lambda_{s}, \lambda_{u}$ are invariants of $\gamma$ under local bilipschitz conjugacy. Under topological conjugacy the numbers $\lambda_{s}, \lambda_{u}$ are no longer invariant, but $n, k$ still are.
Remark. The final condition in the definition may be unnecessary. It seems likely that in the presence of the preceding conditions, each local conjugacy $f$ can be replaced by one which also satisfies the final condition.

Next we turn to the definition of a topological pseudo-Anosov flow. Roughly speaking, $\Phi$ is a topological pseudo-Anosov flow if $\Phi$ has weak stable and unstable foliations, singular
along a collection of pseudohyperbolic orbits, and $\Phi$ has a Markov partition which is expansive in a certain sense. This definition is tailored to serve two purposes. First, it reflects many of the essential dynamic features of a smooth pseudo-Anosov flow, in particular the stable manifold theory and the existence of Markov partitions. Second, it is easy to verify in specific cases, as we shall see in section 3.4.

Let $M$ be a compact, oriented 3 -manifold and $\Phi$ a flow on $M$ without stationary points. We say that $\Phi$ is a topological pseudo-Anosov flow if there exists a finite collection of periodic orbits $\Gamma$, a pair of 2 -dimensional $\Phi$-invariant singular foliations $W^{s}, W^{u}$ called the weak stable and unstable foliations, and a finite set $\mathcal{M}$ called a Markov partition for $\Phi$, satisfying the following conditions:

1. Each $\gamma \in \Gamma$ is pseudo-hyperbolic: for some $\lambda_{s}, \lambda_{u}>1, n \geq 3$, and $k$ with $0 \leq k<n$, there exists a neighborhood $U$ of $\gamma$, and an embedding $f: U \hookrightarrow N_{n k}$ taking $\gamma$ to $\gamma_{n k}$, such that $f$ respects orbits.
2. The foliations $W^{s}, W^{u}$ are regular and transverse away from the pseudohyperbolic orbits, and they agree with the local weak stable and unstable foliations near the pseudohyperbolic orbits.
3. There exists a metric $d$ on $M$ and constants $C>0, \theta>0, a>1$ with the following properties. For any two points $x, y$ in the same leaf of $W^{s}$, if $d(x, y) \leq C$ then there is a proper, monotonic increasing function $s:[0, \infty) \rightarrow \mathbf{R}$ such that $d(x \cdot t, y \cdot s(t)) \leq \theta a^{-t}$ for all $t \geq 0$. A similar condition holds for two points in the same leaf of $W^{u}$.

Before continuing with the definition, let $I^{s}, I^{u}$ be homeomorphic copies of $[0,1]$, and define a flow box to be an embedding $H: I^{u} \times I^{s} \times[0,1] \rightarrow M$ such that $\operatorname{Bottom}(H)=I^{u} \times I^{s} \times 0$ and $\operatorname{Top}(H)=I^{u} \times I^{s} \times 1$ are transverse to $\Phi$, the set $I^{u} \times t \times[0,1]$ is contained in a leaf of $W^{u}$ for each $t \in I^{s}$, the set $t \times I^{s} \times[0,1]$ is contained in a leaf of $W^{s}$ for each $t \in I^{u}$, and the $[0,1]$-orientation on each segment $t \times t^{\prime} \times[0,1]$ agrees with the direction of the flow $\Phi$. An s-subrectangle of $\operatorname{Bottom}(H)$ is a rectangle of the form $J \times I^{s} \times 0$ where $J$ is a subinterval of $I^{u}$; s-subrectangles of $\operatorname{Top}(H)$, and $\mathbf{u}$-subrectangles of $\operatorname{Bottom}(H), \operatorname{Top}(H)$ are similarly defined.
4. $\mathcal{M}$ is a finite set of flow boxes with disjoint interiors, forming the 3-cells of a regular cell decomposition of $M$. For each $H, H^{\prime} \in \mathcal{M}$, each component of $\operatorname{Top}(H) \cap \operatorname{Bottom}\left(H^{\prime}\right)$ is both an s-subrectangle of $\operatorname{Top}(H)$ and a u-subrectangle of Bottom $\left(H^{\prime}\right)$.

To complete the definition, define the transition digraph of $\mathcal{M}$ to be the directed graph $\Gamma$ with vertex set $\mathcal{M}$ and with one edge from $H \in \mathcal{M}$ to $H^{\prime} \in \mathcal{M}$ for each component of $\operatorname{Top}(H) \cap \operatorname{Bottom}\left(H^{\prime}\right)$. A strong component of $\Gamma$ is a directed subgraph $\Gamma^{\prime} \subset \Gamma$ such that there is a directed path in $\Gamma^{\prime}$ from any vertex to any other vertex, and $\Gamma^{\prime}$ is maximal with respect to this property. We say that $\Gamma^{\prime}$ is a $\operatorname{sink}$ if every edge $E$ of $\Gamma-\Gamma^{\prime}$ adjacent to $\Gamma^{\prime}$ points towards $\Gamma^{\prime}$, and $\Gamma^{\prime}$ is a source if every adjacent edge points away from $\Gamma^{\prime}$.
5. The transition digraph of $\mathcal{M}$ has no circular sinks or sources.

There is some redundancy in this definition-as we shall see in the conjecture below, items 3 and 5 are equivalent in the presence of the other conditions.

These two definitions should be related as follows:
Conjecture. Let $M$ be a closed 3-manifold. Every smooth pseudo-Anosov flow on $M$ is a topological pseudo-Anosov flow. Conversely, every topological pseudo-Anosov flow $\Phi$ on $M$ is smoothable, i.e. there exists a smooth structure on $M$ with respect to which $\Phi$ is a smooth pseudo-Anosov flow.

Remark. This conjecture is interesting even in the Anosov case. A lot of work has gone into classifying smooth conjugacy classes of smooth Anosov diffeomorphisms and flows (see [Caw93] and the references there). But so far the existence of a smooth structure on a topological Anosov homeomorphism or flow has been overlooked. For Anosov homeomorphisms of tori, the results of Adler and Weiss can easily be tailored to prove the existence of an invariant smooth structure. In the flow case it seems somewhat harder, but Elise Cawley has told me how to do it, and her methods almost certainly adapt to pseudo-Anosov flows.

We shall suggest a proof of the above conjecture, based on known techniques of Anosov dynamical systems. In order to make this proof rigorous, work is needed to generalize these techniques to pseudo-Anosov dynamical systems.
Remark. One can formulate a higher dimensional analogue of this conjecture, for Anosov diffeomorphisms and flows. It would be very interesting if there were any nonsmoothable examples.

Sketch of a proof. Suppose $\Phi$ is a smooth pseudo-Anosov flow with singular orbits $\Gamma$ and splitting $T M=T \Phi \oplus E^{s} \oplus E^{u}$ on $M-\Gamma$. The stable manifold theory of [HPS77] can be adapted to show that $T \Phi \oplus E^{s}$ is integrable in $M-\Gamma$, yielding a foliation $W^{s}$ in $M$ which is singular along $\Gamma$, and which agrees with the local weak stable foliation near each orbit in $\Gamma$. The singular foliation $W^{u}$ is similarly obtained by integrating $T \Phi \oplus E^{u}$. The foliations $W^{s}, W^{u}$ satisfy the requirements for the weak stable and unstable foliations of a topological pseudo-Anosov flow.

In [Rat73] there is a construction of Markov partitions for transitive Anosov flows. As remarked by Shub in [Shu87] p. 145, transitivity is not needed to construct Markov partitions, as long as there is a local product structure, which follows from the existence of stable and unstable foliations. Thus, the construction of Markov partitions can be carried out for smooth pseudo-Anosov flows.

Suppose that condition 5 fails, so there is, say, a circular sink in $\Gamma$. This corresponds to a cycle of flow boxes $H_{1}, \ldots, H_{k}$ such that $\operatorname{Top}\left(H_{i}\right) \subset \operatorname{Bottom}\left(H_{i+1}\right)$ for all $i \in \mathbf{Z} / k$, which in turn yields a periodic orbit $\gamma$ of $\Phi$ intersecting this cycle of flow boxes. Let $A_{i}$ be the intersection of the unstable manifold of $\gamma$ with $H_{i}$, and let $A=\bigcup A_{i}$. It follows that $A$ is
an embedded, closed annulus in $M$, contained in a leaf of $W^{u}$, invariant under $\Phi$. But this contradicts item 3.

Conversely, suppose $\Phi$ is a topological pseudo-Anosov flow. The first proof that $\Phi$ is smoothable uses Birkhoff sections as in Fried's work [Fri83] together with orbit surgery methods of Goodman [Goo83]. The second proof uses techniques of Cawley [Caw93], as applied to flows [Caw96].

For the first proof, use the Markov partition and apply the methods of [Fri83] to obtain a Birkhoff section for $\Phi$, an embedded surface $S \subset M$ such that $\operatorname{int}(S)$ is transverse to $\Phi, \partial S$ is a union of periodic orbits of $\Phi$, and every orbit of $\Phi$ hits $S$ in bounded time. Then using $S$ as in [Fri83] one shows that the flow $\Phi$ is obtained by orbit surgery from the suspension flow of a pseudo-Anosov homeomorphism. Note that suspension pseudo-Anosov flows are obviously smooth. The methods of [Goo83] (see also [HT80]) may be applied to show that any pseudo-Anosov flow obtained by orbit surgery from a smooth pseudo-Anosov flow is also smooth.

For the second proof, we start by using "super-eigenvectors" of $\mathcal{M}$ to impose coordinates on each rectangle $\operatorname{Bottom}(H)$ for $H \in \mathcal{M}$. Choose an enumeration $\mathcal{M}=\left\{H_{1}, \ldots, H_{k}\right\}$ and let $\mu$ be the $k \times k$ transition matrix, where $\mu_{i j}$ is the number of components of $\operatorname{Top}\left(H_{i}\right) \cap$ $\operatorname{Bottom}\left(H_{j}\right)$. Thus $\mu_{i j}$ is the number of edges in $\Gamma$ from $H_{i}$ to $H_{j}$.

Lemma 3.1.1 (Super-eigenvector lemma). There exists $\lambda>1$, a positive row vector $W$, and a positive column vector $V$ such that $(W \mu)_{i}>\lambda W_{i}$ and $(\mu V)_{i}>\lambda V_{i}$, for $i=$ $1, \ldots, k$.

Proof. Without loss of generality we may assume that the flow boxes are enumerated so that the vertices of each strong component of $\Gamma$ are adjacent in the enumeration, and if there is an edge $H_{i} \rightarrow H_{j}$ with $H_{i}, H_{j}$ in different strong components then $i<j$. Thus, $\mu$ has an upper block decomposition: each strong component of $\Gamma$ corresponds to a block on the diagonal, and all entries below these blocks are zero. A circular strong component of $\Gamma$ corresponds to a diagonal block which is a permutation matrix; condition 5 guarantees that each such permutation block has some nonzero entries above it and some nonzero entries to the right of it.

We construct the column vector $V$; the construction of $W$ is similar. For each nonpermutation block $B_{n}$, the Perron-Frobenius theorem provides a positive column eigenvector for that block, with eigenvalue $\lambda_{n}>1$; put this eigenvector into the positions of $V$ corresponding to the block $B_{n}$. Choose $\lambda>1$ less than each $\lambda_{n}$. We must still define an entry $V_{i}$ corresponding to each $H_{i}$ lying in a circular strong component of $\Gamma$. By condition 5 there exists a directed path from a noncircular strong component to $H_{i}$; let $x_{i}$ be the shortest length of such a path. If $x_{i}=1$ then choose $H_{j}$ to be any vertex in a noncircular strong component such that $H_{j} \rightarrow H_{i}$ is an edge of $\Gamma$, and define $V_{i}=V_{j} /(2 \lambda)$. If $x_{i}>1$, then choose $H_{j}$ to be any vertex such that $x_{j}=x_{i}-1$ and $H_{j} \rightarrow H_{i}$ is an edge, and define
$V_{i}=V_{j} /(2 \lambda)$.
We may now impose coordinates on $\operatorname{Bottom}\left(H_{i}\right)$, making it a $V_{i} \times W_{i}$ matrix, so that for each edge $H_{i} \rightarrow H_{j}$ the first return map from $\operatorname{Bottom}\left(H_{i}\right)$ to $\operatorname{Top}\left(H_{j}\right)$ stretches the $V$-coordinate by at least $\lambda$ and compresses the $W$-coordinate by at least $\lambda$.

These coordinates determine well-defined transverse Hölder structures on $W^{s}, W^{u}$, which agree along the vertical sides of the flow boxes. Moreover, in these structures, the transverse holonomy of $W^{s}$ is exponentially contracting in the backward direction and the holonomy of $W^{u}$ is exponentially contracting in the forward direction (exponential convergence is a well-defined concept in any Hölder structure).

From [Caw93] it follows that for any topological Anosov map $f: T^{2} \rightarrow T^{2}$, given any $f$ invariant transverse Hölder structures on the stable and unstable foliations with exponential contraction properties as above, there is an $f$-invariant smooth structure consistent with these transverse Hölder structures. The same techniques work for a topological Anosov flow [Caw96]. Applying these techniques to a pseudo-Anosov flow $\Phi$, using the transverse Hölder structures on $W^{s}, W^{u}$ constructed above, we obtain a smoothing of $\Phi$.

## 3.2 pA flows

Pseudo-Anosov flows are not well adapted to torally bounded3-manifolds. On closed 3-manifolds, moreover, pseudo-Anosov flows have the disadvantage of requiring one to work with singular foliations. These disadvantages are overcome by the concept of pA flows. The definition is motivated by melding the idea of pseudohyperbolic orbits with two ideas from hyperbolic dynamics: the DA operation, and axiom A flows.

The DA operation is performed on an Anosov flow $\Phi$ by "splitting open" the unstable leaves of a finite collection $\Gamma$ of periodic orbits of $\Phi$, creating a hyperbolic attractor (see below for the definition of attractors and repellers). More precisely, for each $\gamma \in \Gamma$ one alters $\Phi$ on an isolating neighborhood $N(\gamma)$ as shown in figure 3.1a. As proved in [BW83], the result of this operation is an axiom A flow $\Phi^{*}$ having an attractor $\Lambda$ which is a 2 -dimensional lamination. Associated to each $\gamma \in \Gamma$ there is a $\Phi^{*}$-invariant ring $A_{\gamma}$ such that $\operatorname{int}\left(A_{\Gamma}\right) \cap \Lambda=$ $\emptyset, \partial A_{\gamma}$ is a union of hyperbolic periodic orbits in $\Lambda$, and the core of $A_{\gamma}$ is a repelling orbit of $\Phi^{*}$ isotopic to $\gamma$. If $\gamma$ is untwisted then $A_{\gamma}$ is an annulus, otherwise $A_{\gamma}$ is a Möbius band. The union of repelling orbits of $A_{\gamma}$, for $\gamma \in \Gamma$, is a link in $M$ isotopic to $\Gamma$.

One can also split open along stable manifolds of $\Gamma$ to get a flow with a 2-dimensional repeller and isolated attracting orbits (figure 3.1c).

There is also a "double DA" operation [Mos92a], in which one splits along the stable and unstable manifolds of $\Gamma$ simultaneously. In other words, the flow $\Phi$ is altered on $N(\gamma)$ as shown in figure 3.1b. The double DA operation on an Anosov flow produces an Axiom A flow $\Phi^{\#}$ with a 1-dimensional hyperbolic invariant set $I$ (for a sketch of the proof, see proposition 3.2 below). For each $\gamma \in \Gamma$, there is an invariant set $T_{\gamma}$ of $\Phi^{\#}$ having the


Figure 3.1: The unstable DA, double DA, and stable DA operations. Each figure shows a Poincaré section-a local cross-section for the flow-and a phase diagram on the Poincaré section, i.e. a planar flow whose orbits are invariant sets of the first return map of the 3 -dimensional flow. Note: phase lines are not generally leaves of stable or unstable foliations.
structure of a manifold-with-corners fibering over the circle with fiber a square. If $\gamma$ is untwisted this fibration is a product; if $\gamma$ is twisted, the monodromy map is a $180^{\circ}$ rotation on the square. The dynamic structure of $T_{\gamma}$ is as follows. The edges of $T_{\gamma}$ are hyperbolic orbits in $I$. The faces of $T_{\gamma}$ are labelled $\mathbf{s}$ and $\mathbf{u}$, according to whether they are tangent to the stable or unstable directions of the corner orbits. The core of each face is a periodic orbit: an attracting orbit of $\Phi^{\#}$ in each $\mathbf{u}$-face, and a repelling orbit in each s-face. At the core of $T_{\gamma}$ is a hyperbolic orbit, whose unstable manifolds go out to the attracting orbits at the s-face cores, and whose stable manifolds come from the repelling orbits at the $\mathbf{u}$-face cores. All other orbits in $T_{\gamma}$ are transient, going from a repelling orbit in backwards time to an attracting orbit in forwards time. The union of the cores of the $T_{\gamma}$ form a link in $M$ which is isotopic to $\Gamma$.

Roughly speaking, a pA flow is what you get from a pseudo-Anosov flow by doing a double DA operation on each singular orbit. Before turning to the formal definition, we review topological dynamics and Axiom A flows.

First we review the ideas of chain recurrence as developed by Conley [Con78]. Consider a semiflow $\Phi$ on a compact space $X$. An invariant set of $\Phi$ is a subset $\mathcal{I} \subset X$ such that for all $x \in \mathcal{I}$ and $t \in \mathbf{R}$, the point $x \cdot t$ is defined and is in $\mathcal{I}$. The maximal invariant set of $\Phi$ is a compact subset of $X$.

Given $\epsilon, T>0$ and $x, y \in X$, an $\epsilon, T$-chain from $x$ to $y$ is a sequence of flow segments $x_{1} \cdot\left[0, t_{1}\right], \ldots, x_{n} \cdot\left[0, t_{n}\right]$ such that

- $x_{1}=x$
- $x_{n} \cdot t_{n}=y$
- $d\left(x_{i} \cdot t_{i}, x_{i+1}\right)<\epsilon$ for all $i=1, \ldots, n-1$.
- $t_{i}>T$ for all $i=1, \ldots, n$.

The chain recurrent set of $\Phi$ is the set $\mathcal{C}_{\Phi}$ consisting of all $x \in X$ such that for all $\epsilon, T>0$ there exists an $\epsilon, T$-chain from $x$ to $x$. Note that $\mathcal{C}_{\Phi}$ is a closed invariant set of $\Phi$.

A closed $\Phi$-invariant set $I \subset X$ is chain connected if for each $x, y \in I$ and all $\epsilon, T>0$ there exists an $\epsilon, T$-chain from $x$ to $y$. Clearly a chain connected set $I$ is a subset of $\mathcal{C}_{\boldsymbol{\Phi}}$. We say that $I$ is a chain component of $\mathcal{C}_{\Phi}$ if $I$ is a maximal chain connected set. A basic result of topological dynamics is that $\mathcal{C}_{\Phi}$ decomposes into chain components. Chain component are also called basic sets.

Let $C_{1}, C_{2}$ be basic sets. A connecting point from $C_{1}$ to $C_{2}$ is a point $y \in X$ such that for all $x_{1} \in C_{1}, x_{2} \in C_{2}$ and all $\epsilon, T>0$ there exists an $\epsilon, T$ chain from $x_{1}$ to $y$ and one from $y$ to $x_{2}$. The join of $C_{1}$ and $C_{2}$, denoted $\mathcal{J}\left(C_{1}, C_{2}\right)$, is the union of $C_{1} \cup C_{2}$ with all connecting points from $C_{1}$ to $C_{2}$. Given any collection $\mathcal{C}$ of basic sets, define the join $\mathcal{J}(\mathcal{C})=\bigcup\left\{\mathcal{J}\left(C_{1}, C_{2}\right) \mid C_{1}, C_{2} \in \mathcal{C}\right\}$. Note that $\mathcal{J}(\mathcal{C})$ is a $\Phi$-invariant set, and if $\bigcup \mathcal{C}$ is compact then $\mathcal{J}(\mathcal{C})$ is compact.

Define a directed graph $\Gamma_{\Phi}$, called the Lyaponov graph, whose vertices are the basic sets, and with an edge $C_{1} \rightarrow C_{2}$ defined between basic sets $C_{1} \neq C_{2}$ if there exists a connecting point from $C_{1}$ to $C_{2}$. The Lyaponov graph is:

- Transitive: If $C_{1} \rightarrow C_{2}$ and $C_{2} \rightarrow C_{3}$ are directed edges then so is $C_{1} \rightarrow C_{3}$.
- Acyclic: There there are no cycles $C_{1} \rightarrow C_{2} \rightarrow \cdots \rightarrow C_{n} \rightarrow C_{1}$.

Given a subgraph $\Gamma^{\prime} \subset \Gamma_{\Phi}$, let $\mathcal{J}\left(\Gamma^{\prime}\right)$ be the join of the basic sets forming the vertices of $\Gamma^{\prime}$ 。

A subgraph $\Gamma^{\prime} \subset \Gamma_{\Phi}$ is complete if for any two adjacent edges $C_{1} \rightarrow C_{2} \rightarrow C_{3}$ in $\Gamma_{\Phi}$ such that $C_{1}, C_{3} \in \Gamma^{\prime}$, the edges $C_{1} \rightarrow C_{2}$ and $C_{2} \rightarrow C_{3}$ are also in $\Gamma^{\prime}$. Each subgraph $\Gamma^{\prime} \subset \Gamma_{\Phi}$ is contained a unique, minimal complete subgraph called the completion of $\Gamma^{\prime}$, and the join of $\Gamma^{\prime}$ is equal to the join of its completion. If $\Gamma^{\prime}$ is a complete subgraph of $\Gamma_{\Phi}$, then the Lyaponov graph of $\Phi \mid \mathcal{J}\left(\Gamma^{\prime}\right)$ is naturally isomorphic to $\Gamma^{\prime}$.

Now let $\Phi$ be a flow without stationary points on a compact manifold $M$. An isolated invariant set of $\Phi$ is a closed $\Phi$-invariant set $I$ such that for some neighborhood $U$ of $I$ in $M$, the set $I$ is the largest $\Phi$-invariant set contained in $U$, that is $I=\bigcap_{t \in \mathbf{R}} U \cdot t=\{x \in$ $M \mid x \cdot \mathbf{R} \subset U\}$. The neighborhood $U$ is called an isolating neighborhood of $I$. As special cases, we say that $I$ is an attractor if $I=\bigcap_{t \geq 0} U \cdot t$, and $I$ is a repeller if $I=\bigcap_{t \leq 0} U \cdot t$.

Suppose that $U$ is an isolating neighborhood of an isolated invariant set $I$, and $U$ is a codimension- 0 compact submanifold of $M$ whose boundary decomposes as $\partial B=\mathcal{R}_{-} B \cup$ $\mathcal{R}_{+} B$ where $\mathcal{R}_{-} B, \mathcal{R}_{+} B$ have disjoint interiors, $\sigma B=\partial \mathcal{R}_{-} B=\partial \mathcal{R}_{+} B$, the flow exits $B$ along $\operatorname{int}\left(\mathcal{R}_{+} B\right)$, enters along $\operatorname{int}\left(\mathcal{R}_{-} B\right)$, and is externally tangent along $\sigma B$. Then $U$ is called an isolating block for $I$. If $M$ is a 3 -manifold, an isolating block has the natural structure of a sutured manifold in the cusp model. Conley and Easton proved that if $\Phi$ is smooth on some neighborhood of $I$ then $I$ has an isolating block [CE71]. In proposition 3.3 .3 we shall review Bob Williams' explicit construction of isolating blocks for 1 -dimensional hyperbolic invariant sets in dimension 3 [BW83].

Given an isolated invariant set $I \subset M$ such that $\Phi$ is smooth on some isolating neighborhood $U$ of $I$, a hyperbolic splitting of index $k$ on $I$ is a splitting of the bundle $T M \mid I$ into continuous, $\Phi$-invariant sub-bundles $T \Phi \oplus E^{s} \oplus E^{u}$, where $T \Phi$ is the 1 -dimensional tangent bundle for $\Phi, E^{s}$ is a $k$-dimensional stable bundle for $\Phi$, and $E^{u}$ is an $m$ - $k$-1-dimensional unstable bundle, with respect to some Riemannian metric on $M$. If there is a hyperbolic splitting on $I$ then we say that $I$ is a hyperbolic invariant set.

Suppose that $I$ is an isolated hyperbolic invariant set. Choose an isolating block $U$ for $I$. Define the local weak stable lamination of $I$ with respect to $U$, denoted $W_{\text {loc }}^{s}$, to be the set of all $x \in U$ such that $x \cdot t$ is defined for all $t \geq 0$; since $U$ is an isolating block for $I$ it is evident that $x \cdot t$ accumulates in $I$ as $t \rightarrow \infty$. Define the local weak unstable lamination $W_{\text {loc }}^{u}$ similarly. The Stable manifold theorem of Pugh and Shub [HPS77] says that that these are, in fact, laminations: if $I$ has index $k$ then $W_{\text {loc }}^{s}$ is a $k+1$-dimensional lamination, and $W_{\text {loc }}^{u}$ is an $m-k$-dimensional lamination. The laminations $W_{\text {loc }}^{s}, W_{\text {loc }}^{u}$ are transverse, and their intersection is the 1 -dimensional lamination $I$. Given a subset $X \in I$, the set of leaves of $W_{\text {loc }}^{s}$ intersecting $X$ is denoted $W_{\text {loc }}^{s}(X)$.

Note that $W_{\text {loc }}^{s}, W_{\text {loc }}^{u}$ are independent of the choice of isolating block $U$ in the following sense. If $U^{\prime}$ is another isolating block, and if $W_{\text {loc }}^{\prime s}, W_{\text {loc }}^{\prime \prime}$ are the weak stable and unstable laminations of $I$ with respect to $U^{\prime}$, then there are neigborhoods $V \subset U$ of $W_{\text {loc }}^{s} \cup W_{\text {loc }}^{u}$ and $V^{\prime} \subset U^{\prime}$ of $W_{\text {loc }}^{\prime s} \cup W_{\text {loc }}^{\prime u}$, and a diffeomorphism $V \rightarrow V^{\prime}$ that takes each flow line to itself, and takes $W_{\text {loc }}^{s}$ to $W_{\text {loc }}^{\prime s}$ and $W_{\text {loc }}^{u}$ to $W_{\text {loc }}^{\prime \prime}$ preserving the lamination structure. The proof is easy if $U^{\prime} \subset U$ : take the diffeomorphism obtained by flowing $U^{\prime}$ forward until $\mathcal{R}_{+} U^{\prime}$ hits $\mathcal{R}_{+} U$. The proof is also easy if $U^{\prime}$ is a slight perturbation of $U$. In general, use the fact that after replacing $U^{\prime}$ be a slight perturbation, the intersection $U \cap U^{\prime}$ is an isolating block of $I$.

Suppose that $I$ is an index 1 hyperbolic invariant set in an oriented 3-manifold $M$. Consider a periodic orbit $\gamma$ in $I$. Formally, we may regard $\gamma$ as an immersion of an oriented
circle into $M$, mapping as a finite covering space over some embedded oriented circle called an embedded periodic orbit. Note that $\gamma$ preserves orientation in the surface $W_{\text {loc }}^{s}(\gamma)$ if and only if it preserves orientation in the surface $W_{\text {loc }}^{u}(\gamma)$, in which case we say that $\gamma$ is untwisted; otherwise, $\gamma$ is twisted. If $\gamma$ is an embedded untwisted orbit then $W_{\mathrm{loc}}^{s}(\gamma)$ and $W_{\text {loc }}^{u}(\gamma)$ are both annuli; if $\gamma$ is embedded and twisted then these surfaces are Möbius bands. If $\gamma: S^{1} \rightarrow I$ is a twisted periodic orbit, and if $\gamma^{\prime}: S^{1} \rightarrow I$ is the $k$-fold cover of $\gamma$-i.e. $\gamma^{\prime}$ factors as $S^{1} \rightarrow S^{1} \xrightarrow{\gamma} I$ where the first map is a $k$-fold covering map-then $\gamma^{\prime}$ is twisted if and only if $k$ is odd.

A "stable boundary periodic orbit" of $I$ is a periodic orbit $\gamma$ which may be moved off of $I$ by homotoping $\gamma$ into $W_{\text {loc }}^{s}(\gamma)$. To be more explicit we must specify the direction in which $\gamma$ may be homotoped. Let $\gamma \subset I$ be either an untwisted embedded periodic orbit, or the double cover of a twisted embedded periodic orbit; in either case, $\gamma$ is itself untwisted. Let $\tau$ be a transverse orientation of $\gamma$ in $W_{\text {loc }}^{s}(\gamma)$; if $\gamma$ is an embedded twisted orbit then there are two choices for $\tau$; whereas if $\gamma$ is the double cover of a twisted embedded orbit then there is an essentially unique choice of $\tau$. We say that the pair $(\gamma, \tau)$ is a stable boundary periodic orbit of $I$ if the component of $W_{\text {loc }}^{s}(\gamma)-\gamma$ into which $\tau$ points is disjoint from $I$. Note that if $\gamma$ is embedded and untwisted, then it may be a boundary periodic orbit with respect to either, neither, or both of its transverse orientations in $W_{\text {loc }}^{s}(\gamma)$. Unstable boundary periodic orbits of $I$ are similarly defined.

Here are some observations and terminology which describe various types of chain connected, isolated hyperbolic invariant sets on a 3 -manifold $M$. If a chain connected hyperbolic invariant set has index 2 then is an attracting periodic orbit, and if the index is 0 then it is a repelling periodic orbit.

The term "strange attractor" is used in the literature to refer to a hyperbolic invariant set of index 1 which is a boundaryless 2 -dimensional lamination $\Lambda$ tangent to $E^{u}$, that is $\Lambda=W_{\text {loc }}^{u}(\Lambda)$. The transversals are totally disconnected, and if $\Lambda$ is chain connected then the transversals are Cantor sets (which is regarded as strange, I guess). One could also talk about a "strange repeller", though I have never heard the term.

Another common example is a 1 -dimensional hyperbolic invariant set of index 1 ; this is always a 1 -dimensional lamination tangent to $T \Phi$. A local transversal of this lamination is totally disconnected (with a "local product structure" as described below). The local weak stable and unstable laminations are 2 -dimensional, with totally disconnected transversals. If $I$ is chain connected, and if $I$ is not a periodic orbit, then a local transversal is a Cantor set, with a local product structure of the form (Cantor set) $\times$ (Cantor set).

A flow $\Phi$ without stationary points on a smooth, closed manifold $M$ is axiom $A$ if its chain recurrent set is hyperbolic. More generally, an isolated $\Phi$-invariant set $I$ is axiom A if the chain recurrent set of $\Phi \mid I$ is hyperbolic. The main result in the theory of axiom A flows is the Spectral decomposition theorem [Sma67], which says that if $I$ is an axiom A invariant set (e.g. the chain recurrent set of an axiom A flow) then the chain recurrent set of $I$ has finitely many chain components. If $M$ is 3 -dimensional, these all fall into
one of several types: attracting periodic orbits (index 2); repelling periodic orbits (index 0 ); 2-dimensional hyperbolic attractors or "strange attractors"; 2-dimensional hyperbolic repellers; and 1 -dimensional hyperbolic invariant sets. The latter three types are all of index 1.

Now we turn to the new concepts needed to define pA flows. Roughly speaking, a pA flow is like an axiom A flow, in that the chain recurrent set has finitely many chain components-some of the chain components are axiom A, and the remainder are pseudohyperbolic orbits. We put very strong restrictions on the pseudohyperbolic orbits, imprisoning them inside special invariant sets which inhibit their interaction with the other chain components. These invariant sets are the ones which arise in the double DA operation on a pseudo-Anosov flow. We put similarly strong restrictions on the components of $\partial M$.

To imprison a pseudohyperbolic orbit we use a " pA solid torus", which is what you get by doing a double DA operation on a pseudohyperbolic orbit (figure 3.2a). To be precise, fix integers $n \geq 1,0 \leq k<n$. Let $G$ be a regular $2 n$-sided polygon in $\mathbf{C}$ centered on $\mathcal{O}$. Label the sides of $G$ alternately $\mathbf{s}$ and $\mathbf{u}$. Let $\zeta$ be a homeomorphism of $\mathbf{C}$ that preserves $G$, commutes with the symmetry group of $G$, and has the following properties. Each corner of $G$ is a hyperbolic fixed point of $\zeta$ with stable direction tangent to the adjacent s -side of $G$ and unstable direction tangent to the adjacent $\mathbf{u}$-side. Each s-side of $G$ contains a repelling fixed point of $\zeta$, each $\mathbf{u}$-side of $G$ contains an attracting fixed point, and there are no other fixed points on $\partial G$. The origin $\mathcal{O}$ is an $n$-pronged pseudohyperbolic fixed point of $\zeta$, whose unstable manifolds go out to the attractors on $\partial G$ and whose stable manifolds come from the repellers on $\partial G$. All other orbits in $G$ go from a repelling fixed point on $\partial G$ in backwards time to an attracting fixed point on $\partial G$ in forwards time. The map $\zeta$ is smooth except at $\mathcal{O}$. Now take the suspension flow of the map $R_{k / n} \circ \zeta$, a flow defined on the mapping torus of $R_{k / n} \circ \zeta$. The suspension of $G$ is an isolated invariant set for this flow, the local model for a $p A$ solid torus of type $(n, k)$.

To imprison a component of $\partial M$ we use a "pA torus shell" (figure 3.2 b ). To define it, let $G$ be as above, and choose a small, round Euclidean disc $D \subset G$ centered on the origin. Let $c=\partial D$. Let $\hat{\zeta}$ be obtained by altering $\zeta$ near $D$ so that $c$ is preserved, with $2 n$ hyperbolic fixed points on $c$. The map $\hat{\zeta} \mid c$ therefore has $n$ attracting and $n$ repelling fixed points alternating around $c$. In the annulus $A$ bounded by $c$ and $\partial G$, each attractor of $\hat{\zeta} \mid c$ has an unstable manifold going out to an attractor on $\partial G$; similarly for repellers of $\hat{\zeta} \mid c$. All other orbits in $c$ go between two fixed points on $c$, and all other orbits in $A-c$ go between two fixed points on $\partial G$. The annulus $G-\operatorname{int}(D)$ is a closed invariant set of $\hat{\zeta}$. Now throw away $\operatorname{int}(D)$, so $\hat{\zeta}$ is only defined on $\mathbf{C}-\operatorname{int}(D)$. Take the suspension flow of the map $\hat{\zeta}$; the suspension of $G-\operatorname{int}(D)$ is the local model for a pA torus shell of type $n$. The suspensions of the attracting fixed points of $\hat{\zeta} \mid c$ are called boundary attractors of the suspension flow; boundary repellers are similarly defined. Note that a boundary attractor of the suspension flow is not an attractor of the suspension flow on the whole mapping torus; indeed it has


Figure 3.2: A pA solid torus and a pA torus shell are obtained by suspending $R_{k / n} \circ \zeta$, where $\zeta$ is depicted here for $n=3$.
one unstable manifold transverse to the boundary of the mapping torus.
pA solid tori and torus shells are called, collectively, pA torus pieces.
Let $M$ be a smooth, compact 3-manifold with torus boundaries. Let $\Phi$ be a flow on $M$ with no stationary points, and let $\mathcal{C}_{\Phi}$ be the chain recurrent set of $\Phi$. We say that $\Phi$ is a pA flow if the following hold:

1. There exist finitely many pA torus pieces of $\Phi$, all pairwise disjoint.
2. $\Phi$ is smooth off of the pseudohyperbolic orbits, and hence each pseudohyperbolic orbit is contained in a pA solid torus of type $(n, k)$ for some $n \geq 2$ and $k=0, \ldots, n-1$.
3. Every component of $\partial M$ is contained in a pA torus shell of some type $n \geq 1$.
4. Every attracting and repelling orbit of $\Phi$ is contained in some pA torus piece.
5. Let $\mathcal{I}_{\Phi}$ be the union of all chain components of $\Phi$ except for the pseudohyperbolic orbits, attracting orbits, and repelling orbits. Let $\mathcal{J}_{\Phi}=\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$. Then $\mathcal{J}_{\Phi}$ is a 1-dimensional hyperbolic invariant set.
6. For each stable boundary periodic orbit $(\gamma, \tau)$ of $\mathcal{J}_{\Phi}$, there exists a pA torus piece $H$ such that $\gamma$ is a corner orbit of $H$ and $\tau$ points into the adjacent s-face of $H$. A similar statement holds for unstable boundary periodic orbits.
7. There does not exist a transverse bigon for $\mathcal{J}_{\Phi}$, i.e. a smoothly embedded disc-with-two-corners $D \subset M$ with edges $\alpha, \beta$, such that:

- $D$ is transverse to $\Phi$.
- $\alpha=D \cap W_{\text {loc }}^{s}\left(\mathcal{J}_{\Phi}\right)$.
- $\beta=D \cap W_{\text {loc }}^{u}\left(\mathcal{J}_{\Phi}\right)$.
- $\partial \alpha=\partial \beta=D \cap \mathcal{J}_{\Phi}$.

We shall use $\mathcal{A}_{\Phi}$ for the union of attracting orbits, $\mathcal{R}_{\Phi}$ for the union of repelling orbits, and $\mathcal{P}_{\Phi}$ for the union of pseudo-hyperbolic orbits and $\partial M$. We shall prove in theorem 3.3.1 that the attractor $\Lambda_{\Phi}^{u}=\mathcal{J}\left(\mathcal{I}_{\Phi} \cup \mathcal{A}_{\Phi}\right)$ and the repeller $\Lambda_{\Phi}^{s}=\mathcal{J}\left(\mathcal{I}_{\Phi} \cup \mathcal{R}_{\Phi}\right)$ are very full laminations in $M$, whose pared torus pieces are in 1-1 type preserving correspondence with the pA torus pieces of $\Phi$.

Remark. Note that pA torus shells may have any number $n \geq 1$ of prongs. However, we must not allow 1-pronged pA solid tori in the definition, for otherwise the laminations $\Lambda_{\Phi}^{s}, \Lambda_{\Phi}^{u}$ produced in theorem 3.3.1 are not essential.

Remark. If we restrict to the class of pA flows for which all pA solid tori have at least 3 prongs, then there is a natural 1-1 correspondence between (restricted) pA flows and pseudo-Anosov flows on closed, oriented 3 -manifolds, up to isotopy and reparameterization; this correspondence is induced by the double DA operation (see the "proposition" below). We leave the proof to the interested reader. Conditions $4,6,7$, which impose strict conditions on attracting orbits, repelling orbits, boundary periodic orbits, and transverse bigons, are all needed in order for this remark to be true.

Remark. Conditions 4, 6, 7 are also needed to enforce a tight connection between pA flows, dynamic pairs, and essential laminations (theorems 3.3.2 and 3.3.1).
Remark. In condition 5 note that the chain components of $\mathcal{I}_{\Phi}$ form the vertex set of a complete subgraph of the Lyaponov graph $\Gamma_{\Phi}$, and hence $\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$ is defined. The condition that $\mathcal{J}=\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$ be hyperbolic is equivalent to $\mathcal{I}_{\Phi}$ being hyperbolic plus the "transversality condition", which says that for any edge $C_{1} \rightarrow C_{2}$ in the Lyaponov graph where $C_{1}, C_{2} \subset \mathcal{I}$, the weak unstable lamination of $C_{1}$ is transverse to the weak stable lamination of $C_{2}$ at any point where these laminations intersect.

Remark. If $M$ is atoroidal then $\mathcal{I}_{\Phi}$ is chain connected; this is proved by an easy adaptation of the theorem proved in [Mos92a] which says that pseudo-Anosov flows on atoroidal 3 -manifolds are transitive. But if $M$ is not atoroidal then $\mathcal{I}_{\Phi}$ need not be chain connected. For example, starting from an intransitive Anosov flow as constructed e.g. in [FW80], the double DA construction on any collection of periodic orbits produces a pA flow such that $\mathcal{I}_{\Phi}$ is not chain connected.

The following shows how pA flows arise from pseudo-Anosov flows via the double DA operation. Again, this proposition should be regarded more as a conjecture, needing more details to be made rigorous.
"Proposition". Let $\Phi$ be a smooth, transitive pseudo-Anosov flow on a closed, oriented 3-manifold $M$. Let $\Gamma$ be a collection of periodic orbits, including all pseudohyperbolic orbits. Choose pairwise disjoint isolating neighborhoods $N_{\gamma}$ of small radius, for each $\gamma \in \Gamma$. Let $\Phi^{\#}$ be a flow obtained from the double DA operation on $\Phi$, i.e. $\Phi^{\#}$ is obtained by perturbing $\Phi \mid N_{\gamma}$ so that the maximal invariant set $T_{\gamma}$ in $N_{\gamma}$ is a $p A$ torus piece. Then $\Phi^{\#}$ is a $p A$ flow.

Sketch of proof. For each $\gamma \in \Gamma$ there exists a torus $t_{\gamma} \subset N_{\gamma}$ such that the pseudohyperbolic orbit, attracting orbits, and repelling orbits of $T_{\gamma}$ lie inside $t_{\gamma}$, the corner orbits of $T_{\gamma}$ lie outside $t_{\gamma}$, and if $P$ is the connected submanifold of $M$ on the outside of all the tori $t_{\gamma}$ then $P$ is an isolating block for $\Phi^{\#}$; the torus $t_{\gamma}$ is easily sketched in figure 3.2a. In $P-\bigcup_{\gamma} N(\gamma)$ the flow has a hyperbolic splitting, by definition of pseudo-Anosov. Also, in the isolating block $P \cap \bigcup_{\gamma} N(\gamma)$ the flow has a hyperbolic splitting, because the maximal invariant set in this isolating block is the union of corner orbits of pA torus pieces, and these are all hyperbolic. By applying the methods of [HT80], one can use these two hyperbolic splittings to produce a hyperbolic splitting along the maximal invariant set $\mathcal{J}=\mathcal{J}_{\Phi} \#$ in $P$. By using shadowing arguments one can construct a semiconjugacy from $\mathcal{J}$ to $\Phi$, homotopic to the inclusion map $\mathcal{J} \hookrightarrow M$.

The fact that $\mathcal{J}$ is 1 -dimensional follows from transitivity of $\Phi$, using the fact that the singular leaves along which the flow is split are dense in $M$. If $\mathcal{J}$ had a transverse bigon $D$, the semiconjugacy would map the corners of $D$ to an orbit of $\Phi$ along which $W^{s}, W^{u}$ are tangent, an absurdity. The remaining details needed to prove $\Phi^{\#}$ is a pA flow are left to the reader.

Remark. If $\Phi$ is not transitive, then the same proof will work by splitting along a larger set of periodic orbits including the boundary periodic orbits of all chain components of $\Phi$.

## 3.3 pA flows, dynamic pairs, and very full laminations

In this section we define what it means for a pA flow to be carried by a dynamic pair. We also prove theorems which give the relation between pA flows, dynamic pairs, and very full laminations.

The following definition describes the appearance of a regular neighborhood $N=N(\tau)$ of the intersection train track $\tau$ of a dynamic pair $B^{s}, B^{u}$.

A template pair in a sutured manifold $N$, also called a template pair with support $N$, consists of a pair of branched surfaces $B_{N}^{s}, B_{N}^{u} \subset N$, with $\partial B_{N}^{s} \subset \mathcal{R}_{-} N$ and $\partial B_{N}^{u} \subset \mathcal{R}_{+} N$, together with a $C^{0}$ vector field $V$ on $N$, where the quadruple ( $N, B_{N}^{s}, B_{N}^{u}, V$ ) is built by gluing together three different types of pieces: the transitional piece with model $P_{t}$, the diverging piece with model $P_{d}$, and the converging piece with model $P_{c}$. Ignoring for the moment the branched surfaces and the vector field, these models are given as follows (see


Figure 3.3: A bird's eye view, looking down the $z$-direction. Gluing rectangles are drawn with thin lines. Arcs which will be in $\sigma N$ after gluing are drawn with thick lines. The $\mathbf{u}$ and $x$-directions are identified, as are the $\mathbf{s}$ and $y$-directions.
figure 3.3). Pick $\epsilon \in(0,1)$. The transitional piece is

$$
\begin{aligned}
P_{t} & =\left\{(x, y, z) \in \mathbf{R}^{3}\left|0 \leq z \leq 1,|x| \leq(1-\epsilon)^{z},|y| \leq(1-\epsilon)^{1-z}\right\}\right. \\
P_{d} & =\left\{(x, y, z) \in P_{t} \mid z \leq x^{2}-1+\epsilon\right\} \\
P_{c} & =\left\{(x, y, z) \in P_{t} \mid z \geq-y^{2}+1-\epsilon\right\}
\end{aligned}
$$

In words, $P_{d}$ and $P_{c}$ are obtained from $P_{t}$ by gouging out parabolic troths, a troth parallel to the $y$ axis gouged out of the top of $P_{d}$, and an upside down troth parallel to the $x$ axis gouged out of the bottom of $P_{c}$. When the vector field is defined, the incoming boundary of each piece is the set of points where $\partial / \partial z$ points inward, and the outgoing boundary is where $\partial / \partial z$ points outward. Each of these pieces is a manifold with corners, and the gluing rectangles are the faces lying on either $z=0$ or $z=1$. The transitional piece has one incoming and one outgoing gluing rectangle; the diverging piece has one incoming and two outgoing gluing rectangles; and the converging piece has two incoming and one outgoing gluing rectangle. The sutured manifold $N$ is obtained from a collection of transitional pieces, diverging pieces, and converging pieces, by identifying rectangles in pairs, one incoming and one outgoing gluing rectangle in each pair.

The branched surfaces in each model are described as follows (see figure 3.4). In $P_{t}$ take $B_{t}^{s}=\left\{(0, y, z) \in P_{t}\right\}$ and $B_{t}^{u}=\left\{(x, 0, z) \in P_{t}\right\}$. In $P_{d}$, take $B_{d}^{u}=\left\{(x, 0, z) \in P_{d}\right\}$. The branched surface $B_{d}^{s}$ is described as follows. The projection of $P_{d}$ onto the $x, z$ coordinate plane is the set $Y_{d}=\left\{(x, z) \in \mathbf{R}^{2}\left|0 \leq z \leq 1, z \leq x^{2}-1+\epsilon,|x| \leq(1-\epsilon)^{z}\right\}\right.$. Take a properly embedded, oriented train track $\tau_{d} \subset Y_{d}$ such that: every tangent vector is transverse to $\partial / \partial x$; there is one incoming endpoint on the edge $Y_{d} \cap\{|z|=0\}$; there is one outgoing endpoint on each of the two edges $Y_{d} \cap\{|z|=1, x<0\}$ and $Y_{d} \cap\{|z|=1, x>0\}$; and there is one switch, a diverging switch. We now define $B_{d}^{s}=P_{d} \cap \pi_{x z}^{-1}\left(\tau_{d}\right)$ where $\pi_{x z}$ is projection onto the $x, z$ plane. Note that $B_{d}^{s} \cap B_{d}^{u}$ is the oriented train track $\left\{(x, 0, z) \in P_{d} \mid\right.$ $\left.(x, z) \in \tau_{d}\right\}$. In $P_{c}$, take $B_{c}^{s}=\left\{(0, y, z) \in P_{c}\right\}$, and take $B_{c}^{u}=P_{c} \cap \pi_{y z}^{-1}\left(\tau_{c}\right)$ where $\tau_{c}$ is an


Figure 3.4: Stable and unstable branched surfaces in a diverging piece $P_{d}$ and a converging piece $P_{c}$. In order to see the branched surfaces we have changed the viewpoint from figure 3.3; but the reader should imagine what these branched surfaces look like from a bird's eye view, in order to appreciate the difference between $\mathbf{s}$ and $\mathbf{u}$-branched surfaces.
upside down version of $\tau_{d}$, but in the $y z$ coordinate plane. When $N$ is glued together, the edges where branched surfaces intersect gluing rectangles must match up, yielding branched surfaces $B_{N}^{s} \subset N$ and $B_{N}^{u} \subset N$.

The vector fields in each model are described as follows. In $P_{t}$ take $\partial / \partial z$. In $P_{c}$ and $P_{d}$, the vector field $\partial / \partial z$ is not satisfactory because it is not tangent to the branched surfaces; instead, take vector fields which have positive $z$-component, are transverse to the interior of each face, point inward on a face if and only if $\partial / \partial z$ points inward, and are tangent to the branched surfaces, pointing backward along s-branch locus and forward along u-branch locus. When $N$ is glued together, the vector fields along the gluing rectangles must match up, yielding the vector field $V$ on $N$. This finishes the definition of a template pair.
Remark. The vector field $V$ is forward along $\Upsilon B_{N}^{u}$, backward along $\Upsilon B_{N}^{s}$, and tangent to $\tau=B_{N}^{u} \cap B_{N}^{s}$, making $\tau$ an oriented train track.

Remark. There is a deformation retraction $q: N \rightarrow \tau$, called the rectangle collapsing map, whose point inverse images are rectangle fibers. In each model the rectangle fibers are components of intersection of $P_{t}, P_{d}$, or $P_{c}$ with horizontal planes. As with $I$-collapsing maps for train track neighborhoods in surfaces, the map $q$ is not a true homotopy theoreric fibration. The union of intervals parallel to the s-direction define the s-interval fibration
of $N$, and the union of intervals parallel to the $\mathbf{u}$-direction define the $\mathbf{u}$-interval fibration. The map $q$ has two factorizations

$$
N \xrightarrow{q^{s}} B_{N}^{s} \rightarrow \tau, \quad N \xrightarrow{q^{u}} B_{N}^{u} \rightarrow \tau
$$

where $q^{s}, q^{u}$ are deformation retractions collapsing $\mathbf{s}$ and $\mathbf{u}$-intervals, respectively. The vector field $V$ is transverse to each rectangle fiber; the transverse orientation on each fiber induced by $V$ is called the positive transverse orientation.
Remark. For almost all examples, the transitional piece is not necessary in the definition: if $N$ is connected and if there is at least one diverging or converging piece, then $N$ can be resubdivided into only diverging and converging pieces. However, an untwisted or twisted round handle can only be built out of transitional pieces.
Remark. The branched surface $B_{N}^{u}$ is called a "template" in the literature (at least when $\tau$ is transitive and not a circle); see the discussion on templates in the introduction.

Consider a dynamic pair $B^{s}, B^{u}$ with $\tau=B^{s} \cap B^{u}$. We may choose $I$-bundle neighborhoods $N\left(B^{s}\right), N\left(B^{u}\right)$ in the smooth model so that $N(\tau)=N\left(B^{s}\right) \cap N\left(B^{u}\right)$ is a sutured manifold, where $\mathcal{R}_{-} N(\tau)=\partial N(\tau) \cap \partial N\left(B^{u}\right)$ and $\mathcal{R}_{+} N(\tau)=\partial N(\tau) \cap \partial N\left(B^{s}\right)$, and so that $B^{s} \cap N(\tau), B^{u} \cap N(\tau)$ form a template pair in $N(\tau)$. The transitional, diverging, and converging pieces out of which $N(\tau)$ is built may be chosen by taking a finite subset $X \subset \tau$ which divides $\tau$ into arcs, diverging switch neighborhoods, and converging switch neighborhoods, and then cutting $N(\tau)$ up along the rectangles $\left\{R_{x}=q^{-1}(x) \mid x \in X\right\}$, where $q: N(\tau) \rightarrow \tau$ is a rectangle fiber collaping map. We may choose the $I$-fibrations of $N\left(B^{s}\right), N\left(B^{u}\right)$ to be consistent with $\mathbf{s}$ and $\mathbf{u}$-interval fibrations of $N(\tau)$. Note that there is a natural, type preserving correspondence between dynamic torus pieces of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ and of $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$.

We say that a pA flow $\Phi$ is carried by a dynamic pair $B^{s}, B^{u}$ if the following hold:

- $N(\tau)$ is an isolating block for $\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$, with $\Phi$ flowing inward along $\partial_{-} N(\tau)$, outward along $\partial_{+} N(\tau)$, and externally tangent along $\sigma N(\tau)$.
- $\Phi$ is transverse to the rectangle fibers of $N(\tau)$, crossing each fiber in the positive direction.
- $N\left(B^{s}\right)$ is an isolating block for $\Lambda^{s}=\mathcal{J}\left(\mathcal{I}_{\Phi} \cup \mathcal{R}_{\Phi}\right)$, with $\Phi$ flowing outward along $\partial N\left(B^{s}\right)$.
- $N\left(B^{u}\right)$ is an isolating block for $\Lambda^{u}=\mathcal{J}\left(\mathcal{I}_{\Phi} \cup \mathcal{A}_{\Phi}\right)$, with $\Phi$ flowing inward along $\partial N\left(B^{u}\right)$.
- Inclusion induces a type preserving bijection between dynamic torus piece components of $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$ and pA torus pieces of $\Phi$.

Here are our main theorems about pA flows:
Theorem 3.3.1 ( $\mathbf{p A}$ flows yield very full laminations). If $\Phi$ is a $p A$ flow on a compact, oriented 3-manifold $M$ with torus boundaries, then the isolated invariant sets $\Lambda^{s}=$ $\mathcal{J}\left(\mathcal{I}_{\Phi} \cup \mathcal{R}_{\Phi}\right)$ and $\Lambda^{u}=\mathcal{J}\left(\mathcal{I}_{\Phi} \cup \mathcal{A}_{\Phi}\right)$ are very full laminations; these are called the stable and unstable laminations of $\Phi$. Inclusion induces natural, type preserving 1-1 correspondences between the following sets:

- pA torus pieces of $\Phi$
- Components of $\mathfrak{C}\left(M-\Lambda^{s}\right)$
- Components of $\mathfrak{C}\left(M-\Lambda^{u}\right)$

Remark. The pA torus pieces of $\Phi$ are precisely the compact components of $\mathfrak{C}\left(M-\left(\Lambda^{s} \cup\right.\right.$ $\left.\Lambda^{u}\right)$ ). What are the remaining components? They are all noncompact dynamic manifolds of the form (rectangle) $\times \mathbf{R}$, and there are infinitely many of them. Each of these may be thought of as a "homoclinic connection" between pA torus pieces. Each pinched tetrahedron component of $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$ is contained in a (rectangle) $\times \mathbf{R}$ component of $\mathfrak{C}\left(M-\left(\Lambda^{s} \cup \Lambda^{u}\right)\right)$, and each (rectangle) $) \times \mathbf{R}$ contains at most one pinched tetrahedron. There are infinitely many (rectangle) $\times \mathbf{R}$ components which are entirely contained in $N(\tau)$ and hence contain no pinched tetrahedron.

Theorem 3.3.2 (pA flows and dynamic pairs). Let $M$ be a compact, oriented 3-manifold with torus boundaries.
I. Every $p A$ flow on $M$ is carried by some dynamic pair in $M$.
II. Every dynamic pair carries some pA flow.

Moreover, if $\Phi$ is a $p A$ flow with stable and unstable laminations $\Lambda^{s}, \Lambda^{u}$, and if $\Phi$ is carried by a dynamic pair $B^{s}, B^{u}$, then there are natural, $1-1$ type preserving correspondences between the components of $\mathfrak{C}\left(M-\Lambda^{s}\right)$, the components of $\mathfrak{C}\left(M-\Lambda^{u}\right)$, and the dynamic torus piece components of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$.

Remark. The noncompact components of $\mathfrak{C}\left(P-\left(\Lambda^{s} \cup \Lambda^{u}\right)\right)$, which according to the above remark are of the form (rectangle) $\times \mathbf{R}$, may be enumerated in terms of the dynamic train track $\tau=B^{s} \cap B^{u}$ as follows: they are in $1-1$ correspondence with finite train paths of $\tau$ that start at a converging switch and end at a diverging switch.

These theorems will be proved over the next several subsections.

Remark. For application to theorem C of the introduction, statement I of theorem 3.3.2that every pA flow is carried by some dynamic pair-is irrelevant, because theorem C will be proved by first constructing a dynamic pair and then using statement II to construct a pA flow carried by that dynamic pair. Nevertheless a detailed proof of statement I is included, partly for sake of completeness, but more to aid understanding; it wasn't until I wrote out the proof of 3.3 .2 I , in particular step 4 of lemma 3.3 .5 , when I realized that transverse bigons of $\mathcal{J}$ must be ruled out.

### 3.3.1 One-dimensional hyperbolic sets yield template pairs

We start by recalling Williams' construction of template pairs:
Proposition 3.3.3. Let $\mathcal{J}$ be an isolated, 1-dimensional hyperbolic invariant set of a flow $\Phi$ on a compact, oriented 3-manifold $M$. There exists an isolating block $N$ for $\mathcal{J}$, and a template pair $B_{N}^{s}, B_{N}^{u}$ in $N$, with rectangle collapsing map $q: N \rightarrow \tau=B_{N}^{s} \cap B_{N}^{u}$, such that $\Phi \mid N$ is transverse to the rectangle fibers, crossing them in the positive direction.

When this proposition is satisfied we say that the template pair $B_{N}^{s}, B_{N}^{u}$ carries $\mathcal{J}$.
The general construction of templates for one-dimensional isolated hyperbolic invariant sets in 3-manifolds was first described by Birman and Williams [BW83]. A detailed account of this construction is given in [GHS96] § 2.2.1. Proposition 3.3.3 is a mild variation of these results. First we show how 3.3.3 follows from § 2.2.1 of [GHS96], with a few minor comments. Then we will give a fuller sketch of a proof of 3.3.3.

Deriving proposition 3.3.3 from § 2.2.1 of [GHS96]. The definition of templates given in [GHS96] does not exactly match our branched surfaces $B_{N}^{s}, B_{N}^{u}$. First, the behavior of the vector field along $\partial B_{N}^{u}$ and $\partial B_{N}^{s}$ is slightly different, but this part of the conclusion is easily massaged. Second, the construction of [GHS96] requires that $\mathcal{J}$ be chain connected, i.e. transitive, and the conclusion says that the vector field on $B_{N}^{u}$ generates a transitive forward semiflow; however the construction of $B_{N}^{s}, B_{N}^{u}$ goes through without assuming transitivity of $\mathcal{J}$, only that $\mathcal{J}$ is isolated (see next paragraph), yielding a template pair without any transitivity property.

Remark. We say that $\mathcal{J}$ is transitive if there exists a dense orbit. We say that $\tau$ is transitive if there exists a directed path from any $x \in \tau$ to any $y \in \tau$; this is also called strong connectivity. Transitivity of $\mathcal{J}$ is equivalent to transitivity of $\tau$. These conditions imply that there is a transitive forward semiflow on $B_{N}^{u}$, and a transitive backward semiflow on $B_{N}^{s}$, although these semiflows cannot be generated by a single vector field which is simultaneously tangent to $B_{N}^{s}$ and $B_{N}^{u}$.

A sketch of another proof of 3.3.3. This proof uses the same Markov partition ideas used in [GHS96].

Bowen proved in [Bow73] that every transitive hyperbolic invariant set has a "Markov family of local sections". Transitivity is, however, unnecessary for this proof-as observed by Shub ([Shu87], commentary on p. 145), all that is needed is for $\mathcal{J}$ to have a "local product structure", and this is guaranteed by the fact that $\mathcal{J}$ is an isolated hyperbolic invariant set.

Using 1-dimensionality of $\mathcal{J}$ as in [Bow72], the elements of the Markov family may be taken to be the intersections with $\mathcal{J}$ of a finite set $\mathcal{M}$ of smoothly embedded rectangles $I^{u} \times I^{s} \rightarrow M$, where $I^{s}, I^{u}$ are diffeomorphic copies of $[0,1], \partial\left(I^{u} \times I^{s}\right) \cap \mathcal{J}=\emptyset$, and $\mathcal{J} \cap\left(I^{u} \times I^{s}\right)=C^{u} \times C^{s}$ where $C^{s} \subset \operatorname{int}\left(I^{s}\right), C^{u} \subset \operatorname{int}\left(I^{u}\right)$ are totally disconnected. There is a partially defined first return map $f: \bigcup \mathcal{M} \rightarrow \bigcup \mathcal{M}$, whose domain is a union of ssubrectangles of $\bigcup \mathcal{M}$ and whose union is a union of $\mathbf{u}$-subrectangles, so that $f$ maps each $\mathbf{s}$-subrectangle diffeomorphically to a $\mathbf{u}$-subrectangle. This is similar to the concept of a "Markov cell decomposition" used by Farrell and Jones [FJ93].

With a certain amount of careful work, one can arrange moreover that $\mathcal{M}$ has proper overlaps, which means the following: for each $R=I^{u} \times I^{s} \in \mathcal{M}$, the components of $R \cap \operatorname{Domain}(f)$ are proper s -subrectangles, i.e. each component has the form $I^{\prime} \times I^{s}$ where $I^{\prime} \subset \operatorname{int}\left(I^{u}\right)$ is a compact interval; similarly, the components of $R \cap \operatorname{Image}(f)$ are proper $\mathbf{u}$-subrectangles of $R$. By adding extra rectangles we may assume that for each $R, R^{\prime} \in \mathcal{M}$ there is at most one component of $f(R) \cap R^{\prime}$; it follows that there is at most one component of $R \cap f^{-1}\left(R^{\prime}\right)$.

Associated to $\mathcal{M}$ is a directed graph $\Gamma$, the transition digraph, with a vertex $V_{R}$ for each $R \in \mathcal{M}$, and a directed edge $R \rightarrow R^{\prime}$ whenever $f(R) \cap R^{\prime} \neq \emptyset$. By adding extra rectangles, we may assume that $\mathcal{M}$ is generic, which means that each vertex of $\Gamma$ has valence 2 or 3 , and no edge connects two vertices of valence 3. For each directed edge $R \rightarrow R^{\prime}$ choose a flow segment $\rho_{R R^{\prime}}$ going from $R$ to $R^{\prime}$ with interior disjoint from $\cup \mathcal{M}$. The path $\rho_{R R^{\prime}}$ is well-defined up to an isotopy keeping the endpoints in $R, R^{\prime}$ respectively. The digraph $\Gamma$ may be embedded in $M$, taking the vertex $V_{R}$ to a point of $R$, and the edge $R \rightarrow R^{\prime}$ to an embedded path connecting $V_{R}$ to $V_{R}^{\prime}$ and staying in a neighborhood of $R \cup R^{\prime} \cup \rho_{R R^{\prime}}$.

The image of $\Gamma$ may be smoothed, to give a train track $\tau$ (see figure 3.5). Perturb $\tau$ so that each diverging switch $V_{R}$ moves to a point just above $R$, and each converging switch $V_{R}$ moves to a point just below $R$. The components of $\tau-\bigcup \mathcal{M}$ are of three types: a transitional component which is an oriented interval; a diverging component which is a regular neighborhood of a diverging switch; and a converging component which is a regular neighborhood of a converging switch.

Associated to each component of $\tau-\bigcup \mathcal{M}$ we may embed the appropriate piece in $M$, a diverging piece, a converging piece, or a transitional piece. For example, consider a transitional component of $\tau-\bigcup \mathcal{M}$ corresponding to an edge $R \rightarrow R^{\prime}$ where $R$ has only one outgoing edge and $R^{\prime}$ has only one incoming edge. From the overlap condition on $\mathcal{M}$, it follows that there is a flow box $H=[-1,1] \times[-1,1] \times[0,1] \hookrightarrow \mathcal{M}$ intersecting $\mathcal{M}$ in $R=[-1,1] \times[-1+\epsilon, 1-\epsilon] \times 0$ and $R^{\prime}=[-1+\epsilon, 1-\epsilon] \times[-1,1] \times 1$. Now embed


Figure 3.5: Associated to each vertex $V_{R}$ with two outgoing edges $R \rightarrow R_{1}, R \rightarrow R_{2}$, the train track $\tau$ has a diverging switch located just above $R$.
a transitional piece in $M$, using the defining formula for $P_{t}$ in the coordinate system $H$. Similarly, for each diverging component of $\tau-\bigcup \mathcal{M}$ we may embed a diverging piece, and for each converging component we may embed a converging piece. The union of these pieces defines the required isolating block with template pairs, finishing the sketch of the proof of proposition 3.3.3.

### 3.3.2 Local boundary laminations

We study some features of isolated hyperbolic invariant sets: local boundary laminations. In the course of this study we will also learn about the boundary train tracks of a template pair.

Let $\mathcal{J}$ be a 1 -dimensional isolated hyperbolic invariant set of a flow $\Psi$. Choose an isolating block $N$ for $\mathcal{J}$. Let $W_{\text {loc }}^{s}, W_{\text {loc }}^{u}$ be the local stable and unstable laminations of $\mathcal{J}$ with respect to $N$. Define the local stable boundary lamination of $\mathcal{J}$ with respect to $N$ to be the 1 -dimensional lamination $\lambda_{\text {loc }}^{s}=\partial W_{\text {loc }}^{s}$; this is a lamination in the surface $\mathcal{R}_{-} N$. The local unstable boundary lamination is $\lambda_{\text {loc }}^{u}=\partial W_{\text {loc }}^{u}$, a lamination in the surface $\mathcal{R}_{+} N$. Since $W_{\text {loc }}^{s}, W_{\text {loc }}^{u}$ are well-defined independent of $N$, the same is true of $\lambda_{\text {loc }}^{s}, \lambda_{\text {loc }}^{u}$.

A compact 1-dimensional lamination $\lambda$ is said to be finite depth if for each noncompact half-leaf $\ell$ of $\lambda$ there exists a closed leaf $\gamma$ such that for each sequence $x_{i} \in \ell$ diverging to the end of $\ell$, each limit point of $\left(x_{i}\right)$ in $\lambda$ is in $\gamma$.

Proposition 3.3.4. Given a 1-dimensional compact hyperbolic invariant set $\mathcal{J}$ of a flow in an oriented 3-manifold, the boundary laminations $\lambda_{l o c}^{s}=\partial W_{l o c}^{s}$ and $\lambda_{l o c}^{u}=\partial W_{l o c}^{u}$ are finite depth laminations with only finitely many compact leaves. Moreover, the compact leaves of $\lambda_{\text {loc }}^{s}$ are in 1-1 correspondence with the stable boundary periodic orbits of $\mathcal{J}$, and the compact leaves of $\lambda_{\text {loc }}^{u}$ are in 1-1 correspondence with the unstable boundary periodic orbits.

Proof. The "moreover" clause is an easy consequence of the definitions.
By proposition 3.3 .3 we may assume that $N$ is an isolating block for $\mathcal{J}$ supporting a template pair $B_{N}^{s}, B_{N}^{u}$. Let $q: N \rightarrow \tau=B_{N}^{s} \cap B_{N}^{u}$ be the rectangle fiber colapsing map.

Let $\tau^{s}=\partial B_{N}^{s}$, a train track in $\mathcal{R}_{-} N$ carrying $\lambda_{\text {loc }}^{s}$. We may regard $\mathcal{R}_{-} N$ as an $I$-fibered neighborhood of $\tau^{s}$, such that $\lambda_{\text {loc }}^{s}$ is transverse to the $I$-fibers. Similarly, the train track $\tau^{u}=\partial B_{N}^{u} \subset \mathcal{R}_{+} N$ carries $\lambda_{\text {loc }}^{u}$, and $\mathcal{R}_{+} N$ is an $I$-fibered neighborhood of $\tau^{u}$ such that $\lambda_{\text {loc }}^{u}$ is transverse to the $I$-fibers.

There are immersions $\kappa^{s}: \tau^{s} \rightarrow \tau, \kappa^{u}: \tau^{u} \rightarrow \tau$ which are homotopic to $q\left|\tau^{s}, q\right| \tau^{u}$ respectively. By pulling back the orientation on $\tau$ we get singular orientations on $\tau^{s}, \tau^{u}$, whose singularities are the points where $\tau^{s}, \tau^{u}$ are tangent to rectangle fibers of $N$. Each orientation singularity $x$ of $\tau^{u}$ is an orientation source, and each orientation singularity $x$ of $\tau^{3}$ is an orientation sink. Each switch of $\tau^{u}$ is diverging, and each switch of $\tau^{s}$ is converging. The train track $\tau^{u}$ is therefore an unstable train track, and $\tau^{s}$ is a stable train track. Note that under $\kappa^{u}$, each source of $\tau^{u}$ goes to a diverging switch of $\tau$ and each converging switch of $\tau^{u}$ goes to a converging switch of $\tau$; similar comments apply to $\tau^{s}$.

We prove now that $\lambda^{u}$ is finite depth. Any half leaf of $\lambda^{u}$ determines a train path in $\tau^{u}$, a smooth path $f:[0, \infty) \rightarrow \tau^{u}$ which passes over switches infinitely many times. Consider an arbitrary train path $f:[0, \infty) \rightarrow \tau^{u}$. By lemma 2.5.6, $f$ passes over the sources of $\tau^{u}$ at most once. We may therefore truncate $f$ so that it does not pass over a source, and so there is an orientation on $[0, \infty)$ such that $f$ preserves orientation. By compactness of $\tau^{u}$ there exist $s<t \in[0, \infty)$ such that $f(s)=f(t)$, and we obtain an oriented, immersed loop in $\tau^{u}$. For any oriented immersed loop in $\tau^{u}$, if it is not a covering map of an embedded loop in $\tau^{u}$ then the image contains a diverging switch of $\tau^{u}$, a contradiction; therefore, every oriented immersed loop in $\tau^{u}$ covers an embedded loop. Thus, $f$ eventually enters a circle $c$ of $\tau^{u}$. Since there are no diverging switches, once $f$ enters $c$ it can never leave.

If $f$ comes from a half-leaf $\ell$ of $\lambda^{u}$, and if $f$ eventually enters the circle $c$ of $\tau^{u}$, it follows that the accumulation set of $\ell$ is a compact sublamination of $\lambda^{u}$ carried by $c$. Let $\gamma_{c}$ be the maximal compact sublamination of $\lambda^{u}$ carried by $c$. Since $\tau^{u}$ fully carries $\lambda^{u}$ it follows that $\gamma_{c} \neq \emptyset$.

It remains to show that $\gamma_{c}$ is a single compact leaf of $\lambda^{u}$. From the construction of template pairs, the rectangle collapsing map $q: \tau^{u} \rightarrow \tau$ takes $c$ to a loop of $\tau$. Corresponding to this loop is a boundary periodic orbit $\gamma$ of $\mathcal{J}$, and it now follows easily that $\gamma_{c}$ is a compact leaf of $\lambda^{u}$ corresponding to this orbit.

### 3.3.3 Local boundary laminations in the pA case

The discussion in $\S 3.3 .2$ applies to any isolated, 1 -dimensional hyperbolic invariant set $\mathcal{J}$. Now we specialize to the case $\mathcal{J}=\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$ where $\Phi$ is a pA flow. Applying proposition 3.3.3, let $N$ be an isolating block for $\mathcal{J}$ and $B_{N}^{s}, B_{N}^{u}$ a template pair supported on $N$ and carrying $\mathcal{J}$.

For each $\gamma \in \mathcal{A}_{\Phi}$, let $U(\gamma)=\{x \in M \mid x \cdot[0, \infty)$ limits on $\gamma\}$ be the attracting basin of $\gamma$, an open, connected subset of $M$. Note that $U(\gamma) \cap U\left(\gamma^{\prime}\right)=\emptyset$ for $\gamma \neq \gamma^{\prime} \in \mathcal{A}_{\Phi}$. For each $\gamma \in \mathcal{R}_{\Phi}$ there is a similarly defined repelling basin $U(\gamma)$. From the context it should be clear
whether $U(\gamma)$ represents an attracting basin or a repelling basin. Let $U_{\mathcal{A}}=\bigcup_{\gamma \in \mathcal{A}_{\Phi}} U(\gamma)$ and $U_{\mathcal{R}}=\bigcup_{\gamma \in \mathcal{R}_{\phi}} U(\gamma)$.

Each attracting or repelling orbit $\gamma$ of $\Phi$ has an isolating block which is a smooth solid torus $T(\gamma)$, which may be taken to lie in an arbitrarily small neighborhood of $\gamma$. If $\gamma$ is attracting then $\Phi$ flows inward along $\partial T(\gamma)$, and if $\gamma$ is repelling then $\Phi$ flows outward. Let $T_{\mathcal{A}}=\cup T(\gamma)$ for $\gamma \in \mathcal{A}_{\Phi}$, and let $T_{\mathcal{R}}=\cup T(\gamma)$ for $\gamma \in \mathcal{R}_{\Phi}$. At first, we choose the isolating blocks $T(\gamma)$ so small that they are disjoint from $N$. Having made these choices, we alter them as follows.

Note that $\mathcal{R}_{-} N \subset U_{\mathcal{R}}$ and $\mathcal{R}_{+} N \subset U_{\mathcal{A}}$. Let $\partial T_{\mathcal{R}}$ flow forward until it contains $\mathcal{R}_{-} N$, and let $\partial T_{\mathcal{A}}$ flow backward until it contains $\mathcal{R}_{+} N$. More precisely, there exists a smooth function $\rho: \mathcal{R}_{-} N \rightarrow(-\infty, 0)$ such that $x \cdot \rho(x) \in \partial T_{\mathcal{R}}$ for all $x \in \mathcal{R}_{-} N$. There also exists a smooth function $\sigma: \partial T_{\mathcal{R}} \rightarrow(0, \infty)$ such that for each $x \in \mathcal{R}_{-} N$ we have $\sigma(x \cdot \rho(x))=-\rho(x)$. Now replace $T_{\mathcal{R}}$ by the set

$$
T_{\mathcal{R}} \cup \bigcup_{x \in \partial T_{\mathcal{R}}} x \cdot[0, \sigma(x)]
$$

and make similar replacements for the components $T(\gamma)$ of $T_{\mathcal{R}}$. Having done this, we have $\mathcal{R}_{-} N \subset \partial T_{\mathcal{R}}$. Make a similar replacement of $T_{\mathcal{A}}$, so $\mathcal{R}_{+} N \subset \partial T_{\mathcal{A}}$.

We now have $\lambda^{u} \subset \partial T_{\mathcal{A}}$, and so there is a decomposition $\lambda^{u}=\bigcup_{\gamma \in \mathcal{A}_{\Phi}} \lambda_{\gamma}^{u}$ into open and closed sublaminations $\lambda_{\gamma}^{u}=\lambda^{u} \cap \partial T(\gamma)$. There is a similar decomposition $\tau^{u}=\bigcup_{\gamma \in \mathcal{A}_{\Phi}} \tau_{\gamma}^{u}$. Note that $\tau_{\gamma}^{u}$ fully carries $\lambda_{\gamma}^{u}$. There are similar decompositions $\lambda^{s}=\bigcup_{\gamma \in \mathcal{R}_{\Phi}} \lambda_{\gamma}^{s}, \tau^{s}=$ $\bigcup_{\gamma \in \mathcal{R}_{\Phi}} \tau_{\gamma}^{s}$.

Recall that a Reeb lamination on a surface is any lamination contained in a subannulus $A$ of the surface, such that the closed leaves of the lamination are the components $\gamma_{1}, \gamma_{2}$ of $\partial A$, there is at least one leaf in $\operatorname{int}(A)$, each leaf in $\operatorname{int}(A)$ has one end spiralling into $\gamma_{1}$ and the other end spiralling into $\gamma_{2}$, and $\gamma_{1}, \gamma_{2}$ are oriented isotopic with respect to the spiralling orientations; these are the orientations defined by the property that when you go around the curve in the direction of the spiralling orientation, the holonomy map of the lamination is contracting. A Reeb train track in a surface is a stable or unstable train track $\tau$ contained in a subannulus $A$ of the surface, such that the components $\gamma_{1}, \gamma_{2}$ of $\partial A$ are the only loops carried by $\tau, \tau \cap \operatorname{int}(A) \neq \emptyset$, each bi-infinite train path in $\tau$ has one end spiralling around $\gamma_{1}$ and the other end spiralling around $\gamma_{2}$, and $\gamma_{1}, \gamma_{2}$ are oriented isotopic with respect to the orientation restricted from $\tau$.

Lemma 3.3.5 (Local boundary laminations in the pA case). Let $\Phi$ be a pA flow on a compact oriented 3-manifold $M$ with torus boundaries; we adopt the notation in the preceding discussion. Given $\gamma \in \mathcal{A}_{\Phi}$, let $F$ be the unique $\mathbf{u}$-face of a $p A$ torus piece such that $\gamma \subset F$. Let $c_{1}, c_{2}$ be the boundary components of $F$. Let $\gamma_{1}, \gamma_{2}$ be the two components of $\mathcal{R}_{+} N \cap F$, with the notation chosen so that $\gamma_{i}$ is a closed curve in the component of $F-\gamma$ bounded by $c_{i}$. The lamination $\lambda_{\gamma}^{u}$ is a Reeb lamination in $\partial T_{\gamma}$, with closed leaves $\gamma_{1}, \gamma_{2}$. The curve $\gamma_{i}$ with the spiralling orientation is oriented isotopic to $c_{i}$ with the dynamic
orientation. Also, the train track $\tau_{\gamma}^{u}$ is a Reeb train track in $T_{\gamma}$. Given $\gamma \in \mathcal{R}_{\Phi}$, similar statements hold for $\lambda_{\gamma}^{s}$ and $\tau_{\gamma}^{s}$, except that the spiralling orientations are anti-isotopic to the dynamic orientations.

Proof. The properties of $\lambda_{\gamma}^{u}$ and $\tau_{\gamma}^{u}$ are proved in a sequence of steps; proofs for $\lambda_{\gamma}^{s}$ and $\tau_{\gamma}^{s}$ are similar.

Step 1. The leaves $\gamma_{i}, i=1,2$, are the only closed leaves of $\lambda_{\gamma}^{u}$. This follows from the fact that corner orbits of pA torus pieces are the only unstable boundary orbits of $\mathcal{J}$, by definition of a pA flow.

The leaves $\gamma_{i}$ are not isolated in $\lambda_{\gamma}^{+}$. To see why, note that the stable manifold $W_{\text {loc }}^{s}\left(c_{i}\right)$ is divided into two halves by $c_{i}$, one half lying in the $\mathbf{s}$-face of $T$ incident to $c_{i}$; let $V\left(c_{i}\right)$ be the other half of $W^{s}\left(c_{i}\right)$. If $V\left(c_{i}\right) \cap \mathcal{J}$ were empty, then $c_{i}$ would be a stable boundary orbit on both sides, and hence we would have two pA torus pieces intersecting along $c_{i}$. This violates the definition of a pA flow, which requires distinct pA torus pieces to be disjoint. Thus, $V\left(c_{i}\right) \cap \mathcal{J} \neq \emptyset$. Note that in some neighborhood of $\gamma_{i}, \lambda_{\gamma}^{u}$ intersected with that neighborhood is isotopic through $W_{\text {loc }}^{u}$ to $V\left(c_{i}\right) \cap \mathcal{J}$ intersected with a neighborhood of $c_{i}$, and so $\gamma_{i}$ is not isolated.

Step 2. All nearby leaves of $\lambda^{u}$ spiral into $\gamma_{i}$, and the spiralling orientation on $\gamma_{i}$ agrees with the dynamic orientation. This is true because the holonomy of $V\left(c_{i}\right) \cap \mathcal{J}$ around $c_{i}$ is contracting, when you go around $c_{i}$ in the direction of the flow, and the spiralling orientation on $c_{i}$, as a leaf of $V\left(c_{i}\right) \cap \mathcal{J}$, agrees with the dynamic orientation; the same is therefore true of $\gamma_{i}$ as a leaf of $\lambda_{\gamma}^{u}$.

Step 3. The only closed, oriented loops of $\tau_{\gamma}^{u}$ are the loops carrying $\gamma_{1}, \gamma_{2}$. This follows because $\tau_{\gamma}^{u}$ fully carries $\lambda_{\gamma}^{u}$, and since $\tau_{\gamma}^{u}$ is an unstable train track, every closed loop of $\tau_{\gamma}^{u}$ carries a closed leaf of $\lambda_{\gamma}^{u}$; step 3 is therefore a consequence of step 2 .

Step 4. For each nonclosed leaf $\ell$ of $\lambda_{\gamma}^{u}$, each end of $\ell$ spirals into one of $\gamma_{1}$ or $\gamma_{2}$. To see why, under the $I$-collapsing map $\mathcal{R}_{+} N \rightarrow \tau_{\gamma}^{u}$, the leaf $\ell$ maps to a bi-infinite path in $\tau_{\gamma}^{u}$. Since $\tau_{\gamma}^{u}$ is an unstable train track, it follows that each half of $\ell$ eventually winds around a directed loop of $\tau_{\gamma}^{u}$. Applying step 3, each half of $\ell$ eventually spirals around one of the closed leaves $\gamma_{1}, \gamma_{2}$.

Step 5. For each nonclosed leaf $\ell$ of $\lambda_{\gamma}^{u}$, one end of $\ell$ spirals into $\gamma_{1}$ and the other spirals into $\gamma_{2}$.

Suppose that both ends of $\ell$ spiral into one closed leaf, say $\gamma_{i}$. We shall show that this leads to a transverse bigon for $\mathcal{J}$, contradicting the definition of a pA flow (see figure 3.6).

In the torus $\partial T(\gamma)$, one component of $\mathfrak{C}(\partial T(\gamma)-\ell)$ is a monogon $\mu$, a disc with one boundary point removed, so that the two ends of $\ell=\partial \mu$ get closer and closer to each other as you go out the end of $\mu$. The interior of $\mu$ might intersect $\lambda_{\gamma}^{u}$. There are, however, no closed leaves of $\lambda_{\gamma}^{u} \operatorname{in} \operatorname{int}(\mu)$, because each closed leaf of $\lambda^{u}$ lies on some $\mathbf{u}$-face of some pA torus piece. There is, therefore, an innermost leaf in $\mu$, bounding a submonogon of $\mu$ whose interior is disjoint from $\lambda^{u}$. We may therefore assume that $\operatorname{int}(\mu) \cap \lambda^{u}=\emptyset$.

Now let $\mu^{\prime}$ be obtained from $\mu$ by chopping off some neighborhood of the end, so $\mu^{\prime}$ is a bigon with one boundary edge $\beta^{\prime} \subset \ell$ and another short boundary edge $\alpha^{\prime}$, far out the end of $\mu$, with interior disjoint from $\lambda^{u}$. Note that $\mu^{\prime}$ is tranverse to $\Phi$.

The arc $\alpha^{\prime}$ is contained in an arbitrarily small neighborhood of $F$. There exists an embedded rectangle $Q \subset N$ with one edge on $\alpha^{\prime}$, another edge on $\alpha \subset V\left(c_{i}\right)$, and the remaining two edges on $W_{\text {loc }}^{u}(\mathcal{J})$, such that $Q$ is transverse to $\Phi$, and $\operatorname{int}(Q) \cup \operatorname{int}(\alpha) \cup \operatorname{int}\left(\alpha^{\prime}\right)$ is disjoint from $W_{\text {loc }}^{u}(\mathcal{J})$. We may glue $Q$ and $\mu^{\prime}$ along $\alpha^{\prime}$, and smooth along $\alpha^{\prime}$, to obtain a transverse bigon for $\mathcal{J}$, obtaining the contradiction that proves step 5 .

Steps 1-5 together prove the lemma for $\lambda_{\gamma}^{u}$ and $\tau_{\gamma}^{u}$.

### 3.3.4 Proof: pA flows yield laminations

Let $\Phi$ be a pA flow on a manifold with torus boundaries $M$. We prove that $\Lambda^{u}=\mathcal{J}\left(\mathcal{I}_{\Phi} \cup \mathcal{A}_{\Phi}\right)$ is a very full lamination; the proof for $\Lambda^{s}=\mathcal{J}\left(\mathcal{I}_{\Phi} \cup \mathcal{R}_{\Phi}\right)$ is similar. Adopting the notation of $\S 3.3 .3$, there is an isolating block $N$ for $\mathcal{J}=\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$ supporting a template pair $B_{N}^{s}, B_{N}^{u}$. The local weak stable and unstable laminations $W_{\text {loc }}^{s}, W_{\text {loc }}^{u}$ of $\mathcal{J}$ are properly embedded laminations in $N$, with local boundary laminations $\lambda^{s}=\partial W_{\text {loc }}^{s}$ and $\lambda^{u}=\partial W_{\text {loc }}^{u}$. Applying lemma 3.3.5, for each $\gamma \in \mathcal{A}_{\Phi}$, setting $\lambda_{\gamma}^{u}=\lambda^{u} \cap \partial T(\gamma)$ we have:

- $\lambda_{\gamma}^{u}$ is a Reeb lamination in $\partial T(\gamma)$.
- Each closed leaf of $\lambda_{\gamma}^{u}$, equipped with its spiralling orientation, is oriented isotopic to $\gamma$ with its dynamic orientation.

Note that $\Lambda^{u}$ is equal to $W_{\text {loc }}^{u} \cup\left(\lambda^{u} \cdot[0, \infty)\right) \cup \mathcal{A}_{\Phi}$. To prove that $\Lambda^{u}$ is a lamination it therefore suffices to prove that for each $\gamma \in \mathcal{A}_{\Phi}$, the set $\Lambda_{\gamma}^{u}=\left(\lambda_{\gamma}^{u} \cdot[0, \infty)\right) \cup \gamma$ is a lamination in the solid torus $T(\gamma)$, with boundary $\lambda_{\gamma}^{u}$.

We claim that the triple $\left(T(\gamma), \Phi \mid T(\gamma), \lambda_{\gamma}^{u}\right)$ is described up to topological conjugacy as follows. The torus $T(\gamma)$ is the quotient of $\widetilde{T}=\left\{(x, y, z) \in \mathbf{R}^{3} \mid x^{2}+y^{2} \leq e^{2 z}\right\}$ under the map $F(x, y, z)=(e x, e y, z+1)$. The forward semiflow $\Phi \mid T(\gamma)$ is the quotient of the forward semiflow on $\widetilde{T}$ generated by $\partial / \partial z$. The surface $\partial \widetilde{T}$ has an $F$-invariant foliation by curves of intersection with planes parallel to the $y z$-coordinate plane. The lamination $\lambda_{\gamma}^{u} \subset \partial T(\gamma)$ is the quotient of an $F$-invariant sublamination of this foliation. This claim


Figure 3.6: If $\lambda_{\gamma}^{+}$has a nonclosed leaf $\ell$ with both ends spiralling into $\gamma_{i}$, then there is a transverse bigon. The picture shows part of the leaf $W_{\text {loc }}^{u}\left(c_{i}\right)$ and also part of the leaf of $W_{\text {loc }}^{u}(\mathcal{J})$ having $\ell$ as a boundary component. Also shown are several flow lines of $\Phi$ in these leaves. Recall that in any leaf of $W_{\text {loc }}^{u}$, orbits of $\Phi$ converge exponentially in backwards time.
follows from the the above listed facts about $\lambda_{\gamma}^{u}$, and the fact that $T(\gamma)$ is an isolating neighborhood of the attracting orbit $\gamma$.

It follows that $\Lambda_{\gamma}^{u}$ is the quotient of an $F$-invariant sublamination of the foliation of $\widetilde{T}$ by intersections with planes parallel to the $y z$-coordinate plane, thereby proving that $\Lambda_{\gamma}^{u}$ is a lamination with boundary $\lambda_{\gamma}^{u}$.

Since $\Lambda^{u}$ is tangent to $\Phi$, clearly there are no sphere leaves nor Reeb components (in fact, one can use hyperbolicity of $\mathcal{J}$ to show that there are no torus leaves at all).

It remains to examine the components of $\mathfrak{C}\left(M-\Lambda^{u}\right)$, and to prove that they are in 1-1 type preserving correspondence with the pA torus pieces of $\Phi$. Let $C$ be a component of $\mathfrak{C}\left(M-\Lambda^{u}\right)$, which by abuse of notation we may regard as an invariant set of $\Phi$.

For each $x \in C$, we claim that one of the following happens:

- $x$ is contained in some pA torus piece.
- $x \in \operatorname{int}(C)$ and the backwards orbit $x \cdot(-\infty, 0]$ accumulates on some repelling orbit of $\Phi$.
- $x \in \partial C$ and, letting $L$ be the component of $\partial C$ containing $x$, the backwards orbit $x \cdot(-\infty, 0]$ accumulates on some stable boundary periodic orbit of $\Phi$ contained in $L$.

To prove the claim, suppose that $x$ is not already contained in a pA torus piece. If $x \in \operatorname{int}(C)$ then $x \cdot(-\infty, 0]$ must accumulate on a repeller of $\Phi$, and the only possibility is a repelling periodic orbit. If $x$ is contained in the component $L$ of $\partial C$, then the backwards orbit $x \cdot(-\infty, 0]$ must eventually enter $W_{\text {loc }}^{u}$, and since $L$ is a boundary leaf of $\Lambda^{u}$ it follows that $x \cdot(-\infty, 0]$ accumulates on a stable boundary periodic orbit contained in $L$.

In the above claim, each of the three cases picks out a pA torus piece $T_{x}$ such that either $x \in T_{x}$ or $x \cdot(-\infty, 0]$ accumulates on a repelling periodic orbit or a corner orbit of $T_{x}$. Since $T_{x}$ may be regarded as a subset of $\mathfrak{C}\left(M-\Lambda^{u}\right)$, and since $T_{x}$ is connected, it follows that $T_{x} \subset C$. Let $T_{C}=\bigcup_{x \in C} T_{x}$.

We have in fact proved something more: for each $x \in C-T_{C}$, the backward orbit $x \cdot(-\infty, 0]$ accumulates on some s-face $F_{x}$ of $T_{C}$. To see what this implies, for each s-face $F$ of $T_{C}$ attach to $T_{C}$ a collar neighborhood $N(F) \approx F \times[0,1]$ on the outside of $T_{C}$, such that $F \approx F \times 0$, the outer face of $N(F)$ is the annulus $F^{\prime} \approx F \times 1, F^{\prime}$ is transverse to $\Phi$, and $\partial F \times[0,1] \subset \partial C$. Let $N\left(T_{C}\right)$ be obtained from $T_{C}$ by attaching $N(F)$ for each s-face $F$ of $T_{C}$. It now follows that $C$ is obtained from $N\left(T_{C}\right)$ by attaching $F^{\prime} \cdot[0, \infty)$ to each $F^{\prime}$. Since $C$ is connected it follows that $T_{C}$ is connected, and so $T_{C}$ is a pA torus piece. Moreover, $C$ is a u-pared torus piece of the same type as $T_{C}$.

### 3.3.5 Proof: pA flows yield dynamic pairs

Let $\Phi$ be a pA flow on a torally bounded manifold $M$. Apply proposition 3.3 .3 to produce an isolating block $N$ for $\mathcal{J}=\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$ supporting a template pair $B_{N}^{s}, B_{N}^{u} \subset N$ carrying $\mathcal{J}$.

Applying lemma 3.3.5, $\partial B_{N}^{u}$ consists of one Reeb train track $\tau_{\gamma}^{s}$ contained in $T(\gamma)$ for each attracting periodic orbit $\gamma$ of $\Phi$. Let $F, \gamma_{1}, \gamma_{2}$ be as in lemma 3.3.5. For convenience we may assume that $\gamma_{1}, \gamma_{2}$ are the closed loops in $\tau_{\gamma}^{s}$.

We now construct $B^{u}$ by constructing an annulus with tongues in $T(\gamma)$ whose boundary is $\tau_{\gamma}^{u}$, using methods similar to the proof of proposition 2.5.7. Start with the annulus $A \subset F$ bounded by $\gamma_{1}, \gamma_{2}$. Choose an enumeration $x_{1}, \ldots, x_{n}$ of the sources of $\tau_{\gamma}^{u}$. Starting with $x_{1}$, attach a tongue $t_{1}$ to $A$, whose boundary consists of the path $\rho_{11} \subset \tau_{\gamma}^{u}$ connecting $x_{1}$ to $\gamma_{1}$, the path $\rho_{12}$ connecting $x_{1}$ to $\gamma_{2}$, and a maw arc $\beta_{1}$ in $F$ connecting $\gamma_{1}$ to $\gamma_{2}$. Continuing inductively, attach a tongue $t_{k}$ to $A \cup t_{1} \cup \cdots \cup t_{k-1}$, whose boundary consists of the path $\rho_{k 1}$ connecting $x_{k}$ to $\gamma_{1} \cup \rho_{11} \cup \cdots \cup \rho_{k-1,1}$, the path $\rho_{k 2}$ connecting $x_{k}$ to $\gamma_{2} \cup \rho_{12} \cup \cdots \cup \rho_{k-1,2}$, and a maw arc $\beta_{k}$. At each stage, the tangent plane along $\beta_{k}$ may be chosen consistently because of the fact that $\tau_{\gamma}^{u}$ is a Reeb train track. This completes the construction of $B^{u}$.

The neighborhood $N\left(B^{u}\right)$ must be constructed to satisfy several conditions: $N\left(B^{u}\right)$ is an $I$-fibered neighborhood of $B^{u} ; N\left(B^{u}\right)$ is an isolating block for $\Phi$, with $\Phi$ pointing inwards along $\partial N\left(B^{u}\right) ; N(\tau) \subset N\left(B^{u}\right)$ with $\mathcal{R}_{-} N(\tau) \subset \partial N\left(B^{u}\right)$. We construct $N\left(B^{u}\right)$ as follows. Fix an attracting orbit $\gamma$ of $\Phi$. The surface $\mathcal{R}_{+} N(\tau)$ intersects $\partial T(\gamma)$ in an $I$ fibered neighborhood $N\left(\tau_{\gamma}^{u}\right)$. The components of $\mathfrak{C}\left(\partial T(\gamma)-N\left(\tau_{\gamma}^{u}\right)\right)$ consist of one annulus, and a collection of smooth discs $D_{1}, \ldots, D_{n}$ equal in number to the sources of $\tau_{\gamma}^{u}$. Let $D_{k}^{\prime}$ be a smooth, properly embedded disc in $T(\gamma)$ with $\partial D_{k}=\partial D_{k}^{\prime}$, meeting $\mathcal{R}_{-} N(\tau)$ smoothly; this can be achieved by setting $D_{k}^{\prime}=\left\{x \cdot \rho(x) \mid x \in D_{k}\right\}$ where $\rho: D_{k} \rightarrow[0, \infty)$ is an appropriately chosen smooth function whose zero set is $\partial D_{k}$. We can now define $N\left(B^{u}\right)$ to be the manifold with boundary $\mathcal{R}_{-} N(\tau) \cup D_{1}^{\prime} \cup \cdots \cup D_{n}^{\prime}$ containing $N(\tau)$. It is obvious that there is a deformation retraction from $N\left(B^{u}\right) \cap T(\gamma)$ to $B^{u} \cap T(\gamma)$ whose restriction to $N\left(\tau_{\gamma}^{u}\right)$ is the $I$-fiber collaping map onto $\tau_{\gamma}^{u}$; we leave the reader to construct this deformation retraction to be an $I$-collapsing map on all of $N\left(B^{u}\right) \cap T(\gamma)$.

The branched surface $B^{s}$, and its $I$-fibered neighborhood $N\left(B^{u}\right)$, are similarly described.
The only task remaining is to verify that $B^{s}, B^{u}$ is a dynamic pair, and for that the only slightly nonobvious part is that the components of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ which are not dynamic torus pieces are pinched tetrahedra.

By construction, the components of $\mathfrak{C}\left(\partial N\left(B^{s}\right)-N\left(B^{u}\right)\right)$ are discs and annuli, and similarly for $\mathfrak{C}\left(\partial N\left(B^{u}\right)-N\left(B^{s}\right)\right)$. These discs occur in pairs, a disc $D^{s} \subset \mathfrak{C}\left(\partial N\left(B^{s}\right)-\right.$ $\left.N\left(B^{u}\right)\right)$ corresponding to a disc $D^{u} \subset \mathfrak{C}\left(\partial N\left(B^{u}\right)-N\left(B^{s}\right)\right)$ when $\partial D^{s}=\partial D^{u}$. Note that for each $x \in D^{s}$ the flow line $x \cdot[0, \infty)$ hits $D^{u}$ in a unique point; the union of these flow segments forms a topological 3-ball $b$ with boundary $D^{s} \cup D^{u}$ (we have not used irreducibility of $M$ to construct this 3-ball).

This sets up a 1-1 correspondence between the components of $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$ which are 3 -balls and the components of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ which are not dynamic torus pieces, and so the latter must all be topological 3-balls with interval dynamics. Any dynamic manifold with this property, whose faces are $\mathbf{s}$ and $\mathbf{u}$-faces, is a pinched tetrahedron.

### 3.3.6 Proof: Dynamic pairs yield pA flows

Let $B^{s}, B^{u}$ be a dynamic pair on a compact, oriented 3-manifold $M$ with torus boundaries. Let $\tau=B^{s} \cap B^{u}$. Choose $I$-fibered neighborhoods $N\left(B^{s}\right), N\left(B^{u}\right)$ in the smooth model so that $N=N(\tau)=N\left(B^{s}\right) \cap N\left(B^{u}\right)$ is a sutured manifold neighborhood, the $I$-fibrations fit together to give a rectangle fibration $q: N \rightarrow \tau$, and $B_{N}^{s}=N \cap B^{s}, B_{N}^{u}=N \cap B^{u}$ is a template pair in $N$. We shall define a flow $\Phi$, by "induction along skeleta": first we define $\Phi$ on $N$, then on $\mathfrak{C}\left(N\left(B^{s}\right)-N\right)$ and $\mathfrak{C}\left(N\left(B^{u}\right)-N\right)$, finally on $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$. In each case we investigate the behavior of $\Phi$, with an eye towards proving that $\Phi$ is a pA flow.

The flow on $N$. We construct a semiflow with $N$ as an isolating block, whose maximal invariant set $\mathcal{J}$ is a hyperbolic invariant set carried by the template pair $B_{N}^{s}, B_{N}^{u}$.

Choose a finite set $X \subset \tau$ so that the closure of each component $N-q^{-1}(X)$ is a converging piece, diverging piece, or transitional piece $P$. Let $R_{x}=q^{-1}(X)$, so $R_{x}$ is the result of gluing some top rectangle to some bottom rectangle, among all the gluing rectangles of the pieces $P$. Notice that in the standard model for each piece $P$, the bottom gluing rectangles are wider in the $\mathbf{u}$-direction than in the $\mathbf{s}$-direction, and the top gluing rectangles are wider in the $\mathbf{s}$-direction than in the $\mathbf{u}$-direction (see figure 3.3, and also the formulas defining the stadard models $P_{t}, P_{d}, P_{c}$ ). We may therefore choose each gluing map, from a top rectangle to a bottom rectangle, to be a linear map which stretches the u-direction and compresses the s-direction. On each of the standard models $P_{t}, P_{d}, P_{c}$, take the semiflow generated by $\partial / \partial z$, and push these semiflows forward under the gluing to define a smooth semiflow $\Phi$ on $N$. Clearly $N$ is an isolating block for $\Phi$, with maximal invariant set $\mathcal{J} \subset N$, and there is a hyperbolic splitting along $\mathcal{J}$ with stable direction parallel to the s-coordinate and unstable direction parallel to the $\mathbf{u}$-coordinate in each piece. Also, $\mathcal{J}$ is carried by the template pair $B_{N}^{s}, B_{N}^{u}$.

Using symbolic dynamics arguments from [Bow72] or [Fra82], the invariant set $\mathcal{J}$ is 1-dimensional. For each rectangle $R_{x}=I_{x}^{u} \times I_{x}^{s}$, there is a product structure $R_{x} \cap \mathcal{J} \approx$ $C_{x}^{u} \times C_{x}^{s}$, where $C_{x}^{s} \subset \operatorname{int}\left(I_{x}^{s}\right), C_{x}^{u} \subset \operatorname{int}\left(I_{x}^{u}\right)$ are compact and totally disconnected. We have $W_{\mathrm{loc}}^{s}(\mathcal{J}) \cap R_{x}=I_{x}^{u} \times C_{x}^{s}$ and $W_{\mathrm{loc}}^{u}(\mathcal{J}) \cap R_{x}=C_{x}^{u} \times I_{x}^{s}$. There is a $1-1$ correspondence between directed loops of $\tau$ and periodic orbits of $\mathcal{J}$, where a loop $c$ corresponds to an orbit $\gamma_{c}$ so that the cyclic sequences of rectangles $R_{x}$ intersected by $c$ and $\gamma_{c}$ are identical.

We study boundary periodic orbits of $\mathcal{J}$. Let $\lambda^{u}=\partial W_{\text {loc }}^{u}$, a lamination in $\mathcal{R}_{+} N$, and let $\lambda^{s}=\partial W_{\mathrm{loc}}^{s} \subset \mathcal{R}_{-} N$. The surface $\mathcal{R}_{-} N$ is an $I$-fibered neighborhood of the train track $\tau^{s}=\partial \mathfrak{C}\left(B^{s}-\tau\right)$, and $\tau^{s}$ carries $\lambda^{s}$. Similarly $\mathcal{R}_{+} N$ is an $I$-fibered neighborhood of $\tau^{u}=\partial \mathfrak{C}\left(B^{u}-\tau\right)$, and $\tau^{u}$ carries $\lambda^{u}$

Each component $K$ of $\mathfrak{C}\left(B^{s}-\tau\right)$ is an annulus with tongues, by proposition 2.5.7; let $A_{K}$ be the annulus. The component $\tau_{K}^{s}=\partial K$ of $\tau^{s}$ is a stable train track, contained in a component $\mathcal{R}_{-}^{K}$ of $\mathcal{R}_{-} N$, and $\tau_{K}^{s}$ carries $\lambda_{K}^{s}=\lambda^{s} \cap \mathcal{R}_{-}^{K}$. Since $K$ is an annulus with
tongues, it follows that $\tau_{K}^{s}$ has two directed loops, and for each orientation sink $s \in \tau_{K}^{s}$ the backwards directed paths on either side of $s$ go to distinct loops in $\tau_{K}^{s}$. It follows that $\lambda_{K}^{s}$ has exactly two closed leaves, one carried by each directed loop in $\tau_{K}^{s}$; every other orbit of $\lambda_{K}^{s}$ spirals into one loop on one end and the other loop on the other end. In particular, $\lambda_{K}^{s}$ is a connected topological space.

We can now identify the stable and unstable boundary periodic orbits of $\mathcal{J}$. For each dynamic torus piece $T$ of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$, and for each corner circle $c$ of $T$, we may identify $c$ with a directed loop in $\tau$. Corresponding to $c$ is a periodic orbit $\gamma_{c} \subset \mathcal{J}$. There are components $K_{c}^{s}$ of $\mathfrak{C}\left(B^{s}-\tau\right)$ and $K_{c}^{u}$ of $\mathfrak{C}\left(B^{u}-\tau\right)$ such that $c$ is identified with a directed circle in the train track $\partial K_{c}^{s}$ and also in the train track $\partial K_{c}^{u}$, and hence $\gamma_{c}$ is both a stable and an unstable boundary orbit. We shall need even more structure than this, in order eventually to see that $\gamma_{c}$ is a corner orbit of a pA torus piece. Corresponding to the pair $T, c$ is a component $T^{\prime}$ of $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$ and a corner circle $c^{\prime}$ of $T^{\prime}$. Consider a point $x \in X \cap c$. Let $x^{\prime}=R_{x} \cap c^{\prime}$, a corner of $R_{x}$. Let $\mathcal{H}\left(C_{x}^{s} \times C_{x}^{u}\right)$ be the convex hull of $C_{x}^{s} \times C_{x}^{u}$, a subrectangle of $R_{x}$. It follows that $R_{x} \cap \gamma_{c}$ is the corner of $\mathcal{H}\left(C_{x}^{s} \times C_{x}^{u}\right)$ corresponding to $x^{\prime}$.

The flow on $\mathfrak{C}\left(N\left(B^{s}\right)-N\right)$ ) and $\mathfrak{C}\left(N\left(B^{u}\right)-N\right)$ ). Fix a component $K$ of $\mathfrak{C}\left(B^{u}-\tau\right)$. Corresponding to $K$ is a component $N_{K}$ of $\mathfrak{C}\left(N\left(B^{u}\right)-N\right)$, which may be regarded as a regular neighborhood of $K$. We know by proposition 2.5 . 7 that $K$ is a stable annulus with tongues, and so $N_{K}$ is homeomorphic to a solid torus. Let $A \subset K$ be the annulus, with boundary components $\gamma_{1}, \gamma_{2}$. Corresponding to $\gamma_{1}, \gamma_{2}$ are two unstable boundary periodic orbits $c_{1}, c_{2} \subset \mathcal{J}$. The dynamic orientations on $c_{1}, c_{2}$ agree with the dynamic orientations on $\gamma_{1}, \gamma_{2}$, which agree with each other. In the natural manifold-with-corners structure on $N_{K}$, the set $N_{K} \cap N$ is a face. We have already constructed $\Phi$ along this face, and it flows into $N_{K}$ (out of $N$ ). We may now extend $\Phi$ over the whole solid torus $N_{K}$, so that $\Phi$ flows inward on $\partial N_{K}$, and so that the solid torus $N_{K}$ is an isolating neighborhood of an attracting periodic orbit $\gamma$. We are free to choose the dynamic orientation on $\gamma$ at will; choose this orientation so that $\gamma$ is oriented isotopic to $\gamma_{1}$ and $\gamma_{2}$, and hence also to $c_{1}$ and $c_{2}$.

Note that there is an invariant annulus $F_{\gamma}$ of $\Phi$ with core $\gamma$ and with $\partial F_{\gamma}=c_{1} \cup c_{2}$, such that the component of $F_{\gamma}$ containing $c_{i}$ is contained in the unstable manifold of $c_{i}$.

The flow $\Phi$ is extended over $N\left(B^{s}\right)-N$ in a similar manner.

The flow on $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$. Given a component $T$ of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$, let $K_{T}$ be the corresponding component of $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$.

Consider first the case that $T$ is a pinched tetrahedron. Note that $K_{T}$ is a topological ball, with two disc faces meeting along their boundaries at a corner circle. The flow $\Phi$ is already defined on $\partial K_{T}$, entering $K_{T}$ along one disc face and exiting along the other disc
face, and externally tangent along the common boundary curve. Extend $\Phi$ over $K_{T}$ to have interval dynamics.

Next consider the case that $T$ is a torus piece. Corresponding to each face $F$ of $T$ there is an attracting or repelling orbit $\gamma$ and an invariant annulus $F^{\prime}=F_{\gamma}$, whose boundary components are stable and unstable boundary periodic orbits of $\mathcal{J}$. The union of the annuli $F^{\prime}$, over all faces $F$ of $T$, bounds a set $T^{\prime}$ which has the correct manifold-withcorners structure for a pA torus piece of the same type as $T$. The flow $\Phi$ is already defined on a neighborhood of $\partial T^{\prime}$, and it has the correct structure for the flow on a pA torus piece: the corners are hyperbolic orbits and the faces contain, alternately, attracting and repelling periodic orbits. We may now extend $\Phi$ over all of $T^{\prime}$ so that $T^{\prime}$ is a pA torus piece of the correct type.
$\Phi$ is a pA flow. Conditions $1-6$ in the definition of a pA flow are obvious from the construction and the properties of $\mathcal{J}$ already noted. Disjointness of pA torus pieces follows from the fact that for each pA torus piece $T$ and each corner orbit $\gamma$ of $T$, the component of $W_{\text {loc }}^{s}(\gamma)-\gamma$ not lying in $\partial T$ has nonempty intersection with $\mathcal{J}$, and similarly for $W_{\text {loc }}^{u}(\gamma)$, and so $c$ is not a corner orbit of any other pA torus piece.

To check the final condition 7 , suppose by contradiction that there is a transverse bigon $D$ for $\mathcal{J}$.

First we reduce to the case where $D \subset \operatorname{int}\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)$. We know already that $\partial D \subset \operatorname{int}(N)$. By perturbing $D$ we may assume that $D$ is transverse to the surface $F=$ $\partial\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)$. Each component of $F$ bounds a component of $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$, either a 3 -ball or a dynamic torus piece. We may easily push $D$ out of the 3 -balls by isotoping along flow lines.

Consider a dynamic torus piece component $T$ of $\mathfrak{C}\left(M-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$. Let $T^{\prime}$ be the corresponding pA torus piece of $\Phi$, so $T \subset \operatorname{int}\left(T^{\prime}\right)$. We shall show that $D \cap T^{\prime}=\emptyset$, and so $D \cap T=\emptyset$. We know that a collar neighborhood of $\partial D$ is properly embedded in $\mathfrak{C}\left(N-\left(W_{\text {loc }}^{s} \cup W_{\text {loc }}^{u}\right)\right)$, with one edge in $W_{\text {loc }}^{s}$ and the other edge in $W_{\text {loc }}^{u}$. It follows that if $\partial D \cap \partial T^{\prime} \neq \emptyset$ then $\partial D \subset \partial T^{\prime}$. But $D$ is transverse to $\Phi$, and each circle in $\partial T^{\prime}$ that is transverse to $\Phi$ must intersect $\partial T-N$. Therefore if $\partial D \cap \partial T^{\prime} \neq \emptyset$ then $\partial D \not \subset N$, a contradiction. We have shown that $\partial D \cap \partial T^{\prime}=\emptyset$, and so $D \cap \partial T^{\prime}$ is a union of circles in $\operatorname{int}(D)$. But every circle in $\partial T^{\prime}$ that is transverse to $\Phi$ must intersect the corner orbits of $T^{\prime}$, contradicting the fact that $\operatorname{int}(D) \cap \mathcal{J}=\emptyset$. It follows that $D \cap \partial T^{\prime}=\emptyset$, and therefore $D \cap T^{\prime}=\emptyset$, so $D \cap T=\emptyset$.

Next we reduce to the case where $D \subset \operatorname{int}(N)$. By perturbing $D$ we may assume that it is transverse to $\partial N$. Each component of $D \cap \partial N$ is therefore a circle, and each such circle must be contained in $\operatorname{int}\left(\mathcal{R}_{-} N\right)$ or $\operatorname{int}\left(\mathcal{R}_{+} N\right)$, since $D \subset \operatorname{int}\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)$. Consider a component $C$ of $\mathfrak{C}\left(N\left(B^{s}\right)-N\right)$. We know that $C$ is a solid torus isolating neighborhood of a repelling periodic orbit $\gamma$. We also know that $C$ is a manifold with corners, $C \cap N$ is a
face of $C$, and $D \cap \partial C \subset \operatorname{int}(C \cap N)$. Each component of $D \cap C$ is properly embedded in $C$ and transverse to the flow on $C$; it follows that $D \cap \gamma=\emptyset$, because any surface transverse to $\Phi$ and properly embedded in $C$ which intersects $\gamma$ must contain a meridian circle of $\partial C$, but no meridian circle is contained in $C \cap N$. Having shown that $D \cap \gamma=\emptyset$, it follows that we can push $D$ out of $C$ by flowing along trajectories of $\Phi$. The portions of $D$ which were in $C$ are pushed into $N$, and so the number of components of $D \cap \partial N$ are reduced. Repeating this for each component of $\mathfrak{C}\left(N\left(B^{s}\right)-N\right)$ and of $\mathfrak{C}\left(N\left(B^{u}\right)-N\right)$, eventually we have $D \subset \operatorname{int}(N)$.

To finish, we rule out the possibility of a transverse bigon in $N$ by enumerating the three possible types of components of $\mathfrak{C}\left(N-\left(W_{\text {loc }}^{s} \cup W_{\text {loc }}^{u}\right)\right)$.

The first type is the "corner orbit piece", the quotient of the set $[0,1] \times[0,1] \times \mathbf{R} \subset \mathbf{R}^{3}$ with respect to the map $(x, y, z) \rightarrow(x, y, z+1)$, equipped with the flow $(x, y, z) \cdot t=$ $\left(e^{t} x, e^{-t} y, z+t\right)$. The $z$-axis becomes a boundary periodic orbit of $\mathcal{J}$. There is a $1-1$ corresondence between corner orbit pieces, boundary periodic orbits of $\mathcal{J}$, and components of the suture set $\sigma N$ which are identified with corner circles of dynamic torus pieces of $B^{s}, B^{u}$.

The second type is the "acqueduct piece". To describe it, let $f:(-1,1) \rightarrow \mathbf{R}$ be a function which is concave upward, approaching $\infty$ at the endpoints, with $f(0)=0$; the function $f(x)=\sec (\pi x)-1$ will do. Take the set $\left\{(x, y, z) \in \mathbf{R}^{3}| | x|,|y| \leq 1, z \leq\right.$ $f(x)+1, z \geq f(y)-1\}$, with the semiflow generated by $\partial / \partial z$. There is a countable infinity of acqueduct pieces.

The third type is the "leaky acqueduct piece", defined as $\left\{(x, y, z) \in \mathbf{R}^{3}| | x|,|y| \leq\right.$ $1, z \leq f(x)-1, z \geq f(y)+1\}$. There is a $1-1$ correspondence between leaky acqueduct pieces and components of $\sigma N$ that are not corner circles of dynamic torus pieces.

To prove that these are the only possible types, cut open each converging piece, diverging piece, and transitional piece along the laminations, take completions, glue together to get the components of $\mathfrak{C}\left(N-\left(W_{\text {loc }}^{s} \cup W_{\text {loc }}^{u}\right)\right)$, and see what you get.

Obviously none of the three pieces contains a transverse bigon, proving condition 7.

### 3.4 Constructing pseudo-Anosov flows

Having dwelt among pA flows for the last few sections, now we return to pseudo-Anosov flows by proving:

Theorem 3.4.1. Let $M$ be a closed, oriented 3-manifold. Given a dynamic pair $B^{s}, B^{u}$ on $M$, we can construct a pseudo-Anosov flow $\Phi$.

Remark. The pseudo-Anosov flow $\Phi$ constructed in this theorem is said to be carried by the pair $B^{s}, B^{u}$. To say that "we can construct" $\Phi$ means, at the very least, that we can construct a Markov partition for $\Phi$, starting from $B^{s}, B^{u}$ as the input data.

Proof. Recall that in constructing a pA flow, we started with $I$-fibered neighborhoods of $B^{s}, B^{u}$ in the smooth model, whose intersection was a regular neighborhood $N(\tau)$ with rectangle fibration $q: N(\tau) \rightarrow \tau$. Then we constructed a semiflow on $N(\tau)$, for which $N(\tau)$ was an isolating block, whose maximal invariant set was a 1-dimensional hyperbolic invariant set. Finally, we extended this flow to the complement of $N(\tau)$.

Here is a outline of the construction of a pseudo-Anosov flow $\Phi$; details are given below.
Start with $I$-fibered neighborhoods $U\left(B^{s}\right), U\left(B^{u}\right)$ in the cusp model. Let $U(\tau)=$ $U\left(B^{s}\right) \cap U\left(B^{u}\right)$, so that the $I$-fibers of the factors fit together to give a rectangle fibration of $U(\tau)$. Construct a flow $\Phi^{*}$ on $U(\tau)$ for which $U(\tau)$ is a hyperbolic invariant set. Unlike in the construction of pA flows, $\Phi^{*}$ is tangent to the boundary. To be truthful, $\Phi^{*}$ is only a forward semiflow near culverts of $U\left(B^{u}\right)$ and a backward semiflow near culverts of $U\left(B^{s}\right)$, but that will not disturb us. There are stable and unstable foliations $W^{s *}, W^{u *}$ defined on $U(\tau)$. There is a decomposition $\partial U(\tau)=\partial_{\mathbf{s}} U(\tau) \cup \partial_{\mathbf{u}} U(\tau)$, where each s-face is tangent to $W^{s *}$ and transverse to $W^{u *}$, and vice versa for $\mathbf{u}$-faces. The stable foliation $W^{s *}$ therefore induces a foliation of $\partial_{\mathbf{u}} U(\tau)$, whose leaves are just the trajectories of $\Phi^{*}$ (again, there is a singularity near each culvert point); $W^{u *}$ is similarly defined on $\partial_{\mathrm{s}} U(\tau)$.

Construct a "filling map" $\Theta: U(\tau) \rightarrow M$, a surjective map homotopic to inclusion, which is $1-1$ on $\operatorname{int}(U(\tau))$, which identifies faces of $\partial_{\mathbf{s}} U(\tau)$ in pairs, and which identifies faces of $\partial_{\mathbf{u}} U(\tau)$ in pairs. The filling map folds each pair of faces together along cusps. The filling map respects foliations of faces, from which it follows that $\Phi^{*}, W^{s *}$, and $W^{u *}$ induce a flow $\Phi$ and (singular) foliations $W^{s}, W^{u}$ on $M$, the desired pseudo-Anosov flow and its weak stable and unstable foliations. We think of $\Theta$ as collapsing $\mathfrak{C}(M-U(\tau))$ onto some finite 2 -complex, which will be contained in the union of stable and unstable manifolds of the pseudohyperbolic orbits of $\Phi$. The set $U(\tau)$ comes naturally equipped with a Markov partition, which induces a Markov partition of $\Phi$. In some sense, this construction is the reversal of Ratner's construction of Markov partitions [Rat73], which starts with a union of suitable chosen portions of the stable and unstable foliations of a collection of periodic orbits.

Now we turn to the details. Let $U\left(B^{s}\right), U\left(B^{u}\right), U(\tau)$ be as above. We wish to regard these objects as manifolds-with-corners, allowing for new types of singularities. First recall that a culvert edge is an "inverted cusp edge", locally modelled on the set $\{(x, y, z) \mid z \leq$ 0 or $z \geq 0,|x| \geq f(z)\}$ where $f:[0, \infty) \rightarrow[0, \infty)$ is a cusp function. Second we have a new vertex type called an outlet, whose local model is obtained by taking the intersection of the local model for a culvert edge with the set $y \geq 0$. For example, the endpoints of a culvert arc are outlets. The manifolds $U\left(B^{s}\right), U\left(B^{u}\right)$ have a culvert circle for every maw circle of $B^{s}, B^{u}$ respectively. The manifold $U(\tau)$ has a culvert arc for every switch of $\tau$; more precisely, there is a $1-1$ correspondence between converging switches of $\tau$ and culvert arcs of $U(\tau)$ which are subarcs of culvert circles of $U\left(B^{u}\right)$; similarly for diverging switches. There is a rectangle fibration $q_{U}: U(\tau) \rightarrow \tau$ which takes each culvert arc to its corresponding switch. There is a homeomorphism $w: N(\tau) \rightarrow U(\tau)$ with the following properties:

- $w$ preserves rectangle fibrations.
- $w$ is isotopic to the inclusion $\operatorname{map} N(\tau) \hookrightarrow M$.
- $w$ is smooth except over culverts; each arc of $\partial_{\mathbf{u}} N(\tau)$ or $\partial_{\mathbf{s}} N(\tau)$ on which a rectangle fiber is tangent is "unsmoothed" by $w$, to give a culvert arc of $U(\tau)$.
- There is a commutative diagram

where the horizontal maps are rectangle fibrations.
Recall that $\mathcal{R}_{-} N(\tau)$ is "parallel" to the $\mathbf{u}$-direction and $\mathcal{R}_{+} N(\tau)$ is parallel to the sdirection; we define $\partial_{\mathbf{u}} U(\tau)=w\left(\mathcal{R}_{-} N(\tau)\right)$ and $\partial_{\mathbf{s}} U(\tau)=w\left(\mathcal{R}_{+} N(\tau)\right)$.

Let $X=\left\{x_{1}, \ldots, x_{k}\right\}$ be a finite subset of $\tau$ consisting of all switches and two points in the interior of each branch of $\tau$. The closures of components of $\tau-X$ are called edges of $\tau$. We thus regard $\tau$ as a directed graph with vertex set $X$. For each $\boldsymbol{x}_{i} \in X$ we have a rectangle $R_{i}=q_{U}^{-1}\left(x_{i}\right) ;$ let $\mathcal{M}=\left\{R_{i} \mid i=1, \ldots, k\right\}$. For each edge $e=\left(x_{i} \rightarrow x_{j}\right)$ of $\tau$ we have a set $H_{e}=\operatorname{cl}\left(q_{U}^{-1}(\operatorname{int}(e))\right)$. A flow box parameterization of $H_{e}$ is a manifold-with-corners homeomorphism $H_{e} \approx I^{s} \times I^{u} \times[0,1]$ such that

- $\operatorname{Bottom}\left(H_{e}\right)=H_{e} \cap q_{U}^{-1}\left(R_{i}\right) \approx I^{s} \times I^{u} \times 0$.
- $\operatorname{Top}\left(H_{e}\right)=H_{e} \cap q_{U}^{-1}\left(R_{j}\right) \approx I^{s} \times I^{u} \times 1$.
- $H_{e} \cap \partial_{\mathbf{s}} U(\tau) \approx I^{s} \times \partial I^{u} \times[0,1]$.
- $H_{e} \cap \partial_{\mathbf{u}} U(\tau) \approx \partial I^{s} \times I^{u} \times[0,1]$.
- For each $x \times y \in I^{g} \times I^{u}$, the restricted map $x \times y \times[0,1] \rightarrow e$ is orientation preserving.

Note that $\operatorname{Bottom}\left(H_{e}\right)$ is an s-subrectangle of $R_{i}$ and $\operatorname{Top}\left(H_{e}\right)$ is a u-subrectangle of $R_{j}$.
Given a flow box parameterization of $H_{e}$ there is an induced semiflow ( $x, y, s$ ) $t=$ $(x, y, s+t)$, and an induced first return map $\operatorname{Bottom}\left(H_{e}\right) \rightarrow \operatorname{Top}\left(H_{e}\right)$ taking $(x, y, 0)$ to $(x, y, 1)$. Choosing a flow box parameterization for each $H_{e}$, the semiflows piece together to give a semiflow on $U(\tau)$ which is forward along each uu-culvert and backward along each ss-culvert. We shall now impose specific flow box parameterizations, by applying the Supereigenvalue lemma 3.1.1.

We claim that the digraph $\tau$ has no circular sinks or sources. To see why, consider the two train tracks $\tau^{s}, \tau^{u}$ constructed in the last section, and the directed maps $\kappa^{s}: \tau^{s} \rightarrow \tau$,


Figure 3.7: Each component of $\mathfrak{C}\left(U\left(B^{s}\right)-U(\tau)\right)$ is a cloven suu-maw piece, and each component of $\mathfrak{C}\left(U\left(B^{u}\right)-U(\tau)\right)$ is a cloven uss-maw piece. If this example represents an suu-maw piece, the front and back faces are labelled $\mathbf{u}$, and the remaining faces are labelled s.
$\kappa^{u}: \tau^{u} \rightarrow \tau$. If there is a circular $\operatorname{sink} c$ of $\tau$, then depending on whether $c$ is orientation preserving or reversing in $B^{s}$ (or, equivalently, $B^{u}$ ), either $c$ or its double cover lifts to a directed loop $c^{u}$ in $\tau^{u}$. It follows that $c$ corresponds to a boundary periodic orbit $\gamma$ of a pA flow carried by $B^{s}, B^{u}$, and hence $\gamma$ is a corner orbit of some dynamic torus piece component of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$. However, the loop of $\tau$ corresponding to a corner orbit cannot be a sink of $\tau$, a contradiction.

We may now apply lemma 3.1.1 and impose coordinates on each rectangle $R_{i}$, making $R_{i}$ a $V_{i} \times W_{i}$ rectangle. We may also choose flow box parameterizations of each $H_{e}$, so that the induced first return map $\bigcup \mathcal{M} \rightarrow \bigcup \mathcal{M}$ stretches the $\mathbf{u}$-direction and compresses the $s$-direction by a factor of at least $\lambda$. Using these parameterizations we obtained the desired hyperbolic flow on $U(\tau)$.

Now we describe the filling map $\Theta: U(\tau) \rightarrow M$, by describing a collapsing decomposition of $\mathfrak{C}(M-U(\tau))$, and collapsing each decomposition element to a point. Let $M_{0}=\mathfrak{C}(M-$ $\left.\left(U\left(B^{s}\right) \cup U\left(B^{u}\right)\right)\right)$, let $M_{1}^{s}=\mathfrak{C}\left(U\left(B^{s}\right)-U(\tau)\right)$, and let $M_{1}^{u}=\mathfrak{C}\left(U\left(B^{u}\right)-U(\tau)\right)$. First we decompose $M_{0}$, then we decompose $M_{1}^{s}$ and $M_{1}^{u}$.

Each component of $M_{0}$ is a dynamic solid torus or pinched tetrahedron. The collapsing decomposition of a dynamic solid torus of type $(n, k)$ is a fibration by $2 n$-sided polygons. The collapsing decomposition of a pinched tetrahedron is a rectangle fibration, with the ss and uu-cusps as degenerate rectangle fibers.

Consider now $M_{1}^{s}$. For each component $K$ of $\mathfrak{C}\left(B^{s}-\tau\right)$ there is a corresponding component of $M_{1}^{s}$ denoted $U(K)$; this is a $1-1$ correspondence of components. Despite the notation, $U(K)$ is not a neighborhood of $K$; it is more like a " 2 -handle" corresponding to the " 2 -stratum" $K$, in a stratification of $M$. Figure 3.7 together with the next lemma shows how to visualize the manifold-with-corners $U(K)$.

Lemma 3.4.2. The manifold-with-corners $U(K)$ is a cloven suu-maw piece, described as follows. There exists a surface-with-corners $F$, a labelling of each edge of $F$ with the symbol $\mathbf{s}$ or uu, and a map $f: F \times I \rightarrow U(K)$, with the following properties:

- $F$ is a topological annulus.
- One component of $\partial F$ is a smooth circle labelled $\mathbf{s}$.
- The other component of $\partial F$ is a circle with $3 k$ edges for some $k \geq 1$, labelled in order as $\ldots, \mathbf{s}, \mathbf{s}, \mathbf{u u}, \mathbf{s}, \mathbf{s}, \mathbf{u u}, \ldots$
- For each uu-edge $\alpha$ of $F$ and each $x \in \alpha$, the map $f$ collapses $x \times I$ to a single point; otherwise, and $f$ is otherwise 1-1. Also, $f(\alpha \times I)$ is a cusp edge of $U(K)$.
- Each vertex $v$ of $F$ incident to an s-edge and a uu-edge is a corner, and $f(v)$ is an suu-gable of $U(K)$.
- Each vertex $v$ of $F$ incident to two s-edges is a culvert point, locally modelled on the $x, z$-plane intersected with the local model for a culvert. Also, $f(v \times I)$ is a culvert edge of $U(K)$.
- For each s-edge $\beta, f(\beta \times I)$ is a face of $U(K)$, identified with some s-face of $M_{0}$.
- $F_{0}=f(F \times 0)$ and $F_{1}=f(F \times 1)$ are faces of $U(K)$, identified with $\mathbf{u}$-faces of $U(\tau)$.

Similarly, each component of $\mathfrak{C}\left(U\left(B^{u}\right)-U(\tau)\right)$ is a cloven uss-maw piece.
Proof. This obviously follows from the fact that $K$ is an annulus with tongues, the annulus is a face of a solid torus component of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$, one side of the annulus has no tongues, and the other side has at least one tongue.

It follows from this description that if $\beta$ is the s-circle of $F$ then $f(\beta \times I)$ is identified with the s-face of some dynamic solid torus component of $M_{0}$. Similarly, if $\beta, \beta^{\prime}$ are adjacent sedges of $F$ then $f\left(\left(\beta \cup \beta^{\prime}\right) \times I\right)$ is identified with the two s-faces of some pinched tetrahedron component of $M_{0}$. Taking the union of $U(K)$ with all such pinched tetrahedra we obtain a maw piece, so we may regard $U(K)$ as obtained from a maw piece by using a meat cleaver to remove pinched tetrahedra, and hence a "cloven maw piece".

With $f$ specified as in the lemma, we obtain an $I$-fibration of $U(K)$ with fibers of the form $f(x \times I)$ for each $x \in F$. When $x$ is in a uu-edge of $F, f$ collapses $x \times I$ to a single point on a cusp edge of $U(K)$; otherwise $f$ is injective on $x \times I$. We want to use this $I$-fibration as the collapsing decomposition for $U(K)$, but first we must alter it by isotopy so that it respects some of the existing structure on $M_{0}$ and on $U(\tau)$.

First, note that one component of $\partial U(K) \cap M_{0}$ is the face $F=f(\beta \times I)$ where $\beta$ is the s-circle of $F$. The collapsing decomposition of $M_{0}$ induces an interval fibration of


Figure 3.8: The flow on a u-face of a cloven suu-maw piece.
this face, and we require that these intervals be the $I$-fibers of $U(K)$ on $F$. Every other component of $\partial U(K) \cap M_{0}$ is a union of two s-faces of a pinched tetrahedron component of $M_{0}$, whose rectangle fibers intersect these faces in intervals, and we similarly require that these intervals be the $I$-fibers of $U(K)$ on the $\mathbf{s}$-faces.

Next, we claim that the $I$-fibration can be isotoped, relative to the union of s-faces and uu-cusps, so that the map from $F_{0}$ to $F_{1}$ induced by the $I$-fibration respects flow lines. First, note that the two corner orbits $F \cap F_{0}, F \cap F_{1}$ are oriented isotopic through $F$, using the dynamic orientation. Next, every other orbit in $F_{i}$ is attracted in backwards time to the corner orbit, and in forward time it either ends at the uu-cusp, or it hits an outlet where it bifurcates into two orbits following two su-edges, ending at two suu-gables (see figure 3.8). From this description the claim follows easily.

This completes the description of the $I$-fibration on the cloven suu-maw piece $U(K)$, which we take to be the collapsing decomposition of $U(K)$. The collapsing decomposition on a uss-maw piece is constructed similarly.

Remark. An orbit preserving map $F_{0} \rightarrow F_{1}$ which restricts to the identity on the uu-cusps cannot, in general, be made smooth. A primary obstruction to smoothness is the fact that the derivatives of the holonomy of the flow around the corner orbits, i.e. the Lyaponov numbers of the corner orbits, may not be equal. But even if these numbers are equal, there is a secondary obstruction, coming from the fact that the homeomorphism is prescribed to be the identity on the uu-cusps. For example, let $S^{1}=\mathbf{R} / \mathbf{Z}$, and consider the foliation on $S^{1} \times[0,1]$ tangent to the vector field $x d / d x+d / d y$. We leave it as an exercise to check that every diffeomorphism of $S^{1} \times[0,1]$ which respects this foliation restricts to a rigid rotation on $S^{1} \times 1$. These observations indicate why it is hard to smooth a topological pseudo-Anosov or Anosov flow.

Now collapse! Let $\Theta: M \rightarrow M$ be a quotient map from $M$ to itself which is homotopic to the identity, whose nontrivial decomposition elements are the given collapsing
decompositions in $M_{0}, M_{1}^{s}$, and $M_{1}^{u}$. Because the collapsing respects orbits of $\Phi^{*}$ and their orientations, we obtain a well-defined oriented 1-dimensional foliation $\Phi$ of $M$. Assigning an arbitrary continuous parameterization to leaves of $\Phi$, we make $\Phi$ into a flow. The collapsed image of each dynamic solid torus component of $M_{0}$ is clearly a pseudohyperbolic orbit of $\Phi$. For each $\mathbf{u}$-face of $U(\tau)$, each flow line on that $\mathbf{u}$-face is a component of intersection of that $\mathbf{u}$-face with a weak stable leaf of $U(\tau)$; since the collapsing respects flow lines, it follows that the weak stable foliation on $U(\tau)$ induces a foliation $W^{s}$ of $M$ which is singular along the pseudohyperbolic orbits. Similarly, the weak unstable foliation on $U(\tau)$ induces a foliation $W^{u}$ of $M$ which is singular along the pseudohyperbolic orbits. The flow boxes on $U(\tau)$ induce a family of flow boxes $\mathcal{M}$ in $M$. It is now straightforward to check that $\Phi$ is a topological pseudo-Anosov flow, with weak stable and unstable foliations $W^{s}, W^{u}$, and with Markov partition $\mathcal{M}$.

### 3.5 Almost transversality

In [Mos89] and [Mos91] I tried to find a surface $S$ transverse to a pseudo-Anosov flow $\Phi$, with $S$ in a given homology class $\alpha$, assuming $\alpha$ has non-negative intersection number with every periodic orbit of $\Phi$. David Gabai read these papers, and pestered me with questions: "What about this example? What about that example?" I answered with impatience: "This lemma says this; that lemma says that." Finally he asked: "What about that other example?" At which point I realized that other lemma said $1=-1$. In other words, there was a sign error in [Mos89]. Correcting this error in [Mos90] led naturally to the concept of surfaces which are "almost transverse" to pseudo-Anosov flows. The main theorem of [Mos91] says that there is a surface representing $\alpha$ which is almost transverse to $\Phi$.

To say that a surface or foliation is almost transverse to a pseudo-Anosov flow $\Phi$ means that the pseudohyperbolic orbits of $\Phi$ may be "blown up" in a certain manner, producing a new flow $\Phi^{\#}$ which is transverse to the surface or foliation. The main result of this section, theorem 3.5.4, says that if $B^{s}, B^{u}$ is a dynamic pair, if $B$ is a branched surface hierarchy, and if $B^{s}, B^{u}$ is "vertical" with respect to $B$, then a pseudo-Anosov flow carried by $B^{s}, B^{u}$ is almost transverse to a finite depth foliation carried by $B$.

Almost transversality is a delicate property, and so we offer a still useful but much simpler theorem for pA flows, proposition 3.5.3, which says that if $B^{s}, B^{u}$ is vertical with respect to $B$, and if $\Lambda^{s}, \Lambda^{u}$ are the stable and unstable laminations of a pA flow carried by $B^{s}, B^{u}$, then $\Lambda^{s}, \Lambda^{u}$ are vertical with respect to $\mathcal{F}$; in particular, they are transverse to $\mathcal{F}$.

In order to state theorem 3.5.4, our main tasks are: define dynamic blowups of pseudohyperbolic orbits and almost transversality (§3.5.1); and define vertical (§3.5.2). We shall also state proposition 3.5 .5 which gives a "vertical" version of proposition 2.6.2: if an unstable dynamic branched surface $B^{u}$ satisfying the hypotheses of proposition 2.6.2 is vertical with respect to a branched surface hierarchy $B$, then one can constuct a dynamic


Figure 3.9: If $(n, k)=(3,0)$ then there are three nontrivial dynamic blowups, all of type $2 \rightarrow 2$. If $(n, k)=(4,0)$ then: there are four blowups of type $2 \rightarrow 3$ and four of type $3 \rightarrow 2$; there are eight blowups of type $2 \rightarrow 2 \rightarrow 2$; there are four blowups of type $2 \rightarrow 2 \leftarrow 2$; and there are four blowups of type $2 \leftarrow 2 \rightarrow 2$. Only the latter eight are invariant under rotation through angle $\pi$, and so there are eight nontrivial dynamic blowups of a fixed point of type $(4,2)$.
pair which is vertical with respect to $B$.

### 3.5.1 Dynamic blowups of pseudohyperbolic orbits.

Let $f: \mathbf{C} \rightarrow \mathbf{C}$ be the standard model for a pseudohyperbolic fixed point of type $(n, k)$. Let $T=\{(r, \theta) \mid \theta=k \pi / n, k=0, \ldots, 2 n-1\} \subset \mathbf{C}$ be the union of stable and unstable prongs of $f$. Each unstable prong $\theta=2 i \pi / n$ is oriented away from $\mathcal{O}$, and each stable prong $\theta=(2 i+1) \pi / n$ is oriented towards $\mathcal{O}$. The orientation on a prong describes the direction that points move under the first return map of $f$ to that prong. Let $D$ be a small disc around $\mathcal{O}$. Let $T^{\#}$ be any oriented tree that agrees with $T$ outside $D$, such that $T^{\#}$ is invariant under the rotation $R_{k / n}$, and such that each vertex $v$ of $T^{\#}$ is "pseudohyperbolic", meaning that as you go around the edges incident to $v$, the orientations of the edges of $T^{\#}$ alternate pointing toward and away from $v$. There are finitely many ways to choose $T^{\#}$, up to compactly supported isotopy.

In order to discuss examples we define the type of $T^{\#}$ to be a labelled, oriented planar tree obtained from $T^{\#}$ by labelling a vertex with the integer $n$ if it has $2 n$ incident edges, and then throwing away the noncompact edges. When $(n, k)=(3,0),(4,0),(4,2)$ the possibilities are enumerated in figure 3.9. Note that if $n, k$ are relatively prime then there are no nontrivial ways to dynamically blow up a pseudohyperbolic fixed point of type ( $n, k$ ).


Figure 3.10: Some dynamic blowups of a pseudohyperbolic fixed point of type ( 5,0 ).

Given the above data, there is a $C^{0}$ perturbation $f^{\#}$ of $f$, and a continuous map $h: \mathbf{C} \rightarrow \mathbf{C}$ such that:

- $f^{\#}$ leaves $T^{\#}$ invariant.
- For each edge $E$ of $T^{\#}$, the first return map of $f^{\#}$ to $E$ acts as a translation on $\operatorname{int}(E)$, moving points in the direction of the orientation on $E$.
- $h$ collapses the finite edges of $T^{\#}$ to the point $\mathcal{O}$, and $h$ is otherwise $1-1$.
- $h$ is a semiconjugacy from $f^{\#}$ to $f$, i.e. $f \circ h=h \circ f^{\#}$.
- $h$ is close to the identity map in the sup norm, and $h$ equals the identity on $\mathbf{C}-D$.

We say that $f^{\#}$ is obtained from $f$ by dynamically blowing up the pseudohyperbolic fixed point $\mathcal{O}$. Each choice of $T^{\#}$ determines a unique $f^{\#}$, up to conjugation by compactly supported isotopy. There are therefore finitely many ways to dynamically blow up a pseudohyperbolic fixed point, up to conjugation by compactly supported isotopy. The number of ways depends on the type of the pseudohyperbolic fixed point $\mathcal{O}$. An example showing some invariant lines of $f^{\#}$ is given in figure 3.10.

Next we use suspension to define dynamic blowups of pseudohyperbolic orbits of flows.
Given a pseudo-Anosov flow $\Phi$ and a pseudohyperbolic orbit $\gamma$ of type $(n, k)$, a dynamic blowup of $\gamma$ is defined as follows. Choose a Poincaré section for $\gamma$, that is, a disc $D$ transverse to $\Phi$, and a subdisc $D^{\prime} \subset D$, such that $\gamma \cap D^{\prime}=\{x\} \subset \operatorname{int}\left(D^{\prime}\right)$, and there is a continuous first return map $g: D^{\prime} \rightarrow D$, i.e. there is a continuous map $t: D^{\prime} \rightarrow(0, \infty)$ such that $\Phi(x, t(x))=g(x)$ if $x \in D^{\prime}$, and $\Phi(x, s) \notin D$ if $x \in D^{\prime}, 0<s<t(x)$. Let $f: \mathbf{C} \rightarrow \mathbf{C}$ be the standard model for a pseudohyperbolic fixed point of type $(n, k)$. There is an embedding $s:(D, x) \hookrightarrow(\mathbf{C}, \mathcal{O})$ which is a local semiconjugacy from $g$ to $f$, i.e. $f \circ s=s \circ g$ on $D^{\prime}$. Now define a dynamic blowup of $\gamma$ by altering $\Phi$ near $\gamma$ as follows. First, replace $f$ by a
dynamic blowup $f^{\#}$, supported on a tiny subdisc of $s\left(D^{\prime}\right)$. Next, replace $g$ by the dynamic blowup $s^{-1} \circ f^{\#} \circ s$. Finally, alter $\Phi$ so that the first return map $g: D^{\prime} \rightarrow D$ is replaced by $g^{\#}: D^{\prime} \rightarrow D$; this has the effect of altering the generating vector field inside the "mapping torus" $T_{g}=\left\{\Phi(x, s) \mid x \in D^{\prime}, 0 \leq s \leq t(x)\right\}$, and leaving the generating vector field unaltered outside of $T_{g}$.

Dynamic blowups of distinct pseudohyperbolic orbits of $\Phi$ can be performed independently and simultaneously, by choosing sufficiently small Poincaré sections for the pseudohyperbolic orbits so that the mapping tori are pairwise disjoint. We say that the resulting flow $\Phi^{\#}$ is a dynamic blowup of $\Phi$; this terminology allows for the possibility that no orbits are blown up in which case $\Phi^{\#}$ is isotopic to $\Phi$. There is a map $H: M \rightarrow M$ homotopic to the identity, such that $H$ is a semiconjugacy from $\Phi^{\#}$ to $\Phi$, i.e. $H$ takes orbits of $\Phi^{\#}$ to orbits of $\Phi$ preserving orientation, although we do not require $H$ to preserve parameterization. The map $H$ is $1-1$ except over the pseudohyperbolic orbits of $\Phi$. For each pseudohyperbolic orbit $\gamma$, if $H$ is not $1-1$ over $\gamma$ then $H^{-1}(\gamma)$ is a connected union of invariant annuli of $\Phi^{\#}$, glued together along their boundary components, forming an invariant annulus complex associated to $\gamma$. This annulus complex may be viewed as the mapping torus of the finite edges of $T^{\#}$ under the map $f^{\#}$.

Up to isotopy and reparameterization, there are finitely many ways to dynamically blow up a pseudo-Anosov flow $\Phi$, because there are finitely many choices for a dynamic blowup of each pseudohyperbolic orbit of $\Phi$.

Given a pseudo-Anosov flow $\Phi$ and a foliation or lamination $\mathcal{F}$, we say that $\Phi$ is almost transverse to $\mathcal{F}$ if there exists a flow $\Phi^{\#}$, obtained from $\Phi$ by dynamically blowing up certain pseudohyperbolic orbits of $\Phi$, such that $\Phi^{\#}$ is transverse to $\mathcal{F}$. In general the existential quantifier cannot be replaced by a universal quantifier: almost transversality does not mean that $\mathcal{F}$ is transverse to every dynamic blowup of $\Phi$.

### 3.5.2 Vertical dynamic branched surfaces

Let $M$ be a compact, oriented 3-manifold with torus boundaries. We define what it means for a dynamic branched surface or dynamic pair to be "vertical" with respect to a transversely oriented branched surface $\beta$, or to a foliation $\mathcal{F}$ carried by $\beta$.

Let $\beta$ be a transversely oriented branched surface in $M$, such that $P(\beta)=\mathfrak{C}(M-\beta)$ is a product sutured manifold in the cusp model. If $P_{c}(\beta)$ denotes the corner model then there is a sutured manifold homeomorphism $P_{c}(\beta) \approx F \times[0,1]$; now collapse each component of $\partial F \times[0,1]$ to get back to the cusp model, and push forward the tangent planes of each surface $F \times t$ to give a $C^{0}$ tangent plane bundle $\tau_{\beta}$ on $M$ which is an extension of the tangent plane bundle of $\beta$. The transverse orientation on $\beta$ extends to a transverse orientation on $\tau_{\beta}$.

A $C^{0}$ vector field $V$ on $M$ is vertical with respect to $\beta$ if it is transverse to $\tau_{\beta}$ and the direction of $V$ agrees with the transverse orientation on $\tau_{\beta}$. Verticality between $V$ and a
transversely oriented foliation $\mathcal{F}$ is similarly defined. If $\mathcal{F}$ is carried by $\beta$, we may isotop $\mathcal{F}$ so that its tangent plane bundle is arbitrarily close to $\tau_{\beta}$ in the $C^{0}$ topology, and so we have:

Proposition 3.5.1. If $\mathcal{F}$ is a transversely oriented foliation carried by a transversely oriented branched surface $\beta$, and if the vector field $V$ is vertical with respect to $\beta$, then $V$ is vertical with respect to $\mathcal{F}$.

Now consider a dynamic branched surface ( $B, V$ ) in $M$. Suppose that $V$ is vertical with respect to $\beta$, so in particular $B$ and $\beta$ are transverse. A peripheral annulus in $\mathfrak{C}(M-B)$ carried by $\beta$ is a smoothly embedded annulus $A \subset \beta$ such that $A$ is properly embedded in $\mathfrak{C}(M-B)$ and $A$ is isotopic rel boundary in $\mathfrak{C}(M-B)$ to an annulus contained in a face of $\mathfrak{C}(M-B)$.

The dynamic branched surface $(B, V)$ is said to be vertical with respect to $\beta$ if $V$ is vertical with respect to $\beta$ and there is no annulus carried by $\beta$ which is peripheral in $\mathfrak{C}(M-B)$. A dynamic pair $B^{s}, B^{u}$ in $M$, with dynamic vector field $V$, is said to be vertical with respect to $\beta$ if $\left(B^{s}, V\right)$ and $\left(B^{u}, V\right)$ are both vertical with respect to $\beta$. Similarly, $B^{s}, B^{u}$ is vertical with respect to a transversely oriented foliation $\mathcal{F}$ if $V$ is vertical with respect to $\mathcal{F}$ and no smoothly embedded annulus in $\mathcal{F}$ is peripheral in a $\mathfrak{C}\left(M-B^{s}\right)$ or $\mathfrak{C}\left(M-B^{u}\right)$. Finally, given a pA flow $\Phi$ with stable and unstable laminations $\Lambda^{s}, \Lambda^{u}$, we say that $\Lambda^{s}, \Lambda^{u}$ are vertical with respect to $\mathcal{F}$ if $\Phi\left|\Lambda^{s}, \Phi\right| \Lambda^{u}$ are vertical with respect to $\mathcal{F}$, and there are no annuli in leaves of $\mathcal{F}$ which are peripheral in $\mathfrak{C}\left(M-\Lambda^{s}\right)$ or in $\mathfrak{C}\left(M-\Lambda^{u}\right)$.

Proposition 3.5.2. If a dynamic pair $B^{s}, B^{u}$ is vertical with respect to a transversely oriented branched surface $\beta$, then it is also vertical with respect to any foliation $\mathcal{F}$ carried by $\beta$.

Proof. If there is an annulus $A_{1}$ in a leaf of $\mathcal{F}$ which is peripheral in $\mathfrak{C}\left(M-B^{s}\right)$ or $\mathfrak{C}\left(M-B^{u}\right)$, then there is another such annulus $A_{2}$ which is contained in an $I$-bundle neighborhood $N(\beta)$; this follows from the fact that $\mathcal{F} \mid \mathfrak{C}(M-N(\beta))$ is a product foliation. Under the $I$-fiber collapsing map $N(\beta) \rightarrow \beta$ the annulus $A_{2}$ goes to an annulus $A_{3}$ smoothly carried by $\beta$ which is peripheral in $\mathfrak{C}\left(M-B^{s}\right)$ or $\mathfrak{C}\left(M-B^{u}\right)$.

Proposition 3.5.3. Let $\mathcal{F}$ be a transversely oriented foliation of $M$ carried by a transversely oriented branched surface $\beta$. Let $\Phi$ be a $p A$ flow carried by a dynamic pair $B^{s}, B^{u}$, with stable and unstable laminations $\Lambda^{s}, \Lambda^{u}$. If the pair $B^{s}, B^{u}$ is vertical with respect to $\beta$, then $\Lambda^{s}, \Lambda^{u}$ are both vertical with respect to $\mathcal{F}$.

Proof. Suppose there is an annulus $A_{1}$ in a leaf of $\mathcal{F}$ which is peripheral in, say, $\mathfrak{C}\left(M-\Lambda^{s}\right)$. Let $N\left(B^{s}\right)$ be an $I$-fibered neighborhoods with $\Lambda^{s} \subset N\left(B^{s}\right)$ transverse to $I$-fibers. Now take the annulus $A_{1} \cap \mathfrak{C}\left(M-N\left(B^{s}\right)\right)$, and collape $I$-fibers, to get an annulus in a leaf of $\mathcal{F}$ which is peripheral in $\mathfrak{C}\left(M-B^{s}\right)$. Applying the previous proposition finishes the proof. $\diamond$

Now we turn to the main result of this section:
Theorem 3.5.4 (Almost transversality theorem). Let $M$ be a closed, oriented 3-manifold, $\mathcal{F}$ a Reebless, transversely oriented foliation of $M$ transverse to $\partial M$, and $\beta$ a transversely oriented branched surface carrying $\mathcal{F}$. If $B^{s}, B^{u}$ is a dynamic pair in $M$ which is vertical with respect to $\beta$, then there is a pseudo-Anosov flow $\Phi$ carried by $B^{s}, B^{u}$ which is almost transverse to $\mathcal{F}$.

Remark. For each pseudohyperbolic orbit $\gamma$ of $\Phi$, the proof shows that the structure of $\mathcal{F}$ picks out one dynamic blowup of $\gamma$ from among the finitely many choices. More precisely, if $T$ is the dynamic solid torus component of $\mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)$ corresponding to $\gamma$, then Brittenham's theorem on laminated solid tori [Bri93] will be applied to show that $\mathcal{F} \mid T$ satisfies one of two possibilities:

- $\mathcal{F} \mid T$ is a foliation of $T$ by meridian discs.
- Letting $\left.\mathcal{F}\right|_{c} T$ be the sublamination of compact leaves of $\mathcal{F} \mid T$, each leaf of $\left.\mathcal{F}\right|_{c} T$ is an annulus disjoint from the corners of $T$.
In the first case, no dynamic blowup of $\gamma$ is needed. In the second case, the structure of the transversely oriented foliation $\left.\mathcal{F}\right|_{c} T$ determines a dynamic blowup of $\gamma$.
Remark. The proof of this theorem is valid without assuming that $B$ is a branched surface hierarchy, or that $\mathcal{F}$ has finite depth.
Remark. Proposition 3.5 . 3 cannot be strengthened to say that a pA flow carried by $B^{s}, B^{u}$ is transverse to the foliation $\mathcal{F}$, because of almost transversality. On the other hand, the proof of theorem 3.5.4 suggests a way to alter a pA flow to make it transverse to $\mathcal{F}$, by somehow generalizing the notion of almost transversality to pA flows. We will not pursue this issue, preferring to focus only on pseudo-Anosov flows.

Recall that proposition 2.6.2 tells how to construct a dynamic pair starting from an unstable dynamic branched surface. To complement proposition 3.5.3 and theorem 3.5.4 we have a "vertical" version of proposition 2.6.2:
Proposition 3.5.5. Let $M$ be a compact, oriented, torally bounded 3-manifold. Let $\beta$ be a transversely oriented branched surface in $M$, transverse to $\partial M$, and suppose that $\beta$ carries a taut, transversely oriented foliation $\mathcal{F}$ of $M$. Suppose that $(B, V, \mathcal{I})$ is an unstable Markov branched surface in $M$ satisfying the hypotheses of proposition 2.6.2, such that $(B, V)$ is vertical with respect to $\beta$, and each element of the Markov section $\mathcal{I}$ is tangent to $\beta$. Then we may perform the construction of proposition 2.6 .2 so as to produce a dynamic pair $B^{s}, B^{u}$ in $M$ which is vertical with respect to a branched surface carrying $\mathcal{F}$ and obtained by splitting $\beta$.

The proofs of theorem 3.5.4 and proposition 3.5 .5 both require a description of $\mathcal{F}$ restricted to cusped torus pieces, which is contained in $\S 3.5 .3$. The reader who is interested only in proposition 3.5 .5 should read $\S 3.5 .3$ and then go to $\S 3.7$.

### 3.5.3 Foliations of cusped torus pieces

Fix a taut, transversely oriented foliation $\mathcal{F}$, and fix a very full dynamic branched surface $B$ whose dynamic vector field $V$ is circular on each torus piece of $B$, such that $(B, V)$ is vertical with respect to $\mathcal{F}$. By perturbing $B$ we may assume that $B$ is in general position with respect to $\mathcal{F}$, in particular $\Upsilon B$ is transverse to $\mathcal{F}$ except at isolated local minima or maxima. Throughout this section we assume that $B$ is unstable, the stable case being handled similarly.

Under these conditions we describe the restriction of $\mathcal{F}$ to $\mathfrak{C}(M-B)$. We will focus our efforts on a solid torus component $T$ of $\mathfrak{C}(M-B)$, that being the only case needed for theorem 3.5.4.

We first describe $\mathcal{F} \mid T$ on a certain maw piece neighborhood of each cusp of $T$ (lemma 3.5.6), and then on the rest of $T$ (lemma 3.5.7). Consider a properly embedded annulus $A \subset T$ and a component $\nu$ of $\mathfrak{C}(T-A)$ such that $\nu$ is a maw piece with cusp circle $c$ and opposite face $A$. Let $F_{0}, F_{1}$ be the other two faces of $\nu$. An $I$-fibration of $\nu$ is a decomposition of $\nu$ into ordinary $I$-fibers and singular $I$-fibers with the following properties: there exists an annulus $F$ and a quotient map $q: \nu \rightarrow F$ such that $q \mid F_{i}: F_{i} \rightarrow F$ is a homeomorphism for $i=0,1$, each point preimage of $q$ is an $I$-fiber, each singular $I$-fiber is a point on $c$, and each ordinary $I$-fiber is an arc connecting a point of $F_{0}-c$ to a point of $F_{1}-c$. The foliation $\mathcal{F} \mid \nu$ is $I$-parallel if there exists an $I$-fibration of $\nu$ such that each $I$-fiber is contained in a leaf of $\mathcal{F} \mid \nu$.

Lemma 3.5.6. For each cusp circle $c$ of $T$ there exists a properly embedded annulus $A \subset T$ and a component $\nu$ of $\mathfrak{C}(T-A)$ such that:

1. $\nu$ is a maw piece with cusp curve $c$ and opposite face $A$.
2. $\mathcal{F} \mid \nu$ is I-parallel.
3. For every annulus leaf of $\mathcal{F} \mid \nu$, the transverse orientation on that leaf points towards $c$ and away from $A$.
4. $\mathcal{F}$ is transverse to $A$.

Moreover, $\nu$ is maximal with respect to the above properties, in the following sense: if $A^{\prime}$, $\nu^{\prime}$ also satisfy 1-4, then every compact leaf of $\mathcal{F} \mid T$ contained in $\nu \cup \nu^{\prime}$ is contained in $\nu$.

Remark. It follows that $\mathcal{F} \mid A$ is a product foliation whose leaves are arcs connecting opposite components of $\partial A$.
Remark. It follows from 3 that $\mathcal{F} \mid \nu$ has no (Reeb annulus) $\times I$ sublamination, for the two boundary leaves of such a sublamination would point in opposite directions and so one of them would point away from $c$ and towards $A$.


Figure 3.11: Singularities of a foliation along a cusp curve.


Figure 3.12: Cusp discs.

Proof. First we describe the singularities of $\mathcal{F} \mid T$, which occur only along cusps of $T$. Fix a local model for $T$ near a cusp, namely $\left\{(x, y, z) \in \mathbf{R}^{3} \mid z \geq 0,-f(z)<x<f(z)\right\}$ where $f:[0, \infty) \rightarrow[0, \infty)$ is a cusp function. Near a saddle singularity the foliation is locally modelled on the level surfaces of the function $z-y^{2}$, and near an external center singularity the foliation is locally modelled on the level surfaces of $z+y^{2}$ (see figure 3.11). A leaf of $\mathcal{F} \mid T$ is defined to be the completion of a leaf of $\mathcal{F} \mid(T-$ (singularities)). One special type of leaf is a cusp disc, shown in figure 3.12. An ordinary cusp disc is modelled on $z+y^{2}=\epsilon(\epsilon>0)$, intersected with the above local model for a cusp. A pinched cusp disc wraps all around the cusp, approaching a saddle singularity from both sides; a local model is obtained from $z+y^{2}=\epsilon$ by modding out by the action of $(x, y, z) \rightarrow(x, y+\sqrt{\epsilon}, z)$. By Reeb stability a pinched cusp disc is a limit of ordinary cusp discs. Also note that if $L$ is a pinched cusp disc, then a regular neighborhood of $L$ in the leaf of $\mathcal{F}$ containing $L$ is an annulus.

We claim that singularities exist on $c$ if and only if there is a cusp disc intersecting $c$. If a cusp disc exists then its boundary cuts off a disc in $\partial T$ which contains a center singularity, by the Euler-Poincaré index formula. Conversely, suppose there exist singularities on $c$. Since the vector field $V$ points inward on each boundary circle of each face of $T$, it follows that each face of $T$ has a circular trajectory, and so there exists an annulus $H \subset \partial T$ containing $c$ and disjoint from the other cusps of $T$ such that $\partial H$ is a union of trajectories
of $V$. The singular foliation $\mathcal{F} \mid H$ is therefore transverse to $\partial H$. By the Euler-Poincaré index formula the foliation $\mathcal{F} \mid H$ must contain an equal number of saddle and center singularities. Each center singularity is an external center singularity of $\mathcal{F} \mid T$ lying on $c$, near which there exists a cusp disc. This proves the claim.

Now we show that there exist $A, \nu$ satisfying conditions $1-3$, and either $A$ satisfies condition 4 or the following alternative condition:

$$
4^{\prime} . A \text { is a leaf of } \mathcal{F} \mid T .
$$

Case 1: There are no singularities on $c$. Clearly there exist $A, \nu$ contained in an arbitrary neighborhood of $c$ satisfying conditions $1-4$.

Case 2: $c$ contains singularities. Given a (pinched or unpinched) cusp disc $E$, let $t(E)$ be the closure of the simply connected component of $T-E$. Define a partial ordering of cusp discs by $E<E^{\prime}$ if $E \subset t\left(E^{\prime}\right)$. For each cusp disc $E$ there exists a maximal cusp disc $E^{\prime}$ such that $E<E^{\prime}$. Each maximal cusp disc contains a saddle singularity, and so there is a finite number of maximal cusp discs $E_{1}, \ldots, E_{k}$ intersecting $c$. No two of $E_{1}, \ldots, E_{k}$ are comparable with respect to the relation $<$.

Case 2a: The discs $E_{1}, \ldots, E_{k}$ are "cyclically connected" which means that $E_{i}$ and $E_{i+1}$ have a common corner at a saddle singularity for all $i \in \mathbf{Z} / k$ (this case also occurs when $k=1$ and $E_{1}$ is a pinched cusp disc). In the leaf of $\mathcal{F}$ containing $E_{1} \cup \cdots \cup E_{k}$, a regular neighborhood of $E_{1} \cup \cdots \cup E_{k}$ is an annulus. This annulus has a core curve of the form $p=p_{1} * \cdots * p_{k}$ where $p_{i} \subset E_{i}$ connects the two corners of $E_{i}$. If the holonomy of $\mathcal{F} \mid T$ around $p$ on the outside of $t\left(E_{1}\right) \cup \cdots \cup t\left(E_{k}\right)$ is nontrivial, then one can find the desired $A, \nu$ satisfying conditions $1-4$. If the holonomy is trivial then one can find $A, \nu$ satisfying conditions $1-3$ and $4^{\prime}$.

Case 2b: If the discs $E_{1}, \ldots, E_{k}$ are not cyclically connected, one can find $A$ contained in an arbitrary neighborhood of $c \cup\left(t\left(E_{1}\right) \cup \cdots \cup t\left(E_{k}\right)\right)$ and satisfying 1-4.

Having constructed $A$ satisfying $1-3$ and either 4 or $4^{\prime}$, we construct another annulus satisfying $1-4$ and the maximality requirement, as follows. Consider the set of annulus leaves $A^{\prime}$ of $\mathcal{F} \mid T$ such that $A^{\prime}, \nu^{\prime}$ satisfy $1-3$ and $4^{\prime}$ for some $\nu^{\prime}$; the set of such leaves forms a compact sublamination $\lambda$ of $\mathcal{F} \mid T$. If $\lambda=\emptyset$ then any $A$ satisfying 1-4 also satisfies maximality. If $\lambda \neq \emptyset$ then $\lambda$ contains a leaf $A^{\prime}$ bounding a maw piece $\nu^{\prime}$ such that $\lambda \subset \nu^{\prime}$; the leaf $A^{\prime}$ is the "farthest" leaf from $c$ in $\lambda$. The holonomy around $A^{\prime}$ on the outside of $\nu^{\prime}$ must be nontrivial, and so we can find an annulus $A$ just outside of $\nu^{\prime}$ satisfying 1-4 as well as the maximality condition.

For each cusp circle $c$ of $T$, choose an annulus $A_{c}$ bounding a maw piece $\nu_{c}$ satisfying conditions the conclusions of lemma 3.5.6. Truncate $T$ by removing the maw pieces $\nu_{c}$, to produce $T^{\prime}=\mathfrak{C}\left(T-\bigcup_{c} \nu_{c}\right)$. Note that $T^{\prime}$ is a manifold-with-corners, with the same corner structure as a dynamic solid torus. Note also that $\mathcal{F} \mid T^{\prime}$ is transverse to $\partial T^{\prime}$ and to the corner curves of $T^{\prime}$. We describe $\mathcal{F} \mid T^{\prime}$ by using Brittenham's theorem on laminated solid tori (theorem 3.1 of [Bri93]), as follows:

Lemma 3.5.7. Let $T^{\prime}$ be a solid torus embedded in $M$, such that $T^{\prime}$ is a manifold-withcorners of the same type as an essential dynamic solid torus. Suppose that $\mathcal{F}$ is transverse to $\partial T^{\prime}$ and to each corner circle of $T^{\prime}$. Either $\mathcal{F} \mid T^{\prime}$ is a foliation by meridian discs of $T^{\prime}$, or there is a Seifert fibration of $T^{\prime}$ such that:

1. $\mathcal{F} \mid T^{\prime}$ has at least one compact leaf.
2. Each compact leaf is tangent to the Seifert fibration.
3. Each noncompact leaf is simply connected and transverse to the Seifert fibration, and all of its boundary components are noncompact.
4. Each compact leaf is an annulus.
5. Each corner circle of $T^{\prime}$ is a Seifert fiber.

Proof. Suppose that $\mathcal{F} \mid T^{\prime}$ is not a foliation by meridian discs of $T^{\prime}$. The existence of a Seifert fibration satisfying (1-3) is an immediate consequence of Brittenham's theorem on laminated solid tori ([Bri93] 3.1). Property (4) follows from the fact that $\mathcal{F}$ is Reebless and transversely oriented. To prove (5), consider a corner circle $C$ of $T^{\prime}$. If $C$ intersects a compact leaf $A$ of $\mathcal{F} \mid T^{\prime}$ then there exist two points at which $C$ crosses $A$ in opposite directions, contradicting the fact that $\mathcal{F}$ is transversely oriented and transverse to $C$. It follows that $C$ is isotopic to any boundary circle of any compact leaf, in other words $C$ is isotopic to the generic Siefert fiber. By isotoping the Seifert fibration we may arrange that $C$ is actually a fiber.

### 3.6 Constructing almost transverse pseudo-Anosov flows

In this section we prove theorem 3.5.4. Recall the setting: $M$ is a closed, oriented 3-manifold; $\mathcal{F}$ is a taut, transversely oriented foliation of $M$ transverse to $\partial M ; \beta$ is a transversely oriented branched surface in $M$ carrying $\mathcal{F} ; B^{s}, B^{u}$ is a dynamic pair in $M$ which is vertical with respect to $\beta$.

Recall the notation used in theorem 3.4.1 to construct a pseudo-Anosov flow carried by $B^{s}, B^{u}$. There are $I$-fibered neighborhoods $U\left(B^{s}\right), U\left(B^{u}\right)$ in the cusped model whose
intersection $U(\tau)$ is a rectangle fibered neighborhood. The rectangle fiber collapsing map induces a manifold-with-corners homeomorphism

$$
\mathfrak{C}\left(M-\left(U\left(B^{s}\right) \cup U\left(B^{u}\right)\right)\right) \approx \mathfrak{C}\left(M-\left(B^{s} \cup B^{u}\right)\right)
$$

Henceforth the terms "dynamic solid torus", "pinched tetrahedron", "u or s-cusped solid torus" etc. will refer to components of $\mathfrak{C}\left(M-\left(U\left(B^{s}\right) \cup U\left(B^{u}\right)\right)\right), \mathfrak{C}\left(M-U\left(B^{s}\right)\right)$, or $\mathfrak{C}(M-$ $\left.U\left(B^{u}\right)\right)$ as appropriate. Inclusion induces a 1-1 correspondence between dynamic solid torus components of $\mathfrak{C}\left(M-\left(U\left(B^{s}\right) \cup U\left(B^{u}\right)\right)\right)$, components of $\mathfrak{C}\left(M-U\left(B^{s}\right)\right)$, and components of $\mathfrak{C}\left(M-U\left(B^{u}\right)\right)$. If $T$ is a dynamic solid torus, the corresponding components of $\mathfrak{C}(M-$ $\left.U\left(B^{s}\right)\right)$ and $\mathfrak{C}\left(M-U\left(B^{u}\right)\right)$ are denoted $T^{s}, T^{u}$, and these are cusped solid tori of the same type as $T$. The closure of each component of $\mathfrak{C}\left(T^{s}-T\right)$ is an suu-maw piece $\mu$ attached to $T$ along an $\mathbf{s}$-face of $T$, and $\mu$ is the union of a "cloven maw piece" component of $\mathfrak{C}\left(U\left(B^{s}\right)-U(\tau)\right)$ and several pinched tetrahedra of $\mathfrak{C}\left(M-\left(U\left(B^{s}\right) \cup U\left(B^{u}\right)\right)\right)$; a similar statement holds for each component of $\mathfrak{C}\left(T^{u}-T\right)$. Henceforth the term "maw piece of $T$ " refers to one of the maw pieces just described. See the leftmost diagrams in figure 3.15 for an illustration of the notation.

Let $\Phi^{*}$ be the flow on $U(\tau)$ constructed in theorem 3.4.1. Since $B^{s}, B^{u}$ is vertical with respect to $\beta$, and hence with respect to $\mathcal{F}$, it follows easily that $\Phi^{*}$ is transverse to $\mathcal{F}$. By perturbing $U\left(B^{s}\right)$ and $U\left(B^{u}\right)$ we may assume that all culvert curves of $U\left(B^{s}\right)$ and $U\left(B^{u}\right)$ (i.e. all cusp curves of $\mathfrak{C}\left(M-U\left(B^{s}\right)\right)$ and $\mathfrak{C}\left(M-U\left(B^{u}\right)\right)$ ) are in general position with respect to $\mathcal{F}$. Lemma 3.5.6 applies to each cusp of $\mathfrak{C}\left(M-U\left(B^{s}\right)\right)$ and of $\mathfrak{C}\left(M-U\left(B^{u}\right)\right)$.

To understand the idea of theorem 3.5.4, recall how a pseudo-Anosov flow $\Phi$ was constructed from $\Phi^{*}$. The key idea was to define a collapsing of $\mathfrak{C}(M-U(\tau))$ that respects $\Phi^{*}$. The collapsing was specified by cutting $\mathfrak{C}(M-U(\tau))$ into three types of pieces: torus pieces, pinched tetrahedra, and cloven maw pieces. Collapsing decompositions were constructed in each piece, respecting the restriction of $\Phi^{*}$ to the boundary of each piece, and so when each decomposition element was collapsed to a point there was a well-defined flow $\Phi$, the desired pseudo-Anosov flow.

Suppose we try to follow the same general outline, but with the further goal of getting the flow on $M$ to be transverse to $\mathcal{F}$. To do this, we want to construct a collapsing decomposition of $\mathfrak{C}(M-U(\tau))$ that not only respects $\Phi^{*} \mid \partial \mathfrak{C}(M-U(\tau))$, but also respects $\mathcal{F} \mid \mathfrak{C}(M-U(\tau))$. This added constraint presents us with an interesting technical delicacy.

Consider a dynamic solid torus $T$ contained in an s-cusped solid torus $T^{s}$ and a $\mathbf{u}$ cusped solid torus $T^{u}$. Suppose that $\mathcal{F} \mid T^{u}$ has an annulus leaf $L$. Given a uss-maw piece component $\mu$ of $\mathfrak{C}\left(T-T^{u}\right)$, the leaf $L$ could wend its way deeply into $\mu$, and the collapse of $\mu$ would not respect $L$ : the collapsing decomposition of $\mu$ is a foliation by arcs connecting the two faces of $\mu$ adjacent to the cusp, and if $L$ enters $\mu$ then it must cut transversely across some collapsing arcs. The same problem occurs in suu-maw pieces.

In an attempt to solve this problem, one could isotop $\mathcal{F}$ so as to push all annulus leaves of $\mathcal{F} \mid T^{u}$ out of the uss-maw pieces and into $T$. However, this just pushes the problem
out of uss-maw pieces and into suu-maw pieces! There is no room to push annulus leaves of $\mathcal{F} \mid T^{s}$ and $\mathcal{F} \mid T^{u}$ into $T$ simultaneously, for a very simple reason: $T \cap \partial \mathfrak{C}(M-U(\tau))$ is just the union of corner circles of $T$, and there may be an infinite number of annulus leaves whose boundaries are forced into a finite number of corner circles.

In a more serious attempt to solve the problem, we shall "dynamically blow up" the corner circles of $T$, using the pattern of annulus leaves of $\mathcal{F} \mid T$ as a guideline. These corner blowups convert the semiflow $\Phi^{*}$, whose support is $U(\tau)$, into a new semiflow $\Phi^{* \#}$, whose support is denoted $U^{\#}(\tau)$ (see figure 3.15). Now there is enough room to isotope $\mathcal{F}$ so that it looks nice in each solid torus and each maw piece. We can define collapsing decompositions of blownup solid tori, pinched tetahedra, and cloven maw pieces, which respect both $\mathcal{F}$ and $\Phi^{* \#} \mid \mathfrak{C}\left(M-U^{\#}(\tau)\right)$. The collapsing map takes $\mathcal{F}$ to itself and it takes $\Phi^{* \#}$ to the desired flow $\Phi^{\#}$, so that $\mathcal{F}$ and $\Phi^{\#}$ are transverse. The relation between $\Phi^{\#}$ and $\Phi$ is that the pseudohyperbolic orbit $\gamma$ of $\Phi$ corresponding to each dynamic solid torus $T$ is dynamically blown up in a manner determined by the annulus leaves of $\mathcal{F} \mid T$.

Thus we are led to study the foliation $\mathcal{F} \mid T$ in $\S 3.6 .1$. The results are used in $\S 3.6 .2$ to convert $T$ into $T^{\#}$ by dynamically blowing up corner orbits, to isotope $\mathcal{F} \mid T$ to a foliation on $T^{\#}$, and to define a collapsing decomposition on $T^{\#}$. In $\S 3.6 .3$ the isotopy of $\mathcal{F} \mid T$ is extended to an isotopy of all of $\mathcal{F}$, collapsing decompositions are defined on pinched tetrahedra and cloven maw pieces, and the collapse is carried out.

### 3.6.1 Foliations of dynamic solid tori

Fix a dynamic solid torus component $T$ of $\mathfrak{C}\left(M-\left(U\left(B^{s}\right) \cup U\left(B^{u}\right)\right)\right)$, with associated cusped solid tori $T^{s}, T^{u}$, and associated pseudohyperbolic orbit $\gamma$ of $\Phi$. Let $(n, k)$ be the type of $T$ and $\gamma$. In this section we assume that $\mathcal{F} \mid T$ is not a foliation by meridian discs. Fix a Seifert fibration of $T$ satisfying the conclusions of lemma 3.5 .7 with respect to the foliation $\mathcal{F} \mid T$. Let $\mathcal{O}$ be the quotient orbifold of $T$; if $k=0$ then the Seifert fibration is a product and $\mathcal{O}$ is a disc without singular points; if $k \neq 0$ then the Seifert fibration has one singular orbit and the orbifold $\mathcal{O}$ is a disc with one cone point of order $n / \operatorname{gcf}(n, k)$. The constructions that we carry out in this and subsequent sections will all be invariant with respect to the Seifert fibration on $T$.

Recall the conclusions of lemma 3.5.7. First, each leaf of $\left.\mathcal{F}\right|_{c} T$ is a properly embedded annulus in $T$ disjoint from corner circles. Second, for each noncompact leaf $L$, each end of $L$ spirals into some leaf of $\left.\mathcal{F}\right|_{c} T$, and each component of $\partial L$ is noncompact. In lemma 3.6 .1 we obtain further information by taking advantage of the transverse orientation that $\mathcal{F} \mid T$ inherits from $\mathcal{F}$.

Consider a corner circle $C$ of $T$. The faces of $\partial T$ adjacent to $C$ consist of a u-face and an s-face. Viewing $C$ as a codimension- 1 submanifold of $\partial T$, define the us-transverse orientation on $C$ to be the transverse orientation pointing from the $\mathbf{u}$-face to the $\mathbf{s}$-face. Note that the us-transverse orientation on $C$ agrees with the horizontal component of the
tangent vectors of $\Phi^{*}$ near (but not on) $C$ (see the leftmost diagrams in figure 3.15). If $L$ is a leaf of $\left.\mathcal{F}\right|_{c} T$ and $d$ is a component of $\partial L$, the transverse orientation on $L$ in $T$ induces a transverse orientation on $d$ in $\partial T$. We say that $d$ is parallel to a corner circle $C$ of $T$ if $d$ lies in one of the two faces incident to $C$, and the isotopy from $d$ to $C$ takes the transverse orientation on $d$ to the us-transverse orientation on $C$. Each boundary component of each leaf of $\left.\mathcal{F}\right|_{c} T$ is parallel to a unique corner circle of $T$.

Given two corner circles $C, C^{\prime}$ of $T$ and a leaf $L$ of $\left.\mathcal{F}\right|_{c} T$, we say that $L$ is a $\left\{C, C^{\prime}\right\}$ leaf if one component of $\partial L$ is parallel to $C$ and the other is parallel to $C^{\prime}$. The following lemma describes the key structure of $\left.\mathcal{F}\right|_{c} T$ that will be used to determine a dynamic blowup of $\gamma$.

Lemma 3.6.1. The foliation $\left.\mathcal{F}\right|_{c} T$ has the following properties.

1. For all $C, C^{\prime}$, if there exists a $\left\{C, C^{\prime}\right\}$ leaf of $\left.\mathcal{F}\right|_{c} T$ then $C, C^{\prime}$ have "opposite parity" in $\partial T$ which means that $C, C^{\prime}$ are separated by an odd number of faces of $\partial T$; in particular $C \neq C^{\prime}$.
2. For all $C, C^{\prime}$, if there exists a $\left\{C, C^{\prime}\right\}$ leaf of $\left.\mathcal{F}\right|_{c} T$ then $C, C^{\prime}$ do not bound a single face of $T$.
3. For all $C, C_{1}^{\prime}, C_{2}^{\prime}$, if $L_{i}$ is a $\left\{C, C_{i}^{\prime}\right\}$ leaf of $\left.\mathcal{F}\right|_{c} T$ for $i=1,2$, if $L$ is a leaf of $\left.\mathcal{F}\right|_{c} T$, and if $L$ separates $L_{1}$ from $L_{2}$, then there exists $C^{\prime}$ such that $L$ is a $\left\{C, C^{\prime}\right\}$ leaf.
4. For all $C, C^{\prime}$, if $L_{1}, L_{2}$ are $\left\{C, C^{\prime}\right\}$ leaves of $\left.\mathcal{F}\right|_{c} T$, if $L$ is a leaf of $\left.\mathcal{F}\right|_{c} T$, and if $L$ lies in the component of $\mathfrak{C}\left(T-\left(L_{1} \cup L_{2}\right)\right)$ containing $L_{1}$ and $L_{2}$, then $L$ separates $L_{1}$ from $L_{2}$ and $L$ is a $\left\{C, C^{\prime}\right\}$ leaf.

Example. If $T$ has type ( 3,0 ) with corner circles $C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{6}$ ordered cyclically, then lemma 3.6.1 implies that the possible types of annulus leaves are: $\left\{C_{1}, C_{4}\right\},\left\{C_{2}, C_{5}\right\}$, and $\left\{C_{3}, C_{6}\right\}$, and these are mutually exclusive. Compare the fact that a pseudo-hyperbolic orbit of type $(3,0)$ has three non-trivial dynamic blowups.

Proof. Statement (1) follows from the fact that if $L$ is a leaf of $\left.\mathcal{F}\right|_{c} T$ with boundary circles $d, d^{\prime}$ parallel to corner circles $C, C^{\prime}$ respectively, then $d, d^{\prime}$ with their transverse orientations are anti-isotopic in $\partial T$, and so $C, C^{\prime}$ with their us-transverse orientations are anti-isotopic, which implies that they have opposite parity.

Statements (2-4) of the lemma are each a consequence of the following (see figure 3.13):
Sublemma 3.6.2. Let $C$ be a corner circle of $T$, let $F$ be a face of $T$ incident to $C$, let $\mu$ be the maw piece attached to $T$ along $F$, and let $F^{\prime}$ be the other face of $\mu$ incident to $C$. Let $L$ be a leaf of $\left.\mathcal{F}\right|_{c} T$ having a boundary circle $d$ in $F$, let $L^{\prime}$ be the leaf of $\left.\mathcal{F}\right|_{c} \mu$ such that $d \subset \partial L^{\prime}$, and let $d^{\prime}$ be the other component of $\partial L^{\prime}$. Then $d$ is parallel to $C$ if and only if $d^{\prime} \subset F^{\prime}$.


Figure 3.13: $d$ is parallel to $C$ if and only if $d^{\prime} \subset F^{\prime}$.

Proof. Note that $d^{\prime} \not \subset F$, for otherwise $\mathcal{F}$ is tangent to $F$ at some point in the subannulus of $F$ bounded by $d \cup d^{\prime}$, but we know that $\mathcal{F}$ is transverse to $\partial T$. It follows that $d^{\prime}$ is contained in one of the two faces of $\mu$ incident to the cusp circle.

We give the proof in the case that $F$ is a $\mathbf{u}$-face of $T$; the other case is similar. In this case the us-orientation on $C$ points out of $F$. Also, all trajectories of the dynamic vector field in $\mu$ are oriented from the cusp curve towards $F$, and so the transverse orientation on $d^{\prime}$ points towards $F$. From this we obtain the following chain of equivalent statements:

- $d$ is parallel to $C$.
- The transverse orientation on $L^{\prime}$ points towards the component of $\mathfrak{C}\left(\mu-L^{\prime}\right)$ containing $C$.
- $d^{\prime} \subset F^{\prime}$.

Now we prove lemma 3.6.1.
To prove statement (2) suppose there is a leaf $L$ of $\left.\mathcal{F}\right|_{c} T$ with boundary circles $d_{1}, d_{2}$ parallel to corner circles $C_{1}, C_{2}$, such that $C_{1} \cup C_{2}=\partial F$ for some face $F$ of $T$. We show that there is an annulus $\hat{L}$ in a leaf of $\mathcal{F}$ that is peripheral in $T^{s}$ or $T^{u}$, contradicting that $B^{s}, B^{u}$ are vertical with respect to $\mathcal{F}$. For $i=1,2$, let $F_{i}$ be the face of $T$ incident to $F$ along $C_{i}$, let $\mu_{i}$ be the maw piece attached to $T$ along $F_{i}$, and let $F_{i}^{\prime}$ be the face of $\mu_{i}$ incident to $F_{i}$ along $C_{i}$, so $F_{1}^{\prime} \cup F \cup F_{2}^{\prime}$ is a face of $T^{s}$ or $T^{u}$. If $d_{i} \subset F_{i}$, let $L_{i}^{\prime}$ be the leaf of $\left.\mathcal{F}\right|_{c} \mu_{i}$ containing $d_{i}$, and let $d_{i}^{\prime}$ be the other component of $L_{i}^{\prime}$; applying the sublemma it follows that $d_{i}^{\prime} \subset F_{i}^{\prime}$. To construct the annulus $\hat{L}$ that contradicts verticality, there are four
cases, depending on whether $d_{1} \subset F$ or $F_{1}$, and whether $d_{2} \subset F$ or $F_{2}$. If $d_{1}, d_{2} \subset F$ then $\hat{L}=L$; if $d_{1} \subset F$ and $d_{2} \subset F_{2}$ then $\hat{L}=L \cup L_{2}^{\prime}$; if $d_{1} \subset F_{1}$ and $d \subset F$ then $\hat{L}=L_{1}^{\prime} \cup L$; finally if $d_{1} \subset F_{1}$ and $d_{2} \subset F_{2}$ then $\hat{L}=L_{1}^{\prime} \cup L \cup L_{2}^{\prime}$.

To prove statement (3) let $d_{i}$ be the boundary circle of $L_{i}$ parallel to $C$, and let $d$ be the boundary circle of $L$ between $d_{1}$ and $d_{2}$. Let $F_{j}, j=1,2$ be the faces of $T$ incident to $C$, let $\mu_{j}$ be the maw piece attached to $T$ along $F_{j}$, and let $F_{j}^{\prime}$ be the other face of $\mu_{j}$ incident to $C$. If $d_{i} \subset F_{j}$, let $L_{i}^{\prime}$ be the leaf of $\left.\mathcal{F}\right|_{c} \mu_{j}$ containing $d_{i}$, and let $d_{i}^{\prime}$ be the opposite component of $\partial L_{i}^{\prime}$; applying the sublemma we have $d_{i}^{\prime} \subset F_{j}^{\prime}$; similarly, if $d \subset F_{j}$ let $L^{\prime}$ be the leaf of $\left.\mathcal{F}\right|_{c} \mu_{j}$ containing $d$ and let $d^{\prime}$ be the opposite component of $\partial L^{\prime}$. Again there are cases, depending on whether $d_{1} \subset F_{1}$ or $F_{2}$, and whether $d_{2} \subset F_{1}$ or $F_{2}$. If $d_{1}, d_{2} \subset F_{1}$ then $L^{\prime}$ is forced to lie between $L_{1}^{\prime}$ and $L_{2}^{\prime}$ in $\mu_{1}$ and so $d^{\prime}$ lies between $d_{1}^{\prime}$ and $d_{2}^{\prime}$ in $F_{1}^{\prime}$; it follows by the sublemma that $d$ is parallel to $C$. The case where $d_{1}, d_{2} \subset F_{2}$ is similar. If $d_{1} \subset F_{1}$ and $d_{2} \subset F_{2}$ then $d$ is forced to lie either in $F_{1}$ between $d_{1}$ and $C$, or in $F_{2}$ between $d_{2}$ and $C$; we consider only the former case, from which it follows that $L^{\prime}$ is lies between $L_{1}^{\prime}$ and $C$ in $\mu_{1}$, and so $d^{\prime}$ lies between $d_{1}^{\prime}$ and $C$ in $F_{1}^{\prime}$; it follows by the sublemma that $d$ is parallel to $C$. The case where $d_{1} \subset F_{2}$ and $d_{2} \subset F_{1}$ is similar.

To prove statement (4) note that if $L$ does not separate $L_{1}$ from $L_{2}$ then from (3) it follows that $L$ is either a $\{C, C\}$ leaf or a $\left\{C^{\prime}, C^{\prime}\right\}$ leaf, both of which violate (1). Therefore $L$ separates $L_{1}$ from $L_{2}$, and from (3) it follows that $L$ is a $\left\{C, C^{\prime}\right\}$ leaf.

### 3.6.2 Using foliations to specify dynamic blowups

We continue with the notation of the previous section: a dynamic solid torus $T$ of $\mathfrak{C}(M-$ $\left(U\left(B^{s}\right) \cup U\left(B^{u}\right)\right)$ ) such that $\mathcal{F} \mid T$ is not a meridian disc foliation, and the objects accompanying $T$. We use the structure of $\left.\mathcal{F}\right|_{c} T$ to carry out the following steps, which are visualized in figure 3.14:

Step 1 Construct a certain annulus complex $\mathcal{A}=\mathcal{A}_{\gamma}$ in $T$ (figure 3.14a).
Step 2 Dynamically blow up the corners of $T$ to produce a new torus $T^{\#}$, thereby replacing $\Phi^{*}$ with a "blown up" flow $\Phi^{* \#}$ (see figure 3.15, and figure 3.14b).
Step 3 Isotope $\mathcal{F} \mid T$ to give a foliation in $T^{\#}$ with good properties with respect to $\Phi^{* \#}$ (figure 3.14c).

Step 4 Define a collapsing decomposition of $T^{\#}$ whose quotient is $\mathcal{A}$ (figure 3.14d).
In the next section $\S 3.6 .3$, after the collapsing decomposition is extended over pinched tetrahedra and cloven maw pieces, and the collapse is carried out, the annulus complex $\mathcal{A}$ will be precisely the annulus complex arising from some dynamic blowup of $\gamma$, as shown in figure 3.14 d .

Now we begin to implement the above steps.

Consider the foliation $\left.\mathcal{F}\right|_{c} T$. Note that if $L, L^{\prime}$ are two $\left\{C, C^{\prime}\right\}$ leaves, and if $K$ is the component of $\mathfrak{C}\left(T-\left(L \cup L^{\prime}\right)\right)$ containings $L, L^{\prime}$, then by lemma 3.6.1 it follows that each leaf of $\left.\mathcal{F}\right|_{c} K$ is a $\left\{C, C^{\prime}\right\}$ leaf, and by lemma 3.5.7 it follows that each noncompact leaf of $\mathcal{F} \mid K$ is caught between two $\left\{C, C^{\prime}\right\}$ leaves, with two ends spiralling into two different $\left\{C, C^{\prime}\right\}$ leaves. Therefore, for each $C, C^{\prime}$ there is a connected subset $\mathcal{F}\left(C, C^{\prime}\right)$ of $T$ consisting of all $\left\{C, C^{\prime}\right\}$ leaves, together with any noncompact leaves of $\mathcal{F} \mid T$ caught between two $\left\{C, C^{\prime}\right\}$ leaves. Note that $\mathcal{F}\left(C, C^{\prime}\right)=\mathcal{F}\left(C^{\prime}, C\right)$. The foliation $\mathcal{F}\left(C, C^{\prime}\right)$ is either empty, or a single $\left\{C, C^{\prime}\right\}$ leaf, or the subset of $T$ between two "outermost" $\left\{C, C^{\prime}\right\}$ leaves. Note that if $\left\{C_{0}, C_{0}^{\prime}\right\} \neq\left\{C_{1}, C_{1}^{\prime}\right\}$ then $\mathcal{F}\left(C_{0}, C_{0}^{\prime}\right) \cap \mathcal{F}\left(C_{1}, C_{1}^{\prime}\right)=\emptyset$. The results of lemma 3.6.1 can be translated into this new language: if $\mathcal{F}\left(C, C^{\prime}\right) \neq \emptyset$ then $C, C^{\prime}$ are separated by an odd number $\geq 3$ of faces of $T$; and if $\mathcal{F}\left(C, C_{1}\right)$ and $\mathcal{F}\left(C, C_{2}\right)$ are separated by some $\mathcal{F}\left(C_{1}^{\prime}, C_{2}^{\prime}\right)$ then either $C=C_{1}^{\prime}$ or $C=C_{2}^{\prime}$.

Step 1: Constructing the annulus complex $\mathcal{A}$ The annulus complex $\mathcal{A}$ will be dual to the collection of subsets $\left\{\mathcal{F}\left(C, C^{\prime}\right)\right\}$ of $T$, as follows. Let $\mathcal{E}=\bigcup_{C, C^{\prime}} \mathcal{F}\left(C, C^{\prime}\right)$. The annulus complex $\mathcal{A}$ has one circle for each component of $\mathfrak{C}(T-\mathcal{E})$, and one annulus $\mathcal{A}\left(C, C^{\prime}\right)$ for each nonempty component $\mathcal{F}\left(C, C^{\prime}\right)$ of $\mathcal{E}$. Also, $\mathcal{A}$ will be invariant with respect to the Seifert fibration on $T$-in other words, $\mathcal{A}$ will be a union of Seifert fibers. An example is given in figure 3.14a. Here is a precise description.

Consider first a component $K$ of $\mathfrak{C}(T-\mathcal{E})$. There are finitely many leaves in $\left.\mathcal{F}\right|_{c} K$, each of which is an annulus in $\partial K$ where $K$ intersects some $\mathcal{F}\left(C, C^{\prime}\right)$. The Seifert fibration of $T$ restricts to a Seifert fibration of $K$ such that each leaf of $\left.\mathcal{F}\right|_{c} K$ is a union of Seifert fibers, and each noncompact leaf of $\mathcal{F} \mid K$ is transverse to the Seifert fibration. Since $K$ is a solid torus, there is at most one nongeneric Seifert fiber; let $\gamma_{K}$ be the nongeneric fiber if it exists, or a generic fiber in $\operatorname{int}(K)$ otherwise. Also, for each leaf $L$ of $\left.\mathcal{F}\right|_{c} K$, let $\mathcal{A}_{K, L}$ be an annulus transverse to $\mathcal{F} \mid K$ with one boundary circle on $\gamma_{K}$, the other boundary circle in $\operatorname{int}(L)$, such that $\mathcal{A}_{K, L}$ is a union of Seifert fibers.

Next consider some nonempty $\mathcal{F}\left(C, C^{\prime}\right)$. Let $K_{1}, K_{2}$ be the two components of $\mathfrak{C}(T-\mathcal{E})$ incident to $\mathcal{F}\left(C, C^{\prime}\right)$, and let $L_{i}=K_{i} \cap \mathcal{F}\left(C, C^{\prime}\right)$. Let $\mathcal{A}\left(C, C^{\prime}\right)$ be an annulus with the following properties:

- $\mathcal{A}\left(C, C^{\prime}\right) \subset K_{1} \cup \mathcal{F}\left(C, C^{\prime}\right) \cup K_{2}$
- $\mathcal{A}\left(C, C^{\prime}\right) \cap K_{i}=\mathcal{A}_{K_{i}, L_{i}}$
- $\mathcal{A}\left(C, C^{\prime}\right) \cap \mathcal{F}\left(C, C^{\prime}\right)$ is a properly embedded annulus in $\mathcal{F}\left(C, C^{\prime}\right)$, transverse to the foliation there.
- $\mathcal{A}\left(C, C^{\prime}\right)$ is a union of Seifert fibers.

Altogether we get $\mathcal{A}\left(C, C^{\prime}\right) \subset \operatorname{int}(T)$ and $\mathcal{A}\left(C, C^{\prime}\right)$ is transverse to $\mathcal{F} \mid T$. We remark that $\mathcal{A}\left(C, C^{\prime}\right)$ is a function of the unordered pair $\left\{C, C^{\prime}\right\}$, that is $\mathcal{A}\left(C, C^{\prime}\right)=\mathcal{A}\left(C^{\prime}, C\right)$.


Figure 3.14: The foliation $\mathcal{F} \mid T$ determines a dynamic blowup of $\gamma$. Each compact leaf is either a $\left\{C, C_{1}\right\}$ leaf or a $\left\{C, C_{2}\right\}$ leaf. The figure shows a meridian disc of $T$ intersected with the various features. The sublaminations $\mathcal{F}\left(C, C_{1}\right)$ and $\mathcal{F}\left(C, C_{2}\right)$ are drawn with thick lines. Figure (a) shows the annulus complex $\mathcal{A}=\mathcal{A}\left(C, C_{1}\right) \cup \mathcal{A}\left(C, C_{2}\right)$. Figure (b) shows the result of blowing up the corners $C, C_{1}, C_{2}$, converting $T$ into $T^{\#}$. Figure (c) shows $\mathcal{F}\left(C, C_{1}\right)$ and $\mathcal{F}\left(C, C_{2}\right)$ after they have been isotoped into $R\left(C, C_{1}\right)$ and $R\left(C, C_{2}\right)$; components of $\mathfrak{C}\left(T^{\#}-\left(R\left(C, C_{1}\right) \cup R\left(C, C_{2}\right)\right)\right)$ are shaded. Figure (d) shows the result of collapsing $T^{\#}$ onto $\mathcal{A}$.

Define the annulus complex to be

$$
\mathcal{A}=\bigcup_{C, C^{\prime}} \mathcal{A}\left(C, C^{\prime}\right)
$$

and note that it is transverse to $\mathcal{F} \mid T$ and invariant under the Seifert fibration on $T$.
Step 2: Blowing up corners of $T$ Given a corner circle $C$ of $T$, consider the set of corner circles $C^{\prime}$ such that $\mathcal{F}\left(C, C^{\prime}\right) \neq \emptyset$. This set may be enumerated as $C_{1}^{\prime}, \ldots, C_{n(C)}^{\prime}$, where the enumeration is chosen so that $\mathcal{F}\left(C, C_{i}^{\prime}\right)$ separates $\mathcal{F}\left(C, C_{i-1}^{\prime}\right)$ from $\mathcal{F}\left(C, C_{i+1}^{\prime}\right)$ in $T$, and so that the enumeration goes from the $\mathbf{u}$-direction to the $\mathbf{s}$-direction (see figure 3.14a).

For each corner circle $C$ alter $\Phi^{*}$ by doing an $n(C)$-fold corner blowup of $C$. This means that $\Phi^{*}$ is altered by replacing $C$ with an invariant annulus, so that the interior of the annulus has $n-1$ periodic orbits cutting it into $n$ invariant annuli, each with no periodic orbits in the interior (see figures 3.15 and 3.14 b ). We refer to the newly created annuli as corner annuli, and we denote them in order as $H\left(C, C_{1}^{\prime}\right), \ldots, H\left(C, C_{n(C)}^{\prime}\right)$. The altered version of $T$ is denoted $T^{\#}$. These alterations may be done independently on all the dynamic solid tori of $B^{s}, B^{u}$; the altered versions of $\Phi^{*}, U\left(B^{s}\right), U\left(B^{s}\right)$, and $U(\tau)$ are denoted $\Phi^{* \#}, U^{\#}\left(B^{s}\right), U^{\#}\left(B^{u}\right)$, and $U^{\#}(\tau)$.

Note that $H\left(C, C^{\prime}\right)$ is a function of the ordered pair $\left(C, C^{\prime}\right)$, and $H\left(C, C^{\prime}\right) \neq H\left(C^{\prime}, C\right)$. The corner annuli are therefore naturally paired up: associated to each unordered pair $\left\{C, C^{\prime}\right\}$ such that $\mathcal{F}\left(C, C^{\prime}\right) \neq \emptyset$, there is a pair of corner annuli $H\left(C, C^{\prime}\right), H\left(C^{\prime}, C\right)$.

Step 3: Isotop $\mathcal{F} \mid T$ to a foliation of $T^{\#} \quad$ By abusing notation, we will denote the result of this isotopy as $\mathcal{F} \mid T^{\#}$, although the isotopy will not be extended over all of $\mathcal{F}$ until the following section. The goal is to arrange that the restriction of $\mathcal{F} \mid T^{\#}$ to corner annuli is transverse to $\Phi^{* \#}$ (see figure 3.14c). To do this, isotop $\mathcal{F}\left(C, C^{\prime}\right)$ so that the boundary circles parallel to $C$ move into the interior of $H\left(C, C^{\prime}\right)$ and the boundary circles parallel to $C^{\prime}$ move into the interior of $H\left(C^{\prime}, C\right)$. This isotopy can be accomplished so that the restriction of $\mathcal{F}$ to each corner annuli of $\partial T^{\#}$ is transverse to $\Phi^{* \#}$ - to see why, it suffices to note that $\mathcal{F} \mid \partial T$ has no Reeb annulus whose boundary circles are both parallel to the same corner orbit, because the boundary circles of a Reeb annulus are anti-isotopic (see figure 3.16).

Step 4: A collapsing decomposition of $T^{\#}$ with quotient $\mathcal{A}$ First we partition $T^{\#}$ into certain subsets, which will be the preimages of the various circles and annuli of $\mathcal{A}$ under the collapse.

Given corner circles $C, C^{\prime}$ such that $\mathcal{F}\left(C, C^{\prime}\right) \neq \emptyset$, let $R\left(C, C^{\prime}\right)$ be the image of an embedding $S^{1} \times I \times J \hookrightarrow T^{\#}$, with $I=J=[0,1]$, such that:


Figure 3.15: For each dynamic torus piece component $T$ of $\mathfrak{C}\left(M-\left(U\left(B^{s}\right) \cup U\left(B^{u}\right)\right)\right.$, and for each corner orbit $C$ of $T$, the flow $\Phi^{*}$ on $U(\tau)$ can be altered by blowing up $C$ into any number of invariant annuli; we choose the number of invariant annuli to be $n(C)=$ the number of corner orbits $C^{\prime}$ such that $\mathcal{F}\left(C, C^{\prime}\right) \neq \emptyset$ (see figure 3.14 b ). The effect of corner blowups on various objects is indicated with the symbol "\#".


Figure 3.16: The restriction of $\Phi^{* \#}$ and $\mathcal{F}$ to the corner annuli of $T^{\#}$ associated to a corner orbit $C$, viewed from inside $T^{\#}$. In this example $n(C)=2$. When the $C$-parallel circles of $\mathcal{F}\left(C, C_{1}^{\prime}\right), \ldots, \mathcal{F}\left(C, C_{n(C)}^{\prime}\right)$ are isotoped into $H\left(C, C_{1}^{\prime}\right), \ldots, H\left(C, C_{n(C)}^{\prime}\right)$ respectively, the foliation $\mathcal{F} \mid T^{\#}$ is transverse to $\Phi^{* \#}$ on each corner annulus, because the restriction of $\mathcal{F}$ to a subannulus of $\partial T$ bounded by two $C$-parallel circles cannot be a Reeb annulus.

- $S^{1} \times I \times \partial J \approx H\left(C, C^{\prime}\right) \cup H\left(C^{\prime}, C\right)$.
- $S^{1} \times I \times \frac{1}{2} \approx \mathcal{A}\left(C, C^{\prime}\right)$
- $\mathcal{F}\left(C, C^{\prime}\right) \subset S^{1} \times \operatorname{int}(I) \times J$.
- $\mathcal{F} \mid S^{1} \times I \times J$ has the form $f \times J$ where $f$ is a foliation of $S^{1} \times I$ transverse to $\partial\left(S^{1} \times I\right)$.
- The map $S^{1} \times I \times 0 \rightarrow S^{1} \times I \times 1$, given by $(s, t, 0) \mapsto(s, t, 1)$, is a topological conjugacy between the restrictions of $\Phi^{* \#}$.

We make the following additional requirements on how $R\left(C_{0}, C_{0}^{\prime}\right)$ and $R\left(C_{1}, C_{1}^{\prime}\right)$ intersect:

- If $C_{0}=C_{1}=C$ and if $H\left(C, C_{0}^{\prime}\right), H\left(C, C_{1}^{\prime}\right)$ are adjacent corner annuli associated to $C$, then $R\left(C, C_{0}^{\prime}\right) \cap R\left(C, C_{1}^{\prime}\right)$ is a common annulus on the boundary of each, connecting a circle of $\mathcal{A}$ to the circle $H\left(C, C_{0}^{\prime}\right) \cap H\left(C, C_{1}^{\prime}\right)$. Moreover the $J$-directions of $R\left(C, C_{0}^{\prime}\right)$, $R\left(C, C_{1}^{\prime}\right)$ agree along their intersection.
- If $C_{0}, C_{0}^{\prime}, C_{1}, C_{1}^{\prime}$ are all distinct, and if there are no compact leaves between $\mathcal{F}\left(C_{0}, C_{0}^{\prime}\right)$ and $\mathcal{F}\left(C_{1}, C_{1}^{\prime}\right)$, then $R\left(C_{0}, C_{0}^{\prime}\right) \cap R\left(C, C_{1}^{\prime}\right)$ is a circle of $\mathcal{A}$.
- Otherwise, $R\left(C_{0}, C_{0}^{\prime}\right) \cap R\left(C_{1}, C_{1}^{\prime}\right)=\emptyset$.

Define a collapsing decomposition of $T^{\#}$ as follows. On each $R\left(C, C^{\prime}\right) \approx S^{1} \times I \times J$, for each $x \times t \in S^{1} \times I$ the arc $x \times t \times J$ is a decomposition element, which is collapsed onto the point $\boldsymbol{x} \times \boldsymbol{t} \times \frac{1}{2}$ where it intersects $\mathcal{A}\left(C, C^{\prime}\right)$. Note that this decomposition is well-defined where two of the sets $R\left(C_{0}, C_{0}^{\prime}\right), R\left(C_{1}, C_{1}^{\prime}\right)$ intersect. Each component $K$ of $\mathfrak{C}\left(T^{\#}-\bigcup_{C, C^{\prime}} R\left(C, C^{\prime}\right)\right)$ is a solid torus manifold-with-corners, and $\mathcal{F} \mid K$ is a polygon fibration of $K$; each leaf of $\mathcal{F} \mid K$ is a decomposition element, which is collapsed onto the unique point where it intersects $\mathcal{A}$. Again, this decomposition is well-defined where $K$ intersects any $R\left(C, C^{\prime}\right)$.

Note that the collapsing decomposition of $T^{\#}$ respects the restriction of $\Phi^{* \#}$ to corner annuli, and it respects $\mathcal{F} \mid T^{\#}$. Since $\Phi^{* \#}$ and $\mathcal{F}$ are transverse on corner annuli, at the very least it is clear that when $T^{\#}$ is collapsed onto $\mathcal{A}$, we will have $\Phi^{\#} \mid \mathcal{A}$ transverse to $\mathcal{F} \mid \mathcal{A}$.

### 3.6.3 Finishing the construction

The remaining tasks: extend the partially defined isotopy to all of $\mathcal{F}$; and extend the partially defined collapsing decomposition to all of $\mathfrak{C}\left(M-U^{\#}(\tau)\right)$.

The set $\mathfrak{C}(M-U(\tau))$ is decomposed into solid torus pieces, pinched tetrahedra, and cloven maw pieces. Blowing up corner circles of dynamic solid tori converts $U(\tau)$ and $\Phi^{*}$ into $U^{\#}(\tau)$ and $\Phi^{* \#}$, but it has no effect on pinched tetrahedra and cloven maw pieces.

Recall that in step 3 , for each dynamic solid torus $T$ such that $\mathcal{F} \mid T$ is not a meridian disc foliation, we isotoped $\mathcal{F} \mid T$ to give a foliation $\mathcal{F} \mid T^{\#}$, and in step 4 we defined a collapsing decomposition of $T^{\#}$.

Next, given a dynamic solid torus $T$ such that $\mathcal{F} \mid T$ is a meridian disc foliation, set $T^{\#}=T$ and define the collapsing decomposition elements to be the leaves of $\mathcal{F} \mid T$.

To continue the process we carry out the following steps:
Step 5 The isotopy from $\mathcal{F} \mid \bigcup\{T\}$ to $\mathcal{F} \mid \bigcup\left\{T^{\#}\right\}$ is extended to each suu-maw piece $\mu$, so that $\mathcal{F} \mid \mu$ is $I$-parallel, and $\mathcal{F} \mid \partial_{\mathbf{u}} \mu$ is transverse to $\Phi^{* \#} \mid \partial_{\mathbf{u}} \mu$.

Step 6 Repeat step 5 for each uss-maw piece, without disturbing the results of step 5 in the suu-maw pieces. The delicate issue here is that suu-maw pieces and uss-maw pieces are typically not disjoint, intersecting in some number of pinched tetrahedra. Putting together steps $3,5,6$ we get an isotopy from $\mathcal{F} \mid \mathfrak{C}(M-U(\tau))$ to $\mathcal{F} \mid \mathfrak{C}\left(M-U^{\#}(\tau)\right)$. This is easily extended to an isotopy on all of $M$ so that $\mathcal{F} \mid U^{\#}(\tau)$ is transverse to $\Phi^{\#}$.

Step 7 Define collapsing decompositions on pinched tetrahedra, and on cloven maw pieces, so that the collapsings respect both $\mathcal{F}$ and $\Phi^{* \#}$.

Having completed step 7 , we then collapse, taking $\Phi^{* \#}$ to the dynamically blown up pseudoAnosov flow $\Phi^{\#}$. We will then have proved that $\mathcal{F}$ is transverse to $\Phi^{\#}$.

Step 5: suu-maw pieces Let $T^{\#}$ be a dynamic solid torus with blown up corner annuli in $U^{\#}(\tau)$. Let $\mu$ be an suu-maw piece attached to $T^{\#}$ along an s-face $F$, let $F_{1}, F_{2}$ be the $\mathbf{u}$-faces of $\mu$, let $c=F_{1} \cap F_{2}$ be the cusp curve, and let $C_{i}=F \cap F_{i}$ be the corner circles, $i=1,2$. Choose a properly embedded annulus $A \subset \mu$ and a component $\nu$ of $\mathfrak{C}(\mu-A)$ satisfying the conclusions of lemma 3.5.6, in particular $\mathcal{F} \mid \nu$ is $I$-parallel. Note that if $A$ is an annulus leaf of $\mathcal{F} \mid \nu$, and if $A \cap T \neq \emptyset$, then $A \cap T$ is a $\left\{C_{1}, C_{2}\right\}$ leaf, in violation of condition 2 of lemma 3.6.1. Therefore each annulus leaf of $\mathcal{F} \mid \nu$ is contained in $\mu-F$, and so we may assume that $\nu \subset \mu-F$. Let $\mu^{\prime}=\mathfrak{C}(\mu-\nu)$, a (rectangle) $\times S^{1}$ manifold-with-corners.

We claim that before the isotopy from $\mathcal{F} \mid T$ to $\mathcal{F} \mid T^{\#}$, either $\left.\mathcal{F}\right|_{c} \mu^{\prime}=\mathcal{F} \mid \mu^{\prime}$ is a foliation by meridian discs, or there exist disjoint sublaminations $\mathcal{F}\left(\mu, C_{1}\right), \mathcal{F}\left(\mu, C_{2}\right)$ of $\mathcal{F} \mid \mu^{\prime}$ with the following properties:

- $\left.\mathcal{F}\right|_{c} \mu^{\prime} \subset \mathcal{F}\left(\mu, C_{1}\right) \cup \mathcal{F}\left(\mu, C_{2}\right)$.
- Every compact leaf in $\mathcal{F}\left(\mu, C_{i}\right)$ has one boundary circle in $F$ and the other in $F_{i}$; we say that such a leaf cuts off the corner circle $C_{i}$.
- Either $\mathcal{F}\left(\mu, C_{i}\right)$ is empty, or it consists of a single annulus leaf cutting off $C_{i}$, or $\mathcal{F}\left(\mu, C_{i}\right)$ is the region of $\mu^{\prime}$ between two annulus leaves cutting off $C_{i}$.
- We can choose $A$ and $\nu$ so that $A$ is contained in an arbitrarily small neighborhood of $F \cup \mathcal{F}\left(\mu, C_{1}\right) \cup \mathcal{F}\left(\mu, C_{2}\right)$.

To justify this claim, consider a leaf $L$ of $\left.\mathcal{F}\right|_{c} \mu^{\prime}$. The two boundary circles of $L$ cannot lie in the same face of $\mu^{\prime}$, and neither of them can lie in $A$. By maximality of $\nu$, there cannot be one boundary circle in $F_{1}$ and the other in $F_{2}$. It follows that $L$ has one boundary circle in $F$ and the other in $F_{1}$ or $F_{2}$, and so $L$ cuts off $C_{1}$ or $C_{2}$.

The isotopy from $\mathcal{F} \mid T$ to $\mathcal{F} \mid T^{\#}$ moves circle leaves of $\mathcal{F} \mid F$ out of $F$ and into corner annuli of $T^{\#}$. We can now alter this, to obtain an isotopy from $\mathcal{F} \mid(T \cup \mu)$ to $\mathcal{F} \mid\left(T^{\#} \cup \mu\right)$, which moves $\mathcal{F}\left(\mu, C_{1}\right) \cup \mathcal{F}\left(\mu, C_{2}\right)$ out of $\mu$, leaving an $I$-parallel foliation in $\mu$ (see figure 3.17), so that the restriction of $\mathcal{F}$ to the union of corner annuli and $\mathbf{u}$-faces of $\mu$ is transverse to the restriction of $\Phi^{* \#}$.

Step 6: uss-maw pieces Let $\mu_{1}$ be a uss-maw piece. Mimicking step 6, we push annulus leaves of $\mathcal{F} \mid \mu_{1}^{\prime}$ out of $\mu_{1}$, leaving an $I$-parallel foliation in $\mu_{1}$, so that the restriction of $\mathcal{F}$ to the union of corner annuli and s-faces of maw pieces is transverse to the restriction of $\Phi^{* \#}$. However, we must check that this can be done without disturbing the properties established in step 5 for each suu-maw piece $\mu$.

Let $t$ be a component of $\mu_{1} \cap \mu$, so $t$ is a pinched tetrahedron. Let $L$ be a leaf of $\mathcal{F} \mid \mu_{1}$ which is moved by the isotopy in step 6 , so either $L$ is an annulus cutting off a corner circle of $\mu_{1}$, or $L$ is close to such an annulus, or $L$ lies between such an annulus and the corner


Figure 3.17: This figure shows a closeup of the lower portions of figure 3.14a,b,c. If $d$ is a circle leaf of $\mathcal{F} \mid F$ parallel to the corner circle $C_{i}$, and if $d$ is a boundary circle of some annulus leaf $L$ of $\mathcal{F} \mid \mu$ that cuts off $C_{i}$, then we can isotope $\mathcal{F}$ so as to move $L$ out of $\mu$.
circle. Notice that if $L \cap t \neq \emptyset$ then $L$ has nonempty intersection with the uu-cusp of $t$. In particular, $L$ has nonempty intersection with the uu-cusp of $\mu$. In other words, the isotopy that fixes up $\mathcal{F} \mid \mu_{1}$, when looked at from inside $\mu$, only moves points lying on leaves of $\mathcal{F} \mid \mu$ which intersect the uu-cusp circle of $\mu$, and therefore this isotopy does not produce any new annulus leaves of $\mathcal{F} \mid \mu$ cutting off corner circles of $\mu$. It follows that after the isotopy, the foliation $\mathcal{F} \mid \mu$ still satisfies the properties established in step 5 .

Step 7: Collapsing Before defining the collapsing decomposition on the rest of $\mathfrak{C}(M-$ $\left.U^{\#}(\tau)\right)$, consider the isotopy from $\mathcal{F} \mid \mathfrak{C}(M-U(\tau))$ to $\mathcal{F} \mid \mathfrak{C}\left(M-U^{\#}(\tau)\right)$ defined in steps $3,5,6$. After corner blowups but before the isotopy, $\mathcal{F}$ is transverse to $\Phi^{\#}$, and after the isotopy we still have transversality on $\partial \mathfrak{C}\left(M-U^{\#}(\tau)\right)=\partial U^{\#}(\tau)$. Clearly the isotopy can be extended over all of $U^{\#}(\tau)$ so that points are stationary except near $\partial U^{\#}(\tau)$, and so that $\mathcal{F}$ remains transverse to $\Phi^{\#}$ on all of $U^{\#}(\tau)$.

Consider a pinched tetrahedron $t$. We say that $\mathcal{F} \mid t$ is linear if it is conjugate to the foliation of $\mathbf{R}^{3}$ by horizontal planes, restricted to a rectilinear simplex none of whose edges are horizontal. Note that $\mathcal{F} \mid t$ is linear if and only if it has no saddle tangencies or external tangencies in the interior of a cusp edge of $t$. Also note that $\mathcal{F} \mid t$ is linear if and only if the leaf space is an arc $\alpha_{t}$. If $\mathcal{F} \mid t$ is not linear then the leaf space is a tree.

We may alter $U^{\#}(\tau)$ so that $\mathcal{F} \mid t$ is linear for each pinched tetrahedron, as follows. If $\mathcal{F} \mid t$ is not transverse to a uu-cusp edge $\alpha$ of $t$, let $\mu$ be the uss-maw piece containing $t$, so $\alpha$ is an arc connecting the two s-faces of $\mu$. We know that $\mathcal{F} \mid \mu$ is $I$-parallel. By perturbation we may assume that the two endpoints of $\alpha$ do not lie on the same leaf of $\mathcal{F} \mid \mu$. By isotoping the edge $\alpha$ in $\mu$, we may make $\mathcal{F}$ transverse to $\alpha$; in the process we might create a tangency of $\mathcal{F}$ at some endpoint of $\alpha$, but that is not bothersome. Proceeding in the same manner for the ss-cusp edge of $t$, and repeating for each pinched tetrahedron $t$, we may assume that $\mathcal{F} \mid t$ is linear.

Define the collapsing decomposition on a pinched tetrahedron $t$ to be the leaves of the foliation $\mathcal{F} \mid t$.


Figure 3.18: A partial collapse of a pinched tetrahedron $t$ in a maw piece with cusp curve c. Any leaf of $\mathcal{F} \mid t$ intersecting $c$ is collapsed to a point; the union of these points is a segment $\alpha_{t}$. Any leaf of $\mathcal{F} \mid t$ not intersecting $c$ is collapsed to a segment; the union of these segments is a disc. The figure shows trajectories of $\Phi^{* \#}$ on $F_{1}$ before the collapse, and on $F_{1}^{\prime}$ after the collapse. Also shown, as dashed lines in the background, are the trajectory on $F_{2}$ that intersects $t$, and the trajectory on $F_{2}^{\prime}$ that contains $\alpha_{t}$; remaining trajectories on $F_{2}$ and $F_{2}^{\prime}$ are left to the imagination of the viewer.

Now fix a maw piece, for concreteness a uss-maw piece $\mu$ with ss-cusp $c$, corner circles $C_{1}, C_{2}$, and $\mathbf{u}$-face $F$. Let $K=\mathfrak{C}(\mu-$ (pinched tetrahedra) ) be the associated cloven maw piece. Let $F_{i}$ be the s-face of $K$ with $C_{i} \subset F_{i}$.

In describing the collapsing decomposition on $K$, it is inconvenient that $F_{1} \cap F_{2}$ is not a common component of $\partial F_{1}$ and $\partial F_{2}$, and it is also inconvenient that $\Phi^{* \#}$ is not a flow at culvert points of $F_{1}$ and $F_{2}$-since $\mu$ is a uss-maw piece, $\Phi^{* \#}$ is a forward semiflow at these culvert points, but not a backward semiflow. To avoid these minor inconveniences, we alter $K$ by doing a partial collapse of each pinched tetrahedron component $t$ of $\mathfrak{C}(\mu-K)$ (see figure 3.18; imagine a shark with an overbite). First collapse to a point each leaf of $\mathcal{F} \mid t$ which intersects the ss-cusp edge of $t$ (one such leaf is shown in figure 3.18); let $\alpha_{t}$ be the union of these points, a topological arc. Next, for each leaf $L$ of $\mathcal{F} \mid t$ which is disjoint from the ss-cusp edge, let $a_{1}, a_{2}$ be the intersections of $L$ with the two $\mathbf{u}$-faces of $t$, fiber $L$ by arcs connecting $a_{1}$ to $a_{2}$ with a degenerate fiber at each point of $a_{1} \cap a_{2}$, and collapse each fiber (two such leaves $L$ are shown in figure 3.18). Let $K^{\prime}$ be the quotient of $K$ under the partial collapse on each component of $\mathfrak{C}(\mu-K)$. Note that $K^{\prime}$ is a solid torus.

Let $F_{1}^{\prime}, F_{2}^{\prime}$ be the two $\mathbf{u}$-faces of $K^{\prime}$. The surface $F_{i}^{\prime}$ may be regarded as a surface with ordinary corners and reflex corners: for each pinched tetrahedron $t \subset K$, one endpoint of $\alpha_{t}$ is an ordinary corner of $F_{i}^{\prime}$ and the other endpoint is a reflect corner. Let $c^{\prime}=F_{1}^{\prime} \cap F_{2}^{\prime}$,
a component of $\partial F_{1}^{\prime}$ and of $\partial F_{2}^{\prime}$. Let $c_{\text {tet }}^{\prime}=\bigcup_{t} \alpha_{t}$ where the union is taken over all pinched tetrahedra $t \subset K$, and let $c_{\text {cusp }}^{\prime}=\mathfrak{C}\left(c^{\prime}-c_{\text {tet }}^{\prime}\right)$.

The structure of $\Phi^{* \#} \mid F_{i}^{\prime}$ is as follows. The corner circle $C_{i}$ is an orbit. Each component $\alpha_{t}$ of $c_{\text {tet }}^{\prime}$ is contained in an orbit whose backward endpoint is the ordinary corner in $\partial \alpha_{t}$, and whose forward end spirals into $C_{i}$. Every other orbit has a backward endpoint ending transversely on $c_{\text {cusp }}^{\prime}$, and a forward end spiralling into $C_{i}$. There is therefore a $1-1$ correspondence between orbits of $\Phi^{* \#} \mid F_{1}^{\prime}$ and of $\Phi^{* \#} \mid F_{2}^{\prime}$, where noncompact orbits $\ell_{1}, \ell_{2}$ correspond if and only if $\ell_{1} \cap \ell_{2} \neq \emptyset$, and $C_{1}$ corresponds to $C_{2}$. Note that corresponding noncompact orbits $\ell_{1}, \ell_{2}$ have the property that $\ell_{1} \cap \ell_{2}$ is either a point of $c_{\text {cusp }}^{\prime}$ or a component $\alpha_{t}$ of $c_{\text {tet }}^{\prime}$; in particular note that $\ell_{1} \cup \ell_{2}$ is contractible.

The collapsing of $K^{\prime}$ will identify $F_{1}^{\prime}$ and $F_{2}^{\prime}$ homeomorphically. This identification is uniquely determined as follows:

Lemma 3.6.3. There exists a unique homeomorphism $h: F_{1}^{\prime} \rightarrow F_{2}^{\prime}$ with the following properties:

- $h$ is the identity map on $c^{\prime}=F_{1}^{\prime} \cap F_{2}^{\prime}$.
- $h$ takes $C_{1}$ to $C_{2}$ by the unique map compatible with the collapsing decomposition already defined on $F$.
- For each leaf $L$ of $\mathcal{F} \mid K^{\prime}, h$ takes $L \cap F_{1}^{\prime}$ to $L \cap F_{2}^{\prime}$.
- Given corresponding noncompact trajectories $\ell_{i}$ of $\Phi^{* \#} \mid F_{i}^{\prime}, i=1,2$, the map $h$ takes $\ell_{1}$ to $\ell_{2}$; it follows that $x \in \ell_{1}$ is connected to $h(x) \in \ell_{2}$ by a path $q_{x} \subset \ell_{1} \cup \ell_{2}$ whose path homotopy class is well-defined.
- For each simply connected leaf $L$ of $\mathcal{F} \mid K^{\prime}$, and for each $x \in F_{1}^{\prime} \cap L$, let $p_{x}$ be a path in $L$ connecting $x$ to $h(x)$. Then $p_{x}$ is path homotopic to $q_{x}$ in $K^{\prime}$.

Proof. To prove uniqueness, let $\ell_{1}, \ell_{2}$ be corresponding noncompact trajectories. First, note that for each compact leaf $L$ of $\mathcal{F} \mid K^{\prime}$, there is a unique point $x_{i} \in L \cap \ell_{i}$, and so we must have $h\left(x_{1}\right)=x_{2}$. Second, note that for each noncompact leaf $L$, there is a countable infinity of points $x_{i k} \in L \cap \ell_{i}, k \in \mathbf{Z}$. The indexing may be chosen so that if $p_{k}$ connects $x_{1 k}$ to $x_{2 k}$ in $L$, and if $q_{k}$ connect $x_{1 k}$ to $x_{2 k}$ in $\ell_{1} \cup \ell_{2}$, then $p_{k}$ is path homotopic to $q_{k}$ in $K^{\prime}$. In particular, if $k \neq k^{\prime} \in \mathbf{Z}$ then $p_{k} * q_{k^{\prime}}^{-1}$ is $k-k^{\prime}$ times a generator of $H_{1}\left(K^{\prime}\right) \approx \mathbf{Z}$, and hence we must have $h\left(x_{1 k}\right)=x_{2 k}$ for each $k$.

The proof of uniqueness also gives us the definition of $h$. It easy to check that $h$ and $h^{-1}$ are continuous.

It is now straightforward to define a collapsing decomposition of $K^{\prime}$, each element of which is either a point of $c^{\prime}$ or an arc $p_{x}$ in a leaf $L$ of $\mathcal{F} \mid K^{\prime}$ connecting some $x \in F_{1}^{\prime}$ to $h(x) \in F_{2}^{\prime}$, such that $p_{x}$ is path homotopic to $q_{x}$.

Now collapse $K^{\prime}$ for each cloven uss-maw piece $K$, and similarly for each cloven suumaw piece. This completes the collapsing of $\mathfrak{C}\left(M-U^{\#}(\tau)\right)$. By construction, the collapsing respects $\mathcal{F}$ and $\Phi^{* \#}$. Therefore there is an oriented 1-dimensional foliation $\Phi^{\#}$ induced from $\Phi^{* \#}$ by collapse, and by choosing a parameterization we make $\Phi^{\#}$ into a flow. Clearly $\Phi^{\#}$ is a dynamic blowup of $\Phi$. Since $\mathcal{F} \mid U^{\#}(\tau)$ is transverse to $\Phi^{* \#}$ it follows that $\mathcal{F}$ is transverse to $\Phi^{\#}$. This completes the proof of theorem 3.5.4.

### 3.7 Constructing vertical dynamic pairs

In this section we prove proposition 3.5.5. Recall the statement. Let $\beta$ be a transversely oriented branched surface that carries a taut, transversely oriented foliation $\mathcal{F}$ of $M$. Let $(B, V, \mathcal{I})$ be an unstable Markov branched surface such that $(B, V)$ is vertical with respect to $\beta$, and therefore also with respect to $\mathcal{F}$. Suppose that $\mathcal{I}$ is tangent to $\beta$. Suppose also that $(B, V)$ satisfies the hypotheses of proposition 2.6.2: $B$ is very full in $M, B$ carries no closed surfaces, the restriction of $V$ to each component of $\mathfrak{C}(M-B)$ is circular, and no sector of $B$ contains a closed trajectory of $V$.

In order to construct a dynamic pair ( $B^{s}, B^{u}$ ) which is vertical with respect to $\beta$, we go through the proof of proposition 2.6.2 and check every step, verifying that all changes in $(B, V)$ preserve the property that $(B, V)$ is vertical with respect to $\beta$. Also, after constructing $B^{s}$, we verify that no annulus carried by $\beta$ is peripheral in $\mathfrak{C}\left(M-B^{s}\right)$.

In step 1 we constructed a dynamic train track $\tau$ dual to $\mathcal{I}$. Recall how $V$ was altered. The train track $\tau$ was constructed in pieces as $\tau=\bigcup_{I_{i} \in \mathcal{I}} \tau_{i}^{+} \cup \tau_{i}^{-}$(see figure 2.10a). For each $I_{i} \in \mathcal{I}$, we chopped off a tiny neighborhood of each endpoint of $I_{i}$ to obtain a subinterval $I_{i}^{\prime}$, we chose a small number $t>0$, and we constructed $\tau_{i}^{+}$in $I_{i}^{\prime} \cdot[0, t]$. Then $V$ was altered by homotopy in $I_{i}^{\prime} \cdot[0, t]$ to be tangent to $\tau_{i}^{+}$, still having an upward vertical component in the coordinate system $I_{i}^{\prime} \cdot[0, t]$. Since $I_{i}^{\prime}$ is tangent to $B$, before we start the construction of $\tau_{i}^{+}$we can alter the parameterization of $V$ and choose $t$ so that for each $s \in[0, t]$ the horizontal segment $I_{i}^{\prime} \cdot s$ is tangent to the product structure on $\mathfrak{C}(M-\beta)$. Since $V$ still has an upward vertical component after homotopy, it follows that $V$ is still vertical with respect to $\beta$.

In step 2 , the vector field $V$ is altered by perturbation in steps 2 c and 2 d , so verticality is preserved. In step 2 c the branched surface $B$ is altered by splitting along annuli and Möbius bands, introducing cusped torus pieces of type $(2,0)$ and $(2,1)$. Since $B$ was transverse to $\beta$ before splitting along these annuli and Möbius bands, it is clear that the new torus pieces contain no peripheral annuli carried by $\beta$.

Recall that in step 3 , after setting $B=B^{u}$ the construction of $B^{s}$ is carried out. In the present "vertical" setting we need some preliminary work. Consider a component $T$ of $\mathfrak{C}\left(M-B^{u}\right)$. We shall describe in detail how to construct $B^{s} \mid T$. In our construction we will apply lemma 3.5.6-although this lemma was only stated for cusped solid tori, the
proof applies without change for cusped torus shells.
Reviewing the notation of step 3 , the annulus faces and cusp circles of $T$ are enumerated in circular order as $F_{1}, \ldots, F_{N}, c_{1}, \ldots, c_{N}$ so that $c_{n}=\partial F_{n-1} \cap \partial F_{n}$ for each $n \in \mathbf{Z} / N$. The remains of $\tau$ in $\partial T$ is denoted $\tau_{T}=\tau_{T 1} \cup \cdots \cup \tau_{T N}$ where $\tau_{T n}$ is the component intersecting $c_{n}$. The stable train track $\tau_{T n}$ has two circular sources $d_{n 1} \subset F_{n-1}$ and $d_{n 2} \subset F_{n}$, and they are oriented isotopic in $T$. There is a subannulus $R_{n} \subset F_{n}$ with boundary $d_{n 2} \cup d_{n+1,1}$ and interior disjoint from $\tau_{T}$. There is a subannulus $A_{n}^{\prime} \subset \partial T$ containing $c_{n}$ with boundary $d_{n 1} \cup d_{n 2}$, satisfying $\tau_{T} \cap A_{n}^{\prime}=\tau_{T n}$.

In the proof of proposition 2.6 .2 the annulus $A_{n}^{\prime}$ was perturbed to obtain a properly embedded annulus $A_{n} \subset T$ with $\partial A_{n}=d_{n 1} \cup d_{n 2}$, and this annulus $A_{n}$ became part of an annulus-with-tongues component of $\mathfrak{C}\left(B^{s}-B^{u}\right)$. Now we must proceed with much more care, in order to insure that $V$ can be homotoped to be tangent to $A_{n}$ while maintaining verticality. To do this we take lemma 3.5.6, which describes the structure of $\mathcal{F} \mid T$ near each cusp, and use it to describe the structure of $\beta \mid T$ near each cusp.

Recall from lemma 3.5.6 that there exists an annulus which we denote $H_{n}$, and a component $\nu_{n}$ of $\mathfrak{C}\left(T-H_{n}\right)$, such that $\nu_{n}$ is a maw piece with cusp curve $c_{n}$, and:

- $\mathcal{F}$ is transverse to $H_{n}$.
- $\mathcal{F} \mid \nu_{n}$ is $I$-parallel.
- For each annulus leaf of $\mathcal{F} \mid \nu_{n}$, the transverse orientation on that leaf points away from $H_{n}$ and towards $c_{n}$.

Moreover $\nu_{n}$ is maximal, in the sense that if $\nu_{n}^{\prime}$ also satisfies these properties then each annulus leaf of $\mathcal{F} \mid T$ contained in $\nu_{n} \cup \nu_{n}^{\prime}$ is also contained in $\nu_{n}$.

It follows that by perturbing $H_{n}$ and then splitting $\beta$, the restriction of $\beta$ to $\nu_{n}$ satisfies analogous properties:

1. $\beta$ is transverse to $H_{n}$.
2. $\beta \mid \nu_{n}$ is $I$-parallel.
3. For each annulus carried by $\beta \mid \nu_{n}$, the transverse orientation on that annulus points toward $c_{n}$ and away from $H_{n}$.
4. If $\nu_{n}^{\prime}$ also satisfies the above properties then every annulus carried by $\beta \mid\left(\nu_{n} \cup \nu_{n}^{\prime}\right)$ is also carried by $\nu_{n}$.

To clarify 2 , we say that $\beta \mid \nu_{n}$ is $I$-parallel if there is an integrable line field on $\nu_{n}$ tangent to $\beta$, which is uniquely integrable except along $\Upsilon \beta \mid \nu_{n}$, such that each integral curve connects the two $\mathbf{u}$-faces of $\nu_{n}$.

Now there is a technical glitch. We will need the following property, which unfortunately may not be true:
5. $\nu_{n} \cap\left(d_{n 1} \cup d_{n 2}\right)=\emptyset$.

This property says intuitively that all annuli carried by $\beta \mid T$ that "cut off" the cusp $c_{n}$ are "close to" $c_{n}$. We will need this property in order to prove that $\beta$ carries no annulus that is peripheral in $\mathfrak{C}\left(M-B^{s}\right)$ (c.f. the proof of lemma 3.6.1-2, and the relation of that property to nonexistence of peripheral annuli). To establish this property we will isotope $\mathcal{F}$ and $\beta$. There are two cases:

Case 1: $\beta \mid \nu_{n}$ carries no annulus Each cusp disc carried by $\beta \mid \nu_{n}$ is obviously disjoint from $d_{n 1} \cup d_{n 2}$. By mimicking the proof of lemma 3.5.6, we may choose $H_{n}$ to be contained in a neighborhood of the union of $c_{n}$ and all maximal cusp discs intersecting $c_{n}$, in which case $\nu_{n}$ is disjoint from $d_{n 1} \cup d_{n 2}$.

Case 2: $\beta \mid \nu_{n}$ carries some annulus By splitting $\beta$ we may assume that the annuli carried by $\beta \mid \nu_{n}$ are pairwise disjoint. Let $A$ be the outermost such annulus, and let $L$ be the outermost annulus leaf of $\mathcal{F} \mid T$ carried by $A$.
Holonomy Claim. The holonomy of $\mathcal{F}$ around $L$, on the side away from $c_{n}$, going around the generator of $\pi_{1}(L)$ in the positive direction, is repelling.

Accepting the claim for the moment, we show how to use it to fix up $\beta$. Suppose that $d_{n i} \subset \nu_{n}$, let $C_{n i}$ be the corresponding corner circle of $\nu_{n}$, and let $d_{n i}^{\prime}$ be a circle in $\partial T-\nu_{n}$ very close to $C_{n i}$. Orient $d_{n i}^{\prime}$ so that it is oriented isotopic to $d_{n i}$. By the holonomy claim, it follows that $d_{n i}^{\prime}$ is positively transverse to $\beta$. Of course, $d_{n i}$ is also positively transverse to $\beta$.

Now we do an ambient isotopy of $\beta$ taking $d_{n i}^{\prime}$ to $d_{n i}$. More precisely, choose a smooth annulus $\alpha$ in $\partial T$ with $d_{n i} \cup d_{n i}^{\prime} \subset \alpha$. Note that we can homotope $V$ in $\alpha$ so that every trajectory in the region between $d_{n i}$ and $d_{n i}^{\prime}$ crosses $d_{n i}^{\prime}$ in forward time and is asymptotic to $d_{n i}$ in backward time; this follows from the fact that each annulus carried by $\beta \mid \nu_{n}$ is transversely oriented away from $c_{n}$, and so each circle carried by the train track $\beta \mid \alpha$ is transversely oriented from $d_{n i}$ towards $d_{n i}^{\prime}$; also, no obstructions to this homotopy are presented by the train track $\tau$, which does not intersect the region between $d_{n i}$ and $d_{n i}^{\prime}$. We can therefore choose a parameterization $\alpha \approx[0,1] \times S^{1}$ so that every vector in $V \mid \alpha$ points into the first quadrant, except those based on $d_{n i}$ which point straight upward, and every tangent line of $\beta \mid \alpha$ has either negative slope or infinite slope. Define a map $\Theta: M \rightarrow M$ isotopic to the identity which equals the identity outside a small neighborhood of $\alpha$, preserves $\alpha$, preserves the $S^{1}$-factor of $\alpha$, and takes $d_{n i}^{\prime}$ to $d_{n i}$. Note that $V$ is vertical with respect to $\Theta(\beta)$, and all the other good properties of $\beta$ are shared by $\Theta(\beta)$. In addition, we have arranged that $\Theta(\beta)$ satisfies property 5 , and we replace $\beta$ by $\Theta(\beta)$.

To prove the holonomy claim, consider the component $K$ of $\mathfrak{C}\left(T-\left.\mathcal{F}\right|_{c} T\right)$ such that $A \subset \partial K$ and $K$ is on the side of $A$ away from $c_{n}$. Note that $K$ is a solid torus, $\left.\mathcal{F}\right|_{c} K$


Figure 3．19：In the annulus $\alpha$ ，the map $\Theta$ takes $d_{n i}^{\prime}$ to $d_{n i}$ ．The branched surface $\beta$ is replaced by $\Theta(\beta)$ ．Note that $V$ is still vertical with respect to $\Theta(\beta)$ ．
consists of a finite number of annuli $A=A_{0}, \ldots, A_{n}$ located in $\partial K$, and every noncompact leaf of $\left.\mathcal{F}\right|_{c} K$ is simply connected with one end spiralling into each of $A_{0}, \ldots, A_{n}$. We may regard $K$ as a solid torus manifold-with-corners of type ( $m, l$ ), obtained from a $2 m$-gon as the mapping torus of a rotation through angle $2 \pi l / m$, with $m \geq 2$. In order to prove the claim it suffices to show that $V$ may be perturbed to a vector field $V^{\prime}$ having a closed trajectory $\gamma \operatorname{in} \operatorname{int}(K)$-for, by circularity of $V \mid T$, it follows that $\gamma$ is a positive multiple of the core of $T$. First perturb $V$ so that it is smooth in $K$, and hence uniquely integrable. If it were true that every trajectory entering $K$ through $A_{0}$ exited through some other face, it would follow that $(m, l)=(2,0)$ and $V$ exits $K$ through the face $A_{1}$, contradicting maximality of $\nu_{n}$. It follow therefore that some trajectory of $V$ in $K$ is forward infinite, accumulating on some point $p \in K$. By perturbing $V$ we can create a closed trajectory through $p$.

Having established property 5 , we can perturb $V$ to satisfy the following additional property:
6. Each trajectory of $V \mid \nu_{n}$ is a compact arc from a point of $H_{n}$ to a point of $c_{n}$.

If this property were not true, then there would exist two annuli carried by $\beta \mid \nu_{n}$ and a circular trajectory of $V$ between these two annuli. However, since the transverse orientation on both annuli points away from $c$ and toward $H_{n}$, we can perturb $V$ between the two annuli so that any trajectory piercing one annulus also pierces the other.

Now construct the annulus $A_{n}$ as follows. Let $A_{n}^{\prime \prime}$ be the annulus in $\partial \mathfrak{C}\left(T-\nu_{n}\right)$ bounded by $d_{n 1} \cup d_{n 2}$ and containing $H_{n}$. Perturb $A_{n}^{\prime \prime}$ to obtain a properly embedded annulus $A_{n}$ in $T$, disjoint from $\nu_{n}$, with boundary $d_{n 1} \cup d_{n 2}$, and so that $A_{n}$ is transverse to $\beta \mid T$.

Note that the vector field $V$ is tangent to the boundary circles of $A_{n}$, but not necessarily to $A_{n}$ itself. To fix this, note that each component of $\mathfrak{C}\left(A_{n}-\beta\right)$ is of index 0 , and so we can extend $V \mid \partial A_{n}$ to a vector field $V_{A_{n}}$ tangent to $A_{n}$ and transverse to $\beta \mid A_{n}$, pointing in the positive direction with respect to the transversely oriented plane field $\tau_{\beta}$. Now we can choose a small neighborhood of $A_{n}$, and an isotopy of $V$ supported in that neighborhood, so after the isotopy the restriction of $V$ to $A_{n}$ becomes equal to $V_{A_{n}}$.

Finally, we attach a tongue $t_{n}$ for each point of $\tau_{T_{n}} \cap c_{n}$, exactly as in the proof of proposition 2.6.2, to produce an annulus with tongues $B_{T_{n}}^{s}$ as in the proof of proposition 2.6.2. The construction of $t_{n}$ works because of property 6 ; since the construction requires only a perturbation of $V$, verticality of $V$ with respect to $\beta$ is preserved.

We now construct $B^{s}=\bigcup_{T} \bigcup_{n} B_{T n}^{s}$ exactly as in proposition 2.6.2, and note that $V$ is tangent to $B^{s}$.

To complete the proof of proposition 3.5 .5 we must check that $\beta$ carries no annulus which is peripheral in $\mathfrak{C}\left(M-B^{s}\right)$. Suppose that such an annulus $A^{\prime}$ exists. Let $T^{\prime}$ be the torus piece component of $\mathfrak{C}\left(M-B^{s}\right)$ containing $A^{\prime}$. Let $T$ be the corresponding component of $\mathfrak{C}\left(M-B^{u}\right)$. Let $F^{\prime}$ be the face of $T^{\prime}$ containing $\partial A^{\prime}$. Let $c$ be the cusp of $T$ corresponding
to $F^{\prime}$. There is an annulus component $F$ of $T \cap F^{\prime}$ and a maw piece component $\mu_{F}$ of $\mathfrak{C}(T-F)$ such that $c$ is the cusp curve of $\mu_{F}$. Similarly, there is an annulus component $A$ of $T \cap A$ and a maw piece component $\mu_{A}$ of $\mathfrak{C}(T-A)$ such that $c$ is the cusp curve of $\mu_{A}$. Choose an annulus $H \subset T$ bounding a maw piece $\nu$ with cusp curve $c$ such that $H, \nu$ satisfy 1-6. Since $\nu \subset \mu_{F}$ it follows that $A \not \subset \mu_{F}$, and so by maximality of $\nu$ there is an annulus $A_{1}$ carried by $\beta \mid T$ which cuts off $c$ such that $A_{1} \subset \mu_{A}$ but the transverse orientation on $A_{1}$ points towards $c$. Now we can mimic the arguments of lemma 3.6.1 to obtain a contradiction: either $A_{1}$ has a boundary circle in $\mu_{A}$ which contradicts that trajectories of $V \mid \mu_{A}$ all go from $F$ to $c$; or $A_{1} \cap T^{\prime}$ extends to a properly embedded annulus $A_{1}^{\prime}$ in $T^{\prime}$ which leads to a similar contradiction in some maw piece contained in $T^{\prime}$.

This finishes the proof of proposition 3.5.5.

## Chapter 4

## Sutured manifolds

In this section the theory of dynamic pairs and pseudo-Anosov flows is extended to the setting of sutured manifolds. The statements and proofs in this section share many features with those in $\S 2$ and $\S 3$, and we use this opportunity to be brief when we can refer to those sections for details. There are, however, many interesting differences which we will be highlighted.

In $\S 4.1$ we generalize branched surface hierarchies to the suured manifold setting, and in $\S 4.2$ we generalize dynamic branched surfaces. The sutured manifold definition of dynamic pairs is developed in $\S 4.3-4.5$. In $\S 4.6$ we generalize the results of $\S 2.5$, using dynamic train tracks to analyze the branched surfaces of a dynamic pair. The most important new features of a dynamic pair $B^{s}, B^{u}$ in a sutured manifold $P$ are the boundary train tracks $\beta^{s}=\partial B^{s} \subset \mathcal{R}_{+} P$ and $\beta^{u}=\partial B^{u} \subset \mathcal{R}_{-} P$. These train tracks and their properties are investigated in $\S 4.7$. In $\S 4.10$ we generalize pA flows and their stable and unstable laminations to the setting of sutured manifolds.

One interesting pedagogical feature is that the simplest sutured 3 -manifolds are simpler by many degrees than the simplest closed or torally bounded 3 -manifolds. Big guns were used to construct just a single example of a dynamic pair in a closed 3 -manifold. By contrast, in $\S 4.8-4.9$ we use our bare hands to construct simple and instructive examples of dynamic pairs in sutured 3-manifolds, starting with Dan Asimov's "round handle" = (square) $\times S^{1}$ [Asi75].

In $\S 4.11$ we generalize the results of $\S 2.6$, constructing a dynamic pair from a Markov unstable branched surface. However, an interesting new feature arises, present in the Dynamic Pair Theorem of the introduction but not yet in chapter 2, namely the emergence of incompressible, nonperipheral tori. The sutured manifold version of proposition 2.6.2constructing dynamic pairs from Markov branched surfaces-yields either a dynamic pair or a family of nonperipheral incompressible tori bounding a Seifert fibered submanifold. The manner in which these tori arise, in an attempt to construct dynamic pairs, is one of
the most interesting features of the sutured manifold theory. This feature will, of course, also arise in later sections in an attempt to construct dynamic pairs that are vertical with respect to sutured manifold hierarchies.

### 4.1 Hierarchies in sutured manifolds

In order to formulate branched surface hierarchies in sutured manifolds we need a new class of branched surfaces, designed specially for sutured manifolds. Roughly speaking, an " $\mathcal{R}$ branched surface" in a sutured manifold $Q$ is a branched surface containing $\mathcal{R} Q$ and transverse to $\gamma Q$.

To be more precise, let $Q$ have the corner model for sutured manifolds. An $\mathcal{R}$ branched surface in $Q$ is a smooth, transversely oriented 2-complex $B \subset Q$ such that:

- $\mathcal{R} Q \subset B$, and the transverse orientation on $B$ agrees with the transverse orientation on $\mathcal{R} Q$.
- Each point $x \in B-\partial B$ has a neighborhood in $B$ which is a union of smooth 2-discs embedded in $Q$, all tangent at $x$.
- $B$ is transverse to $\gamma Q$, i.e. in some collar neighborhood $U(\gamma Q)=\gamma Q \times[0,1)$ we have $B \cap U(\gamma Q)=\partial B \times[0,1)$, where $\partial B=B \cap \gamma Q$ is a train track in $\gamma Q$.
- $B$ is groomed in the sense that for each annulus component $A$ of $\mathfrak{C}(\gamma Q-\partial B)$, the transverse orientation points into $A$ along one boundary circle and out of $A$ along the other.
- $B$ has generic branching.

As in the sutureless case, there is an $I$-bundle neighborhood $N(B)$, a sutured manifold $P(B)=\operatorname{cl}(Q-N(B))$, and an $I$-collapsing map $Q \rightarrow Q$, whose obvious definitions and properties are left to the reader. Note that as in the sutureless case, $\gamma P(B)=\operatorname{Fr}_{v} P(B) \cup$ $(P(B) \cap \gamma Q)$.

A branched surface hierarchy $B_{0} \subset \cdots \subset B_{K}$ in a sutured manifold $Q$ is defined word for word like in a manifold with torus boundaries, except that $B_{0}=\mathcal{R} Q$ and each $B_{k}$ is an $\mathcal{R}$ branched surface; in particular, $\mathfrak{C}\left(Q-B_{K}\right)$ is required to be a product sutured manifold. Equivalence among branched surface hierarchies in sutured manifolds is also defined exactly as for manifolds with torus boundaries.

Note that if $M$ is a 3 -manifold with torus boundaries, if $B_{0} \subset \cdots \subset B_{K}$ is a branched surface hierarchy in $M$, and if we fix $k$ and define $B_{i}^{\prime} \subset P\left(B_{k}\right)$ to be the union of $\mathcal{R} P\left(B_{k}\right)$ with the closure of the pullback of $B_{k+i}$, then $B_{0}^{\prime} \subset \cdots \subset B_{K-k}^{\prime}$ is a branched surface hierarchy in $P\left(B_{k}\right)$.


Figure 4.1: A local model for the dynamic vector field near a boundary switch of an unstable, dynamic branched surface. $\mathcal{R}_{+} P$ corresponds to a horizontal plane in $\mathbf{R}^{3}$ and the dynamic vector field corresponds to $d / d z$. The branch locus, as it moves away from $\mathcal{R}_{+} P$, slants towards the one-sheeted side; this is necessary in order for $d / d z$ to point forward along the branch locus.

### 4.2 Dynamic branched surfaces in sutured manifolds

Let $P$ be a sutured manifold. An unstable dynamic branched surface in $P$ is a branched surface $B^{u} \subset P$ and a nowhere zero, $C^{0}$ vector field $V$ on $P$, with the following properties:

- $\partial B^{u} \subset \mathcal{R}_{+} P$.
- $(P, V)$ is a dynamic sutured manifold.
- $V$ is tangent to $B^{u}$.
- $V$ points forward along $\Upsilon B^{u}$.

Note that $\partial B^{u}$ is a train track in $\mathcal{R}_{+} P$. The set of switches of $\partial B^{u}$ is the same as $\Upsilon B^{u} \cap$ $\mathcal{R}_{+} P$; these points are called boundary switches of $B^{u}$. Figure 4.1 explains the appearance of $V$ near a boundary switch. Figure 4.2 shows a Reeb annulus of $\partial B^{u}$, an annulus $A \subset \mathcal{R}_{+} P$ such that $\partial B^{u} \cap A$ is a Reeb train track in $A$.

A stable dynamic branched surface $B^{s} \subset P$ is similarly defined, except that $\partial B^{s} \subset \mathcal{R}_{-} P$, and the conditions on $V$ along $\Upsilon B^{s}$ are reversed. To see the picture near $\Upsilon B^{s} \cap \mathcal{R}_{-} P$, turn figure 4.1 upside down. Boundary Reeb annuli for $B^{s}$ are defined just as for $B^{u}$.

The complementary dynamic manifold. Consider an unstable dynamic branched surface $B^{u} \subset P$ in a sutured manifold. Note that $P^{u}=\mathfrak{C}\left(P-B^{u}\right)$ is naturally a dynamic


Figure 4.2: A Reeb train track.
manifold, with vector field obtained by pulling back $V$ via the overlay map $P^{u} \mapsto P$, and with faces labelled $\mathbf{b}, \mathbf{m}, \mathbf{p}$, or $\mathbf{u}$ depending on whether the image lies in a $\mathbf{b}, \mathbf{m}$, or $\mathbf{p}$-face of $P$ or in $B^{u}$. The overlay map restricted to $\mathbf{b}$ and $\mathbf{m}$-faces is a homeomorphism. The edges of $P^{u}$ are: uu-cusps mapping to maw components of $B^{u}$; pu-corners mapping to paths in $\partial B^{u}$; and $\mathbf{p b}$ and $\mathbf{m b}$-corner circles mapping homeomorphically to similar circles in $P$. Each corner of $P^{u}$ is a puu-gable mapping to a boundary switch.

Consider a u-face $A$ of $P^{u}$. Note that the dynamic vector field points outward on $A$; this is obvious at points of $\partial A$ mapping into uu-cusps and pu-corners of $P^{u}$, and it is also obvious for puu-gables as a glance at figure 4.1 shows. Since $P$ is oriented so is $P^{u}$, and therefore so is $A$. The vector field on $A$ is nowhere zero, and so from the Euler-Poincaré index formula it follows that $A$ is an annulus or torus. In most situations tori will not occur, so $A$ is almost always an annulus. If $A$ is an annulus we say that $V \mid A$ is circular if there is a homotopy equivalence $A \mapsto S^{1}$ such that the map from each trajectory in $A$ to $S^{1}$ has positive derivative. Pulling back the standard generator of $H_{1}\left(S_{1}\right) \approx \mathbf{Z}$ we obtain a generator of $H_{1}(A)$, called the positive generator.

For example, suppose $T$ is a component of $P^{u}$ which is a $\mathbf{u}$-cusped torus piece. If $V$ is circular on $T$ then $V$ is circular on all annulus $\mathbf{u}$-faces of $T$. Moreover, the positive generators on the $\mathbf{u}$-faces of $T$ are all homologous to the same positive element of $H_{1}(T)$; if $T$ has type $(n, k)$ this element is $n / \operatorname{gcf}(n, k)$ times the positive generator of $H_{1}(T)$.

Consider a uu-cusp circle $c$ of $P^{u}$, and let $F_{1}, F_{2}$ be the $\mathbf{u}$-faces on either side of $c$. The inclusion $c \hookrightarrow F_{i}$ induces an isomorphism $H_{1}(c) \approx H_{1}\left(F_{i}\right)$, and we can ask whether the composition $H_{1}\left(F_{1}\right) \approx H_{1}\left(F_{2}\right)$ preserves positive generators. If positive generators are not preserved then we say that $c$ is incoherent, otherwise $c$ is coherent. For example, if $V$ is circular on a cusped torus $T$ then the cusp circles of $T$ are all coherent.

Similarly, suppose $A$ is an annulus $\mathbf{p}$-face of $P^{u}$ such that $\partial A=\gamma_{1} \cup \gamma_{2}$, and suppose that $V$ is circular on the $\mathbf{u}$-face $F_{i}$ containing $\gamma_{i}$. The inclusion maps of $\gamma_{i}$ into $A$ and $F_{i}$ induce
homology isomorphisms, and by composition we obtain an isomorphism $H_{1}\left(F_{1}\right) \approx H_{1}\left(F_{2}\right)$. If this does not preserve positive generators then we say that $A$ is incoherent, and otherwise $A$ is coherent.

Dynamic splitting. We need to define dynamic splitting in the sutured manifold context. Let $(B, V)$ be an unstable dynamic branched surface in a sutured manifold $P$. Let $p: N(B) \rightarrow B$ be the projection map of an $I$-fibered neighborhood of $B$. First we define a splitting surface to be any surface with corners $F \subset N(B)$ tranverse to the $I$-fibers such that $\partial F$ decomposes into compact 1-manifolds with disjoint interior $\partial_{v} F \cup \partial_{i} F \cup \partial_{\partial} F$, where $\partial_{v} F=F \cap \partial_{v} N(B), \partial_{\partial} F=F \cap \partial P, \partial_{i} F$ is properly embedded in $N(B)$, and $F$ has cusps wherever $\partial_{i} F$ and $\partial_{v} F$ meet and corners wherever $\partial_{\partial} F$ meets the other two. We say moreover that $F$ is a dynamic splitting surface if, letting $V^{\prime}$ be the lift of $V$ under the submersion of $F$ to $B$, the vector field $V^{\prime}$ enters $F$ at each point of $\partial_{v} F$, is externally tangent at each endpoint of $\partial_{v} F$, and leaves $F$ at each point of $\partial_{i} F \cup \partial_{\partial} F$. The branched surface $B_{F}$ obtained by splitting along $F$ is defined as in the unsutured case: choose an $I$-fibered neighborhood $N(F) \subset N(B)$ whose fibers are subsets of fibers of $N(B)$, then collapse each $I$-fiber of $N\left(B_{F}\right)=\operatorname{cl}(N(B)-N(F))$, by perturbing the map $p \mid N\left(B_{F}\right)$ so that it maps each $I$-fiber of $N\left(B_{F}\right)$ to a separate point. Dynamic splitting of stable dynamic branched surfaces is similarly defined. We have:

Proposition 4.2.1. Any branched surface obtained by dynamic splitting of an unstable dynamic branched surface on a sutured manifold is an unstable dynamic branched surface, and similarly for stable.

As a special case of a dynamic splitting surface, let $B \subset P$ be an unstable dynamic branched surface, and let $A$ be a smoothly embedded annulus in $B$ such that one component of $\partial A$ lies in $\partial B \subset \partial_{\mathbf{p}} P$ and the other component of $A$ is a cusp circle of $B$. The annulus $A$ is called a maw-boundary annulus of $B$. Splitting along $A$ produces an unstable dynamic branched surface $B^{\prime}$. Note that the train track $\partial B^{\prime}$ is obtained from $\partial B$ by splitting along the circle $c=A \cap \partial_{\mathbf{p}} P$, resulting in an annulus $\mathbf{p}$-face $F$ of $\mathfrak{C}\left(P-B^{\prime}\right)$ whose boundary components are pu-corners.

We will also need the opposite operation. Start with an unstable dynamic branched surface $B^{\prime} \subset P$ and an annulus $\mathbf{p}$-face $F$ of $\mathfrak{C}\left(P-B^{\prime}\right)$ whose boundary circles are pucorners. Choose a parameterization $F \approx S^{1} \times[0,1]$ so that for each fiber (point) $\times[0,1]$, at least one of the endpoints is not a boundary switch. Choose a regular neighborhood $N(F) \subset \mathfrak{C}\left(P-B^{\prime}\right)$ and extend the parameterization $N(F) \approx\left(S^{1} \times[0,1]\right) \times[0,1]$ so that for each fiber $(($ point $) \times[0,1]) \times($ point $))$ at least one of the endpoints is not in $\Upsilon B^{\prime}$. Now collapse each of these fibers, by using a quotient map from $P$ to itself that is homotopic to the inclusion and is the identity outside of a neighborhood of $N(F)$. Let $B$ be the image of $B^{\prime}$ under this collapsing. We say that $B$ is obtained from $B^{\prime}$ by collapsing the $\mathbf{p}$-annulus
$F$. Notice that $B^{\prime}$ is obtained by splitting along a maw-boundary annulus of $B$, namely the image of $N(F)$ under the collapsing.

Suppose $B^{\prime}$ is obtained by splitting along a maw boundary annulus $A$ of $B$, resulting in an annulus p-face $F$ of $\mathfrak{C}\left(P-B^{\prime}\right)$. Let $c$ be the cusp circle on $\partial A$. Note that the dynamic vector field $V$ is circular on the $\mathbf{u}$-faces of $\mathfrak{C}(P-B)$ incident to $c$ if and only if $V$ is circular on the $\mathbf{u}$-faces of $\mathfrak{C}\left(P-B^{\prime}\right)$ incident to $F$; if this is the case, note moreover that $c$ is coherent if and only if $F$ is coherent.

Pairing a stable and an unstable dynamic branched surface. Suppose that $P$ is a sutured manifold, $B^{s}, B^{u}$ is a pair of branched surfaces in general position, and $V$ is a vector field on $P$ such that $\left(B^{u}, V\right)$ is an unstable dynamic branched surface, $\left(B^{s}, V\right)$ is a stable dynamic branched surface. Set $\tau=B^{s} \cap B^{u}$, so $V$ is necessarily tangent to the train track $\tau$. Then $Q=\mathfrak{C}\left(P-\left(B^{s} \cup B^{u}\right)\right)$ is naturally a dynamic manifold, with vector field obtained by pulling back $V$ via the overlay map $Q \mapsto P$, and with faces labelled $\mathbf{b}, \mathbf{p}, \mathbf{m}, \mathbf{s}$, or $\mathbf{u}$ depending on whether they map to $\mathbf{b}, \mathbf{p}$, or $\mathbf{m}$-faces of $P$ or to $B^{s}$ or $B^{u}$. The uu-cusps of $Q$ map to $\Upsilon B^{u}-\tau$, the ss-cusps to $\Upsilon B^{s}-\tau$, the su-corners to $\tau$, the pu-corners to $\partial B^{u}$, the $\mathbf{m s}$-corners to $\partial B^{s}$, and the remaining corners to similarly labelled corners of $P$. The puu-gables map to boundary switches of $B^{u}$, the mss-gables to boundary switches of $B^{s}$, the suu-gables to points of $B^{s} \cap \Upsilon B^{u}$, and the uss-gables to points of $B^{u} \cap \Upsilon B^{s}$.

In order for $B^{s}, B^{u}$ to be a dynamic pair on $P$ we shall require each component of $Q$ to have "simple dynamics". As in the case of a torally bounded manifold we formalize this definition by enumerating the allowed types of components of $Q$. Besides torus pieces and maw pieces, the components of $Q$ are closely related to products over surfaces-with-corners, whose description we now turn to.

### 4.3 Index of an even surface-with-corners

A surface-with-corners $F$ is even if it is compact, connected, oriented, and each boundary component has an even number of corners (cusps are ignored in determining whether $F$ is even). If $F$ is even, the index of $F$ is an integer defined as an obstruction, as follows. Let $L \rightarrow F$ be the tangent line bundle over $F$, an oriented $S^{1}$ bundle. Choose two symbols arbitrarily, and label each side of $F$ by one of the two symbols, so that the labels alternate at each corner and do not alternate at each cusp; this is possible because $F$ is even. This is called a corner alternating labelling of $F$. Sometimes the two labels will be $\|$ and $\perp$, other times they will be $\mathbf{s}$ and $\mathbf{u}$. Choosing the labels to be "||" and " $\perp$ ", pick a line field along $\partial F$ which is tangent to each $\|$ edge and transverse to each $\perp$ edge. Local models for the line field at each corner and cusp are shown in figure 4.3. The line field on $\partial F$ defines a section of $L$ along $\partial F$. The obstruction to extending this section over all of $F$ lives in $H^{2}(F, \partial F ; \mathbf{Z})$, which is canonically isomorphic to $\mathbf{Z}$, because $F$ is a compact, connected, oriented surface.


Figure 4.3: A line field along the boundary of a surface with corners.

The obstruction is therefore an integer index $(F)$, and it is well-defined independent of the choice of corner alternating labelling. The following formula for $\operatorname{index}(F)$ is a variation on the Euler-Poincaré index formula:

$$
\operatorname{index}(F)=2 \chi(F)-\frac{1}{2} \#(\text { corners })-\#(\text { cusps })
$$

When $F$ has no boundary we get $\operatorname{index}(F)=2 \chi(F)$, which is consistent with the fact that the tangent line bundle is doubly covered by the unit tangent bundle of $F$ (it might be more consistent to divide $\operatorname{index}(F)$ by 2 , obtaining a half-integer, but we do not wish to do so).

Remark. We could also define an index for an arbitrary surface-with-corners, replacing the tangent line bundle $L \rightarrow F$ by the bundle of unordered pairs of distinct tangent lines, which up to bundle homotopy is quadruply covered by the unit tangent bundle of $F$; we would therefore obtain an index whose value would equal $4 \chi(F)-\#$ (corners) - $2 \#$ (cusps). Again it might be more consistent to divide this number by 4 , obtaining a quarter integer.

Note that the only even surfaces-with-corners that have positive index are: the sphere (index $=4$ ); the disc (index $=2$ ); the cusped monogon, a disc with one cusp and no corners (index $=1$ ); and the uncusped bigon, a disc with no cusps and two corners (index $=1$ ). The only examples with index 0 are: the torus; the annulus; the rectangle, a disc with four corners; the one-cusped triangle, a disc with one cusp and two corners; and the cusped bigon, a disc with two cusps. In general, when referring to a surface with corners, if no cusps or corners are specified we assume they do not exist. For example, unless otherwise specified an "annulus" has no cusps or corners.

If $F$ is an even surface-with-corners, and if $e$ is an edge of $F$ with two endpoints at corners, we define a new surface with corners $F / e$ by pinching $e$ : a neighborhood of $e$ is replaced by a new neighborhood which has one cusp and no corners, as shown in figure 4.4. For example, pinching an edge of a rectangle produces a one-cusped triangle, and pinching
pinch edge


Figure 4.5: Some cylinders. The base of (a) is a hexagon. The base of (b) is a disc whose boundary has no corners, and with a dynamic orientation equal to the boundary orientation.
of $c$ using the two symbols $\mathbf{s}$ and $\mathbf{u}$. If $c$ has no corners, then choose one of the following: label $c$ with the symbol $\mathbf{b}$; or label $c$ with one of the two orientations on $c$ and call it the "dynamic orientation". Now take the topological manifold $S \times[0,1]$, and make it into a dynamic manifold as follows. The face $S \times 0$ is labelled $\mathbf{m}$, and $S \times 1$ is labelled $\mathbf{p}$. For each edge $\alpha \subset \partial S$ labelled $\mathbf{u}$, the face $\alpha \times[0,1]$ is labelled $\mathbf{u}$; similarly for the label $\mathbf{s}$. If a component $c$ is labelled $\mathbf{b}$ then the face $c \times[0,1]$ is labelled $\mathbf{b}$. If a component $c$ has a dynamic orientation then the face $c \times\left[0, \frac{1}{2}\right]$ is labelled $\mathbf{s}$, the face $c \times\left[\frac{1}{2}, 1\right]$ is labelled $\mathbf{u}$, and the us-circle $c \times \frac{1}{2}$ is given a dynamic orientation so that the projection $c \times \frac{1}{2} \rightarrow c$ is orientation preserving; note that $S \times[0,1]$ is given a manifold-with-corners structure so that $c \times \frac{1}{2}$ is a corner edge, and hence the smooth structure on $S \times[0,1]$ is not the product structure, as long as some $c$ is given a dynamic orientation. It is easy to define a smooth vector field $V$ which makes $S \times[0,1]$ into a dynamic 3 -manifold. For instance, if $c$ has a dynamic orientation, we can choose $V$ in a neighborhood of $c \times \frac{1}{2}$ so that $c \times \frac{1}{2}$ is a hyperbolic orbit with the adjacent s-face as a stable manifold and the adjacent $\mathbf{u}$-face as an unstable manifold; then extend $V$ over the rest of the manifold so that each forward trajectory either spirals into some hyperbolic corner circle or ends on the $\mathbf{p}$-face, and similarly for backward trajectories. We have defined a dynamic manifold $C$ called a cylinder over $S$ (see figure 4.5). The surface $S$ is also called the base of the cylinder, and we say that $C$ is a cylinder over $S$.

Let $C$ be a cylinder over $S$. We showed above how to pinch an edge of a surface-withcorners; now we show how to pinch certain corner edges of $C$. Given a ps-edge $\alpha$ of $C$ which connects two psu-corners, $\alpha$ can be pinched to create a uu-cusp edge connecting a puu-gable to an suu-gable, as shown in figure 4.6. To do this explicitly, let $N(\alpha)$ be a regular neighborhood of $\alpha$ in $C$ with rectangular frontier $R$, foliate $R$ by arcs parallel to $\alpha$, then remove $\operatorname{int}(N(\alpha))$ from $C$ and collapse each arc of $R$ to a point. An mu-edge of $C$ can be similarly pinched, creating an ss-cusp edge connecting an mss-gable to a uss-gable.


Figure 4.6: Pinching a ps-edge.

Suppose that $S$ is an even surface-with-corners without cusps. We wish to define a drum with base $S$ or a drum over $S$. Start with a cylinder $C$ over $S$.

If index $(S) \neq 0$, a drum over $S$ is defined by pinching all $\mathbf{p s}$ and mu-edges of $C$. See figure 4.7 for an example of a drum over a hexagon. Note that figure 4.5(b) is a drum, since there are no $\mathbf{p s}$ or $\mathbf{m u}$ edges to pinch.

If index $(S)=0$ then $S$ is either an annulus or a rectangle, and we consider these cases separately.

When $S$ is an annulus then no pinching is done and $C$ is itself a drum. Different labellings of $S$ give different types of drums, which we name as follows (see figure 4.8). Suppose first that both boundary circles of $S$ are assigned dynamic orientations. If the boundary circles of $S$ are oriented isotopic in $S$, then $C$ is called a coherent annulus drum, whereas if the boundary circles of $S$ are anti-isotopic in $S$, then $C$ is called an incoherent annulus drum - these two drums have the same manifold-with-corners structure, but they have different dynamics: in a coherent annulus drum the us-circles are oriented isotopic, but in an incoherent annulus drum the us-circles are anti-isotopic. Suppose next that one boundary circle of $S$ is labelled $\mathbf{b}$, and the other is assigned a dynamic orientation; in this situation no pinching is done and $C$ itself is a drum, called a half-annulus drum, which has one $\mathbf{u s}$-circle and one $\mathbf{b}$-face. The remaining case, where both boundary circles of $S$ are labelled $\mathbf{b}$, is given no special name (this drum is the same as an (annulus) $\times I$ sutured manifold-however, if such a sutured manifold ever occurs in a sutured manifold hierarchy, the hierarchy can always be simplified in a trivial manner).

Finally, if $S$ is a rectangle, a drum over $S$ is specified by making some choices (see figure 4.9). One may pinch the two ps-edges of $C$; or one pinch the entire $\mathbf{p}$-face of $C$, foliating that face by arcs parallel to the ps-edges and collapsing each arc, creating a uu-cusp edge


Figure 4.7: Three views of a drum over a hexagon. The top face is labelled $\mathbf{p}$, the bottom face $\mathbf{m}$, the three faces adjacent to the top are labelled $\mathbf{u}$, and the three faces adjacent to the bottom are labelled $\mathbf{s}$. In the "bird's eye" view, the $\mathbf{p}$ and $\mathbf{u}$ faces face upward toward the viewpoint, and the $\mathbf{m}$ and $\mathbf{s}$ faces face downward; the $\mathbf{u s}$ edges form the visual contour.


Figure 4.8: The base of a coherent annulus drum has oriented isotopic boundary circles. The base of an incoherent annulus drum has anti-isotopic boundary circles. The base of a half annulus drum has one oriented boundary circle and one $\mathbf{b}$-boundary circle. To understand the different dynamics, one should visualize trajectories in the $\mathbf{s}, \mathbf{u}$, and $\mathbf{b}$-faces.


Figure 4.9: Bird's eye views of four types of drums over a rectangle. They are classified by which of the $\mathbf{m}$ and $\mathbf{p}$-faces are totally pinched: (a) no total pinching; (b) m-pinched; (c) p-pinched; (d) pm-pinched, i.e. a pinched tetrahedron.
connecting two uus-gables. Similarly one may pinch the two mu-edges of $C$; or one may pinch the entire $\mathbf{m}$-face of $C$ thereby creating an ss-cusp edge connecting two ssu-gables. There are, therefore, four different types of drums over a rectangle, distinguished by two choices for pinching the $\mathbf{p}$-face and two choices for pinching the $\mathbf{m}$-face. If both the $\mathbf{p}$ and $\mathbf{m}$-face are entirely pinched, the resulting dynamic manifold is a pinched tetrahedron.

This completes the definition of a drum. In general, if $S$ is a surface-with-corners of genus $g$ with $k$ boundary components $c_{1}, \ldots, c_{k}$, with an edge labelling of $S$ chosen as above, the type of $S$ is a sequence of the form $\left(g ; w_{1}, \ldots, w_{k}\right)$, where the symbol $w_{i}$ describes $c_{i}$ in one of the following manners:

- $w_{i}$ is a positive integer and $2 w_{i}$ is the number of corners in $c_{i}$; or
- $w_{i}$ is the symbol $\mathbf{b}$ for "bare"; or
- $w_{i}$ is the symbol $0^{+}$, which means $c_{i}$ has no corners and has dynamic orientation matching the induced boundary orientation; or
- $w_{i}$ is the symbol $0^{-}$, which means $c_{i}$ has no corners and dynamic orientation opposite to the induced boundary orientation.

The symbols $0^{+}$and $0^{-}$are simply a convenient way to encode the choice of dynamic orientation. If the genus $g$ is understood, it may be omitted from the notation. Thus a coherent annulus drum has an annulus base of type $\left(0^{+}, 0^{-}\right)$, and an incoherent annulus drum has an annulus base of type $\left(0^{+}, 0^{+}\right)$or $\left(0^{-}, 0^{-}\right)$.

Here is another way to view a drum $D$ with base $S$ (assuming, when $S$ is a rectangle, that $S$ has no totally pinched $\mathbf{p}$ or $\mathbf{m}$-faces). Define a standard decomposition of $D$ into level surfaces as follows (with some imagination this can be visualized in figure 4.7). There is an embedding $D \rightarrow \mathbf{R} \times[0,1]$ so that each vector in the dynamic vector field on $D$ maps
to a vector with positive component in the $[0,1]$ direction, and the projection onto $[0,1]$ is a submersion with level surfaces $E_{t}=D \cap\left(\mathbf{R}^{2} \times t\right)$. For $t \in[0,1 / 4)$, the structure of $E_{t}$ does not change, although the s-sides of $E_{t}$ get longer as $t$ increases, to accomodate the fact that the dynamic vector field points inward along the ss-cusp edges of $D$. The surface $E_{1 / 4}$ contains the uss-gables of $D$, and as $t$ passes the value $1 / 4$ each ss-cusp of $E_{t}$ is "unpinched" to become a u-edge. For $t \in(1 / 4,1 / 2), E_{t}$ is an isomorphic copy of $S$, such that each dynamically oriented circle of $S$ is labelled s in $E_{t}$. The surface $E_{1 / 2}$ contains the us-circles of $Q$, and as $t$ passes the value $1 / 2$ the s-circles of $E_{t}$ become ucircles. For $t \in(1 / 2,3 / 4), E_{t}$ is again an isomorphic copy of $S$, but now each dynamically oriented circle in $S$ is labelled $\mathbf{u}$ in $E_{t}$. As $t$ increases from $1 / 4$ to $3 / 4$, the noncircular u-edges get longer and the noncircular s-edges get shorter. The surface $E_{3 / 4}$ contains the suu-gables of $Q$, as as $t$ passes the value $3 / 4$ each s-edge of $E_{t}$ is pinched to a uu-cusp. For $t \in(3 / 4,1]$ the structure of the surfaces $E_{t}$ does not change, although the u-edges get shorter to accomodate the fact that the dynamic vector field points outward along uu-cusp edges of $D$.

The following obvious lemma gives a useful way to recognize drums.
Lemma 4.4.1 (Recognizing drums). Suppose that $Q$ is a dynamic manifold, and $Q$ is not a drum over a rectangle. Then $Q$ is a drum if and only if, for each us-circle $c$ of $Q$, there exists a properly embedded annulus $A_{c} \subset Q$, and a component $D_{c}$ of $\mathfrak{C}\left(Q-A_{c}\right)$, such that:

- Labelling $A_{c}$ with the symbol $\mathbf{b}$, we have that $D_{c}$ is a half-annulus drum with us-circle c.
- If $c \neq c^{\prime}$, then $A_{c} \cap A_{c^{\prime}}=\emptyset$, and so $D_{c} \cap D_{c^{\prime}}=\emptyset$.
- The dynamic manifold $\mathfrak{C}\left(Q-\bigcup_{c} D_{c}\right)$ has interval dynamics, where the union is taken over all us-circles $c$.
- Q has no pm, ps, or mu-edges.
- Each uu-edge of $Q$ connects an suu-gable to a puu-gable.
- Each ss-edge of $Q$ connects a uss-gable to an mss-gable.


### 4.5 The definition of a dynamic pair in a sutured manifold

Define a dynamic pair of branched surfaces in a sutured 3-manifold $P$ to be a pair $B^{s}, B^{u}$ of branched surfaces in general position, together with a $C^{0}$ vector field $V$ on $P$, such that the following conditions hold:

- $(P, V)$ is a dynamic sutured manifold.
- $B^{u}$ and $B^{s}$ are unstable and stable branched surfaces on $P$ with respect to $V$. It follows that $Q=\mathfrak{C}\left(P-\left(B^{s} \cup B^{u}\right)\right)$ is a dynamic manifold, with dynamic vector field obtained by pulling back $V$ via the overlay map $Q \rightarrow P$.
- The vector field $V$ is smooth on $P$ except along $\Upsilon B^{s}$ where it has locally unique backward trajectories, and along $\Upsilon B^{u}$ where it has locally unique forward trajectories.
- $Q$ has simple dynamics. Each component of $Q$ is one of the following: an essential dynamic torus piece with a circular vector field; a drum whose base has index $\leq 0$ (which includes a pinched tetrahedron); or a maw piece $\mu$ with the following two properties:
- $\mu$ is attached to some dynamic torus piece, which means: if $\mu$ is an suu-maw piece, then the s-face of $\mu$ is identified with an s-face of some dynamic torus piece; similarly if $\mu$ is a uss-maw piece.
- $\mu$ is boundary parallel which means: if $\mu$ is an suu-maw piece then there is a smoothly embedded annulus $A \subset B^{u}$ with one boundary component on the maw circle of $\mu$ and the other boundary component in $\mathcal{R}_{+} P$, and $A \cap B^{s}=\emptyset$; similarly if $\mu$ is a uss-maw piece. Note that $A$ is a maw-boundary annulus.
- Transience of forward trajectories. For each component $K$ of $\mathfrak{C}\left(B^{u}-B^{s}\right)$, either there is a $\mathbf{u}$-face $F$ of some torus piece such that $F \subset K$ and $F$ is a sink of $K$, or each forward trajectory in $K$ is finite and ends at some point on $\mathcal{R}_{+} P$.
- Transience of backward trajectories. For each component $K$ of $\mathfrak{C}\left(B^{s}-B^{u}\right)$, either there is an s-face $F$ of some torus piece such that $F \subset K$ and $F$ is a source of $K$, or each backward trajectory in $K$ is finite and ends at some point on $\mathcal{R}_{-} P$.
- Separation of torus pieces. The union of torus piece components of $Q$ has no face gluings.
- $B^{u}, B^{s}$ have no boundary Reeb annuli.
- No component of $Q$ is a coherent annulus drum.

The final axiom plays a special role, which will be elucidated in $\S 4.12$. If $B^{s}, B^{u}$ satisfies all of the above properties except that coherent annulus drums are allowed, we will say that $B^{s}, B^{u}$ is a dynamic pair with coherent annulus drums. The main result of $\S 4.12$ will describe how to deal with coherent annulus drums, either converting them into torus pieces and producing a true dynamic pair, or using them to produce a nonperipheral incompressible torus.

Remark. If $\mathcal{R} P=\emptyset$ this definition clearly reduces to the definition for torally bounded manifolds.

Remark. As in the torally bounded case, in the presence of the other properties the property Separation of torus pieces is equivalent to the nonexistence of corner gluings among torus piece components.
Remark. A Reeb annulus $R$ on a surface $F$ has the following unpleasant property: if $\gamma$ is a simple closed curve in $F$ which intersects $\partial R$ essentially, then it is impossible to isotope $\gamma$ so that it has essential intersection with the train track in $R$; "essential intersection" means that no segment of $\gamma$ is path homotopic to a smooth train path. This pathology is the main reason for prohibiting boundary Reeb annuli in dynamic pairs: in the gluing step, we will need to be able to acheive essential intersection.

There are, nonetheless, some situations in which it is useful to allow Reeb annuli, albeit under very strict controls. For example, they are used in the Franks-Williams construction of intransitive Anosov flows [FW80]. We will touch on this issue later ( $\S ? ?$ ), showing how our methods can be used to reproduce the Franks-Williams construction.

Remark. Coherent annulus drums are avoided because in the induction step they can give rise to boundary Reeb annuli. Regarding the train track in a Reeb annulus as an unstable train track, we see that Reeb annuli and coherent annuli are closely related, since in both cases the two boundary circles are oriented isotopic; the only difference is that Reeb annuli have interior branches and coherent annuli have no interior branches.

In truth, coherent annulus drums are not all that bad, for they can always be eliminated, although accomplishing this task is somewhat delicate; see $\S 4.12$.

Remark. Note the very specific usage of the predicate "is attached to", in the context of the phrase "the maw piece $\mu$ is attached to the torus piece $T$ ". This means that if $F$ is the unique face of $\mu$ not adjacent to the cusp edge of $\mu$, then for some face $F^{\prime}$ of $T$, the annuli $F, F^{\prime}$ map homeomorphically onto the same annulus in $M$. Figure 4.14 gives an example where a maw piece is not "attached to" any solid torus piece.

Given a dynamic pair $B^{s}, B^{u}$ in a sutured manifold $\Pi$, the boundary train tracks are $\beta^{s}=\partial B^{s}=B^{s} \cap \mathcal{R}_{-} \Pi$ and $\beta^{u}=\partial B^{u}=B^{u} \cap \mathcal{R}_{+} \Pi$.

A train track $\rho$ in an oriented surface $F$ is essential if each component of $\mathfrak{C}(F-\rho)$ has nonpositive index; for purposes of computing index, $\mathfrak{C}(F-\rho)$ is a surface-with-corners, and it is even, since there are no corners. The property of essentiality specifically rules out a component of $\mathfrak{C}(F-\rho)$ which is a disc with smooth boundary or with one boundary cusp.

Lemma 4.5.1. $\beta^{s}$ is essential in $\mathcal{R}_{-} \Pi$, and similarly $\beta^{u}$ is essential in $\mathcal{R}_{+} \Pi$.
Proof. Each component of $\mathfrak{C}\left(\mathcal{R}_{-} \Pi-\beta^{s}\right)$ is the $\mathbf{m}$-face of some drum component of $\mathfrak{C}(\Pi-$ $\left(B^{s} \cup B^{u}\right)$ ). The base of the drum has nonpositive index, and the $\mathbf{m}$-face has the same index as the base.

Further properties of $\beta^{s}$ and $\beta^{u}$ will emerge in the next sections. Ideally we would like to say that $\beta^{s}$ is a stable train track and $\beta^{u}$ is unstable, but this may not be true in general. In proposition 4.7.1 we will see that it is true after mild alteration of the dynamic pair.

### 4.6 Dynamic train tracks and properties of dynamic pairs

In this section we generalize the results of $\S 2.5$ to the context of sutured manifolds. The proofs follow the same ideas as in $\S 2.5$, and when possible we will refer to the proofs in that section, with emphasis on the differences.

The main result will describe the branched surfaces of a dynamic pair $B^{s}, B^{u}$ in a sutured manifold $P$. Recall that when $P$ is a torallyl bounded manifold, that is when $\mathcal{R} P=\emptyset$, all the components of $\mathfrak{C}\left(P-B^{s}\right)$ and $\mathfrak{C}\left(P-B^{u}\right)$ are cusped torus pieces. However, when $\mathcal{R} P \neq \emptyset$ there is a new class of components, which we now describe.

A dynamic 3-manifold $H$ is called a $\mathbf{u}$-cusped product if there is a compact, connected, oriented surface $F$ and a homeomorphism $H \approx F \times[0,1]$ such that:

- $F \times 0$ is an $\mathbf{m}$-face of $H$.
- Each component of $\partial F \times[0,1]$ is a $\mathbf{b}$-face of $H$.
- The surface $F \times 1$ is subdivided into $\mathbf{p}$-faces and $\mathbf{u}$-faces, with the $\mathbf{u}$-faces contained in $\operatorname{int}(F) \times 1$.
- Every forward trajectory $\gamma$ in $H$ has one of three fates: $\gamma$ ends at a point on a p-face; $\gamma$ ends at a point on a uu-cusp; or $\gamma$ is infinite, accumulating on a $\mathbf{u}$-face.
- The index of the surface $F$ and of each p-face of $H$ is nonpositive.
- Each $\mathbf{u}$-face of $H$ is an annulus on which the vector field is circular.
- Incoherence of cusp circles. Each uu-cusp circle of $H$ is incoherent.

Remark. Recall from $\S 4.2$ that each $\mathbf{u}$-face of $H$ must be an annulus or torus; the definition of a u-cusped product has the effect of ruling out a torus and restricting the dynamic vector field to be circular on an annulus.

Remark. As proposition 4.6 .1 will show, given a dynamic pair $B^{s}, B^{u}$ in a sutured manifold $P$, each component of $\mathfrak{C}\left(P-B^{u}\right)$ adjacent to $\mathcal{R}_{-} P$ is a $\mathbf{u}$-cusped product.
Remark. There is a close relation between coherent cusp circles and Reeb annuli. In proposition 4.6.1, given a dynamic pair $B^{s}, B^{u} \subset P$, the fact that $\partial B_{s}$ has no Reeb annuli in $\partial_{\mathbf{m}} P$ will be used to prove coherence of cusp circles for $\mathbf{u}$-cusped products in $\mathfrak{C}\left(P-B^{u}\right)$, and this argument will be reversed in proposition 4.11.1. Note that in a cusped torus piece with a circular vector field, every cusp circle is coherent.

Remark. As a consequence of the definition, all backward trajectories that are not contained in a $\mathbf{u}$-face are finite, and hence end on the $\mathbf{m}$-face of $H$. For if there were an infinite backward trajectory not on a u-face, and if $x$ were an accumulation point of that backward trajectory, then the forward trajectory of $x$ would be infinite and would not accumulate on a $\mathbf{u}$-face. Thus it is appropriate to say that a $\mathbf{u}$-cusped product has simple dynamics.

These ideas may be used to recognize a $\mathbf{u}$-cusped product. Suppose that $Q$ is a dynamic manifold with smooth vector field, having only $\mathbf{m}, \mathbf{b}, \mathbf{p}$, and $\mathbf{u}$-faces, such that there is a single $\mathbf{m}$-face, each $\mathbf{b}$-face is an annulus with one $\mathbf{m b}$-circle and one $\mathbf{p b}$-circle on the boundary, and each backward trajectory not contained in a $\mathbf{u}$-face terminates on the $\mathbf{m}$ face. It follows easily that $Q$ satisfies the first four properties of a u-cusped product: the product structure on $Q$ can be constructed from the trajectories of the flow. However, circularity of the vector field on annulus u-faces, and Incoherence of cusp circles must be checked separately.

Given a sutured manifold $P$ and an unstable dynamic branched surface $B \subset P$, we say that $B$ is very full in $P$ if each component of $\mathfrak{C}(P-B)$ is a $\mathbf{u}$-cusped torus piece or a u-cusped product. Any component which is disjoint from $\mathcal{R} P$ is a u-cusped torus piece, and any component intersecting $\mathcal{R} P$ is a $\mathbf{u}$-cusped product.

An s-cusped product, and a very full stable dynamic branched surface, are similarly defined.

The following proposition generalizes proposition 2.5 .1 to the setting of sutured manifolds:

Proposition 4.6.1. Let $B^{s}, B^{u}$ be a dynamic pair in a sutured manifold $P$, let $Q=\mathfrak{C}(P-$ $\left.\left(B^{s} \cup B^{u}\right)\right), P^{s}=\mathfrak{C}\left(P-B^{s}\right), P^{u}=\mathfrak{C}\left(P-B^{u}\right)$. Then:

- $B^{u}$ is very full in $P$.
- Inclusion induces a type preserving, 1-1 correspondence between the set of cusped torus piece components of $P^{u}$ and the set of dynamic torus piece components of $Q$. If $D \subset P^{u}$ and $C \subset Q$ are corresponding components, then each component $\mu$ of $\mathfrak{C}(K-C)$ is an suu-maw piece attached to $C$ by identifying the $\mathbf{s}$-face of $\mu$ with an s-face of $C$. Either $\mu$ is itself a component of $Q$, or $\mu$ is cut into pinched tetrahedra by $B^{s}$ (see figure 2.8).
- The dynamic vector field on each cusped torus piece of $P^{u}$ is circular.
- $B^{u}$ does not carry a closed surface.
- If $\sigma$ is a sector of $B^{u}$ containing a periodic trajectory of the dynamic vector field, then $\sigma$ is an annulus or Möbius band, and on at least one side of $\sigma$ the adjacent component of $P^{u}$ is a $\mathbf{u}$-cusped product.


## Similar statements hold for $B^{s}$.

Remark. This proposition describes how $B^{s}$ cuts u-cusped torus pieces into dynamic torus pieces and suu-maw pieces. In $\S 4.7$ we shall describe how u-cusped products are cut into drums (and other pieces), after some minor alterations are performed on the dynamic pair.

As with the previous version, the proof again depends on the idea of a dynamic train track.

Let $P$ be a sutured manifold and let $(B, V)$ be an unstable dynamic branched surface in $P$. The definition of a dynamic train track $\tau$ in $B$ is very similar to the torally bounded case: $\tau$ is embedded in $B$ disjoint from $\partial B$; there is a dynamic vector field $V^{\prime}$ on $B$ tangent to $\tau$; the set of converging switches of $\tau$ is $\tau \cap \Upsilon B ; V^{\prime}$ is smooth on $B$ except at diverging switches of $\tau$. Also, an analogue of Transience of forward trajectories is satisfied, as follows: for each component $K$ of $\mathfrak{C}(B-\tau)$, one of two things happens:

- There exists a smooth surface $A \subset K$ such that $\partial A \subset \partial K$ and $A$ is a sink of $K$; or
- Each forward trajectory in $K-\tau$ is finite and ends at a point of $K \cap \partial B$.

Each sink is a surface of Euler characteristic zero. We say that $\tau$ fills up $B$ if each sink is a ring. As we shall see in proposition 4.6.4, each component $K$ of the first type is a ring with tongues. A component $K$ of the second type is boundary adjacent, and the structure of these components will be described in section 4.7. The definitions obviously adapt to stable branched surfaces.

As a consequence of the definitions we have:
Proposition 4.6.2. If $B^{s}, B^{u}$ is a dynamic pair in a sutured manifold $P$ and $\tau=B^{s} \cap B^{u}$, then $\tau$ is a dynamic train track filling up both $B^{s}$ and $B^{u}$.

Recall the previous definition of a cusped, unstable branched surface $B$. We generalize this definition to the sutured manifold setting by allowing the vector field $V$ to point outward on certain components of $\partial B$ and to be tangent with cusps on other components of $\partial B$. The following is the sutured manifold analogue of lemmas 2.5.3-2.5.5:

Lemma 4.6.3. If $P$ is a sutured manifold, $B \subset P$ is an unstable dynamic branched surface, and $\tau \subset B$ is a dynamic train track, then:

- $\mathfrak{C}(B-\tau)$ is a cusped, unstable branched surface.
- $\partial \mathfrak{C}(B-\tau)-\mathcal{R}_{+} P$ is an unstable train track.
- The remains of $\tau$ in $\mathfrak{C}(P-B)$ is a stable train track.

The following is an analogue of proposition 2.5.7:
Proposition 4.6.4. Let $B \subset P$ be an unstable, dynamic branched surface, and let $\tau \subset B$ be a dynamic train track. Then $\tau$ fills up $B$ if and only if, for each component $K$ of $\mathfrak{C}(B-\tau)$, either $K$ is a ring with tongues, or $K \cap \partial B \neq \emptyset$.

Proof. The "if" direction is clear. For the "only if" direction, if $\tau$ fills up $B$, and if $K$ is a component of $\mathfrak{C}(B-\tau)$ such that $K \cap \partial B=\emptyset$, then the arguments of proposition 2.5.7 apply to show that $K$ is a ring with tongues.

Proof of proposition 4.6.1. Consider a component $K$ of $\mathfrak{C}\left(P-B^{u}\right)$. We must show that $K$ is either a u-cusped torus piece with circular vector field, or a u-cusped product.

If $K$ is disjoint from $\mathcal{R} P$, follow the proof of 2.5.1. First show that $K$ contains a torus piece $C$. Then show that each component $\mu$ of $\mathfrak{C}(K-C)$ is a uss-maw piece: if $\mu$ is itself a component of $Q$, then $\mu$ cannot be a torus piece because it shares a $\mathbf{u}$-face with the torus piece $C$, so the only possibility remaining is that $\mu$ is a uss-maw piece; whereas if $\mu$ is not a component of $Q$ then the proof of 2.5 .1 shows that $\mu$ is a uss-maw piece cut into pinched tetrahedra by $\mu \cap B^{u}$. It follows that $K$ is a $\mathbf{u}$-cusped torus piece; circularity of the vector field on $K$ is obvious.

If $K$ is not disjoint from $\mathcal{R} P$, it follows from the definition of a dynamic pair that all backward trajectories disjoint from $\mathbf{u}$-faces terminate in $\mathcal{R}_{-} P$. To show that $K$ is a $\mathbf{u}$-cusped product, the only nonobvious facts remaining are that each annulus $\mathbf{u}$-face has a circular vector field, and that cusp circles of $K$ are incoherent. Let $\tau_{K}^{u}$ be the remains of $\tau^{u}$ in $K$.

Consider an annulus $\mathbf{u}$-face $A$ of $K$. Note that $\tau_{A}=\tau_{K} \cap A$ has only diverging switches, and that the points of $\partial \tau_{A}$ all point outward along $\partial A$. It follows as in lemma 2.5.6 that each immersed circle of $\tau_{A}$ is an embedded circular source, and these circles are pairwise disjoint. If $\tau_{A}$ has a unique circular source then the vector field on $A$ is clearly circular. Suppose that $\gamma_{0}, \ldots, \gamma_{n}$ are the circular sources in $\tau_{A}, n \geq 1$, and let $R_{i}$ be the subannulus of $A$ with boundary $\gamma_{i-1} \cup \gamma_{i}$; we assume the notation chosen so that the $R_{i}$ have disjoint interiors. From proposition 4.6 .4 it follows that each $R_{i}$ is the sink of a ring with tongues component of $\mathfrak{C}\left(B^{u}-\tau\right)$, and so the vector field on the annulus $R_{i}$ is circular. It follows that the vector field on $R_{1} \cup \cdots \cup R_{n}$ is circular, as is the vector field on $A$.

Consider a uu-cusp circle $c$ of $K$, with adjacent $\mathbf{u}$-faces $A, A^{\prime}$. We prove that $c$ is incoherent. Let $\gamma \subset A, \gamma^{\prime} \subset A^{\prime}$ be the circular sources of $\tau_{A}$ and $\tau_{A^{\prime}}$ closest to $c$, and let $H$ be the annulus in $A \cup A^{\prime}$ bounded by $\gamma \cup \gamma^{\prime}$ and containing $c$. It suffices to prove that $\gamma$ and $\gamma^{\prime}$ are not oriented isotopic through $H$. From lemma 4.6.3, it suffices to prove that the spiralling orientations on $\gamma$ and $\gamma^{\prime}$ are not oriented isotopic through $H$. Regarding $H$ as a smooth annulus by smoothing the cusp $c$, consider the train track $\tau_{K} \cap H$. Each of $\gamma, \gamma^{\prime}$ is a circular source of $\tau_{K} \cap H$. Note that $\tau_{K} \cap c \neq \emptyset$, because no maw circle of $B^{u}$ is disjoint from $B^{s}$, except for the cusp circles of suu-maw pieces, all of which are contained
in $\mathbf{u}$-cusped torus pieces. It now follows that $\tau_{K} \cap \operatorname{int}(H) \neq \emptyset$. Since $\gamma, \gamma^{\prime}$ are closest to $c$, each backward trajectory of $\tau_{K}$ starting in $H \cap A$ eventually hits $\gamma$, and each backward trajectory starting in $H \cap A^{\prime}$ eventually hits $\gamma^{\prime}$. Therefore, it suffices to prove that $H$ is not a Reeb annulus.

Move $H$ slightly off of $\partial_{\mathbf{u}} K$, keeping the boundary in $B^{s}$, and flow backward to $\mathcal{R}_{-} P$, eventually reaching an annulus $H^{\prime} \subset \mathcal{R}_{-} P$. Some convergence of backward trajectories may occur, nevertheless it is clear that $\partial H^{\prime} \subset \beta^{s}, \operatorname{int}\left(H^{\prime}\right) \cap \beta^{s} \neq \emptyset$, and $\operatorname{int}\left(H^{\prime}\right) \cap \beta^{s}$ contains no cycles. From the definition of a dynamic pair we have that $H^{\prime}$ is not a Reeb annulus, and therefore $H$ is not a Reeb annulus, completing the proof of Incoherence of cusp circles for $c$.

This completes the proof of the first three statements of the proposition.
As in the proof of proposition 2.5.1, the remaining two statements of proposition 4.6.1 follow from an analysis of circular sinks and sources in $\tau$. Unlike the case of a manifold with torus boundaries, circular sinks and sources are not prohibited, but they are strictly regulated:

Lemma 4.6.5 (Circular sinks and sources). If $\gamma$ is a circular sink of $\tau$, then there exists an annulus or Möbius band sector $\sigma$ of $B^{s}$ such that $\gamma \subset \sigma$ and $\sigma$ satisfies the conclusion of lemma 4.6.1, namely: on at least one side of $\sigma$ the adjacent component of $P^{s}$ is an s-cusped product. A similar conclusion holds for a circular source of $\tau$.

Accepting this lemma for the moment, we prove the remaining statements of proposition 4.6.1.

Suppose that $B^{u}$ carries a closed surface $S$. Following the proof of statement 4 of proposition 2.5.1, it follows that $S$ contains a circular $\operatorname{sink} \gamma$ of $\tau$. Note that all forward trajectories starting in $S$ are infinite and stay in $S$ for all future time. However, let $R \subset B^{u}$ be a smoothly embedded annulus or Möbius band with core curve $\gamma$. At least one of the components of $\mathfrak{C}(R-\gamma)$ lies in an s-cusped product component $K$ of $P^{s}$, according to lemma 4.6.5. All forward trajectories in an s-cusped product which are disjoint from the $\mathbf{u}$-faces must end on $\mathcal{R}_{+} P$, but some of these trajectories in $K$ lie in $S$, contradicting that forward trajectories starting in $S$ are infinite.

Next suppose that $B^{u}$ has a sector $\sigma$ containing a periodic trajectory $\gamma$ of $V$. If $\gamma \subset \tau$ then $\gamma$ is a circular source of $\tau$ and the desired conclusion for $\sigma$ follows directly from lemma 4.6.5. If $\gamma \not \subset \tau$ then, as in the proof of statement 5 of proposition 2.5.1, $\gamma \cap \tau=\emptyset$. Let $K$ be the component of $\mathfrak{C}\left(B^{u}-\tau\right)$ containing $\gamma$. Since $K$ contains an infinite forward trajectory disjoint from $K-\tau$, it follows from proposition 4.6 .4 that $K$ is an annulus with tongues. There are, moreover, no tongues, because the attaching curve of the first tongue would intersect $\gamma$, contradicting that $\gamma \cap \Upsilon B^{u}=\emptyset$. Therefore, $K$ is a common $\mathbf{u}$-face of some torus piece component $T$ and some boundary parallel uss-maw piece component $\mu$ of $Q$. The component of $P^{u}$ on the side of $K$ facing $\mu$ is clearly a $\mathbf{u}$-cusped product, completing the proof of proposition 4.6.1.

Proof of lemma 4.6.5. Let $\gamma$ be a circular sink of $\tau$. Since $\tau$ has no diverging switches in $\gamma$ it follows that $\gamma \cap \Upsilon B^{s}=\emptyset$ and so $\gamma \subset \sigma$ for some sector $\sigma$ of $B^{s}$. By proposition 1.5.1 the sector $\sigma$ is either an annulus, Möbius band, or disc, but the latter is ruled out because the existence of a periodic trajectory in a disc implies the existence of a zero of the dynamic vector field by the Euler-Poincaré index formula, a contradiction.

Let $R$ be a smoothly embedded annulus or Möbius band in $B^{u}$ having core $\gamma$. The surface $\mathfrak{C}(R-\gamma)$ has one or two components, depending on whether or not $R$ is a Möbius band. Given a component $C$ of $\mathfrak{C}(R-\gamma)$, let $K_{C}$ be the component of $\mathfrak{C}\left(B^{u}-\tau\right)$ containing $C$. Applying proposition 4.6.4 to $K$, there are two cases.

Case 1: There exists a component $C$ of $\mathfrak{C}(R-\gamma)$ such that $K_{C}$ is a ring with tongues. By definition of a dynamic pair, the sink of $K_{C}$ is a $\mathbf{u}$-face $A$ of some torus piece $T$, and clearly $\gamma$ is a component of $\partial A$, so $\gamma$ is a corner circle of $T$. Let $A^{\prime}$ be the s-face of $T$ adjacent to $\gamma$, and let $\mu$ be the suu-maw piece attached to $T$ along $A^{\prime}$. The component of $\mathfrak{C}\left(B^{s}-\tau\right)$ containing $A^{\prime}$ is a ring with tongues, but in fact there are no tongues: if there were tongues, then the boundary of the first tongue attached to $A^{\prime}$ would contain a branch of $\tau$ intersecting $\gamma$ in a diverging switch, contradicting that $\gamma$ is a circular sink. It follows that $\mu$ is a component of $Q$, and so $\mu$ is boundary parallel. Thus, $\gamma$ (and therefore $\sigma$ ) lies on the boundary of an s-cusped product component of $P^{s}$.

Case 2: For each component $C$ of $\mathfrak{C}(R-\gamma), K_{C}$ is not a ring with tongues. It follows that all forward trajectories in $K_{C}-\tau$ are finite and end on $K_{C} \cap \partial B^{u}$, and this implies that the component of $P^{s}$ containing $K_{C}$ is an s-cusped product.

### 4.7 Boundary train tracks of a good dynamic pair

In this section we study the "peripheral" structure of a dynamic pair $B^{s}, B^{u}$ in a sutured manifold $P$. That is, letting $\tau=B^{s} \cap B^{u}$, we study the structure of the unstable cusped branched surface $\hat{B}^{u}=\mathbb{C}\left(B^{u}-\tau\right)$ and the boundary train track $\beta^{u}=\partial B^{u}$; also the structure of $\hat{B}^{s}=\mathfrak{C}\left(B^{s}-\tau\right)$ and $\beta^{s}=\partial B^{s}$. Let $\hat{\tau}^{u}$ be the remains of $\tau$ in $\hat{B}^{u}$; by lemma 4.6 .3 we have that $\hat{\tau}^{u}$ is an unstable train track. There is a disjoint union

$$
\partial \hat{B}^{u}=\hat{\tau}^{u} \coprod \beta^{u}
$$

Similarly,

$$
\partial \hat{B}^{s}=\hat{\tau}^{s} \coprod \beta^{s}
$$

where $\hat{\tau}^{s}$, the remains of $\tau$ in $\hat{B}^{s}$, is a stable train track. Let $\hat{B}_{\partial}^{u}$ be the union of components of $\hat{B}^{u}$ which have nonempty intersection with $\partial B^{u}$, and let $\hat{\tau}_{\partial}^{u}=\hat{\tau}^{u} \cap \hat{B}_{\partial}^{u}$, so we have $\partial \hat{B}_{\partial}^{u}=\hat{\tau}_{\partial}^{u} \amalg \partial B^{u}$.

We would like to say, roughly speaking, that $\hat{B}_{\partial}^{u}$ has the topological structure of $\hat{\tau}^{u}$ crossed with an interval. In particular we would like to say that $\beta^{u}$ is an unstable train track
isomorphic to $\hat{\tau}^{u}$. Unfortunately these statements fail in the presence of boundary parallel maw pieces. Associated to a boundary parallel suu-maw piece $\mu$ are two periodic cycles of $\hat{\tau}^{u}$ (the corner circles of $\mu$ ) but only one circle in $\beta^{u}$ (contained in the maw-boundary annulus associated to $\mu$ ). What goes wrong is that trajectories on the two $\mathbf{u}$-faces of $\mu$ converge at the uu-cusp circle. If this were the only thing that could go wrong that would still be "good", and it would follow that $\beta^{u}$ is an unstable train track. Unfortunately, there might be other convergences of trajectories occurring along $\Upsilon \hat{B}^{u}$ to mess things up, but this can be cured with a little dynamic splitting, as proposition 4.7.1 will show.

To state the proposition, we need a new type of dynamic manifold. If $\mu$ is a boundary parallel suu-maw piece of some dynamic pair, and if $A$ is the maw-boundary annulus of $\mu$, splitting along $\mu$ results in a dynamic manifold $H$ which is a solid torus with four faces labelled $\mathbf{u}, \mathbf{s}, \mathbf{u}, \mathbf{p}$, such that the core of each face is isotopic to the core of $H$, and the two us-circles are oriented isotopic; we call $H$ a split suu-maw piece. Similarly, splitting along the maw-boundary annulus of a uss-maw piece produces a dynamic manifold called a split uss-maw piece.

Consider a dynamic pair $B^{s}, B^{u}$ in a sutured manifold $P$. We say that $B^{s}, B^{u}$ is a good dynamic pair if, after splitting along all maw-boundary annuli, converting all boundary parallel maw pieces into split maw pieces, the resulting pair $B^{\prime s}, B^{\prime u}$ has the following property: setting $\tau=B^{\prime s} \cap B^{\prime u}$, all components of $\hat{B}^{\prime s}=\mathfrak{C}\left(B^{\prime s}-\tau\right)$ incident to $\mathcal{R}_{-} P$ are topological products, and all components of $\hat{B}^{\prime u}=\mathfrak{C}\left(B^{\prime u}-\tau\right)$ incident to $\mathcal{R}_{+} P$ are topological products. Let $\hat{B}_{\partial}^{\prime u}$ be the union of components of $\hat{B}^{\prime u}$ intersecting $\mathcal{R}_{+} P$, and let $\hat{B}_{\partial g}^{\prime s}$ be the union of components of $\hat{B}^{\prime s}$ intersecting $\mathcal{R}_{-} P$. Let $\hat{\tau}^{u}=\partial \hat{B}_{a}^{\prime u}-\mathcal{R}_{+} P$, and let $\hat{\tau}^{s}=\partial \hat{B}^{\prime s}-\mathcal{R}_{-} P$. Goodness is equivalent to the existence of a homeomorphism $\hat{B}_{\partial}^{\prime u} \approx \hat{\tau}^{u} \times[0,1] \approx \hat{\tau}^{u} \approx \hat{\tau}^{u} \times 0$, and similarly for $\hat{B}^{\prime}{ }_{\partial}$.

Proposition 4.7.1 (Splitting for goodness sake). Any dynamic pair $B^{s}, B^{u}$ in a sutured manifold $P$ may be altered by dynamic splitting, along splitting surfaces disjoint from $\tau=B^{s} \cap B^{u}$, so that after the splitting the pair $B^{s}, B^{u}$ becomes good. If $B^{s}, B^{u}$ is good, then $\beta^{u}$ is naturally an unstable train track, and $\beta^{s}$ is naturally a stable train track.

For future reference, we point out some features of a good dynamic pair $B^{s}, B^{u}$. If $\mu$ is a boundary parallel suu-maw piece of $B^{s}, B^{u}$, and if $A$ is the maw-boundary annulus associated to $\mu$, then $A \cup \mu$ is called a cusped boundary parallelism of $B^{u}$. We also define an uncusped boundary parallelism of $B^{u}$ to be the image $p$ of a smooth immersion $f: S^{1} \times$ $[0,1] \rightarrow B^{u}$ such that:

- For some smooth circle $c \subset \hat{\tau}^{u}, p$ factors as

$$
S^{1} \times[0,1] \approx c \times[0,1] \subset \hat{\tau}^{u} \times[0,1] \approx \hat{B}_{\partial}^{\prime u} \mapsto \hat{B}^{u} \mapsto B^{u},
$$

where $\hat{\tau}^{u} \times[0,1] \approx \hat{B}_{\partial}^{\prime u}$ is the homeomorphism given in the definition of a good
dynamic pair, $\hat{B}_{\partial}^{\prime \mu} \rightarrow \hat{B}^{u}$ is the inverse collapsing map of maw-boundary annulus splitting, and $\hat{B}^{u} \mapsto B^{u}$ is the overlay map.

- $p$ is disjoint from any cusped boundary parallelism.

The image of $c$ in $\tau$ is a periodic orbic $\gamma$, and either $\gamma$ is an embedded, untwisted orbit in $\tau$ or $\gamma$ is a double cover of an embedded twisted orbit. We say that $p$ is attached to $\gamma$. From the definitions it follows that if $B^{s}, B^{u}$ is good then inclusion induces a 1-1 correspondence between circular sinks of the unstable train track $\beta^{u}$ and boundary parallelisms of $B^{u}$ (cusped or uncusped). Similar statements apply to $B^{s}$.

Proof of proposition 4.7.1. We begin by proving that if $B^{s}, B^{u}$ is good then $\beta^{u}$ is naturally an unstable train track. Let $\beta^{\prime u}=B^{\prime u} \cap \mathcal{R}_{+} P$. Use the product structure on $\hat{B}_{\partial}^{\prime u}$ to transfer the singular orientation on $\hat{\tau}^{u}$ to a singular orientation on $\beta^{\prime u}$, with makes $\beta^{\prime u}$ an unstable train track. Note that $\beta^{\prime u}$ is obtained from $\beta^{u}$ by splitting along boundary circles of maw-boundary annuli of $B^{u}$. Moreover if $A$ is the annulus component of $\mathfrak{C}\left(\mathcal{R}_{+} P-\beta^{\prime u}\right)$ resulting from one of these these maw-boundary annulus splittings, then $A$ is incoherent, when regarded as a $\mathbf{p}$-face of $\mathfrak{C}\left(P-B^{\prime u}\right)$. In particular, the two boundary circles of $A$, regarded as circles in the train track $\beta^{\prime u}$, are oriented isotopic. When all such annuli $A$ are collapsed, therefore, the singular orientation on $\beta^{\prime u}$ induces a well-defined singular orientation on $\beta^{u}$, with respect to which $\beta^{u}$ is unstable. The fact that $\beta^{s}$ is stable follows similarly.

We now describe how to split $B^{u}$ so that $\hat{B}_{\partial}^{\prime u} \approx \hat{\tau}^{u} \times[0,1]$.
Let $K$ be the union of s-cusped product components of $\mathfrak{C}\left(P-B^{s}\right)$. Let $U$ be a closed regular neighborhood in $K$ of $\partial_{\mathbf{m}} K \cup \partial_{s} K$, chosen so that $L=\operatorname{Fr}(U)$ is a smooth surface transverse to the dynamic vector field and to $\Upsilon \hat{B}^{u}$. We may assume that the product structure on $K$ restricts to a product structure $U \approx L \times[0,1]$ so that $\partial_{\mathbf{m}} K \cup \partial_{\mathbf{s}} K \approx L \times 0$ and $L \approx L \times 1$, and so that $\hat{B}^{u} \cap U \approx \hat{\tau}^{u} \times[0,1]$. Let $\phi: \partial_{m} K \cup \partial_{s} K \rightarrow L$ be the homeomorphism induced by the product structure on $U$. The map $\phi$ can be chosen to smooth out ss-cusp curves and to be smooth everywhere else. Let $\tau_{0}=L \cap \hat{B}^{u}$, and note that $\phi\left(\hat{\tau}^{u}\right)=\tau_{0}$.

Note that each forward trajectory in $\operatorname{cl}(K-U)$ is finite and ends on $\mathcal{R}_{+} P$; in particular this is true for each trajectory starting from $L$. The trajectories starting from $x, y \in L$ are said to converge if $x \cdot s=y \cdot t$ for some $s, t>0$, from which it follows that $x \cdot(s+h)=y \cdot(t+h)$ if $h \geq 0$. The first point where these trajectories converge lies on $\Upsilon \hat{B}^{u}$.

For each suu-maw piece $\mu$, the set $\mu \cap L$ is an annulus, and there is a homeomorphism between the two components $c, c^{\prime}$ of $\partial(\mu \cap L)$ so that if $x \in c, x^{\prime} \in c^{\prime}$ correspond under the homeomorphism then the trajectories starting at $x, x^{\prime}$ converge at the cusp circle of $\mu$.

Choose pairwise disjoint regular neighborhoods $W(s) \subset L$ of the switches $s \in \tau_{0}$, with the following property: if $\alpha, \alpha^{\prime}$ are the two components of $\left(W(s) \cap \tau_{0}\right)-s$ on the two-

follows that $\hat{B}_{\partial}^{\prime u}$ is homeomorphic to $\hat{\tau}^{u} \times[0,1]$. Doing similar splittings on $B^{s}$, we obtain a good dynamic pair.

### 4.8 Simple examples of dynamic pairs in sutured manifolds

When we introduced dynamic pairs in manifolds with torus boundaries, it was difficult to give any immediate examples. We finally constructed examples in section 2.7 using the bombast of proposition 2.6.2.

By contrast, in the sutured manifold setting there are many simple examples which we can view directly.
Example (Product sutured manifold). Every product sutured manifold $P$ over a surface $F$ of nonpositive Euler characteristic has a dynamic pair which is empty, because $P$ itself is a drum over $F$ where each component of $\partial F$ is labelled $\mathbf{b}$.
Example (The round handle). Let $H$ be a solid torus whose sutures consist of four longitudinal circles. As an isolating block, this object was dubbed the round handle by Asimov [Asi75]. Setting $I=J=[-1,1]$ we have $H=I \times J \times S^{1}$ with $\sigma H=\partial I \times \partial J \times S^{1}$, $\mathcal{R}_{-} H=I \times \partial J \times S^{1}$, and $\mathcal{R}_{+} H=\partial I \times J \times S^{1}$. There is a dynamic pair $B^{s}, B^{u}$ with $B^{s}=0 \times \partial I \times S^{1}$ and $B^{u}=\partial I \times 0 \times S^{1}$. The train track $\tau=B^{s} \cup B^{u}=0 \times 0 \times S^{1}$ is a circle, and has two possible choices for the dynamic orientation. There are four components of $Q$ : two half-annulus drums of type $\left(0^{+}, \mathbf{b}\right)$; and two half-annulus drums of type $\left(0^{-}, \mathbf{b}\right)$. There are four uncusped boundary parallelisms, two each in $B^{s}$ and $B^{u}$, and all four are untwisted. Another view of the round handle is given in figure 4.11, which shows the result of a sutured manifold decomposition $H \stackrel{D}{\rightsquigarrow} H^{\prime}$ where $D$ is a meridian rectangle of $H$. The remains $B^{\prime s}, B^{\prime u} \subset H^{\prime}$ of $B^{s}, B^{u}$ are also shown. In figure 4.12, one of the four components of $H^{\prime}-\left(B^{\prime s} \cup B^{\prime u}\right)$ is glued up to form one of the four components of $H-\left(B^{s} \cup B^{u}\right)$.

Example (Twisted round handle). This is a variation on the round handle, obtained from figure 4.11 by gluing the top and bottom using an affine homeomorphism which rotates the $x$ and $y$ directions through $180^{\circ}$. Note that $B^{s}$ and $B^{u}$ are now Möbius bands (instead of annuli). Also there are just two uncusped boundary parallelisms, one in $B^{s}$ and one in $B^{u}$, and both are twisted. And there are only two components of $Q$, each a half-annulus drum, one of type $\left(0^{+}, \mathbf{b}\right)$ and the other of type $\left(0^{-}, \mathbf{b}\right)$.
Example (Untwisted, polygonal round handle). Let $H$ be a solid torus sutured manifold whose sutures consist of $2 n$ longitudinal circles, $n \geq 2$. There is a meridinal $2 n$-gon $D$ giving a sutured manifold decomposition $H \xrightarrow{D} H^{\prime}$ where $H^{\prime}$ is a product sutured manifold over a disc. There is a dynamic pair $B^{s}, B^{u}$ in $H$, which intersects $D$ in the pattern shown in figure 4.13. The entire dynamic pair is obtained from $D \times[0,1]$ by gluing $(x, 0)$ to $(x, 1)$ for each $x \in D$; in other words, take figure 4.13 and cross with $S^{1}$. There are $2 n$ cusped boundary parallelisms, $n$ in $B^{s}$ and $n$ in $B^{u}$. The completed complementary components


Figure 4.11: A sutured manifold $H^{\prime}$ obtained by decomposing the round handle $H$ along a meridian rectangle $D . H^{\prime}$ is a product over a disc, embedded in 3 -space as a "skewed cube". It is viewed from a point just off the positive $z$-axis, looking down the $z$-axis towards the origin. The scars $D^{+}$and $D^{-}$are the top and bottom faces of $H^{\prime}$, parallel to the $x, y$-plane. To obtain $H$, glue $D^{+}$to $D^{-}$by an affine map that shrinks the $x$-direction and stretches the $y$-direction. One pair of vertical faces is parallel to the $x$-axis and tilted to face upward; these glue up to give $\mathcal{R}^{+} H$. The other pair of vertical faces is parallel to the $y$-axis and tilted to face downward; these glue up to give $\mathcal{R}^{-} H$. The suture $\sigma H^{\prime}$ is the visual contour of the figure, an octagon, where $d / d z$ is tangent to $\partial H^{\prime}$ (it is best to think of the cusp model for $H^{\prime}$ ). The remains of $B^{s}$ in $H^{\prime}$ is $B^{\prime s}$, the intersection of $H^{\prime}$ with the $x, z$-plane, and similarly for $B^{\prime u}$. The arc $\tau^{\prime}=B^{\prime s} \cap B^{\prime u}$, which is the remains of $\tau$, is the intersection of $H^{\prime}$ with the $z$-axis. Choose the dynamic vector field on $H$ so that when pulled back to $H^{\prime}$ it coincides with $d / d z$, and hence $\tau^{\prime}$ points "upward".


Figure 4.12: The lower right component of $H^{\prime}-\left(B^{\prime s} \cup B^{\prime u}\right)$ is a dynamic manifold with interval dynamics (see ( $\mathrm{a}, \mathrm{b}$ ) ), shown in the "bird's eye" convention: the $\mathbf{p}$ and $\mathbf{u}$ faces form the top of the object viewed; the $\mathbf{m}$ and $\mathbf{s}$ faces form the bottom; each $\mathbf{b}$ face is tangent to the line of sight, and (converting to the cusp model) may be collapsed to a pm-edge. The intersection of $D^{+}$with the $\mathbf{p}$ face forms a rectangle, as does the intersection of $D^{-}$ with the $\mathbf{m}$ face. These rectangles are glued to yield a component of $H-\left(B^{s} \cup B^{u}\right)$ shown in (c), which is a half-annulus drum of type ( $0^{-}, \mathbf{b}$ ). Note that the upper left component of $H^{\prime}-\left(B^{\prime s} \cup B^{\prime u}\right)$ also yields a half-annulus drum of type $\left(0^{-}, \mathbf{b}\right)$, and the other two components yield half-annulus drums over annuli of type $\left(0^{+}, \mathbf{b}\right)$.
of the dynamic pair are: $2 n$ half-annulus drums, $n$ of type $\left(0^{+}, \mathbf{b}\right)$ and $n$ of type $\left(0^{-}, \mathbf{b}\right)$; a dynamic solid torus of type ( $n, 0$ ); and $2 n$ boundary parallel maw pieces, half of type uss and half of type suu. When $n=2$ then $H$ is an untwisted round handle, and we have constructed a dynamic pair different from the one described earlier-in the present construction, the dynamic pair has a dynamic solid torus piece, as well as four maw pieces, two of type uss and two of type suu; such pieces were not present in the previous construction. When $n=1$ the construction can also be carried out, but the resulting solid torus piece is not essential, violating the definition of a dynamic pair; on the other hand, if $n=1$ then $H$ is a product sutured manifold over an annulus and so has an empty dynamic pair.

Example (Twisted, polygonal round handle). This is a variation on the untwisted polygonal round handle, obtained from a polygon $D$ as in figure 4.13 by taking $D \times[0,1]$ and gluing $(x, 1)$ to $(f(x), 0)$, where $f: D \rightarrow D$ is a rotation through angle $2 \pi k / m$ with $0 \leq k<m$. The complementary components of the dynamic pair consist of: a dynamic solid torus of type ( $m, k$ ); a collection of half-annulus drums numbering $2 \operatorname{gcf}(m, k)$, half of type $\left(0^{+}, \mathbf{b}\right)$ and half of type $\left(0^{-}, \mathbf{b}\right)$; and a collection of boundary parallel maw pieces numbering $2 \operatorname{gcf}(m, k)$, half of type uss and half of type suu. There are $2 \cdot \operatorname{gcf}(m, k)$ cusped boundary parallelisms, half in $B^{u}$ and half in $B^{s}$.

Example (Torus shell). This is another variation on the untwisted, polygonal round handle, obtained by removing a regular neighborhood of the core curve of the dynamic solid




Figure 4.15: The sutured manifold $\Pi$, a product over a disc. The scars are glued, with vertex matchings as labelled, to form $P$.
decomposition $P \underset{\sim}{s} \Pi$ where $S$ is a disjoint union of two rectangles $D$ and $E$, and $\Pi$ is a product over a disc. Figure 4.15 shows $\Pi$ with the scars of $D$ and $E$, following the convention of figure 4.11 , so we regard $\Pi$ as a subset of $\mathbf{E}^{3}$ and the figure shows a bird's eye view, looking down along the $z$-axis.

Figure 4.16 shows branched surfaces $B_{\Pi}^{s}, B_{\Pi}^{u}$ in $\Pi$. Note that $B_{\Pi}^{s}$ intersects each scar in a single arc connecting the two $\mathbf{m}$-sides of the scar, therefore $B_{\Pi}^{s}$ glues up to give a branched surface $B^{s} \subset P$. Similarly $B_{\Pi}^{u}$ glues up to give $B^{u} \subset P$. It is also easy to see that $B^{s}$ is a stable branched surface and $B^{u}$ is unstable.

Note that $\Pi, B_{\Pi}^{u}$, and $B_{\Pi}^{s}$ may also be constructed by gluing the $\mathbf{p}$-face of a converging piece to the $\mathbf{m}$-face of a diverging piece (see figures 3.3 and 3.4).

The dynamic manifold $\mathfrak{C}\left(\Pi-\left(B_{\Pi}^{s} \cup B_{\Pi}^{u}\right)\right)$ has eight components, shown in figure 4.17. The components are labelled according to segments of $\sigma \Pi$ with endpoints at two of the letters a-g. When $\Pi$ is cut along $B_{\Pi}^{s} \cup B_{\Pi}^{u}$, each of the scars $D^{+}, D^{-}, E^{+}, E^{-}$is cut into four quarters. Each component of $\mathfrak{C}\left(\Pi-\left(B_{\Pi}^{s} \cup B_{\Pi}^{u}\right)\right)$ contains a " + " quarter scar and a "-" quarter scar. Each "+" quarter scar glues to some "-" quarter scar, to form $\mathfrak{C}\left(P-\left(B^{s} \cup B^{u}\right)\right)$. The $\overline{a a}$ and $\overline{e e}$ components each glue to themselves, forming two halfannulus drums of type $\left(0^{-}, \mathbf{b}\right)$, as in figure 4.12. The remaining six components glue up to form a drum over an annulus of type ( $2, \mathbf{b}$ ), as shown in figure 4.18.

There are two uncusped boundary parallelisms in $B^{u}$ and two in $B^{s}$, namely the $\mathbf{u}$-faces and s-faces of the two half-annulus drums of type $\left(0^{-}, \mathbf{b}\right)$.

Many interesting examples in handlebodies can be constructed by generalizing this example. Start with any finite collection consisting of $n$ converging pieces and $n$ diverging pieces. Choose a bijection from the set of $\mathbf{p}$-gluing rectangles to the set of $\mathbf{m}$-gluing rectangles, and identify each pair by a gluing homeomorphism which stretches one direction and compresses the other. If the result of gluing is connected then it is a sutured manifold


Figure 4.16: Branched surfaces $B_{\Pi}^{u}$ (a) and $B_{\Pi}^{s}$ (b) in $\Pi$. Some scar sides are suppressed for simplification, but the scar corners are shown in (c). The intersection train track $\tau_{\Pi}=B_{\Pi}^{s} \cap B_{\Pi}^{u}$ is shown in (c).


Figure 4.17: The eight components of $\mathfrak{C}\left(\Pi-\left(B_{\Pi}^{s} \cup B_{\Pi}^{u}\right)\right)$. Each component is a dynamic manifold with interval dynamics, shown in the bird's eye view as described in figure 4.12. To understand the lettering on quarter-scar corners, superimpose figure 4.15 with figure 4.16(c).


Figure 4.18: The six components $\overline{g f}, \overline{f h}, \overline{h c}, \overline{c b}, \overline{b d}, \overline{d g}$ glue up to form a drum over an annulus of type ( $2, \mathbf{b}$ ).
$P$ whose underlying topological manifold is a handlebody of genus $1+n$. The branched surfaces in figure 3.4 glue up to form a dynamic pair in $P$.

### 4.10 Flows on sutured manifolds

In this section we generalize pA flows and their stable and unstable laminations to the setting of sutured manifolds; we shall not attempt to generalize pseudo-Anosov flows. The main result which will be used outside of this section is:

Theorem 4.10.1. Suppose that the sutured manifold $P$ has a dynamic pair. Then $P$ is irreducible, each face of $P$ is incompressible, and any two distinct components of $\gamma_{T} P$ are nonisotopic.

This is a consequence of the following generalizations of the results of $\S 3$ :

- Every dynamic pair of branched surfaces on a sutured manifold $P$ carries a pA flow (theorem 4.10.4).
- The stable and unstable laminations are essential laminations in the category of sutured manifolds (theorem 4.10.3).

The main work of this section is formulating the definitions carefully; once that is accomplished, the theorems are easy generalizations of theorems in $\S 3$.

Let $P$ be a sutured manifold. Let $\Phi$ be a semiflow on $P$ with no stationary points, generated by a $C^{\infty}$ vector field $V$, such that $V$ points inward along $\mathcal{R}_{-} P$, outward along $\mathcal{R}_{+} P$, and tangentially along $\gamma P$. Let $\mathcal{C}_{\Phi}$ be the chain recurrent set of $\Phi$. Let $\mathcal{D}_{\Phi}$ be the maximal invariant set of $\Phi$, the set of all $x \in P$ such that $x \dot{t}=\Phi(x, t)$ is defined for all $t \in \mathbf{R}$. Note that $\mathcal{C}_{\Phi} \subset \mathcal{D}_{\Phi}$ and both are closed subsets of $P$.

We wish to define what it means for $\Phi$ to be a pA flow in $P$. In crafting the definition, we keep in mind the requirement that $\Phi$ have stable and unstable laminations $\Lambda^{s}, \Lambda^{u}$ with boundaries satisfying $\partial \Lambda^{s}=\lambda^{s} \subset \mathcal{R}_{-} P$ and $\partial \Lambda^{u}=\lambda^{u} \subset \mathcal{R}_{+} P$. In order for $\Lambda^{s}, \Lambda^{u}$ to be essential laminations in $P$, we need for the boundary laminations $\lambda^{s}, \lambda^{u}$ to be essential laminations in the surfaces $\mathcal{R}_{-} P, \mathcal{R}_{+} P$ respectively.

To begin the definition of pA flows on a sutured manifold, we first repeat almost verbatim all of the conditions in the original definition except for item 6 concerning boundary periodic orbits:

1. There exist finitely many pA invariant sets for $\Phi$, all pairwise disjoint.
2. $\Phi$ is smooth off of the pseudohyperbolic orbits contained in pA solid tori.
3. Each component of $\gamma_{T} P$ is a face of a pA torus shell of $\Phi$.
4. Each attracting or repelling orbit of $\Phi$ is contained in some pA invariant set.
5. Let $\mathcal{A}_{\Phi}$ be the union of attracting orbits, $\mathcal{R}_{\Phi}$ the union of repelling orbits, and $\mathcal{P}_{\Phi}$ the union of pseudo-hyperbolic orbits and $\gamma_{T} P$. Let $\mathcal{I}_{\Phi}$ be the union of the remaining chain components of $\mathcal{C}_{\Phi}$, the index 1 hyperbolic components. Define $\mathcal{J}_{\Phi}=\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$. Then the invariant set $\mathcal{J}_{\Phi}$ is a 1-dimensional hyperbolic invariant set.
6. There does not exist a transverse bigon for $\mathcal{J}_{\Phi}$.

The original definition of a pA flow contained a restriction on the behavior of boundary periodic orbits, item 6. In the present setting there is a more complicated set of restrictions on local boundary laminations. To formulate them, apply proposition 3.3 .3 we obtain an isolating block $N$ for $\mathcal{J}_{\Phi}$, and local stable and unstable laminations $W_{\text {loc }}^{s}, W_{\text {loc }}^{u}$ of $\mathcal{J}_{\Phi}$ with respect to $N$. Consider the local boundary laminations

$$
\begin{align*}
& \lambda_{\mathrm{loc}}^{s}=W_{\mathrm{loc}}^{s} \cap \mathcal{R}_{-} N  \tag{4.1}\\
& \lambda_{\mathrm{loc}}^{u}=W_{\mathrm{loc}}^{u} \cap \mathcal{R}_{+} N \tag{4.2}
\end{align*}
$$

Define a decomposition of $\lambda_{\text {loc }}^{s}$ as follows:

$$
\begin{align*}
\lambda_{\infty}^{s} & =\left\{x \in \lambda_{\text {loc }}^{s} \mid x \cdot t \quad \text { is defined for all } t \in(-\infty, 0]\right\}  \tag{4.3}\\
\lambda_{f}^{s} & =\lambda_{\text {loc }}^{s}-\lambda_{\infty}^{s} \tag{4.4}
\end{align*}
$$

Note that $\lambda_{\infty}^{s}$ and $\lambda_{f}^{s}$ are open and closed sublaminations of $\lambda_{\text {loc }}^{s}$. Openness of $\lambda_{f}^{s}$ is obvious. Since backward orbits of points in $\lambda_{\infty}^{s}$ must accumulate on the compact repeller $\mathcal{J}\left(\mathcal{R}_{\Phi} \cup \mathcal{P}_{\Phi}\right)$, openness of $\lambda_{\infty}^{s}$ follows. We similarly define a decomposition of $\lambda_{\text {loc }}^{u}$ into open and closed sublaminations $\lambda_{\infty}^{u} \cup \lambda_{f}^{u}$.

The next requirement in defining a pA flow $\Phi$ is:
7. Each closed leaf of $\lambda_{\infty}^{s}$ or $\lambda_{\infty}^{u}$ is contained in a side of a pA invariant set of $\Phi$.

Before formulating the final requirements for defining a pA flow, we must pause to study the "stable and unstable laminations" of $\Phi$ :

$$
\begin{align*}
\Lambda^{s} & =\operatorname{cl}\left(W_{\mathrm{loc}}^{s} \cdot(-\infty, 0]\right)  \tag{4.5}\\
\Lambda^{u} & =\operatorname{cl}\left(W_{\mathrm{loc}}^{u} \cdot[0, \infty)\right) \tag{4.6}
\end{align*}
$$

We need to generalize part of theorem 3.3 .1 by saying that $\Lambda^{s}, \Lambda^{u}$ are in fact laminations:
Lemma 4.10.2. $\Lambda^{s}, \Lambda^{u}$ are laminations in $P$, they are transverse to each other, their intersection is $\mathcal{J}_{\Phi}$, and their boundaries $\lambda^{s}=\partial \Lambda^{s}, \lambda^{u}=\partial \Lambda^{u}$ are laminations of $\mathcal{R}_{-} P$, $\mathcal{R}_{+} P$ respectively.

Proof. Obviously $W_{\text {loc }}^{s}, W_{\text {loc }}^{u}$ are laminations, they are transverse, and their intersection is $\mathcal{J}_{\Phi}$. It is also obvious that $\Lambda^{s} \cap \Lambda^{u}=W_{\text {loc }}^{s} \cap W_{\text {loc }}^{u}$. We must prove that $\Lambda^{u}$ is a lamination near any point of $\operatorname{cl}\left(\lambda_{\text {loc }}^{u} \cdot[0, \infty)\right)$; the proof for $\Lambda^{s}$ is similar.
$\Lambda^{u}$ is obviously a lamination near points obtained by flowing $\lambda_{f}^{u}$ forwards to $\mathcal{R}_{+} P$. More precisely, there is a continuous function $t_{x} \in \mathbf{R}_{+}$defined for $x \in \lambda_{f}^{u}$ such that $\boldsymbol{x} \cdot t$ is defined for $t \in\left[0, t_{x}\right]$ and $x \cdot t$ is not defined for $t>t_{x}$. Since $W_{\text {loc }}^{u}$ is a lamination near $\lambda_{f}^{u}$, it follows easily that $\Lambda^{u}$ is a lamination near points of $\left\{x \cdot t \mid x \in \lambda_{f}^{u}, t \in\left[0, t_{x}\right]\right\}$. Note in particular that $\lambda^{u}=\left\{x \cdot t_{x} \mid x \in \lambda_{f}^{u}\right\}$ is a 1-dimensional lamination in $\mathcal{R}_{+} P$.

To prove that $\Lambda^{u}$ is a lamination near points of $\operatorname{cl}\left(\lambda_{\infty}^{u} \cdot[0, \infty)\right)$ we adapt the arguments of lemma 3.3.5 and theorem 3.3.1. The lamination $\lambda_{\infty}^{u}$ decomposes into open and closed sublaminations of the form $\lambda_{\gamma}^{u}$, one for each attracting orbit $\gamma$ of $\Phi$, where $\lambda_{\gamma}^{u}$ is the subset of $\lambda_{\infty}^{u}$ contained in the attracting basin of $\gamma$.

For each $\gamma$ we check that $\Lambda^{u}$ is a lamination near points of $\operatorname{cl}\left(\lambda_{\gamma}^{u} \cdot[0, \infty)\right)$. Let $F$ be the $\mathbf{u}$-face of the pA invariant set of $\Phi$ containing $\gamma$. Let $\gamma_{1}, \gamma_{2}$ be the components of $\lambda_{\gamma}^{u} \cap F$. As in step 1 of lemma 3.3.5, the only compact leaves of $\lambda_{\gamma}^{u}$ are $\gamma_{1}, \gamma_{2}$; however it is now possible for one or both of $\gamma_{1}, \gamma_{2}$ to be isolated leaves. Nonetheless, steps $2-5$ of lemma 3.3.5 go through as stated. We now break into two cases, depending on whether there exists a noncompact leaf of $\lambda_{\gamma}^{u}$.

Case 1: There exists a noncompact leaf. Any such leaf spirals into $\gamma_{1}$ on one end and $\gamma_{2}$ on the other end. Just as in lemma 3.3.5 it follows that $\lambda_{\gamma}^{u}$ is a Reeb lamination. Now construct lamination charts exactly as in theorem 3.3.1.

Case 2: There does not exist a noncompact leaf. In this case $\lambda_{\gamma}^{u}=\gamma_{1} \cup \gamma_{2}$ and we have $\operatorname{cl}\left(\left(\lambda_{\gamma}^{u} \cdot[0, \infty)\right)=F\right.$ which is obviously a lamination.

To formulate the final requirements for a pA flow we describe some types of components of the dynamic manifold $\mathfrak{C}\left(P-\left(\Lambda^{s} \cup \Lambda^{u}\right)\right)$. If $Q$ is a drum over a labelled, even surface-withcorners $S$ (as in section 4.4), by removing the uu and ss-cusps of $Q$ we obtain a dynamic manifold called a pared drum over $S$. A pared drum over $S$ is a coherent annulus drum if $S$ is an annulus with oriented isotopic boundary components (in which case there were no cusps to remove). A dynamic manifold $\mu$ is called a split maw piece if $\mu$ is a product $R \times S^{1}$ where $R$ is a rectangle with sides labelled either usup or susm, such that the two us-circles are oriented isotopic.

To complete the definition of a pA flow $\Phi$, we require:
8. Each component of $\mathfrak{C}\left(P-\left(\Lambda^{s} \cup \Lambda^{u}\right)\right)$ having nonempty intersection with $\mathcal{R}_{-} P \cup \mathcal{R}_{+} P$ is one of the following:

- A pared drum whose base has non-positive index, but not a coherent annulus drum.
- A split maw piece, one of whose faces is also a face of a pA invariant set of $\Phi$.

9. The boundary laminations $\lambda^{s}, \lambda^{u}$ have no Reeb annuli.

We could go on from here to generalize very full laminations to the sutured manifold setting, but that would take us too far afield. We shall be content to say that pA stable and unstable laminations are essential:

Theorem 4.10.3. If $\Phi$ is a $p A$ flow on a sutured manifold $P$, and if $\Lambda^{s}, \Lambda^{u}$ are the stable and unstable laminations of $\Phi$, then $\Lambda^{s}, \Lambda^{u}$ are essential in the following sense:

- $\Lambda^{s}, \Lambda^{u}$ have no sphere leaves, no Reeb components, and no half-Reeb components.
- The manifolds-with-corners $\mathfrak{C}\left(P-\Lambda^{s}\right)$ and $\mathfrak{C}\left(P-\Lambda^{u}\right)$ are irreducible, and each of their faces are incompressible and end incompressible.

To define a half-Reeb component, let $H=\left\{(x, y) \in \mathbf{R}^{2} \mid\|(x, y)\| \leq 1, y \geq 0\right\}, \partial_{0} H=$ $\{(x, y) \in H \mid y=0\}, \partial_{+} H=\operatorname{cl}\left(\partial H-\partial_{0} H\right)$. A half-Reeb component of $\Lambda \subset P$ is the image of an embedding $H \times S^{1} \hookrightarrow P$ such that $\partial_{0} H \times S^{1}$ is the intersection with $\partial P, \partial_{+} H \times S^{1}$ is a leaf of $\Lambda$, and the intersection of $\Lambda$ with $\partial_{0} H \times S^{1}$ is a Reeb lamination of an annulus, each of whose leaves extends to a leaf of $\Lambda$ accumulating on $\partial_{+} H \times S^{1}$.

We also have a generalization of theorem 3.3.2. Given a pA flow $\Phi$ and a good dynamic pair $B^{s}, B^{u}$ on a sutured manifold $P$, we say that the pair $B^{s}, B^{u}$ carries the flow $\Phi$ if the following hold:

- $N(\tau)$ is an isolating block for $\mathcal{J}\left(\mathcal{I}_{\Phi}\right)$, with $\Phi$ flowing inward along $\partial_{-} N(\tau)$, outward along $\partial_{+} N(\tau)$, and externally tangent along $\sigma N(\tau)$.
- $\Phi$ is transverse to the rectangle fibers of $N(\tau)$, crossing each fiber in the positive direction.
- $N\left(B^{s}\right)$ contains $\Lambda^{s}$, with $\Phi$ flowing outward along $\partial N\left(B^{s}\right) \cap \operatorname{int}(P)$ and inward along $\partial N\left(B^{s}\right) \cap \mathcal{R}_{-} P$.
- $N\left(B^{u}\right)$ contains $\Lambda^{u}$, with $\Phi$ flowing inward along $\partial N\left(B^{u}\right) \cap \operatorname{int}(P)$ and outward along $\partial N\left(B^{u}\right) \cap \mathcal{R}_{+} P$.
- Inclusion induces a type preserving $1-1$ correspondence between components of $\mathrm{cl}(P-$ $\left.\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$ which are not pinched tetrahedra and components of $\mathfrak{C}\left(P-\left(\Lambda^{s} \cup\right.\right.$ $\left.\Lambda^{u}\right)$ ) which are not pared drums over rectangles.

Remark. A "pared drum over a rectangle" can have one of several structures, depending on whether the drum is unpinched, p-pinched, $\mathbf{m}$-pinched, or $\mathbf{p m}$-pinched; in the latter case we obtain a pared pinched tetrahedron, which is the same thing topologically as a
(rectangle) $\times \mathbf{R}$, as we saw already in our study of torally bounded manifolds. The correspondence in the last item matches solid torus pieces to dynamic solid tori, torus shell pieces to dynamic torus shells, drums to pared drums (over surfaces which are not rectangles), and boundary parallel maw pieces to split maw pieces. The relationship between the remaining components of $\mathfrak{C}\left(P-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right.$ ) (rectangle drums), and the remaining components of $\mathfrak{C}\left(P-\left(\Lambda^{s} \cup \Lambda^{u}\right)\right)$ (pared rectangle drums) is just like for manifolds with torus boundaries: all but finitely many pared rectangle drums are entirely contained in $N\left(B^{s}\right) \cup N\left(B^{u}\right)$, and the remaining ones are in 1-1 correspondence with the rectangle drums of $\mathfrak{C}\left(P-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$.

Theorem 4.10.4. Given a sutured manifold $P$, each good dynamic pair of branched surfaces in $P$ carries a $p A$ flow, and each $p A$ flow is carried by a good dynamic pair.

Before proving theorems 4.10 .4 and 4.10 .3 we use them to prove theorem 4.10.1:
Proof of theorem 4.10.1. Let $P$ be a sutured manifold with a dynamic pair $B^{s}, B^{u}$. By proposition 4.7 .1 we may assume that $B^{s}, B^{u}$ is good. By theorem 4.10 .4 the pair $B^{s}, B^{u}$ carries a pA flow $\Phi$, and by theorem 4.10 .3 the stable and unstable laminations $\Lambda^{s}, \Lambda^{u}$ of $\Phi$ are essential.

The proof now follows the lines of [GO89] very closely.
Here is a sketch of the proof that faces of $P$ are incompressible. Let $N^{s}=\mathfrak{C}\left(P-\Lambda^{s}\right)$. Suppose that $D$ is a compressing disc for $P$. Perturb $D$ so that it is transverse to $\Lambda^{s}$, and consider the lamination $\mu=D \cap \Lambda^{s}$. Since each face of $N^{s}$ is incompressible, and since there are no Reeb components, we can isotope $D$ so that no component of $\mathfrak{C}(D-\mu)$ is a disc whose boundary is a circle leaf of $\mu$. If $\mu$ had a circle leaf bounding a disc $E$, it would follow by an Euler characteristic argument that some component of $\mathfrak{C}(E-\mu)$ is a disc bounded by a circle leaf of $\mu$, or an end compression of $N^{s}$ which is vertical near the ends, neither of which exist. Therefore, $\mu$ has no circle leaves. If $\mu$ had a noncompact leaf it would follow by recurrence that $\mu$ has a circle leaf, therefore every leaf of $\mu$ is a compact, properly embedded arc. If $\mu \neq \emptyset$ then it follows by an Euler characteristic argument that some component of $\mathfrak{C}(D-\mu)$ is a bigon. Since $N^{s}$ is boundary irreducible and there are no half Reeb components, we can isotop $D$ to get rid of $\mu$ completely. But now $D$ is a compressing disc for a face of $N^{s}$, a contradiction.

Here is a sketch of the proof that distinct components $T_{0}, T_{1}$ of $\gamma_{T} P$ must be nonisotopic. Note that $\mathcal{R} P=\emptyset$. If $T_{0}, T_{1}$ were isotopic it would follow that $P \approx$ (torus) $\times[0,1]$. Now $T_{i}$ is contained in a pared torus shell component of $N^{s}$, and the cusps are isotopic to curves on $T_{i}$ called the "degeneracy locus". Let $c_{0} \subset T_{0}$ be an essential simple closed curve which is not isotopic to a component of the degeneracy locus. Let $A \subset P$ be a properly embedded annulus with one boundary component on $c_{0}$ and the other on $c_{1} \subset T_{1}$. By mimicking the above operations we may isotop $A$ so that it is transverse to $\Lambda^{s}$ and the lamination $\mu=A \cap \Lambda^{s}$ is essential in $A$; in particular, the index of each component of $\mathfrak{C}(A-\mu)$
is nonpositive. From this it follows that each index is 0 , because the sum of the indices must equal $2 \chi(A)=0$. In particular, the component of $\mathfrak{C}(A-\mu)$ containing $c_{0}$ must be a compact annulus $A_{0}$. Since $\Lambda^{s}$ separates $T_{0}$ from $T_{1}$ it follows that $\mu \neq \emptyset$, and so $\partial A_{0}-c_{0}$ is a smooth curve in $\Lambda^{s}$. Therefore $c_{0}$ is isotopic to a component of the degeneracy locus, a contradiction.

Proof of theorem 4.10.4. We adapt the proof of theorem 3.3.2. Let $B^{s}, B^{u}$ be a dynamic pair in $P$. Let $\tau=B^{s} \cap B^{u}$. Choose $I$-fibered neighborhoods $N\left(B^{s}\right), N\left(B^{u}\right)$ intersecting in $N=N(\tau)$, so that $B_{N}^{s}=N \cap B^{s}, B_{N}^{u}=N \cap B^{u}$ is a template pair in $N(\tau)$. Let $\tau^{s}=\partial B_{N}^{s}, \tau^{u}=\partial B_{N}^{u}$.

Define the flow $\Phi \mid N(\tau)$ exactly as in theorem 3.3.2. Consider a component $K$ of $\mathfrak{C}\left(B^{u}-\tau\right)$, and extend $\Phi$ over the corresponding component $N(K)$ of $\mathfrak{C}\left(N\left(B^{u}\right)-N(\tau)\right)$ in two cases, as follows.

Consider first the case where $K$ contains a u-face $F$ of a solid torus piece of $\mathfrak{C}(P-$ $\left.\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$. It follows that $K$ is either an annulus with tongues or just an annulus. In either case $N(K)$ is a solid torus and $\Phi \mid N(K)$ is defined just as in theorem 3.3.2, pointing inward along $\partial N(K)$ and with every orbit accumulating on an attracting periodic orbit $\gamma$ at the core of $N(K)$, such that $\gamma$ is oriented isotopic to the boundary circles of the sink of $K$.

In the second case it follows that, after splitting $K$ along unstable maw-boundary annuli, we obtain $\tau_{K}^{u} \times[0,1]$ where $\tau_{K}^{u} \times 0 \approx K \cap \tau^{u}$ is a component of $\tau^{u}$, and $\tau_{K}^{u} \times 1 \subset \mathcal{R}_{+} P$. The branched surface $K$ may be reconstructed from $\tau_{K}^{u}$ by collapsing certain annulus components of $\mathfrak{C}\left(R_{+} P-\left(\tau_{K}^{u} \times 1\right)\right)$. Consider $F_{K}=N(K) \cap \mathcal{R}_{+} N$, a s-face of $\mathcal{R}_{+} N$ which may be regarded as an $I$-bundle neighborhood of the train track $\tau_{K}^{u} \times 0$. There is an embedding $F_{K} \times[0,1] \hookrightarrow P$ which is an $I$-bundle neighborhood of $\tau_{K}^{u} \times[0,1]$ so that $F_{K} \times 0 \approx F_{K}$ and $F_{K} \times 1 \approx\left(F_{K} \times[0,1]\right) \cap \mathcal{R}_{+} P$. Moreover, $N(K)$ may be reconstructed from $F_{K} \times[0,1]$ by adding a regular neighborhood of each collapsing annulus. Construct a flow on $F_{K} \times[0,1]$ which enters along $F_{K} \times 0$, exits along $F_{K} \times 1$, is tangent to the gluing annuli, and enters along the remaining portion of $\partial F_{K} \times[0,1]$; then extend the flow over the neighborhoods of collapsing annuli in the obvious way.

This defines the extension of $\Phi$ over $N\left(B^{u}\right)$, and the extension over $N\left(B^{s}\right)$ is defined similarly. The flow $\Phi$ is extended over torus piece components of $\mathfrak{C}\left(P-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$ exactly as in theorem 3.3.2. The remaining components of $\mathfrak{C}\left(P-\left(N\left(B^{s}\right) \cup N\left(B^{u}\right)\right)\right)$ are all pinched tetrahedra, maw pieces, and drums, and in each case $\Phi$ is extended to have interval dynamics.

The proof that $\Phi$ is a pA flow is left to the reader.
The converse direction, constructing a dynamic pair from a pA flow, is also left to the reader.

Proof of theorem 4.10.3. This proof follows closely the lines of theorem 1 of [GO89].

Let $\Phi$ be a pA flow. Let $B^{s}, B^{u}$ be a dynamic pair carrying $\Phi$. Then $B^{s}, B^{u}$ carry the laminations $\Lambda^{s}, \Lambda^{u}$ respectively. The main idea is to use proposition 4.6.1 to show that $B^{s}, B^{u}$ are essential in the appropriate sense, generalizing the definition of an essential branched surface in [GO89] definition 1.2. Now mimic section 2 of [GO89].

Remark. An alternative approach to the above proof is to describe the components of $\mathfrak{C}\left(P-\Lambda^{s}\right)$ and $\mathfrak{C}\left(P-\Lambda^{u}\right)$ explicitly. Each component $C$ of $\mathfrak{C}\left(P-\Lambda^{s}\right)$, say, can be described as follows.

Suppose first that $C \not \subset N\left(B^{s}\right)$, and so there is some component of $\mathfrak{C}\left(P-N\left(B^{s}\right)\right)$ contained in $C$, corresponding to a component $H$ of $\mathfrak{C}\left(P-B^{s}\right)$. Unlike in the case of a manifold with torus boundaries, we cannot simply pare the ss-cusps of $H$ to obtain $C$; that would not allow for the possibility of ideal bigon components of $\mathfrak{C}\left(\mathcal{R}_{-} P-\lambda^{s}\right)$, and there are usually infinitely many of these. Instead we partially pare the ss-cusps, and blow up the remainder to form ideal bigon $\mathbf{m}$-faces. That is, there exists a certain compact subset $A$ contained in the union of ss-cusps of $H$, including the endpoints of all ss-edges, such that $C$ is obtained from $H$ by the following process: remove $A$ from $H$; for each component $\alpha$ of $\partial_{\mathbf{s s}} H-A$, blow up the open segment $\alpha$ to form an $\mathbf{m}$-face of the form $\mathbf{R} \times[0,1]$, whose boundary consists of two $\mathbf{m s}$-lines. These $\mathbf{m}$-faces are ideal bigon components of $\mathfrak{C}\left(\mathcal{R}_{-} P-\lambda^{s}\right)$. From this description it is obvious that each face of $C$ is incompressible and end incompressible.

If $C \subset N\left(B^{s}\right)$, then $C$ is an $I$-bundle over a surface, and so each face of $C$ is incompressible and end incompressible.

### 4.11 Markov branched surfaces yield dynamic pairs

In this section we generalize the results of $\S 2.6$. We extend the definition of Markov branched surfaces to the category of sutured manifolds. In proposition 4.11 .1 we generalize proposition 2.6.2: starting from a Markov branched surface $B$ in a sutured manifold $P$, we produce a dynamic pair with coherent annulus drums in $P$. Simple examples show that this result cannot be improved, but in the following section we will study the general problem of eliminating coherent annulus drums, with somewhat surprising results.

Let $P$ be a sutured manifold and $(B, V)$ an unstable dynamic branched surface in $P$, with $V$ generating a semiflow $\phi$ on $P$. A Markov section for $\phi$ is a finite collection $\mathcal{I}$ of smooth, embedded arcs in $B$ such that:

- Each $I \in \mathcal{I}$ is transverse to $V$.
- For each $I \in \mathcal{I}$, one of the following is true:

$$
\begin{aligned}
& -I \subset \partial B \text { and } \operatorname{int}(I) \cap \Upsilon B=\emptyset . \\
& -I \subset \Upsilon B \text { and } \operatorname{int}(I) \cap \partial B=\emptyset .
\end{aligned}
$$

$$
-\operatorname{int}(I) \cap(\Upsilon B \cup \partial B)=\emptyset
$$

- For each $I \neq I^{\prime} \in \mathcal{I}$, we have $I \cap I^{\prime} \subset \partial I \cap \partial I^{\prime}$.
- $\partial B \subset \bigcup \mathcal{I}$. Let $\mathcal{I}_{\partial}=\{I \in \mathcal{I} \mid I \subset \partial B\}$ and let $\mathcal{I}^{\circ}=\mathcal{I}-\mathcal{I}_{\partial}$.
- I is a cross section. For each $x \in B-\partial B$ there exists $t>0$ such that $\phi(x, t) \subset \bigcup \mathcal{I}$. The first return time $t_{x}>0$ is therefore defined for each $x \in B-\partial B$; by convention, we extend $t_{x}$ to a function on $x \in \bigcup \mathcal{I}^{\circ}$ by defining $t_{x}=0$ if $x \in \bigcup \mathcal{I}^{\circ} \cap \partial B$, but we do not define $t_{x x}$ for other points in $\partial B$. The first return map $\phi\left(x, t_{x}\right)$ is therefore defined for $x \in \bigcup \mathcal{I}^{\circ}$.
- The Markov property. For any point $x \in \bigcup \mathcal{I}^{\circ}$ lying on the boundary of some element of $\mathcal{I}$, the first return point $\phi\left(x, t_{x}\right)$ is also a boundary point of some element of $\mathcal{I}$.

If $\phi$ has a Markov section $\mathcal{I}$ then $(B, V, \mathcal{I})$ is called a Markov branched surface. The train track $\partial B \subset \mathcal{R}_{+} P$ is not necessarily an unstable train track, although as the proof of proposition 4.11.1 will show we can alter $B$ naturally so that $\partial B$ is unstable.

Proposition 4.11.1. Let $(B, V, \mathcal{I})$ be a Markov unstable branched surface in a compact, oriented sutured manifold $P$. Suppose that:

- $B$ is very full in $P$.
- $\partial B$ is an essential train track in $\mathcal{R}_{+} P$.
- $\partial B$ has no Reeb annuli in $\mathcal{R}_{+} P$.
- $V$ is circular in each $\mathbf{u}$-cusped torus piece of $\mathfrak{C}(P-B)$.
- B does not carry a closed surface.
- If $\sigma$ is an annulus or Möbius band sector of $B$, and if $\sigma$ contains a periodic trajectory of $V$, then on at least one side of $\sigma$ the adjacent component of $\mathfrak{C}(P-B)$ is a u-pared product.

Then we may construct a dynamic pair with coherent annulus drums $B^{s}, B^{u} \subset P$ so that $B^{u}$ is obtained from $B$ by dynamic splitting.

We also need a "vertical" version. Let $\beta$ be a transversely oriented $\mathcal{R}$ branched surface in $P$, carrying a taut transversely oriented foliation $\mathcal{F}$. Verticality of $V$ with respect to $\beta$ is defined just as in a torally bounded manifold. Verticality of $B$ (resp. of a dynamic pair $B^{s}, B^{u}$ ), is defined to mean that the dynamic vector field is vertical, and no annulus $A$ carried by $\beta$ is peripheral in $\mathfrak{C}(P-B)$ (resp. $\mathfrak{C}\left(P-B^{s}\right)$ or $\left.\mathfrak{C}\left(P-B^{u}\right)\right)$, that is, if $A$ is properly embedded in $K$ then $A$ is not isotopic rel boundary into a face.

Proposition 4.11.2. In the above proposition, if $(B, V)$ is vertical with respect to $\beta$, and if $\mathcal{I}$ is tangent to $\beta$, then we may perform the construction so that $B^{s}, B^{u}$ is vertical with respect to $\beta$.
Remark. In the setting of proposition 4.11.1, the boundary train track $\partial B \subset \mathcal{R}_{+} P$ does not automatically have an unstable singular orientation. This can happen even if there is a dynamic pair $B^{s}, B^{u}=B$, as long as the dynamic pair is not good. Thus, given an annulus component $A$ of $\mathcal{R}_{+} P-\partial B$ whose boundary components are smooth circles of $\partial B$, it makes no sense to ask whether $A$ is incoherent, and so it is difficult to give an a priori condition on $B$ which tells when coherent annulus drums may arise in $B^{s}, B^{u}$. Nevertheless, in the course of the proof we will see how to predict when coherent annulus drums will arise.

For example, suppose $H_{1}, H_{2}$ are two round handles and $B_{i}^{s}, B_{i}^{u} \subset H_{i}$ is the dynamic pair depicted in figure 4.11. Choose an annulus $\mathbf{b}$-face $b_{i}$ of $H_{i}$. Contruct a sutured manifold $P$ from $H_{1} \cup H_{2}$ by gluing $b_{1}$ and $b_{2}$ so that the $\mathbf{m}$ and $\mathbf{p}$-labels on boundary circles of $b_{1}, b_{2}$ are compatible. Note that $P$ is a hexagonal round handle. Let $B^{s}=B_{1}^{s} \cup B_{2}^{s}, B^{u}=B_{1}^{u} \cup B_{2}^{u}$. Note that $B^{u}$ is an unstable dynamic branched surface in $P$ satisfying the hypotheses of proposition 4.11.1, and $\mathfrak{C}\left(P-B^{u}\right)$ has an annulus $\mathbf{p}$-face $A$ whose boundary circles are pu-corners-the face $A$ has nontrivial intersection with the gluing locus. There is a choice in gluing $b_{1}$ to $b_{2}$, and if the gluing is done so that the positive orientations of the dynamic solid tori in $H_{1}$ and $H_{2}$ are compatible in $P$, then $A$ is coherent. Note also that the pair $B^{s}, B^{u}$ is a dynamic pair having one coherent annulus drum, whose $\mathbf{p}$-face is $A$. As we know from figure 4.13, the sutured manifold $P$ does have a true dynamic pair $B^{\prime s}, B^{\prime u}$, but the relationship between this pair and $B^{s}, B^{u}$ is not entirely clear at this stage. We will take up this issue in §4.12.

Proof of proposition 4.11.1. We follow the same outline as in proposition 2.6.2, borrowing from the proof of that proposition when we can, emphasizing the differences otherwise.

Step 1: From Markov section to dual dynamic train track. As before, start with an enumeration $\mathcal{I}=\left\{I_{1}, \ldots, I_{L}, \ldots I_{M}\right\}$. Choose the enumeration so that $\mathcal{I}^{\circ}=\left\{\mathcal{I}_{1}, \ldots, \mathcal{I}_{L}\right\}$ and $\mathcal{I}_{\partial}=\left\{I_{L+1}, \ldots, I_{M}\right\}$. For each $I_{i} \in \mathcal{I}$ construct $\tau_{i}^{-}, \tau_{i}^{+}$as before, the only caveat being that if $I_{i} \in \mathcal{I}_{\partial}$ then $\tau_{i}^{+}=\emptyset$. Now let $\tau^{*}=\bigcup_{i}\left(\tau_{i}^{-} \cup \tau_{i}^{+}\right)$, an oriented train track in $B$, which points forward along $\Upsilon B$.

The train track $\tau^{*}$ is not yet the dual dynamic train track of $\mathcal{I}$-note that $\partial \tau^{*}=\tau^{*} \cap \partial B$ consists of an outward pointing endpoint in each $I_{i} \in \mathcal{I}_{\partial}$. We correct this as follows.

A point $x \in \tau^{*}$ is called immortal if there exists an infinite, forward directed path in $\tau^{*}$ starting at $x$; otherwise $x$ is mortal. There are several equivalent formulations of this concept: $x$ is immortal if and only if there is a forward directed path from $x$ that ends in a nontrivial strong component of $\tau^{*} ; x$ is mortal if and only if there is an upper bound to the length of every forward directed path starting at $x ; \boldsymbol{x}$ is mortal if and only if every maximal, forward directed path starting at $x$ ends in $\partial \tau^{*}$.

Let $\tau$ be the set of immortal points in $\tau^{*}$. Note first that $\tau$ is a train track, in fact $\tau$ is a union of closed branches of $\tau^{*}$. Moreover, $\tau \cap \partial B=\emptyset$, because each point of $\tau^{*} \cap \partial B$ is mortal. We must show that $\tau$ is a dynamic train track in $B$.

Each converging switch of $\tau$ is obviously a converging switch of $\tau^{*}$, and the latter all lie in $\Upsilon B$. Conversely, consider a point $p \in \tau \cap \Upsilon B$, so $p \in \tau^{*} \cap \Upsilon B$. Following the proof of proposition 2.6.2 it follow that $p$ is a converging switch of $\tau^{*}$. Since $p \in \tau$ it follows that $p$ is an immortal point of $\tau^{*}$. Observe that every immortal converging switch in $\tau^{*}$ has a neighborhood in $\tau^{*}$ consisting of immortal points. It follows that $p$ is a converging switch of $\tau$, as required.

The homotopy from $V$ to $V^{\prime}$ is easily constructed. The only difficult thing to check is Transience of forward trajectories; once that is done, since $B$ carries no closed surfaces it will follow that $\tau^{*}$ fills up $B$.

The statement Transience of forward trajectories for $\tau$ is concerned with components of $\mathfrak{C}(B-\tau)$. First we consider an arbitrary component $K$ of $\mathfrak{C}\left(B-\tau^{*}\right)$, a branched surface with cusps (corresponding to diverging switches of $\tau^{*}$ ) and corners (corresponding to points of $\tau^{*} \cap \partial B$ ). We carry out the process of removing tongues, and see what happens. For each cusp $c$ of $K$, if $\gamma^{\prime}$ is an arc transverse to $V$ forming the boundary of a regular neighborhood of $c$, then the connectivity argument of proposition 2.5 .7 can be carried out in the current setting to show that the forward trajectory through any $x \in \gamma^{\prime}$ eventually hits $\Upsilon K \cup(K \cap \partial B)$ at some point $y(x)$, and $\gamma=\left\{y(x) \mid x \in \gamma^{\prime}\right\}$ is the attaching curve for some tongue $T(c)$. Now remove $T(c)-\gamma$ from $K$, and also remove interior points of the 1-manifold $\gamma \cap \partial B$; the result, denoted $K^{\prime}$, is a branched surface with cusps and corners. Notice that either $\gamma \subset \partial B$ in which case $T(c)$ is a component of $K$ and $K^{\prime}=K-T(c)$, or $K^{\prime} \cap \partial B \neq \emptyset$; in the latter case, the removed tongue may join components of $K^{\prime}$. In particular, $K^{\prime}$ may have a different number of components than $K$. Repeating this process, eventually we reduce to a branched surface $R$ with no cusps. There are, moreover, no corners, because if there is a corner then $\partial R \cap \tau^{*}$ has an endpoint and so must have a cusp, contradiction. In addition $R \cap \partial B=\emptyset$, because each component of $K \cap \partial B$ is a union of half-arcs of $\mathcal{I}$, and if any of this lives in $R$ then there must be corners, a contradiction. It follows that $\partial R$ is a union of smooth circles in $\tau^{*}-\partial B$.

There are now two cases, depending on whether $R=\emptyset$.
If $R=\emptyset$ then clearly each forward trajectory in $K$ ends on $K \cap \partial B$.
If $R \neq \emptyset$ then one can mimic the proof of proposition 2.6.2 to show that $\Upsilon R=\emptyset$ and so each component of $R$ is an annulus or Möbius band; tori and Klein bottles are ruled out by hypothesis on $B$. Since $R \cap \partial B=\emptyset$ it also follows that as tongues are attached, the attaching locus cannot intersect $\partial B$, and so the tongue cannot join distinct components. Since $K$ is connected it follows that $R$ is connected. Also, $K \cap \partial B=\emptyset$. Thus, $K$ is a ring with tongues.

Note that $\mathfrak{C}(B-\tau)$ is obtained from $\mathfrak{C}\left(B-\tau^{*}\right)$ by cutting open along (the remains of) mortal branches in $\tau^{*}$. Equivalently, $\mathfrak{C}\left(B-\tau^{*}\right)$ is obtained from $\mathfrak{C}(B-\tau)$ by gluing
together along (the remains of) mortal branches. Under such gluings, components whose forward trajectories end on $\partial B$ are glued to other such components. It follows that ring-with-tongues components of $\mathfrak{C}\left(B-\tau^{*}\right)$ are never involved in gluing, and so each component of $\mathfrak{C}(B-\tau)$ is a ring with tongues, or its forward trajectories end on $\partial B$.

This completes the proof of Transience of forward trajectories for $\tau$, and so $\tau$ is a dynamic train track in $B$. As remarked above, $\tau$ fills up $B$. As in the proof of proposition 2.6.2, the alterations on $V$ needed for the construction of $\tau$ preserve circularity in each pared solid torus component of $\mathfrak{C}(P-B)$.

Step 2a: Each ring without tongues component $K$ of $\mathfrak{C}(B-\tau)$ is boundary parallel. In other words, if $K$ is a ring with tongues component of $\mathfrak{C}(B-\tau)$, and if $K$ has no tongues, then there exists a u-pared product component $C$ of $\mathfrak{C}(P-B)$ such that $K$ lies in a $\mathbf{u}$-face of $C$. To prove this, following the proof of step 2a in proposition 2.6.2 we see, letting $\sigma$ be the sector of $B$ containing $K$, that $\sigma$ contains a periodic trajectory of $V$. It follows that $\sigma$, and therefore also $K$, lies in a u-face of a u-pared product component of $\mathfrak{C}(P-B)$.

Step 2b: Eliminating extraneous circular sinks of $\tau$. Let $\gamma$ be a circular sink of $\tau$. We say that $\gamma$ is extraneous if the following is true: if $R$ is a smoothly embedded annulus or Möbius band in $B$ with core $\gamma$, then each component of $\mathfrak{C}(R-\gamma)$ is contained in a ring with tongues component of $\mathfrak{C}(B-\tau)$.

We remark that if $\tau$ is the intersection train track of a dynamic pair in $P$ then $\tau$ has no extraneous circular sinks, by propositions 4.6.5 and 4.6.1.

If $\tau$ has an extraneous circular sink, remove it by following the method of step 2 b in proposition 2.6.2. The result is still a dynamic train track filling up $B$, and by repeating this operation we eventually obtain a dynamic train track filling up $B$ which has no extraneous circular sinks.

Step 2c: Splitting rings having tongues on both sides. This step is an exact repetition of step 2 c of proposition 2.6.2: for any component $K$ of $\mathfrak{C}(B-\tau)$ which is obtained from a ring $R$ by attaching tongues, if each side of $R$ has tongues attached to it, then $B$ should be split along a slight enlargement of $R$.

Step 2d: Splitting lonely face orbits. As in step 2c of proposition 2.6.2, let $T$ be a component of $\mathfrak{C}(M-B)$, let $A$ be a face of $T$, and let $\tau_{A}=\tau \cap A$; also let $\tau_{A}^{*}=\tau^{*} \cap A$. As before, the train track $\tau_{A}^{*}$ consists of one or two embedded, periodic trajectories, and every other trajectory goes from a periodic trajectory to a component of $\partial A$. Clearly the periodic trajectories in $\tau_{A}^{*}$ consist entirely of immortal points of $\tau^{*}$, and so these trajectories are contained in $\tau_{A}$.

If $\tau_{A}$ contains only one periodic trajectory $\gamma$, then $\tau$ should be split along a regular neighborhood $K(\gamma) \subset \gamma$, as in proposition 2.6 .2 (see figure 2.11).

After completing this operation, inclusion of sinks induces a $1-1$ correspondence between ring-with-tongues components of $\mathfrak{C}(B-\tau)$ and faces of torus piece components of $\mathfrak{C}(P-B)$.

Now we must go beyond the proof of 2.6.2, and do some further alterations of $B$ for "goodness sake" (compare proposition 4.7.1).

Step 2e: Splitting to make $B$ good. Let $c$ be a maw circle in some pared torus piece $T$ of $\mathfrak{C}(P-B)$, and suppose that $c \cap \tau=\emptyset$, in other words each point of $c \cap \tau^{*}$ is mortal. It follows that the forward trajectory from each point on $c$ ends on $\partial B$, and so there is a smoothly immersed annulus in $B$ with one boundary circle going to $c$ and the other mapped into $\partial B$; this is the maw-boundary annulus associated to $c$. Also, each $\mathbf{u}$-face of $T$ incident to $c$ contains an annulus with one boundary component on $c$, the other on $\tau$, with interior disjoint from $\tau$. Let $Y_{c}$ be the union of the these annuli with the maw-boundary annulus associated to $c$; we call this the boundary parallelism associated to $c$. Strictly speaking we think of $Y_{c}$ as an immersion in $B$ whose domain is $Y \times S^{1}$, where $Y$ denotes a triod, a union of three segments meeting at a common boundary point.

We say that $B$ is good if the following holds:

- For each maw circle $c$ in a pared torus piece which is disjoint from $\tau$, the associated boundary parallelism $Y_{c}$ is embedded.
- For each component $K$ of $\mathfrak{C}(P-B)$ intersecting $\partial B$, let $\tau_{K}$ be the remains of $\tau$ in $K$. If $K$ is split along all maw-boundary annuli, then the result is a cusped unstable branched surface which is homeomorphic to $\tau_{K} \times[0,1]$.

If $B$ were the unstable branched surface of a good dynamic pair, then $B$ would be good. We shall, therefore, describe dynamic splittings of $B$ supported in $B-\tau$ which are designed to make $B$ good. These splittings are very similar to the ones used in proposition 4.7.1, whose proof we will rely on.

First consider the immersion $Y \times S^{1} \rightarrow B$ defining $Y_{c}$, for some $c$. Let $\gamma_{1}, \gamma_{2}$ be the two boundary components of $Y \times S^{1}$ mapping to $\tau$. Note that the image of $\gamma_{1} \cup \gamma_{2}$ is disjoint from the image of $\left(Y \times S^{1}\right)-\left(\gamma_{1} \cup \gamma_{2}\right)$, and so there are regular neighborhoods of $\gamma_{1}, \gamma_{2}$ in $Y$ which are mapped disjoint from $\left(Y \times S^{1}\right)-\left(\gamma_{1} \cup \gamma_{2}\right)$; let $\gamma_{1}^{\prime}$, $\gamma_{2}^{\prime}$ be the boundary circles of these two regular neighborhoods. There is a homeomorphism $\gamma_{1}^{\prime} \approx \gamma_{2}^{\prime}$ such that if $x_{1}, x_{2}$ correspond then the forward trajectories from $x_{1}$ and $x_{2}$ converge at $c$. If $Y \times S^{1} \rightarrow B$ is not an embedding, then there must exist two points $x, y \in \gamma_{1}^{\prime} \cup \gamma_{2}^{\prime}$ which do not correspond under the homeomorphism $\gamma_{1}^{\prime} \approx \gamma_{2}^{\prime}$ whose forward trajectories nonetheless converge in $\Upsilon B$. It follows that $Y_{c}$ contains a maw component $c^{\prime}$ of $B$ distinct from $c$ and disjoint from $\tau$. Flowing forward from $c^{\prime}$ to $\partial B$ one obtains a dynamic splitting surface of $B$. Now split $B$ along each such splitting surface; after these splittings it follow that $Y_{c}$ is embedded.

We note that for if $c, c^{\prime}$ are distinct boundary parallel maw circles in pared torus pieces $T, T^{\prime}$ of $\mathfrak{C}(P-B)$, then the cycles of $\tau \cap Y_{c}$ are distinct from the cycles in $\tau \cap Y_{c^{\prime}}$. For if there were a cycle $\gamma$ in common, then $\gamma$ would be a periodic trajectory in a sector $\sigma$ of $B$, but one side of $\sigma$ would lie in $T$ and the other side in $T^{\prime}$, contradicting the hypothesis on $B$.

Now let $K$ be a component of $\mathfrak{C}(P-B)$ intersecting $\partial B$. We now know that the mawboundary annuli in $K$ are embedded, and it makes sense to split $K$ along them; let $K^{\prime}$ be the result. If $K^{\prime}$ is not a topological product, then we may define splittings of $B$ just as in proposition 4.7.1. After these splittings, $K^{\prime}$ is a product, and $B$ is good.

The following lemma, an immediate consequence of goodness of $B$, will be useful below.
Lemma 4.11.3. If $K$ is a $\mathbf{u}$-pared product component of $\mathfrak{C}(P-B)$, if $F$ is an annulus $\mathbf{u}$-face of $\mathfrak{C}(P-B)$, and if $c$ is a component of $\partial F$, then either $c \cap \tau \neq \emptyset$ or $c$ is a pu-corner circle of $K$.

Remark. This lemma need not be true for $\mathbf{u}$-faces of a torus piece component $T$ of $\mathfrak{C}(P-B)$, because $T$ may have a uu-cusp circle that is disjoint from $\tau$.

Step 3: Constructing the stable branched surface. Inspired by goodness, we define $B^{s}$. For any component $K$ of $\mathfrak{C}(P-B)$, let $\tau_{K}$ be the remains of $\tau$ in $K$. We construct $B_{K}^{s}$, the remains of $B^{s}$ in $K$.

If $K$ is a cusped torus piece, the construction of $B_{K}^{s}$ follows exactly the proof of proposition 2.6.2; the only contrast is that if $c$ is a cusp circle of $K$ disjoint from $\tau$ then the ring-with-tongues component of $B_{K}^{s}$ corresponding to $c$ will have no tongues.

Let $K$ be a u-pared product component of $\mathfrak{C}(P-B)$. First we construct $B_{K}^{s}$ without regard to the dynamic vector field. Choose a product structure $K \approx F \times[0,1]$ where $\partial_{\mathbf{m}} K=F \times 0$ and $\partial_{\mathbf{b}} K=\partial F \times[0,1]$. We therefore have $\tau_{K} \subset F \times 1$. Identifying $F \times 1 \approx F$ we regard $\tau_{K}$ as a subset of $F$, and we have a cusped stable branched surface $B_{K}^{\prime s}=\tau_{K} \times[0,1] \subset K$. Next consider an annulus $A$ in a $\mathbf{u}$-face of $K$ such that $\partial A \subset \tau_{K}$. By step 2 it follows that $\operatorname{int}(A) \cap \tau_{K}=\emptyset$. There is a component of $\mathfrak{C}\left(K-B_{K}^{\prime s}\right)$ of the form $A \times[0,1]$ where $A \times 1 \approx A, \partial A \times[0,1] \subset B_{K}^{\prime s}$, and $A \times 0 \subset \partial_{\mathbf{m}} K$. Now collapse the annulus $A \times 0$ as described in $\S 4.2$, creating a boundary parallel uss-maw piece whose $\mathbf{u}$-face is $A$. Doing this collapsing for each $A$, we obtain the branched surface $B_{K}^{s}$.

Now we repair the construction of $B_{K}^{s}$ to account for the dynamic vector field $V$. First, since $V$ is continuous, we may perturb $V$ slightly near the $\mathbf{u}$-faces of $K$ so that $V$ is tangent to $B_{K}^{s}$ near those faces. Since $V$ points backward along each switch of $\tau_{K}$ then $V$ points backward along the branch locus of $B_{K}^{s}$ in a neighborhood of the $\mathbf{u}$-faces of $K$. We may assume that this is all true in $F \times[1-\epsilon, 1]$, and also that $B_{K}^{s}$ intersects $F \times(1-\epsilon)$ in a train track $\tau_{1-\epsilon}$, and $V$ points upward on $\tau_{1-\epsilon}$. Note that all backward trajectories of $V$ starting on $F \times(1-\epsilon)$ end on $\partial_{\mathbf{m}} K$. Now redefine $B_{K}^{s}$ in $F \times[0,1-\epsilon]$ by letting $\tau_{1-\epsilon}$ flow backward along $V$, and then smoothing along $F \times(1-\epsilon)$. This is still not right, because $V$
will not point backward along the branch locus in $F \times[0,1-\epsilon]$, but will instead be tangent to the branch locus. To fix this, perturb $V$ slightly near the branch locus in $F \times[0,1-\epsilon]$ so that it points backward along the branch locus. Finally, when an annulus $A$ is collapsed as in the last paragraph, we may alter $V$ by homotopy so that it behaves correctly.

Now set $B^{s}$ to be the union of the overlay images of $B_{K}^{s}$, as $K$ varies over all components of $\mathfrak{C}(P-B)$. Set $B^{u}=B$. We verify that $B^{s}, B^{u}$ satisfies all axioms of a dynamic pair, except for the prohibition on coherent annulus drums.

Transience of forward trajectories follows from lemma 4.6.4, and Transience of backward trajectories follows from construction of $B^{s}$.

By hypothesis there are no Reeb annuli in $\partial B^{u}$. For each u-pared product component $K$ of $\mathfrak{C}\left(P-B^{u}\right)$, it follows from Coherence of cusp circles for $K$ that the train track $\tau_{K}$ has no Reeb components in the union of $\mathbf{u}$-faces of $K$ (see the proof of proposition 4.6.1), and it follows that $\partial B^{s}$ has no Reeb annuli.

Now we prove that the components of $Q=\mathfrak{C}\left(P-\left(B^{s} \cup B^{u}\right)\right)$ have the correct types, and we prove Separation of torus pieces.

Consider first those components of $Q$ that are contained in a u-pared torus piece $T$ of $\mathfrak{C}\left(P-B^{u}\right)$. By construction of $B^{s}$ each of these is a dynamic torus piece, pinched tetrahedron, or suu-maw piece. Furthermore any suu-maw piece is incident to a cusp circle in $T$ which is disjoint from $\tau_{T}$, and so the maw piece is boundary parallel. The nonexistence of face gluings is obvious from construction.

Consider next a component $C$ of $Q$ such that $C$ is contained in a u-pared product $K$ of $\mathfrak{C}\left(P-B^{u}\right)$. We consider two cases, depending on whether $C \cap \partial_{\mathbf{p}} K$ is empty.

Case 1: $C \cap \partial_{\mathbf{p}} K \neq \emptyset$. We shall verify that $C$ is a drum whose base has nonpositive index, by using lemma 4.4.1. Let $\gamma$ be a us-corner circle of $C$, let $A_{\mathbf{u}}$ be the $\mathbf{u}$-face incident to $\gamma$, and let $A_{\mathbf{s}}$ be the incident s-face. By lemma 4.11.3, the opposite boundary circle of $A_{\mathbf{u}}$ is a pu-circle. By construction of $B_{K}^{s}$, the opposite boundary circle of $A_{\mathbf{s}}$ is an ms-circle. It follows that the annulus $A_{\mathbf{s}} \cup A_{\mathbf{u}}$ may be perturbed, producing a properly embedded annulus $A_{\gamma} \subset C$, such that $A_{\gamma}$ cuts off a half annulus drum $D_{\gamma}$ containing $\gamma$. Moreover, each trajectory in $C$ that is not contained in a face incident to some us-circle ends at $\partial_{p} K$ in forward time and at $\partial_{m} K$ in backward time. Also, since each uu-cusp curve of $K$ has nontrivial intersection with $\tau_{K}$, it follows that each uu-cusp curve of $C$ has a puu-gable at one endpoint and an suu-gable at the other. By construction of $B_{K}^{s}$, each ss-cusp curve of $C$ has a mss-gable at one endpoint and a uss-gable at the other. Applying lemma 4.4.1 it follows that $C$ is a drum. Noting that $\partial_{\mathbf{p}} C$ is a component of $\mathfrak{C}\left(\mathcal{R}_{+} P-\beta^{u}\right)$ and that $\beta^{u}$ is essential in $\mathcal{R}_{+} P$, it follows that the base of $C$ has nonpositive index.

Note moreover that $C$ is a coherent annulus drum if and only if the $\mathbf{p}$-face of $C$ is an coherent annulus component of $\mathfrak{C}\left(\mathcal{R}_{+} P-\beta^{u}\right)$; in particular, if $\mathfrak{C}\left(\mathcal{R}_{+} P-\beta^{u}\right)$ has no coherent annuli then there are no coherent annulus drums.

Case 2: $C \cap \partial_{\mathbf{p}} K=\emptyset$. Let $K \approx F \times 1$ give the u-pared product structure of $K$. By construction of $B_{K}^{s}$ it follows that $\partial_{\mathbf{u}} C=C \cap F \times 1$ is a component of $\mathfrak{C}\left((F \times 1)-\tau_{K}\right)$, and is disjoint from $\partial_{\mathbf{p}} K$. We analyze such components.

Let $G$ be an annulus $\mathbf{u}$-face of $K$, and let $\tau_{G}$ be the remains of $\tau$ in $G$. Recall that $\tau_{G}$ has either one or two cycles, and every forward trajectory in $\tau_{G}$ that is disjoint from a cycle ends in $\partial G$. It follows that each component of $\mathfrak{C}\left(G-\tau_{G}\right)$ has one of three types:

Type I An annulus with boundary in $\tau_{G}$.
Type II A one-cusped triangle, each edge adjacent to the cusp being in $\tau_{G}$, and the third edge being in $\partial G$.

Type III An annulus with one boundary circle in $\tau_{G}$ and the other boundary circle in $\partial G$.
Type I is clearly disjoint from $\partial_{\mathbf{p}} K$. Also, by lemma 4.11 .3 type III intersects $\partial_{\mathbf{p}} K$. Each component of $\mathfrak{C}\left((F \times 1)-\tau_{K}\right)$ disjoint from $\partial_{\mathbf{p}} K$ is therefore a union of pieces of types I and II. The only way for such a union to be connected is if it consists of a single type I piece, or two type II pieces glued along their common third edge to form a uu-cusp.

By construction of $B_{K}^{s}$, if $\partial_{\mathbf{u}} C$ is a single type I piece then $C$ is a boundary parallel uss-maw piece, and by step 2 it follows that $C$ is attached to some dynamic torus piece of $Q$. Also, if $\partial_{\mathbf{u}} C$ is a union of two type II pieces glued along their common third edge then $C$ is a p-pinched drum over a rectangle (figure 4.9-c).

We have checked the axioms for $B^{s}, B^{u}$ to be a dynamic pair with coherent annulus drums.

Proof of proposition 4.11.2. In the setting of the previous proof, suppose now that $\beta$ is a transversely oriented $\mathcal{R}$ branched surface in $P$ carrying a foliation, $(B, V)$ is vertical with respect to $\beta$, and $\mathcal{I}$ is tangent to $\beta$. We show that $B^{s}, B^{u}$ may be constructed to be vertical with respect to an $\mathcal{R}$-branched surface obtained by splitting $\beta$, by borrowing from the proof of proposition 3.5 .5 in section 3.7. In steps $1-3$ several alterations are performed on ( $B, V$ ), and we must check that these steps may be done so as to preserve verticality.

In steps 1 and 2, verticality is preserved just as it is in proposition 3.5.5.
Step 3 contains the construction of $B^{s}$, and $V$ is altered to be tangent to $B^{s}$. Consider a component $K$ of $\mathfrak{C}(P-B)$.

If $K$ is a $\mathbf{u}$-cusped torus piece, construct $B_{K}^{s}$ exactly as in proposition 3.5.5. Recall that in that construction, the branched surface $\beta$ was altered by isotopy to establish the property that for any annulus $A$ carried by $\beta \mid K$, if $A$ cuts off a cusp circle $c$ of $K$, then $A$ is close to $c$.

Similarly, if $K$ is a u-cusped product, $\beta$ should be isotoped so that if $A$ is an annulus carried by $\beta \mid K$, and if $A$ cuts off a cusp circle $c$ of $K$, then $A$ is close to $c$, so close that if $\gamma$ is a circle carried by $\tau_{K}$, then $\gamma$ does not lie between $A$ and $c$. The proof that this may be done is similar to the case where $K$ is a u-cusped torus piece.

Now construct $B_{K}^{\prime s}$ exactly as in the proof of proposition 4.11.1. First construct $B_{K}^{\prime s}$ near $\tau_{K}$, and perturb $V$ slightly to be tangent to that portion of $B_{K}^{\prime s}$ near $\tau_{K}$. Then construct the rest of $B_{K}^{\prime s}$ by flowing backward along $V$. Then perturb $V$ slightly to point backward along the branch locus of $B_{K}^{\prime s}$. Since these alterations on $V$ are all perturbations, it follows that $V$ is still vertical with respect to $\beta$.

For the final step, consider an annulus $A$ in a $\mathbf{u}$-face of $K$ such that $\partial A \subset \tau_{K}$ and $\operatorname{int}(A) \cap \tau_{K}=\emptyset$. Corresponding to $A$ there is a component $H$ of $\mathfrak{C}\left(K-B_{K}^{\prime s}\right)$ of the form $H \approx A \times[0,1]$ where $A=\partial_{\mathbf{u}} H \approx A \times 1 \approx A, H \cap B_{K}^{\prime s}=\partial_{\mathbf{s}} H \approx \partial A \times[0,1]$, and $H \cap \partial_{\mathbf{m}} K=\partial_{\mathbf{m}} H \approx A \times 0$. We must collapse of $\partial_{\mathbf{m}} H$ to convert $H$ into a boundary parallel uss-maw piece, and we must check that this can be done preserving verticality of $V$. To do this, split $\beta$ so that $\beta \mid H$ is $I$-parallel near $\partial_{\mathbf{m}} H$, and then we may easily collapse $\partial_{\mathbf{m}} H$ preserving verticality of $V$. Do this simultaneously for each $A$, completing the construction of $B_{K}^{s}$.

This completes the definition of the dynamic pair $B^{s}, B^{u}$. By construction, the dynamic vector field $V$ is vertical with respect to $\beta$, and no annulus carried by $\beta$ is peripheral in $\mathfrak{C}\left(P-B^{u}\right)$. As in the proof of proposition 3.5.5, one can prove that no annulus carried by $\beta$ is peripheral in $\mathfrak{C}\left(P-B^{s}\right)$, making use of the fact that each annulus carried by $\beta$ and cutting off a cusp circle of $\mathfrak{C}\left(P-B^{u}\right)$ is close to that cusp circle.

### 4.12 Eliminating coherent annulus drums

Proposition 4.11 .1 shows how to take a Markov branched surface in a sutured manifold $P$ and produce a dynamic pair with coherent annulus drums on $P$. In this section we attempt to eliminate the coherent annulus drums, in the hope of producing a true dynamic pair. But a surprise is in store: there is an obstruction to eliminating coherent annulus drums. When the obstruction is nontrivial, something nice happens in exchange: one can construct a family of nonperipheral, incompressible tori, such that each complementary component has either a dynamic pair or a Seifert fibration.

An example To motivate the construction, recall that in the remark following proposition 4.11.1 we exhibited a dynamic pair with a coherent annulus drum in a hexagonal round handle $H$, and we compared this pair to a true dynamic pair in $H$.

Mimicking this example, consider a coherent annulus drum $D$ in some dynamic pair $B^{s}, B^{u}$. Here is a recipe for converting $D$ into a solid torus piece of type (3,0).

Step 1: Inserting sectors Add an annulus sector $\sigma^{s}$ to $B^{s}$, and add an annulus sector $\sigma^{u}$ to $B^{u}$, as shown in figure $4.19 \mathrm{a}-\mathrm{b}$. More precisely, let $F_{1}^{\mathbf{s}}, F^{\mathbf{m}}, F_{2}^{\mathbf{s}}, F_{2}^{\mathbf{u}}, F^{\mathbf{p}}, F_{1}^{\mathbf{u}}$ be the faces of $D$ in circular order, with superscripts denoting labels. Let $c_{i}=F_{i}^{\mathbf{s}} \cap F_{i}^{\mathbf{u}}$,


Figure 4.19: To eliminate a coherent annulus drum, insert the s-annulus sector $\sigma^{\mathbf{s}}$ and the $\mathbf{u}$-annulus sector $\sigma^{\mathbf{u}}$ (a-b), and then collapse the two split maw pieces (b-c), resulting in a dynamic solid torus of type $(3,0)$ and two boundary parallel maw pieces.
$i=1,2$, be the us-circles of $D$. Let $A_{1}^{\mathbf{s}}$ be a smoothly embedded annulus in $B^{\mathbf{s}}$ such that $A_{1}^{\mathrm{s}} \cap D=c_{1}$ and $A_{1}^{\mathrm{s}}$ is contained in an arbitrarily small neighborhood of $c_{1}$. Let $A_{1}^{\mathbf{u}}, A_{2}^{\mathrm{s}}, A_{2}^{\mathbf{u}}$ be similarly defined.

To define $\sigma^{s}$, take the annulus $A_{1}^{\mathbf{s}} \cup F_{1}^{\mathbf{u}} \cup F^{\mathbf{p}} \cup F_{2}^{\mathbf{u}} \cup A_{2}^{\mathbf{s}}$ and perturb it so that its intersection with $D$ is a properly embedded annulus with one boundary circle on each of $F_{1}^{\mathbf{u}}, F_{2}^{\mathbf{u}}$, and so that $\sigma^{s}$ intersects $B^{s}$ tangentially along the two circles $\partial A_{1}^{\mathbf{s}}-c_{1}, \partial A_{2}^{\mathbf{s}}-c_{2}$. Define $\sigma^{u}$ similarly by perturbing the annulus $A_{1}^{\mathbf{u}} \cup F_{1}^{\mathbf{s}} \cup F^{\mathbf{m}} \cup F_{2}^{\mathbf{s}} \cup A_{2}^{\mathbf{u}}$.

We must assign an orientation to each of the four circles $c_{i}^{s}=\sigma^{s} \cap F_{i}^{\mathbf{u}}, c_{i}^{u}=\sigma^{\mathbf{u}} \cap F_{i}^{\mathbf{u}}$, $i=1,2$. Since $D$ is a coherent annulus drum, the circles $c_{1}, c_{2}$ have isotopic orientations, and so we may assign orientations so that the six circles $c_{i}, c_{i}^{\mathbf{s}}, c_{i}^{\mathbf{u}}, i=1,2$, have isotopic orientations.

From the construction it follows that $\sigma^{s}, \sigma^{u}$ cut $D$ into a dynamic solid torus of type ( 3,0 ), a split uss-maw piece, and a split suu-maw piece.

Step 2: Collapsing maw pieces As described in §4.2, collapse the split maw pieces, to form boundary parallel uss and suu-maw pieces (figure 4.19b-c).

The result Let $B_{1}^{s}, B_{1}^{u}$ be the new pair of branched surfaces, and $T$ the dynamic solid torus, resulting from performing steps 1 and 2 on the drum $D$. Note that when $\sigma^{u}$ is added to $B^{u}$, there is an suu-maw piece $\mu$ with corner circles $c_{1}, c_{1}^{u}$, and with cusp circle $\partial A_{1}^{\mathbf{u}}-c_{1}$. If $\Upsilon B^{s} \cap F_{1}^{\mathrm{s}} \neq \emptyset$ then $\mu$ is divided into pinched tetrahedra, otherwise $\mu$ is undivided. Similar comments apply to the uss-maw piece with corner circles $c_{1}, c_{1}^{\mathrm{s}}$, and to the two maw pieces having $c_{2}$ as a corner circle.

Let us suppose now that $D$ was the only coherent annulus drum in $B^{s}, B^{u}$, and so the pair $B_{1}^{s}, B_{1}^{u}$ has no coherent annulus drums. Is $B_{1}^{s}, B_{1}^{u}$ a dynamic pair?

Here is a potentially fatal problem. The definition of a dynamic pair rules out corner gluings among dynamic torus pieces, but it does not rule out corner gluings between a dynamic torus piece and a coherent annulus drum. In the pair $B^{s}, B^{u}$, either of the corner circles $c_{1}, c_{2}$ of $D$ could be glued to a corner circle of some dynamic torus piece. After steps 1 and 2 are carried out, either $c_{1}$ or $c_{2}$ could be the locus of a corner gluing among the dynamic torus pieces of $B_{1}^{s}, B_{1}^{u}$, and so $B_{1}^{s}, B_{1}^{u}$ would not be a dynamic pair.

Another example Figure 4.20 shows an example where this phenomenon occurs: an octagonal round handle $P$, obtained by gluing an ordinary round handle $h$ and a hexagonal round handle $h^{\prime}$ along a suture. There is a dynamic pair with one coherent annulus drum in $P$. Carrying out steps 1 and 2 produces the pair shown in figure 4.14 , but the two dynamic solid tori in this pair share a corner circle, violating the ban on corner gluing. As remarked in figure 4.14, by removing certain annulus sectors the problem is corrected, producing a true dynamic pair in $P$.

Yet another example Still more complicated problems can arise. For another example, let the ordinary round handle $h$ and the hexagonal round handle $h^{\prime}$ be glued along two sutures, as shown in 4.21a, to form a sutured manifold $P$ homeomorphic to (annulus) $\times S^{1}$. One component of $\partial P$ has two $\mathbf{m}$ and two $\mathbf{p}$-faces, the other component has one $\mathbf{m}$ and one $\mathbf{p}$-face. These faces are all annuli whose cores are homotopic to the $S^{1}$ factor of $P$. The dynamic pairs in $h, h^{\prime}$ combine to form a dynamic pair in $P$ with two coherent annulus drums; again we assume that the gluing is done so that the positive generators of $H_{1}(h)$ and $H_{1}\left(h^{\prime}\right)$ are identified in $H_{1}(P)$. Elimination of these drums using steps 1,2 above will produce a pair with a chain of three dynamic solid tori $T_{1}, T_{2}, T_{3}$ such that for each $i \in \mathbf{Z} / 3$ the tori $T_{i}$ and $T_{i+1}$ share a corner circle. No removal of annulus sectors will produce a dynamic pair, but-pursuing the analogy with figure 4.14 one finds that, after removal of certain annulus sectors, there is a nonperipheral, incompressible torus $F$ such that the resulting pair is a dynamic pair in the sutured manifold $\mathfrak{C}(P-F)$ (figure 4.21b).

The following proposition describes the general method for eliminating coherent annulus drums. If $P$ is a sutured manifold, a torus $T \subset \operatorname{int}(P)$ is said to be nonperipheral if $T$ is not isotopic to a b-torus face of $P$. Note that a nonperipheral torus $T$ is allowed to be isotopic to a torus component of $\partial P$, as long as that component is not a $\mathbf{b}$-face.

Proposition 4.12.1. Let $B^{s}, B^{u}$ be a dynamic pair with coherent annulus drums in a sutured manifold $P$. There is a construction which produces a family $T$ of nonperipheral, incompressible tori in $P$ and, for each component $P^{\prime}$ of $\mathfrak{C}(P-T)$, either a Seifert fibration of $P^{\prime}$ or a dynamic pair $B^{\prime s}, B^{\prime u}$ in $P^{\prime}$. The pair $B^{s}, B^{u}$ is obtained from $B^{\prime s}, B^{\prime u}$ by the following operations: dynamic splitting of maw-boundary annuli; removal of annulus



Figure 4.21: In this example, an ordinary round handle and a hexagonal round handle are glued along two sutures (the dotted lines). If steps 1 and 2 are carried out to eliminate the two coherent annulus drums, the resulting pair is seen to have three violations of the ban on corner gluings. The offending annulus sectors can be removed so that, after $P$ is cut along a nonperipheral, incompressible torus (the dotted circle), the resulting sutured manifold has a dynamic pair.
sectors; insertion of annulus sectors by perturbing annuli in $B^{\prime s} \cup B^{u}$; collapsing split maw pieces.

If $B^{s}, B^{u}$ is vertical with respect to a branched surface hierarchy in $P$, then then the construction may be done so that $B^{\prime s}, B^{\prime u}$ is vertical.

The proof of this proposition will involve a series of operations on the pair $B^{s}, B^{u}$ taking it outside the realm of dynamic pairs. The intermediate objects, called "protodynamic pairs" for lack of better terminology, will be like ordinary dynamic pairs except that we allow complementary pieces which are Seifert fiber spaces, and we also allow face and corner gluings. The proof will then follow several steps:

1. Starting from a dynamic pair with coherent annulus drums, produce a proto-dynamic pair.
2. Remove certain annulus sectors to produce a new proto-dynamic pair having no face or corner gluings.
3. Add certain annulus sectors, putting the boundary of each Seifert piece in a standard form.
4. Cut open along peripheral tori of Seifert pieces, so that each component has either a dynamic pair or a Seifert fibering.

In practice, steps 3 and 4 will be reversed.
To define proto-dynamic pairs and implement this outline, we must first define the "Seifert pieces" of a proto-dynamic pair. A labelled Seifert manifold is an oriented dynamic manifold ( $N, V$ ) together with a Seifert fibering on $N$, such that:

- The corner locus of $N$ is a union of corner circles, each of which is a Seifert fiber.
- Each face $F$ of $N$ is a union of Seifert fibers.
- Each $\mathbf{u}, \mathbf{s}, \mathbf{p}$, and $\mathbf{m}$-face is an annulus.
- Each b-face is a torus or annulus, and each annulus $\mathbf{b}$-face has one $\mathbf{b m}$-boundary circle and one bp-boundary circle.
- The flow on $N$ generated by $V$ is circular in the following sense: with respect to some Riemannian metric on $N$, the dot product of $V$ with the tangent vector of the Seifert fibration is everywhere positive.

As a consequence of the last condition, if $c$ is a corner circle then the orientations induced by $V$ and by the Seifert fibering are identical, and if $F$ is a face then $V \mid F$ generates a circular flow whose direction agrees with the Seifert fibering.

If $N$ is a labelled Seifert manifold, then the base space of the Seifert fibration may be regarded as an oriented 2 -orbifold with corners $O$. The index of $O$ is a rational number defined by the formula

$$
\operatorname{index}(O)=2 \chi(O)-\frac{1}{2} \cdot \#(\text { corners })-2 \sum_{n \geq 2} k(n) \cdot\left(1-\frac{1}{n}\right)
$$

where $k(n)$ is the number of cone points labelled $\mathbf{Z} / n$.
We need the following easily established fact:
Proposition 4.12.2. Let $S$ be a Seifert fiber space over a base orbifold $O$. If $O$ is not $a$ disc with $\leq 1$ cone point, then each component of $\partial S$ is incompressible. If $O$ is not an annulus with zero cone points, then any two distinct components of $\partial S$ are non-isotopic.

Example. Suppose $N$ is a connected, labelled Seifert manifold with nonempty boundary and with base orbifold $O$. Then index $(O)>0$ if and only if $O$ is a disc satisfying one of the following: $O$ has no corners and at most one cone point; $O$ has one corner and either no cone points or one $\mathbf{Z} / 2$ cone point; $O$ has two or three corners and no cone points.
Example. Suppose $T$ is a dynamic solid torus of type ( $n, k$ ) with circular vector field; as noted above, $T$ is a labelled Seifert fiber space. Letting $g=\operatorname{gcf}(n, k)$, the base orbifold is a polygon with $2 g$ corners and with one interior cone point labelled by the group $\mathbf{Z} /(n / g)$;
when $k=0$ there is no cone point. The index is $2-g-2 \cdot\left(1-\frac{g}{n}\right)=(-1+2 / n) \cdot g$. It follows that $T$ is essential as a dynamic solid torus if and only if the base of $T$ has nonpositive index, which happens if and only if $n \geq 2$.
Example. A dynamic torus shell is a labelled Seifert fiber space whose base has nonpositive index.

Example. A coherent annulus drum is a labelled Seifert fiber space whose base is a hexagon with no cone points, and with edges labelled smsupu in circular order; the index equals -1 . An incoherent annulus drum cannot be given the structure of an essential labelled Seifert fiber space, because the two us-circles are anti-isotopic.
Example. Splitting a boundary parallel suu-maw piece along the maw-boundary annulus results in a split suu-maw piece, which is a labelled Seifert fiber space, whose base is a rectangle with no cone points and sides labelled usup; the index equals 0 . The base of a split uss-maw piece is similarly described.
Example. A half annulus drum (figure 4.8) is a labelled Seifert fiber space, whose base is a pentagon with no cone points and sides labelled bmsup; the index equals $-1 / 2$.

A proto-dynamic pair in a sutured manifold $P$ is a pair of branched surfaces $B^{s}, B^{u}$ in $P$, in general position with respect to each other, and a $C^{0}$ vector field $V$ on $P$, such that:

- $(P, V)$ is a dynamic manifold.
- $\left(B^{s}, V\right)$ is a stable dynamic branched surface in $P$, and $\left(B^{u}, V\right)$ is an unstable dynamic branched surface.
- $V$ is smooth, except along $\Upsilon B^{s}$ where backward trajectories are locally unique, and along $\Upsilon B^{u}$ where forward trajectories are locally unique.
- Each component of $Q=\mathfrak{C}\left(P-\left(B^{s} \cup B^{u}\right)\right)$ is a labelled Seifert manifold or a drum, whose base has nonpositive index.
- Each us-circle of $Q$ is contained in a Seifert component; in particular, the only drums which contain us-circles are half annulus drums and coherent annulus drums.
- For each component $K$ of $\mathfrak{C}\left(B^{u}-B^{s}\right)$, either there exists a $\mathbf{u}$-face $F$ of some Seifert component of $Q$ such that $F \subset K$ and $F$ is a sink of the forward semiflow on $K$, or every forward trajectory in $K$ is finite and ends at a point of $\mathcal{R}_{+} P$. A similar propery holds for components of $\mathfrak{C}\left(B^{s}-B^{u}\right)$.
- The boundary train tracks have no Reeb annuli.
- $B^{s}$ and $B^{u}$ carry no tori or Klein bottles.

Proof of proposition 4.12.1. Let $B_{0}^{s}, B_{0}^{u}$ be a dynamic pair with coherent annulus drums in $P$.

Step 1: A proto-dynamic pair First we cut off the us-circles of drums. For each drum component $D$ of $\mathfrak{C}\left(P-\left(B_{0}^{s} \cup B_{0}^{u}\right)\right)$ whose base is not an annulus, let $A_{D} \subset D$ be a union of properly embedded annuli cutting off pairwise disjoint half annulus drums, one for each us-circle of $D$. For each incoherent annulus drum $D$, let $A_{D}$ be a single properly embedded annulus dividing $D$ into two half annulus drums. If $D$ is a coherent annulus drum or half-annulus drum let $A_{D}=\emptyset$. Let $\mathcal{A}=\bigcup_{D} A_{D}$. Let $P^{\prime}=\mathfrak{C}(P-\mathcal{A})$ and label the scars of $\mathcal{A}$ with the symbol $\mathbf{b}$, so $P^{\prime}$ is a sutured manifold. At the very end of the proof of proposition 4.12 .1 we will reglue along $\mathcal{A}$.

Next, split $B_{0}^{s}$ and $B_{0}^{u}$ along all maw-boundary annuli, converting boundary parallel maw pieces into split maw pieces, producing a pair $B_{1}^{s}, B_{1}^{u}$.

By the list of examples above, it follows that each component of $\mathfrak{C}\left(P^{\prime}-\left(B_{1}^{s} \cup B_{1}^{u}\right)\right)$ which is not a drum is an essential labelled Seifert manifold: a dynamic torus piece, split maw piece, coherent annulus drum, or half-annulus drum. It also follows from the construction that each us-circle of $\mathfrak{C}\left(P^{\prime}-\left(B_{1}^{s} \cup B_{1}^{u}\right)\right)$ is contained in a Seifert component. The rest of the properties of a proto-dynamic pair are easily checked for $B_{1}^{s}, B_{1}^{u}$ in $P^{\prime}$.

Step 2: Removing offending sectors If the pair $B_{1}^{s}, B_{1}^{u}$ has any corner or face gluings, we will find certain annulus or Möbius band sectors of $B_{1}^{s}$ or $B_{1}^{u}$ to take the blame. These sectors will consist mostly of the loci of face gluings. After removing finitely many sectors we will produce a new proto-dynamic pair having neither corner nor face gluings.

First we analyze corner gluings, by refining the proof of proposition 2.4.1. Let $Q_{1}=$ $\mathfrak{C}\left(P^{\prime}-\left(B_{1}^{s} \cup B_{1}^{u}\right)\right)$, and let $Q_{1}^{S}$ be the union of Seifert components of $Q_{1}$. When corner circles of $Q_{1}^{S}$ are glued in $P^{\prime}$, the glued circles have the same type: either ms, pu, or us. When two ms or two pu-corner circles are glued, they produce a circle component of the boundary train track $\partial B_{1}^{s}$ or $\partial B_{1}^{u}$. When two or more us-corner circles are glued, they produce a periodic orbit $\gamma$ of $\tau_{1}=B_{1}^{s} \cap B_{1}^{u}$.

Suppose $\gamma$ is a periodic orbit of $\tau_{1}$ obtained from gluing us-circles of $Q_{1}^{S}$. Recall that the overlay map $Q_{1}^{S} \rightarrow P^{\prime}$ has a factorization $Q_{1}^{S} \rightarrow X \rightarrow P^{\prime}$ such that one of the following occurs:

- The map $Q_{1}^{S} \rightarrow X$ identifies two corner circles of $Q_{1}^{S}$ to a single circle $c$ in $X$, and the map $X \rightarrow P^{\prime}$ restricted to $c$ is a parameterization of $\gamma$; or
- The map $Q_{1}^{S} \rightarrow X$ identifies one corner circle of $Q_{1}^{S}$ to a single circle $c \subset X$ by a double covering map, and the map $X \rightarrow P^{\prime}$ restricted to $c$ is a parameterization of $\gamma$.

The orbit $\gamma$ has a normal bundle, which is a smoothly immersed solid torus $h: H \rightarrow P^{\prime}$ such that $h \mid \operatorname{Core}(H)$ parameterizes $\gamma$. We may choose the normal bundle to have two
properly embedded rings $A^{s}, A^{u} \subset H$ intersecting transversely along Core $(H)$ so that $A^{s} \subset$ $h^{-1}\left(B_{1}^{s}\right)$ and $A^{u} \subset h^{-1}\left(B_{1}^{u}\right)$. Note that $\gamma$ is twisted if and only if $A^{s}, A^{u}$ are Möbius bands, and $\gamma$ is untwisted if and only if $A^{s}, A^{u}$ are annuli. A quadrant of $\gamma$ is a component of $\mathfrak{C}\left(H-\left(A^{s} \cup A^{u}\right)\right)$. Locally at each $p \in \operatorname{Core}(H)$ there are four "quadrants", but if $\gamma$ is twisted then each opposite pair of "quadrants" at $p$ is actually a single quadrant of $\gamma$. With this understanding, even when $\gamma$ is twisted we shall sometimes abuse terminology by referring to the "four quadrants of $\gamma$ ".

Consider a quadrant $q$ of $\gamma$ and a corner circle $c$ of $Q_{1}^{S}$. We say that $q$ arises from $c$ if $c$ has arbitrarily small neighborhoods $N$ in $Q_{1}^{S}$ such that, abusing notation by identifying $N$ with its overlay image in $P^{\prime}$, we have $h^{-1}(N) \cap q$ contains a neighborhood of Core( $H$ ) in $q$.

Lemma 4.12.3. Exactly one of the following is true, for the corner gluing $\gamma$ :
Type (a) All four quadrants arise from corner circles of $Q_{1}^{S}$.
Type (b) Exactly two adjacent quadrants arise from corner circles of $Q_{1}^{S}$. The other two quadrants arise from corner circles of divided maw pieces $\mu, \mu^{\prime}$. Moreover, $\mu \cap \mu^{\prime}=A_{\gamma}$ is an annulus contained in a sector $\sigma$ of $B_{1}^{s}$ or $B_{1}^{u}, \gamma$ is a component of $\partial A_{\gamma}$, the other component $c$ of $\partial A_{\gamma}$ is a component of $\partial \sigma$, and $A_{\gamma}$ is a neighborhood of $c$ in $\sigma$.

If $\gamma$ is twisted, then $\gamma$ must be a type (a) corner gluing.
Proof. We prove the lemma when $\gamma$ is untwisted; the twisted case is similar. Since $\gamma$ is the image of a corner gluing, then $\gamma$ has at least two quadrants arising from corner gluings.

Suppose that $\gamma$ has an opposite pair of quadrants $q, q^{\prime \prime}$ which arise from corner circles. Let $q^{\prime}$ be one of the other two quadrants. Suppose that the notation is chosen so that $q, q^{\prime}$ are separated by $A^{s}$ and $q^{\prime}, q^{\prime \prime}$ are separated by $A^{u}$. Since $B_{1}^{u}$ does not intersect $q$-that is, $\left(q-A^{u}\right) \cap h^{-1}\left(B_{1}^{u}\right)=\emptyset$-it follows that $B_{1}^{u}$ does not intersect $q^{\prime}$. Since $B_{1}^{s}$ does not intersect $q^{\prime \prime}$, it follows that $B_{1}^{s}$ does not intersect $q^{\prime}$. Therefore, $q^{\prime}$ arises from a corner circle.

It follows from this argument that either all four quadrants of $\gamma$ arise from corner circles, or only two adjacent ones do. In the latter case, suppose that quadrants $q^{\prime \prime}, q^{\prime \prime \prime}$ arise from corner circles and $q, q^{\prime}$ do not. Assume that $q, q^{\prime}$ are separated by $A^{u}$; the $A^{s}$ case is similar. Since $B_{1}^{u}$ does not intersect $q^{\prime \prime}$ or $q^{\prime \prime \prime}$, it follows that $B_{1}^{u}$ does not intersect $q$ or $q^{\prime}$, and so $B_{1}^{s}$ must intersect both $q$ and $q^{\prime}$. From this it follows, by definition of a proto-dynamic pair, that each component of $Q_{1}$ intersecting $q$ or $q^{\prime}$ is a pinched tetrahedron. The union of the pinched tetrahedra intersecting $q$ (resp. $q^{\prime}$ ) forms a divided maw piece $\mu$ (resp. $\mu^{\prime}$ ), by applying proposition 4.6.1.

It may seem possible that $\mu=\mu^{\prime}$, and $q, q^{\prime}$ arise from the two different us-circles of this maw piece; but in that case the $\mathbf{u}$-face of the maw piece would close up to form a torus carried by $B_{1}^{s}$, which is impossible. Thus, $\mu \neq \mu^{\prime}$.

Let $\alpha \subset \mu, \alpha^{\prime} \subset \mu^{\prime}$ be the $\mathbf{u}$-faces which intersect $A^{u}$ nontrivially. It is easy to see that $\mu \cap \mu^{\prime}=\alpha \cap \alpha^{\prime}$, and this intersection is an annulus $A_{\gamma}$. The remaining statements of the lemma follow easily.

We have the following corollary, which is really just a restatement of proposition 2.4.1:
Corollary 4.12.4. If corner gluings exist, then face gluings exist.
Now we use face gluings to construct certain annulus or Möbius band sectors. Suppose that $F_{0}$ is the annulus or Möbius band locus of a face gluing of $Q_{1}^{S}$. We assume, say, that $F_{0} \subset B_{1}^{s}$, and we construct an annulus or Möbius band sector of $B_{1}^{s}$. Note that $F_{0}$ arises in one of two ways: two annulus s-faces of $Q_{1}^{S}$ are glued homeomorphically to produce the annulus $F_{0}$; or one annulus s-face is glued to itself, mapping to the Möbius band $F_{0}$ by a double cover.

Consider a boundary component $\gamma$ of $F_{0}$. If $\gamma \subset \partial B_{1}^{s}$ then $\gamma$ is the boundary of a sector of $B_{1}^{s}$. If $\gamma \not \subset \partial B_{1}^{s}$ then $\gamma$ is the locus of a us-corner gluing. If $\gamma$ is a type (a) corner gluing, then $F_{0}$ meets the locus of another s-face gluing at $\gamma$. Add this face to $F_{0}$, forming a larger annulus or Möbius band $F_{1}$ composed of the loci of two face gluings. Continue in this manner, forming an increasing sequence $F_{0} \subset F_{1} \subset F_{2} \subset \cdots$ consisting of larger and larger annuli or Möbius bands embedded in a sector of $B_{1}^{s}$, each $F_{i}$ a union of s-face gluings. This sequence must eventually stop, because there are only finitely many components of $B_{1}^{s}-B_{1}^{u}$. Let $F_{n}$ be the last term of the sequence.

Note that $\partial F_{n} \neq \emptyset$, for otherwise $F_{n}$ is a torus or Klein bottle carried by $B_{0}^{s}$, a contradiction. If $\gamma$ is a component of $F_{n}$, it follows that $\gamma$ is either a component of $\partial B_{1}^{s}$ or a type (b) corner gluing. For each $\gamma$ of the latter type, add the annulus $A_{\gamma}$ to $F_{n}$; let $F$ be the result. Clearly $F$ is an annulus or Möbius band sector of $B_{1}^{s}$.

Now remove $\operatorname{int}(F)$ from $B_{1}^{s}$, to produce a new branched surface $B_{1}^{\prime s}$. It is mostly easy to check that $B_{1}^{\prime s}, B_{1}^{u}$ is still a proto-dynamic pair. Some divided maw pieces of $\mathfrak{C}\left(P^{\prime}-\left(B_{1}^{s} \cup B_{1}^{u}\right)\right)$ have been glued to form larger divided maw pieces of $\mathfrak{C}\left(P^{\prime}-\left({B^{\prime s}}_{1}^{\prime} \cup B_{1}^{u}\right)\right)$. Some Seifert pieces of $\mathfrak{C}\left(P^{\prime}-\left(B_{1}^{s} \cup B_{1}^{u}\right)\right)$ have been glued along faces to form larger Seifert pieces of $\mathfrak{C}\left(P^{\prime}-\left(B_{1}^{\prime s} \cup B_{1}^{u}\right)\right)$. The only slight subtlety is checking that the new Seifert pieces are essential. To see why, note that the base orbifolds of the new Seifert pieces are obtained by edge gluings of base orbifolds of old Seifert fiber pieces. Since index is additive under such gluings, and since indices are nonpositive before gluing, it follows that indices are nonpositive after gluings, and so all new Seifert pieces are essential. The remaining properties of a proto-dynamic pair are easily checked for $B_{1}^{\prime s}, B_{1}^{u}$.

We need a measure of complexity which is reduced by passage from the pair $B_{1}^{s}, B_{1}^{u}$ to the pair $B_{1}^{\prime s}, B_{1}^{u}$. This complexity is defined to be the number of orbits which are either sources, sinks, or components of the dynamic train track, plus the number of circle components of $\partial B_{1}^{s}$ and $\partial B_{1}^{u}$. Note that the sector $F \subset B_{1}^{s}$ contains at least one such circle: for each component $c$ of $\partial F$, either $c$ is a component of $\partial B_{1}^{s}$, or there is a type (b)
corner gluing $\gamma \subset F$ such that $\partial A_{\gamma}=c \cup \gamma$. In addition, $F$ may contain additional circle components of $\tau$ which are type (a) corner gluings. Removing $\operatorname{int}(F)$ removes all these circles, and does not add new ones; complexity is therefore reduced.

Now iterate this operation: if $B^{\prime s}, B_{1}^{u}$ has any face or corner gluings, there is an annulus or Möbius band sector of $B_{1}^{\prime s}$ or $B_{1}^{u}$ which may be removed, further reducing the complexity. Since the complexity is a non-negative integer, the operation can be repeated only finitely many times, arriving at a proto-dynamic pair $B_{2}^{s}, B_{2}^{u}$ which has no corner or face gluings.

Step 3: Cutting open along tori Let $Q_{2}=\mathfrak{C}\left(P^{\prime}-\left(B_{2}^{s} \cup B_{2}^{u}\right)\right)$ and let $Q_{2}^{S}$ be the union of Seifert pieces of $Q_{2}$. Some components of $Q_{2}^{S}$ are solid tori, and some are peripheral torus shells, i.e. torus shells one of whose boundary components is a $\mathbf{b}$-torus of $P^{\prime}$.

For each component $C$ of $Q_{2}^{S}$ which is neither a solid torus nor a peripheral torus shell, cut $P^{\prime}$ open along tori contained in $C$ as follows. If $C \approx T^{2} \times[0,1]$ is a torus shell, let $T_{C} \approx T^{2} \times \frac{1}{2}$. If $C$ is not a torus shell, let $T_{C}$ be obtained by isotoping $\partial C$ slightly into the interior of $C$. If $C$ is a solid torus or peripheral torus shell component of $Q_{2}^{S}$ let $T_{C}=\emptyset$. Let $\mathcal{T}=\bigcup_{C} T_{C}$. Let $P^{\prime \prime}=\mathfrak{C}\left(P^{\prime}-\mathcal{T}\right)$, and note that $P^{\prime \prime}=\mathfrak{C}(P-(\mathcal{A} \cup \mathcal{T}))$. Each scar of $\mathcal{T}$ in $P^{\prime \prime}$ should be labelled with the symbol $\mathbf{b}$, and so $P^{\prime \prime}$ is a sutured manifold.

Associated to each component $C$ of $Q_{2}^{S}$ which is neither a solid torus nor a torus shell, there is a component $P_{C}^{\prime \prime}$ of $P^{\prime \prime}$ such that:

- $P_{C}^{\prime \prime}$ is a deformation retract of $C$.
- $P_{C}^{\prime \prime}$ is a Seifert fiber space with boundary components labelled $\mathbf{b}$.
- $P_{C}^{\prime \prime}$ is disjoint from $B_{2}^{s} \cup B_{2}^{u}$.

Let $P_{S}^{\prime \prime}=\bigcup_{C} P_{C}^{\prime \prime}$, the Seifert part of $P^{\prime \prime}$; each component of $\partial P_{S}^{\prime \prime}$ is a face labelled $\mathbf{b}$. Let $P_{D}^{\prime \prime}=P^{\prime \prime}-P_{S}^{\prime \prime}$, the dynamic part of $P^{\prime \prime}$.

Notice that:

- $B_{2}^{s}, B_{2}^{u}$ is a proto-dynamic pair in $P_{D}^{\prime \prime}$.
- Each labelled Seifert piece of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{2}^{s} \cup B_{2}^{u}\right)\right)$ is a solid torus or a peripheral torus shell.
- The pair $B_{2}^{s}, B_{2}^{u}$ has no face or corner gluings.

As a consequence of the latter, if $\gamma$ is a us-corner circle of some component $C$ of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\right.$ $\left(B_{2}^{s} \cup B_{2}^{u}\right)$ ), then $\gamma$ does not double cover a twisted orbit of $\tau=B_{2}^{s} \cap B_{2}^{u}$, and so each of the other three quadrants of $\gamma$ besides the $C$ quadrant have nonempty intersection with $B_{2}^{s} \cup B_{2}^{u}$. More precisely, the quadrant of $\gamma$ adjacent to $C$ through $B_{2}^{s}$ has nonempty intersection with $B_{2}^{s}$, the quadrant adjacent to $C$ through $B_{2}^{u}$ has nonempty intersection with $B_{2}^{u}$, and the remaining quadrant has nonempty intersection with both $B_{2}^{s}$ and $B_{2}^{u}$.

Step 4: Inserting new sectors Now we describe how to insert new sectors and collapse to form boundary parallel maw pieces, converting $B_{2}^{s}, B_{2}^{u}$ into a pair $B_{3}^{s}, B_{3}^{u}$ which is a true dynamic pair in $P_{D}^{\prime \prime}$.

Until further notice, fix a solid torus or torus shell component $H$ of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{2}^{s} \cup B_{2}^{u}\right)\right)$. Let $d$ be a cross-section of $H$ so that $\partial d$ intersects the edges of $H$ efficiently, with side labellings on $d$ inherited from face labellings of $H$. Thus, $H$ is the mapping torus of $d$ under some rotational monodromy acting on $d$. Let $c$ be the component of $\partial d$ which is not $\mathbf{a} \mathbf{b}$-circle. Each edge of $c$ is a segment labelled $\mathbf{b}, \mathbf{m}, \mathbf{p}, \mathbf{u}$, or $\mathbf{s}$. Each $\mathbf{b}$-edge is incident to a $\mathbf{p}$ and an $\mathbf{m}$-edge. Each $\mathbf{u}$-edge is incident to a $\mathbf{p}$ and an s-edge. Each $\mathbf{s}$-edge is incident to an $\mathbf{m}$ and a $\mathbf{u}$-edge. Let $n_{\mathbf{m}}$ be the number of $\mathbf{m}$-edges in $c, n_{\mathbf{u}}$ the number of $\mathbf{u}$-edges, etc. It follows that

$$
n_{\mathbf{m}}+n_{\mathbf{u}}=n_{\mathbf{p}}+n_{\mathbf{s}}
$$

Let $n$ be this number. Choose $k$ so that the angle of rotation of the monodromy is $2 \pi k / n$. We now break into cases, depending on the values of $n$ and $n_{\mathbf{m}}$, and on whether $H$ is a solid torus.

In all cases which actually occur, we will insert annulus or Möbius band $\mathbf{u}$ and $\mathbf{s}$-faces, which intersect transversely along circles. The inserted sectors and their intersection circles will be composed of Seifert fibers of $H$. As in any dynamic pair, orientations must be assigned to the intersection circles; the assigned orientation will always agree with the orientation of Seifert fibers.

Case 0: $n=0$. In this case every face of $H$ is labelled $\mathbf{b}$, which is impossible.

Case 1: $n=1$. There are two subcases, depending on whether $n_{\mathbf{m}}=0$ or 1 .
Case 1a: $n_{\mathbf{m}}=0$. In this case the labelling of $c$ must be su. If $H$ is a solid torus we reach a contradiction, because $H$ is not essential. If $H$ is a torus shell, then $H$ is a dynamic torus shell of type 1 , and no sectors need be added.

Case 1b: $n_{\mathbf{m}}=1$. The labelling of $c$ must be mbpb, in cyclic order. This labelling has no cyclic symmetries, and so $k=0$.

If $H$ is a solid torus, we reach a contradiction as follows. No sectors could have been removed from $H$, because, for example, an annulus or Möbius band $\mathbf{u}$-sector would have its boundary attached to the $\mathbf{p}$-face, cutting off a Seifert piece of positive index; but at no time during steps $1-3$ do we encounter positive index. Therefore, $H$ is a Seifert component of $\mathfrak{C}\left(P^{\prime}-\left(B_{1}^{s} \cup B_{1}^{u}\right)\right)$. But each Seifert component of $\mathfrak{C}\left(P^{\prime}-\left(B_{1}^{s} \cup B_{1}^{u}\right)\right)$ with a b-annulus face is a half annulus drum, and $H$ is not a half annulus drum.

If $H$ is a torus shell, insert an $\mathbf{s}$ and a $\mathbf{u}$-annulus sector as follows (see figure 4.22). Let $F_{\mathbf{m}}, F_{\mathbf{p}}$ be the $\mathbf{m}$ and $\mathbf{p}$-faces, and let $F_{\mathbf{b} 1}, F_{\mathbf{b} 2}$ be the two $\mathbf{b}$-annulus faces. Take the


Figure 4.22: An mbpb-torus shell. Inserting sand $\mathbf{u}$-annulus sectors and then collapsing split maw pieces produces a dynamic torus shell, two boundary parallel maw pieces, and two half annulus drums.
annulus $F_{\mathbf{b} 1} \cup F_{\mathbf{p}} \cup F_{\mathbf{b} 2}$ and perturb it to a properly embedded annulus in $H$ with boundary in $\operatorname{int}\left(F_{\mathbf{m}}\right)$, producing an annulus s-sector $\sigma_{\mathbf{s}}$. Construct $\sigma_{\mathbf{u}}$ similarly by perturbing $F_{\mathbf{b} 1} \cup$ $F_{\mathbf{m}} \cup F_{\mathbf{b} 2}$. The perturbations should be chosen so that $\sigma_{\mathbf{s}}$ intersects $\sigma_{\mathbf{u}}$ transversely along two circles, one near each $\mathbf{b}$-annulus face. As remarked above, orientations are assigned to these circles which agree with the Seifert fiber orientation (we will not repeat this remark again).

The components of $\mathfrak{C}\left(H-\left(\sigma_{\mathbf{s}} \cup \sigma_{\mathbf{u}}\right)\right)$ are: a dynamic torus shell of type 1 , two half annulus drums, a split uss-maw piece, and a split suu-maw piece. Now collapse the two split maw pieces to form true, boundary parallel maw pieces.

Case 2: $n=2$. There are three subcases, depending on whether $n_{\mathbf{m}}=0,1$, or 2 .
Case 2a: $n_{\mathbf{m}}=0$. In this case the labelling is susu, and we can have $k=0$ or 1 . It follows that $H$ is a dynamic solid torus of type $(2,0)$ if $k=0$, or type $(2,1)$ if $k=1$, or a dynamic torus shell of type 2 if $k=0$, or type 1 if $k=1$. No sectors need be inserted.

Case 2b: $n_{\mathbf{m}}=1$. In this case the labelling is bmsup. It follows that $k=0$. If $H$ is a solid torus, then $H$ is a half annulus drum, and we leave it alone: no sectors are inserted.

Suppose $H$ is a torus shell, i.e. a "half annulus drum with a hole". We insert an s and a $\mathbf{u}$-annulus sector, as follows (see figure 4.23). Given a label $\mathbf{x}$, let $F_{\mathbf{x}}$ be the annulus face of $H$ labelled $\mathbf{x}$. Let $c$ be the us-circle of $H$. Let $A_{\mathbf{u}} \subset B_{2}^{u}$ be a smoothly embedded annulus with one boundary circle on $c$ such that $A_{\mathbf{u}} \cap H=c$, and $A_{\mathbf{u}}$ is contained in an arbitrarily small neighborhood of $c$. Let $d_{\mathbf{u}}$ be the component of $\partial A_{\mathbf{u}}$ opposite $c$. Let $A_{\mathbf{s}} \subset B_{2}^{s}$ and $d_{\mathbf{s}} \subset \partial A_{\mathbf{s}}$ be similarly defined.


Figure 4.23: A "half annulus drum with a hole", labelled bmsup. Inserting annulus sectors $\sigma_{\mathbf{s}}$ and $\sigma_{\mathbf{u}}$, and then collapsing split maw pieces, produces a dynamic torus shell, two boundary parallel maw pieces, a half annulus drum, and two divided maw pieces.

To describe the added s-sector, consider the annulus $F_{\mathbf{b}} \cup B_{\mathbf{p}} \cup F_{\mathbf{u}} \cup A_{\mathbf{s}}$, and perturb this to obtain an annulus s-sector $\sigma_{\mathrm{s}}$ which has one boundary component in $\operatorname{int}\left(F_{\mathbf{m}}\right)$, the other boundary component meeting $B_{2}^{s}$ tangentially at $d_{\mathbf{s}}$, and intersecting $F_{\mathbf{u}}$ transversely along a circle $\gamma_{\mathbf{u}}$.

The added annulus $\mathbf{u}$-sector $\sigma_{\mathbf{u}}$ is similarly obtained by perturbing $F_{\mathbf{b}} \cup B_{\mathbf{m}} \cup F_{\mathbf{s}} \cup A_{\mathbf{u}}$, so that $\sigma_{\mathbf{u}}$ intersects $F_{\mathrm{s}}$ transversely along a circle $\gamma_{\mathrm{s}}$.

When the sectors $\sigma_{\mathbf{s}}, \sigma_{\mathbf{u}}$ are added, they cut $H$ into a dynamic torus shell of type 2, a split suu-maw piece, a split uss-maw piece, and a half annulus drum. Now collapse the two split maw pieces, to obtain true maw pieces which are boundary parallel.

Let $A_{\mathrm{s}}^{\prime}=\operatorname{cl}\left(\sigma_{\mathbf{s}}-H\right)$, the perturbed image of $A_{\mathbf{s}}$. Notice that $A_{\mathbf{s}}^{\prime} \cup A_{\mathbf{s}}$ form the $\mathbf{s}$-faces of a uss-maw piece $\mu$, whose $\mathbf{u}$-face is the subannulus of $F_{\mathbf{u}}$ bounded by $\gamma_{\mathbf{u}} \cup c$. We claim that $\mu$ is divided into one or more pinched tetrahedra. This follows from the fact that the quadrant of $c$ between $A_{\mathbf{s}}$ and $F_{\mathbf{u}}$ has nonempty intersectin with $B_{2}^{u}$. Similarly, $A_{\mathbf{u}}$ together with its perturbed image in $\sigma_{\mathbf{u}}$ are the $\mathbf{u}$-faces of a divided suu-maw piece whose s-face has boundary $\gamma_{\mathbf{s}} \cup c$.

Case 2c: $n_{\mathbf{m}}=2$. In this case the labelling is $\mathbf{m b p b m b p b}$. The rotational symmetry group of this labelling is order 2 , so we can have $k=0$ or 1 .

Suppose first that $H$ is a solid torus (figure 4.24). If $k=0$, insert a $\mathbf{u}$-annulus sector connecting the two $\mathbf{p}$-faces, and an s-annulus sector connecting the two $\mathbf{m}$-faces, intersecting transversely along an untwisted orbit. These sectors cut $H$ into four half annulus drums. If $k=1$, proceed similarly, except that the added sectors are Möbius bands intersecting along a twisted orbit; only two half annulus drums are created.

Suppose next that $H$ is a torus shell. If $k=0$, insert two $\mathbf{u}$-annuli sectors, one for each $\mathbf{m}$-face, and two $\mathbf{s}$-annulus sectors, one for each $\mathbf{p}$-face, as done in case 1 b . These sectors cut off a dynamic torus shell of type 2 containing the $\mathbf{b}$-torus, as well as four half annulus drums, two split uss-maw pieces, and two split suu-maw pieces. Now collapse to form two


Figure 4.24: Inserting annulus sectors, when the boundary labelling is mbpbmbpb. In a torus shell, after inserting sectors the split maw pieces should be collapsed to form boundary parallel maw pieces.
uss and two suu boundary parallel maw pieces. If $k=1$ proceed similarly, except that one inserts only one $\mathbf{u}$-annulus and one $\mathbf{s}$-annulus sector, cutting off a dynamic torus shell of type 1 containing the b-torus, two half annulus drums, one split uss-maw piece, and one split suu-maw piece. Collapse to form a uss and an suu boundary parallel maw piece.

Case 3: $n \geq$ 3. We insert one $\mathbf{u}$-annulus sector for each $\mathbf{m}$-face of $H$, and one $\mathbf{s}$-annulus sector for each $\mathbf{p}$-face. The rotation parameter $k$ can be anything from 0 to $n-1$, and all constructions must be done equivariantly with respect to the $2 \pi k / n$ rotational monodromy.

Consider an $\mathbf{m}$ face $F$ of $H$. Let $F_{1}, F_{2}$ be the faces of $H$ adjacent to $F$. Each of $F_{1}, F_{2}$ is labelled $\mathbf{b}$ or $\mathbf{s}$. The description of the added $\mathbf{u}$-sector depends on how many of $F_{1}, F_{2}$ are labelled $\mathbf{b}$ (see figure 4.25).

Case 3a: Two adjacent b-faces. This is like case 1b. The next two faces adjacent to $F_{1}$ and $F_{2}$ are $\mathbf{p}$-faces. Take the annulus $F_{1} \cup F \cup F_{2}$ and perturb it to obtain a properly embedded $\mathbf{u}$-annulus sector with boundary in the interior of the $\mathbf{p}$-faces (figure 4.25a).

Case 3b: One adjacent b-face. This is like case 2b. Say $F_{1}$ is a b-face and $F_{2}$ is an s-face. Let $c$ be the us-circle contained in $\partial F_{2}$. Let $A \subset B_{2}^{u}$ be a smoothly embedded annulus with one boundary circle on $c$ such that $A \cap H=c$, and $A$ is contained in an arbitrarily small neighborhood of $c$. Let $d$ be the component of $\partial A$ opposite $c$. Take the annulus $F_{1} \cup F \cup F_{2} \cup A$, and perturb it to obtain a u-annulus sector attached tangentially to $B_{2}^{u}$ along $d$ and with opposite boundary circle in the interior of a p-face of $H$ (figure 4.25 b ).


Figure 4.25: For each $\mathbf{m}$-face of $H$, insert a $\mathbf{u}$-annulus sector (thickened line), whose description depends on the number of $\mathbf{b}$-faces incident to the $\mathbf{m}$-face. Similarly, for each s-face, insert an s-annulus sector.

Case 3c: No adjacent b-faces. This is like figure 4.19. Let $c_{i}$ be the us-circle contained in $\partial F_{i}$. Let $A_{i} \subset B_{2}^{u}$ be a smoothly embedded annulus with $c_{i} \subset \partial A_{i}$ as in case 3 b , and let $d_{i}$ be the component of $\partial A_{i}$ opposite $c_{i}$. Take the annulus $A_{1} \cup F_{1} \cup F \cup F_{2} \cup A_{2}$, and perturb it to obtain a $\mathbf{u}$-annulus sector attached tangentially to $B_{2}^{u}$ along $d_{1} \cup d_{2}$ (figure 4.25c).

In a similar manner, for each $\mathbf{p}$-face of $H$, we attach an annulus $\mathbf{s}$-sector to $B_{2}^{s}$.
Intersections between added sectors are described as follows. If $A_{\mathbf{s}}, A_{\mathbf{u}}$ are $\mathbf{s}$ and $\mathbf{u}$ sectors that have been added, and if they each contain a perturbation of the same $\mathbf{b}$-face $A_{\mathbf{b}}$, then the perturbations should be chosen so that $A_{\mathbf{s}}$ and $A_{\mathbf{u}}$ intersect transversely along a circle. No other intersections between added sectors are allowed.

When these $\mathbf{u}$ and s-sectors are attached, they cut $H$ into pieces as follows. If $H$ is a solid torus, there is a dynamic solid torus of type $(n, k)$; if $H$ is a torus shell, there is a dynamic torus shell of type $n / \operatorname{gcf}(n, k)$. Each $\mathbf{b}$-face of $H$ is contained in a half annulus drum. Each $\mathbf{m}$-face is adjacent to a split uss-maw piece, and each $\mathbf{p}$-face is adjacent to a split suu-maw piece.

Now collapse the split maw pieces to obtain ordinary, boundary parallel maw pieces. A complete example, with boundary labelling mbpusmsupb, is shown in figure 4.26.

For each solid torus or torus shell component $H$ of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{2}^{s} \cup B_{2}^{u}\right)\right)$, we have described how to add sectors and collapse split maw pieces. Doing this for each $H$, we obtain a pair of branched surfaces $B_{3}^{s}, B_{3}^{u}$ in $P_{D}^{\prime \prime}$. We remark that any two split maw pieces are disjoint, because there are no face gluings among the Seifert components of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{2}^{s} \cup\right.\right.$ $\left.B_{2}^{u}\right)$ ). Therefore, the split maw pieces among all the different $H$ 's may all be collapsed simultaneously.


Figure 4.26: Inserting annulus sectors in $H$ with boundary labelling mbpusmsupb. After inserting sectors, all split maw pieces should be collapsed.

It is almost obvious that $B_{3}^{s}, B_{3}^{u}$ is a dynamic pair in $P_{D}^{\prime \prime}$-without coherent annulus drums. We check only that there are no corner and face gluings among dynamic torus pieces, and that all maw piece components of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$ are boundary parallel.

We consider maw pieces first. For each $\mathbf{u}$-annulus face that is inserted, a split uss-maw piece is created, which is then collapsed to form a boundary parallel maw piece; similarly, each inserted $\mathbf{s}$-annulus face gives rise to a boundary parallel maw piece. We claim that there are no other maw piece components of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$-the one possible exception to this claim is when a $\mathbf{u}$-annulus sector is attached tangentially to $B_{2}^{u}$, creating an suu-maw piece, but this maw piece is divided into pinched tetrahedron components of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$. Thus, all maw piece components of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$ are boundary parallel.

Now we consider face gluings among the dynamic torus pieces of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$. If face gluings exist then corner gluings exist; it is therefore sufficient to rule out corner gluings. Let $c$ be a us-circle of some dynamic torus piece $H$ of $B_{3}^{s}, B_{3}^{u}$. If $c$ is a us-circle of some component of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{2}^{s} \cup B_{2}^{u}\right)\right)$, then $c$ is not involved in any corner gluings. If $c$ is not a us-circle of a component of $\mathfrak{C}\left(P^{\prime \prime}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$, then $c$ lies in the interior of at least one newly added sector, say an annulus $\mathbf{s}$-sector $A_{\mathbf{s}}$. If $c \subset A_{\mathbf{s}} \cap A_{\mathbf{u}}$ where $A_{\mathbf{u}}$ is also a newly added sector, then by construction the four quadrants of $c$ are: the dynamic torus piece $H$, two boundary parallel maw pieces, and a half annulus drum. It follows that $c$ is not involved in a corner gluing of dynamic torus pieces. If $c \subset A_{\mathbf{s}} \cap B_{2}^{u}$, then three of the quadrants of $c$ are: the dynamic torus piece $H$, an suu-boundary parallel maw piece, and a uss-maw piece divided into pinched tetrahedra by sectors of $B_{2}^{u}$. The fourth quadrant of $c$ lies between the two maw pieces quadrants, and has nonempty intersection with $B_{2}^{u}$, and
therefore $c$ is not involved in any corner gluings.
This shows that $B_{3}^{s}, B_{3}^{u}$ is a dynamic pair in $P_{D}^{\prime \prime}$. Now we take $P^{\prime \prime}=P_{D}^{\prime \prime} \cup P_{S}^{\prime \prime}$ and reglue along $\mathcal{A}$, to obtain $\mathfrak{C}(P-\mathcal{T})=P_{D} \cup P_{S}$, where $P_{S}=P_{S}^{\prime \prime}$ and $P_{D}$ is obtained from $P_{D}^{\prime \prime}$ by gluing along $\mathcal{A}$. Note that each component of $P_{S}$ is an essential Seifert fiber space, and $B_{3}^{s}, B_{3}^{u} \subset P_{D}$.

Now we show that $B_{3}^{s}, B_{3}^{u}$ is a dynamic pair in $P_{D}$. The only nontrivial thing to check is that when drum components of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$ are glued along $\mathcal{A}$, none of the resulting components of $\mathfrak{C}\left(P_{D}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$ are coherent annulus drums. The only way that an annulus drum can be formed is when two half annulus drum components $C_{1}, C_{2}$ of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{3}^{s} \cup B_{3}^{u}\right)\right)$ are glued along $\mathbf{b}$-annulus faces $A_{i} \subset C_{i}$, where $A_{1}, A_{2}$ are the scars of some component of $\mathcal{A}$. By construction of $B_{3}^{s}, B_{3}^{u}$, we know that $A_{i}$ is a $\mathbf{b}$-annulus face of a solid torus or torus shell component $C_{i}^{\prime}$ of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{2}^{s} \cup B_{2}^{u}\right)\right)$. By construction of $B_{2}^{s}, B_{2}^{u}$, we know that $A_{i}$ is a $\mathbf{b}$-annulus face of a half annulus drum component $C_{i}^{\prime \prime}$ of $\mathfrak{C}\left(P_{D}^{\prime \prime}-\left(B_{1}^{s} \cup B_{1}^{u}\right)\right)$. Now the Seifert fiberings in $C_{i}^{\prime \prime}, C_{i}^{\prime}, C_{i}$ all agree along $A_{i}$, and the orientation of the us-circles of $C_{i}^{\prime \prime}, C_{i}$ agree with the direction of the Seifert fiberings. By step 1 , we know that $C_{1}^{\prime \prime}$ and $C_{2}^{\prime \prime}$ are glued by identifying $A_{1}$ and $A_{2}$ to form an annulus drum component of $\mathfrak{C}\left(P-\left(B_{1}^{s} \cup B_{2}^{u}\right)\right)$, and this annulus drum is incoherent, so the directions of the Seifert fiberings of $A_{1}$ and $A_{2}$ do not agree under gluing. Therefore, the annulus drum obtained from $C_{1}$ and $C_{2}$ by identifying $A_{1}$ and $A_{2}$ is incoherent.

To finish the proof of proposition 4.12.1, we check that if the original pair $B^{s}, B^{u}$ is vertical with respect to an $\mathcal{R}$-branched surface $\beta$ in $P$, then after splitting of $\beta$ the pair $B_{3}^{s}, B_{3}^{u}$ is also vertical. It is straightforward to check, in steps $1-3$, that the alterations performed on the dynamic vector field $V$ preserve verticality. In step 4 , when sectors are inserted and split maw pieces are collapsed, the alterations on $V$ and $\beta$ are very similar to the alterations done in the proof of proposition 3.5 .5 when inserting annuli of $B^{s}$. The dynamic vector field for $B^{s}, B^{u}$ is therefore vertical, and peripheral annuli are easily ruled out.

## Part II

## Constructing dynamic pairs (not included)

## Chapter 5

## The basis step: Handel-Miller theory

## Chapter 6

## The gluing step

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