

Preliminary Version

Automorphisms of Surfaces
after Nielsen and Thurston (vol. 2)

A. J. Casson
Notes by Steven A. Bleiler

University of Texas at Austin
Spring 1983

8/30 Casson

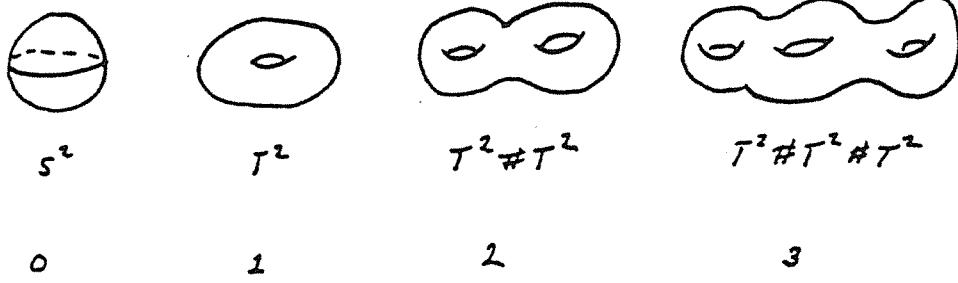
Automorphisms of Surfaces

We will be studying the work of Nielsen's Thurston on the automorphism of surfaces

Recall: The classification of closed surfaces

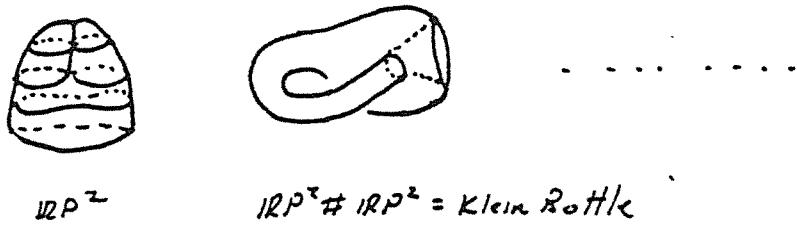
A closed surface is homeomorphic to one of the following

Orientable



genus

Inorientable



Our emphasis will be on orientable surfaces, usually denoted by F (from the German "fläche"). The genus of F , i.e. the number of torus summands, will be denoted by g . An automorphism of a surface F we will take to be a homeomorphism $h: F \rightarrow F$.

At first it would seem that there is a difference in considering automorphisms of surfaces up to isotopy or up to homotopy, but there is the

Thm Homotopic automorphisms of a closed orientable surface are isotopic.

The isotopy classes of automorphisms of a given surface form a group under composition, the classes of orientation preserving automorphisms form a subgroup of index 2 called the mapping class group of F .

8/30 CASSON

2

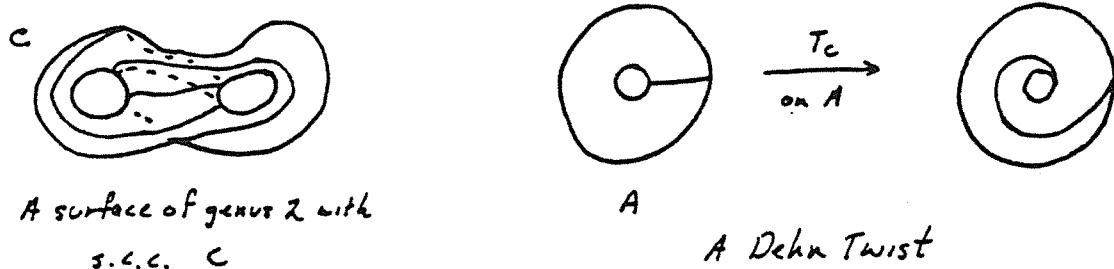
This group has been historically denoted by Mg where $g = \text{genus } F$. We will occasionally denote this group by $\text{Aut}(F)$ or $\text{Aut}_+(F)$; the entire group of surface automorphisms we will denote by $\text{Aut}_\pm(F)$.

Summary of Results

Thm: The group $\text{Aut}(F)$ is finitely generated

Recall: Dehn twist

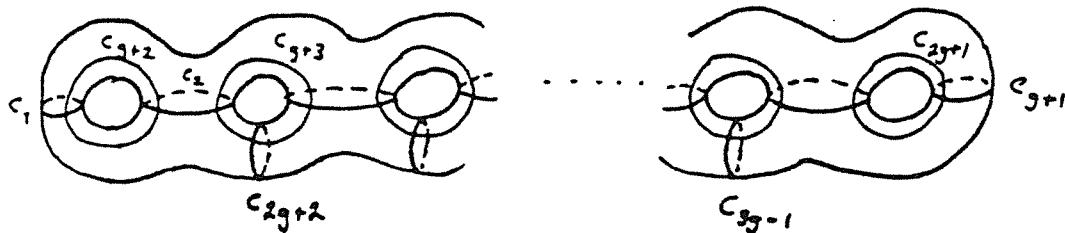
Suppose that C is a simple closed curve in an orientable F , then C has a neighborhood A homeomorphic to an annulus. $A \subseteq \{[r, \theta] \mid 1 \leq r \leq 2\}$



Define the Dehn twist in C to be the automorphism $T_C : F \rightarrow F$ given by the identity off A and

$$[r, \theta] \mapsto [r, \theta + 2\pi r] \quad \text{on } A.$$

Thm: (Dehn, Lickorish) $\text{Aut } F$ is generated by all Dehn twists. Moreover, the Dehn twists $T_{C_1}, \dots, T_{C_{3g-1}}$ in the curves illustrated below generate $\text{Aut } F$



Thm: (Humphries) $T_{C_1}, \dots, T_{C_{2g+2}}$ suffice

8/30 - 9/1, Casson

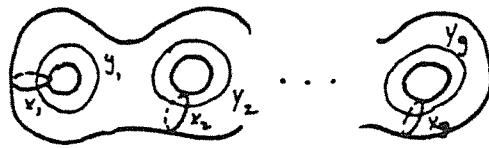
Thm: (McCool J. Alg 1975; geometric proof Hatcher & Thurston Topology 1980)
This group is actually finitely presented.

Remark: Composing Dehn twists would not seem to give much insight into automorphisms,
but this is the approach of Nielsen & Thurston

Recall: For an orientable surface F of genus g , the fundamental group of F
is given by

$$\pi_1(F) = \langle x_1, \dots, x_g, y_1, \dots, y_g \mid [x_1, y_1] \cdot [x_2, y_2] \cdots [x_g, y_g] = 1 \rangle$$

where the curves x_i, y_i are given by (ignoring base points)



Thm (Nielsen) If genus(F) ≥ 1 , then

$$\text{Aut}_\pm(F) \cong \frac{\text{Aut}(\pi_1(F))}{\text{Inner Automorphisms}} = \text{Outer automorphisms}(\pi_1(F))$$

An aside: The automorphisms of the torus T^2

Identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$, then $\text{Aut}_\pm T^2 \cong \text{Out}(\mathbb{Z} \oplus \mathbb{Z}) = \text{Aut}(\mathbb{Z} \oplus \mathbb{Z})$

$$\text{But } \text{Aut}(\mathbb{Z} \oplus \mathbb{Z}) \cong \text{GL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \mid p, q, r, s \in \mathbb{Z} \text{ and } ps - qr = \pm 1 \right\}$$

Let $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be in $\text{GL}_2(\mathbb{Z})$, then A can be considered as a linear map

$\mathbb{R}^2 \rightarrow \mathbb{R}^2: x \mapsto Ax$. Since $p, q, r, s \in \mathbb{Z}$, A maps \mathbb{Z}^2 to itself and hence induces a continuous map $h_A: T^2 \rightarrow T^2$. Notice that the map h_A preserves orientation $\Leftrightarrow \det A = \pm 1$. Further, as $A \in \text{GL}_2(\mathbb{Z})$, h_A is a homeomorphism with inverse $h_{A^{-1}}$; and $h_A^*: \pi_1(T^2) \rightarrow \pi_1(T^2)$ has matrix A .

9/1 Casson

4.

We wish to understand the behavior of h_A . Assume $\det A = 1$

Consider the characteristic polynomial of A , i.e. $t^2 - (p+s)t + \det A = 0$
 As $\det A = 1$ and $p+s = \text{trace}(A)$ we can write this polynomial as $t^2 - \text{tr}(A)t + 1$
 There are 3 cases as the eigenvalues of A ; λ, λ^{-1} ; are

- 1) complex i.e. $\text{tr}(A) = 0, \pm 1, \pm 2$ call h_A periodic
- 2) ± 1 i.e. $\text{tr}(A) = \pm 2$ call h_A reducible
- 3) distinct reals i.e. $|\text{tr}(A)| > 2$ call h_A Anosov

We'll analyse each of these in turn

Case 1) λ, λ^{-1} complex

Apply the Cayley-Hamilton Theorem to each possibility for $\text{tr} A$, and you'll see
 that $h_A^{12} = 1$

$$a) \text{tr } A = 0 \Rightarrow A^2 + I = 0 \Rightarrow A^2 = -I \Rightarrow A^4 = I$$

$$b) \text{tr } A = 1 \Rightarrow A^2 - A + I = 0, \text{ multiplication by } A + I \Rightarrow A^3 + I = 0 \Rightarrow A^6 = I$$

$$c) \text{tr } A = -1 \Rightarrow A^2 + A + I = 0, \quad " \quad A - I \Rightarrow A^3 - I = 0 \Rightarrow A^3 = I$$

Case 2) $\lambda = \lambda^{-1} = \pm 1$

Claim: h_A leaves an essential simple closed curve invariant (altho it maybe
 flips the string orientation of the curve)

Pf: A has rational eigenvalues $\Rightarrow A$ has a rational eigenvector

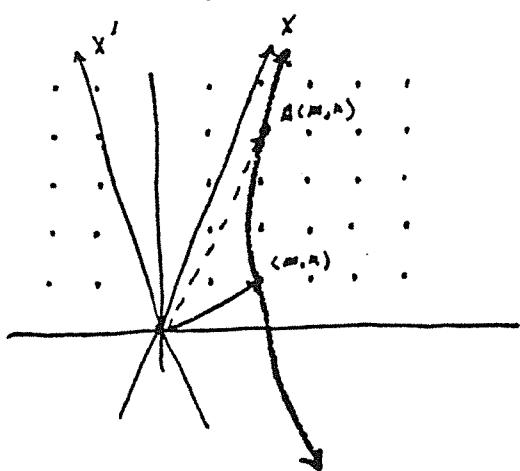
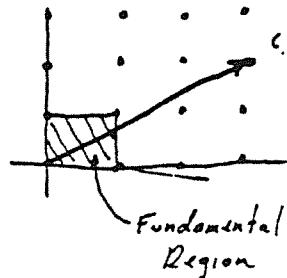
$\Rightarrow A$ has an integral eigenvector $x = (p, q)$

Look in lattice and note that this vector projects to
 a (p, q) curve on T^2 .

Remark: If h_A is orientation preserving, h_A is a Dehn twist, if
 orientation reversing h_A is $-I \cdot$ (Dehn Twist)

Case 3) λ, λ^{-1} distinct real, say $|\lambda| > 1 > |\lambda^{-1}|$ and real eigenvectors x, x'

In this case h_A has infinite order; moreover,
 no essential s.c.c. is left invariant. This is
 because if one considers the eigenspaces as the
 "coordinate axes", applying powers of A multiplies
 the x -coordinate by λ , and divides the x' coord-
 inate by λ . Hence the track of a point (m, n)
 lies on some hyperbola with the eigenspaces
 as asymptotes.



9/1. Casson

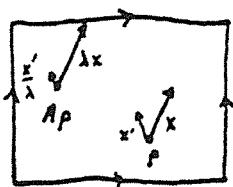
Notice that for large k , choosing a geodesic representative of the isotopy class of $h_A^k(s.c.c.)$ yields a curve nearly "parallel" to x ; while $h_A^{-k}(s.c.c)$ becomes nearly "parallel" to x' .

So we have shown that if $h_A^m(s.c.c) = h_A^n(s.c.c)$, then $m=n$.

Remark: All s.c.c.'s under discussion are essential; i.e. they do not bound discs in F .

Notice also that translating x, x' yields vector fields $\mathcal{F}, \mathcal{F}'$ and h_A carries \mathcal{F} to $\lambda \mathcal{F}$ and \mathcal{F}' to $\lambda^{-1} \mathcal{F}'$.

Pix:



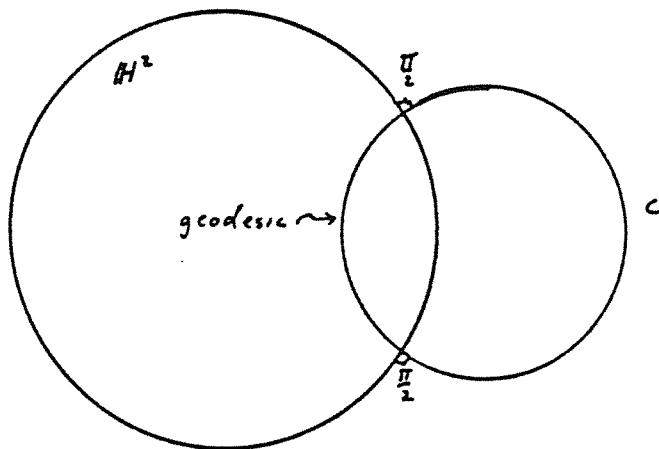
Fundamental Region

Thm (Nielsen, Thurston) Let $h: F \rightarrow F$ be an automorphism of a closed orientable surface. Then one of the following occurs

- 1) h is periodic; i.e. h^n is isotopic to 1 for some n . (N.B. this implies (Nielsen) h is isotopic to g with $g^n=1$)
- 2) h is reducible; i.e. $h \cong g$ such that g leaves some disjoint union of s.c.c's invariant, altho the string orientation may reverse and the components permuted.
- 3) h is pseudo-Anosov; i.e. For any s.c.c. $C \in F$, $h^m(C) = h^n(C)$
 $\Rightarrow m=n$

§1 The Hyperbolic Plane \mathbb{H}^2

The Poincaré model: Let D be the unit disc in the Euclidean \mathbb{R}^2 . Define \mathbb{H}^2 as $\text{Int } D^2$; ∂D is said to be the circle at ∞ and is not in \mathbb{H}^2 .



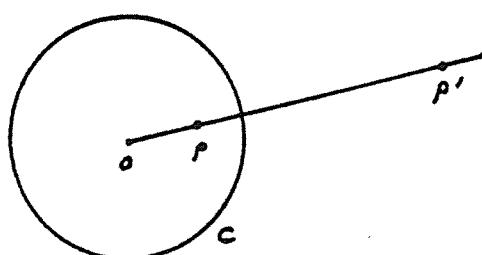
A geodesic (or straight line) in \mathbb{H}^2 is $C \cap \mathbb{H}^2$ where C is an Euclidean circle in \mathbb{R}^2 meeting ∂D orthogonally (we also include straight lines thru O but consider these as circles centered at " ∞ "). The Intermediate Value Theorem asserts that between any two points of \mathbb{H}^2 there exists a unique geodesic.

The angle between geodesics is defined to be the Euclidean angle between the circles defining them.

A small digression: Inversions in Euclidean geometry

Let C be a circle in $\mathbb{R}^2 \cup \{\infty\}$ with center O and radius r

Define inversion in C as the function which carries a point P to the unique point P' on the ray OP such that $\overline{OP} \cdot \overline{OP'} = r^2$ and takes O to ∞ and ∞ to O .



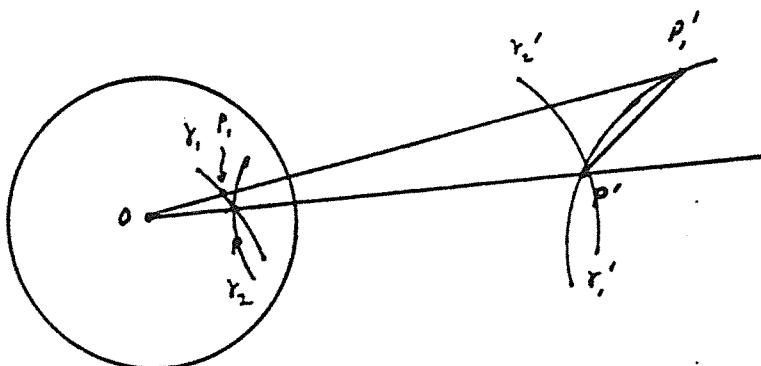
Notice that inversion in C defines an involution on $\mathbb{R}^2 \cup \{\infty\}$. An inversion centered at ∞ is ordinary reflection in a line.

9/3 Casson

Lemma 1.2 Inversions preserve angles (but reverse orientation)

Lemma 1.3 Inversions carry circles to circles (regarding Euclidian lines as circles thru ∞)

Here is a proof of 1.2, the other proof is left as an exercise



Observe that it is enough to show that the angle between $\gamma, \gamma' \overline{OP}$ equals the angle between $\gamma', \gamma' \overline{OP}$. For this use similar triangles

$$\overline{OP} \cdot \overline{OP'} = \overline{OP_1} \cdot \overline{OP'_1} \Rightarrow \frac{\overline{OP}}{\overline{OP_1}} = \frac{\overline{OP'}}{\overline{OP_1}} \Rightarrow \triangle OPI_1 \text{ is similar to } \triangle OPI'_1$$

Now letting $P_1 \rightarrow P$ establishes the desired equality.

Define the reflection in the geodesic $C \cap \mathbb{H}^2$ to be Euclidian inversion in C

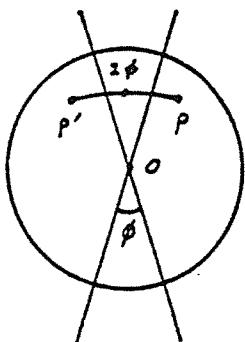
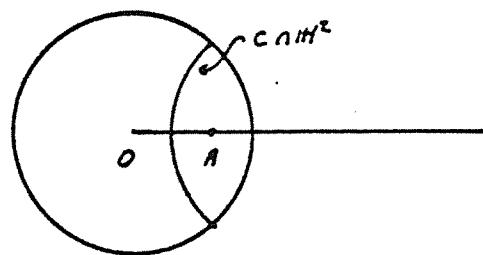
Observe that inversion in C maps $D \rightarrow D$ because C meets ∂D in right angles

Further define an isometry of \mathbb{H}^2 as a product of reflections in geodesics, and observe that isometries preserve angles between geodesics (up to sign)

Lemma 1.4 The group of isometries acts transitively on \mathbb{H}^2 , and the stabilizer of a point A in \mathbb{H}^2 is isomorphic to $O(2)$.

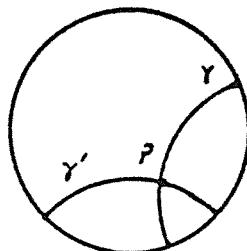
9/3 - 9/5 Coxson

Proof: Let O be the center of D , A any point of \mathbb{H}^2 . Then A can be carried to O by a single reflection in a geodesic $C \cap \mathbb{H}^2$, C having center on the ray OA . So any 2 points of \mathbb{H}^2 are "connected" by an isometry that is a product of at most two reflections.



In light of this it follows that the stabilizers of any two points in \mathbb{H}^2 are isomorphic via conjugation. Thus we need only determine $\text{Stab}(O)$. This group contains the reflections in all lines through O , and any rotation about O can be expressed as the product of two reflections, and these generate $O(2)$.

To see that $O(2)$ is the entire stabilizer of O , notice that isometries extend in a unique manner to the circle at ∞ . So it is enough to show that any isometry extending to γ on the circle at ∞ is actually the identity. But for any point P in \mathbb{H}^2 , the intersection of geodesics γ, γ' notice that the "ends" of γ, γ' are fixed, hence γ, γ' are fixed, and hence P .



11.

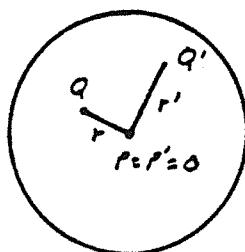
Lemma 1.5 Isometries leave $\frac{ds}{1-r^2}$ invariant where r = Euclidian distance

from the center of D and $ds = d$ (Euclidian distance)

Proof Let $P' = f(P)$, $Q' = f(Q)$

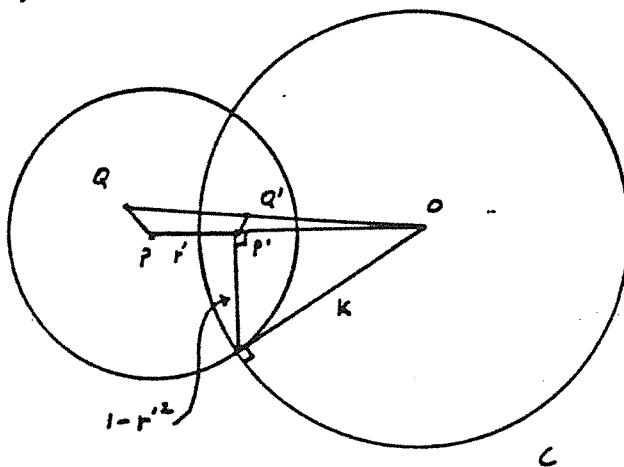
$$\text{We must show } \frac{P'Q'}{1-r'^2} \approx \frac{PQ}{1-r^2}$$

Notice this is true for $P = P' = \text{center of } D$



9/5 Casson

Thus it is enough to check when $P = \text{center}$, P' arbitrary, $f = \text{reflection}$
in $C \cap \mathbb{H}$, where C has center O and radius k , carrying P to P'



For Q close to P

$$\frac{P'Q'}{PQ} \approx \frac{OP'}{OP} = \frac{k^2}{(OP)^2} = \frac{1 - r'^2}{1 - r^2}$$

geometry
inversion rule

Hence $\frac{P'Q'}{1 - r'^2} \approx \frac{PQ}{1 - r^2}$

II.

Exercise: Show that geodesics w.r.t this metric really are geodesics

Remark: \exists an elegant proof.

Def The hyperbolic metric on \mathbb{H}^2 is given by $\frac{2ds}{1 - r^2}$

Convention: Greek letters denote hyperbolic distance, Roman Euclidian distance

Examples:

1. Hyperbolic distance $OP = \rho = \int_0^r \frac{2dx}{1-x^2} = 2 \tanh^{-1} r$

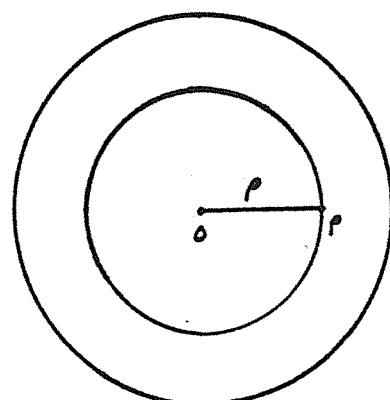
$$\Rightarrow r = \tanh \frac{\rho}{2}$$

2. The circle centered at O , hyperbolic radius ρ

has circumference

$$2\pi \cdot \frac{2r}{1 - r^2} = \frac{4\pi r}{1 - r^2} = 2\pi \cdot 2 \tanh \frac{\rho}{2} \cdot \cosh^2 \frac{\rho}{2}$$

$$= 2\pi \sinh \rho$$



3. The circle of the previous example has area

$$\begin{aligned} \int_0^r 2\pi \sinh z dz &= 2\pi (\cosh r - 1) \\ &= \int_0^r \frac{4\pi x}{(1-x^2)^2} \cdot 2 dx = 4\pi \left(\frac{1}{1-r^2} - 1 \right) = 2\pi \left(\frac{2r^2}{1-r^2} \right) \end{aligned}$$

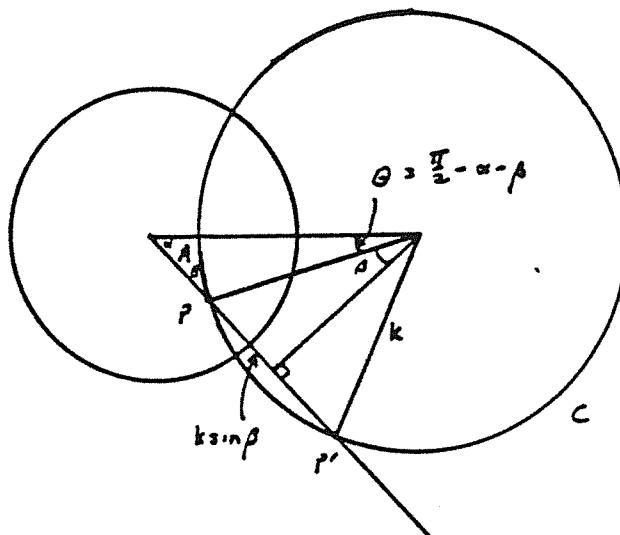
Theorem 1.6 (Gauss-Bonnet) A geodesic triangle in \mathbb{H}^2 with angles α, β, γ has area $\pi - (\alpha + \beta + \gamma)$

Proof wlog the α vertex is at O , and as we can subdivide our triangle into right triangles, $\gamma = \pi/2$.

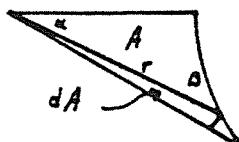
$$\text{Notice: } k \sin \beta d\theta = r d\alpha$$

$$\Rightarrow \frac{d\theta}{d\alpha} = \frac{r}{k \sin \beta}$$

$$= \frac{r}{\frac{1}{2}(1-r)} = \frac{2r^2}{1-r^2}$$



Further note:



$$\frac{dA}{d\alpha} = \frac{2r^2}{1-r^2}$$

$$\text{Thus } A = \Theta + \text{constant} = \frac{\pi}{2} - \alpha - \beta + \text{constant}$$

$$\text{Now choosing } \alpha = \beta \Rightarrow \frac{\pi}{2} = \beta \Rightarrow \text{constant} = 0$$

9/5 - 9/7 Coxson

Remark: Theorem 1.6 can be generalized

Cor 1: An n -gon with angles $\alpha_1, \dots, \alpha_n$ has area $(n-2)\pi - (\alpha_1 + \dots + \alpha_n)$

Cor 2: If the vertices of an n -gon are all on the circle at ∞ , the area is $(n-2)\pi$

9/7

Lemma 1.7 Let $h: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be an orientation preserving isometry, $h \neq I$. Then exactly one of the following occur

(1) h has a unique fixed point in \mathbb{H}^2 , no fixed points on the circle at ∞ . Call these elliptic

(2) h has a unique fixed point on the circle at ∞ , none in \mathbb{H}^2 . Call these parabolic

(3) h has exactly two fixed points on the circle at ∞ and none in \mathbb{H}^2 . Call these hyperbolic

Remark: This lemma holds for $h: \mathbb{H}^n \rightarrow \mathbb{H}^n$ and the proof given below holds.

Proof Recall Brower's Theorem: $h: D^2 \rightarrow D^2$ has a fixed point

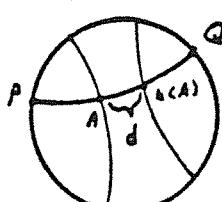
Case 1: If there is an interior fixed point, wlog this point is the center of D . Hence h is a rotation and we are in case (1) above

Case 2: If h fixes two (or more) points P and Q on the circle at ∞ , h leaves the unique geodesic PQ invariant.

$h|_{\text{geodesic}}$ is translation by hyperbolic distance d .

Call this invariant geodesic the axis of h .

This determines h on the circle at ∞ as perpendicular geodesics go to perpendicular geodesics, and hence h has exactly 2 fixed points on the circle at ∞ , hence (3) above



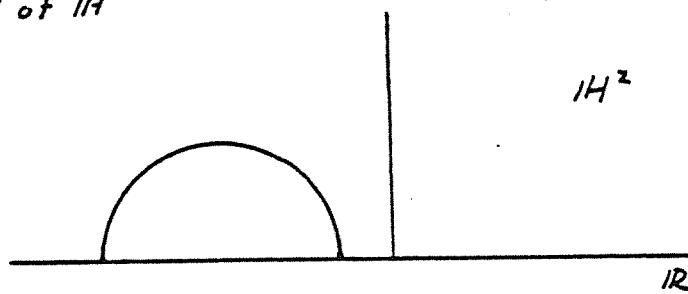
All other h 's fall into case (2) above.

The Half-plane Model of \mathbb{H}^2

Instead of the disc D we can apply a sequence of inversions and use the upper half plane in \mathbb{C} as our model of \mathbb{H}^2

In this model:

- circle at $\infty = \mathbb{R} \cup \{\infty\}$
- geodesics = circles meeting \mathbb{R} in $\frac{\pi}{2}$ angles; vertical lines
- invariant metric = ds/y where $y = \text{imaginary part}$



Theorem 1.8 The group of orientation preserving isometries of \mathbb{H}^2 is isomorphic to $PSL_2(\mathbb{R})$.

$$\text{Recall: } SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

$$PSL_2(\mathbb{R}) = \frac{SL_2(\mathbb{R})}{\pm I}$$

A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $PSL_2(\mathbb{R})$ acts on the upper half plane

$$\text{via } z \rightarrow \frac{az + b}{cz + d} \quad \text{Note: This preserves the real axis.}$$

Proof The generators of $PSL_2(\mathbb{R})$ have two forms

First: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, z \rightarrow z + b$ This is a translation parallel to the

real axis and can be realized by a product of reflections in vertical geodesics.

Second: $\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}, z \rightarrow -a^2/z$ This is a product of inversion in a

9/9 - 9/13 Casson

Proof (cont) circle centered at 0 of radius α with reflection in the Y-axis.

Hence $PSL_2(\mathbb{H}) \subseteq \text{Aut}(\mathbb{H}^2)$. Follow the steps backward and it's easy to check (Exercise) that any orientation preserving h can be represented by a matrix in $PSL_2(\mathbb{H})$.

11.

Remark: An exercise in linear algebra recovers Theorem 1.7

9/12

We wish to develop conditions for an isometry to be elliptic, hyperbolic, and parabolic

Given $z \xrightarrow{h} \frac{az+b}{cz+d}$. The equation for fixed points

$$z = \frac{az+b}{cz+d} \Rightarrow cz^2 + (d-a)z - b = 0$$

The discriminant of this equation is $(d-a)^2 + 4bc$. Recall $ad-bc=1$ and so $(d-a)^2 - 4(ad-1) = (d+a)^2 - 4$ is the discriminant.

As $d+a$ is the trace of the matrix representation we obtain

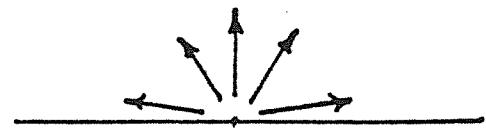
$$(\text{tr})^2 - 4 = \text{discriminant}$$

Case 1: $|\text{trace}| < 2 \Rightarrow$ conjugate complex roots \Rightarrow exactly one fixed point in $\mathbb{H}^2 \Rightarrow h$ elliptic and represented by a rotation in Poincaré Disc model

Case 2: $|\text{trace}| > 2 \Rightarrow$ 2 real roots \Rightarrow two fixed points on circle at $\infty \Rightarrow h$ hyperbolic. To represent h , use the upper half plane model with $0, \infty$ as fixed points. Axis is then the imaginary axis

$$z \rightarrow \frac{az+b}{cz+d} \text{ fixing } 0, \infty \Rightarrow b=c=0$$

$$\Rightarrow \text{Matrix is } \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, z \rightarrow a^2 z; \text{ a } \underline{\text{dilation}}$$

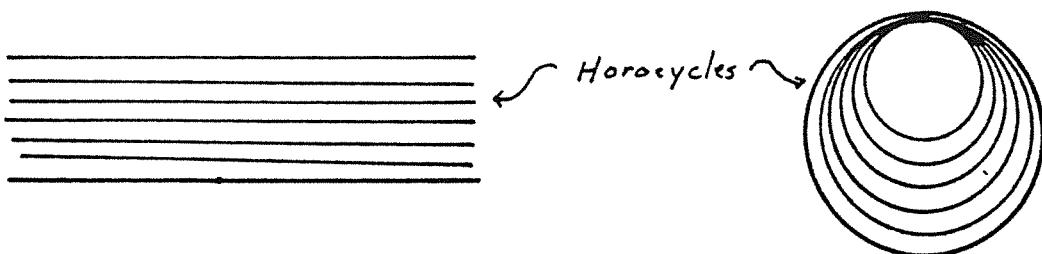


Case 3: $|\text{trace}| = 2 \Rightarrow 2 \text{ equal real roots} \Rightarrow \text{unique fixed point}$
 on the circle at $\infty \Rightarrow h \text{ parabolic. To represent } h \text{ use the half plane model with } \infty \text{ the fixed point.}$

$$z \rightarrow \frac{az+b}{cz+d} \text{ fixing } \infty \Rightarrow c=0 \Rightarrow \text{matrix is } \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

But $a+a^{-1}=2 \Rightarrow a=\pm 1 \Rightarrow \text{matrix for general parabolic fixing } \infty$
 is given by $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

But this is just $z \rightarrow z+b$, horizontal translation parallel to real axis.
 So parabolic fixing ∞ preserves horizontal levels in the half plane model. These are called horocycles. In the Poincaré disc model



horocycles are Euclidian circles tangent to the circle at ∞ .

Remark: The group of orientation preserving isometries of $\mathbb{H}^3 \cong PSL_2(\mathbb{C})$

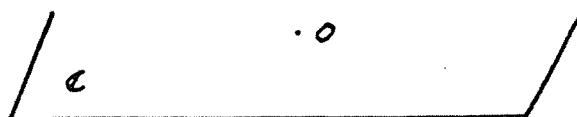
Use the upper half-space model of \mathbb{H}^3 . $PSL_2(\mathbb{C}) = \overline{\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}}$

$\pm I$

such act on the "floor" plane \mathbb{C} by

$$z \rightarrow \frac{az+b}{cz+d}$$

Notice the real axis is not preserved
 as a, b, c, d are complex



9/13 - 9/15 Cossor

15.

Now $PSL_2(\mathbb{C})$ is generated by $z \rightarrow z+b$, $z \rightarrow -\frac{a^2}{z}$; $a, b \in \mathbb{C}$

Notice $z \rightarrow z+b$ can be realized by 2 reflections in vertical hyperplanes in Euclidian space.

$z \rightarrow -\frac{a^2}{z}$ can be realized by inversion in a Euclidian sphere

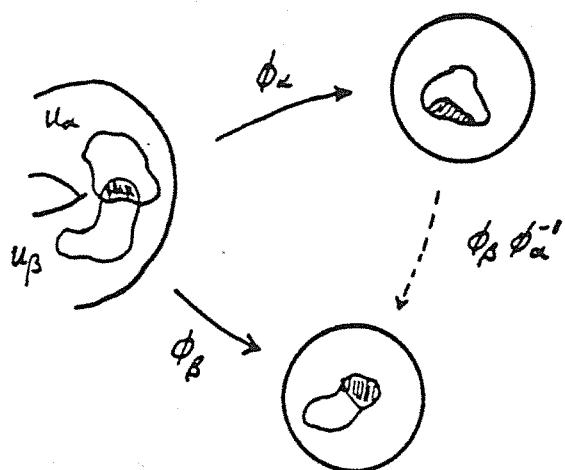
So these extend uniquely to isometries of \mathbb{H}^3 , hence any $A \in PSL_2(\mathbb{C})$ extends. As before the uniqueness of the extension comes from an intersecting geodesics argument.

§ 2 Hyperbolic Structures on Surfaces

Def A hyperbolic structure on F is determined by an atlas of charts

$\phi_\alpha: U_\alpha \rightarrow \mathbb{H}^2$ such that $\phi_\beta \phi_\alpha^{-1}(\phi_\alpha(U_\alpha \cap U_\beta)) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$

is the restriction of an orientation preserving isometry of \mathbb{H}^2 .

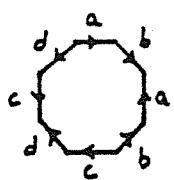


We usually require F & the atlas to be oriented

9/15 Cesson

Example: Every closed oriented F of genus > 1 has a hyperbolic structure

The surface F is made by identifying edges of a $4g$ -gon (so that all vertices are identified).

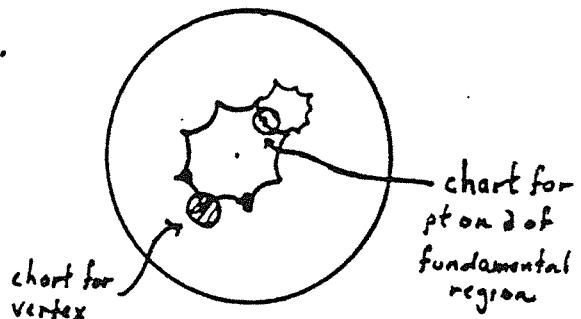


Consider a regular geodesic $4g$ -gon in the Poincaré disk model of \mathbb{H}^2

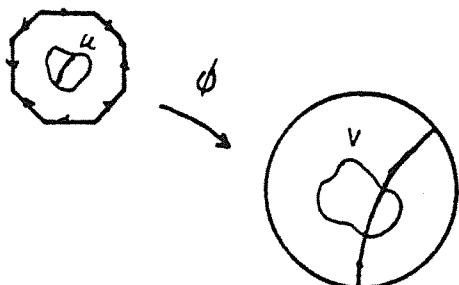
concentric with D^2 . The angle sum of a small $4g$ -gon $\sim (4g-2)\pi$ is greater than 2π if $g > 1$.

For a large $4g$ -gon the angle sum ~ 0 , so we can find a $4g$ -gon with angle sum 2π .

As the sides have the same length, we can glue sides by orientation preserving isometries. F has a hyperbolic atlas. We use the unique isometries joining edges to make charts for the edges. As the angle sum is 2π we get a chart for the vertices



Remark: In the construction the $4g$ -gon needn't be regular. In our case, moving the fundamental $4g$ -gon by the gluing isometries gives a tiling of \mathbb{H}^2 (Poincaré).



Geodesics in F are defined locally, i.e. we can pull back the geodesics in \mathbb{H}^2 via the chart maps.

9/15 Casson

Def A hyperbolic surface F is complete if F is complete as a metric space
(The metric induced by pulling back the metric in \mathbb{H}^2)

Lemma 2.1 (half of Hopf-Rinot) In a complete hyperbolic surface, all geodesics can be extended indefinitely.

Proof Suppose we have a bounded arc in the surface which is geodesic. Take a sequence of points tending toward an end. This sequence is Cauchy \Rightarrow \exists an endpoint. Take a chart with this endpoint the center of Poincaré disk. Now we can extend the geodesic and pull back this extension to the manifold via the chart map. Taking a maximal extension finishes the proof.

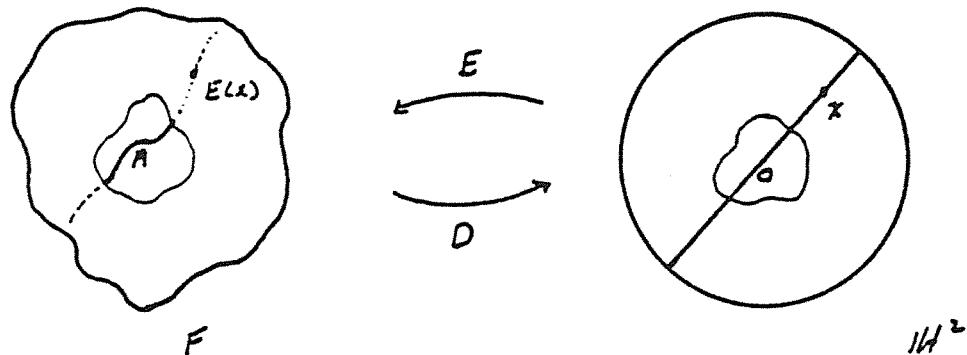


11.

Theorem 2.2 Any complete, connected, simply-connected hyperbolic surface is isometric to \mathbb{H}^2

Remark: This implies that the universal cover of a compact hyperbolic surface is \mathbb{H}^2

Proof: We construct maps D, E as below



E : Exponential

Chose A in F and a chart neighborhood $\phi : U \rightarrow \mathbb{H}^2$ such that $\phi(A) = o$

For $x \in \mathbb{H}^2$ extend the geodesic $\varphi^{-1}(ox)$, define

$E(x)$ = point on $\varphi^{-1}(ox)$ with $\text{dist}(A, E(x)) = \text{dist}(o, x)$

$D = \text{Developing}$

Claim: There exists a unique map $D : F \rightarrow \mathbb{H}^2$ such that

- 1) D is a local isometry
- 2) $D|_U = \phi$

D is constructed by "analytic continuation"

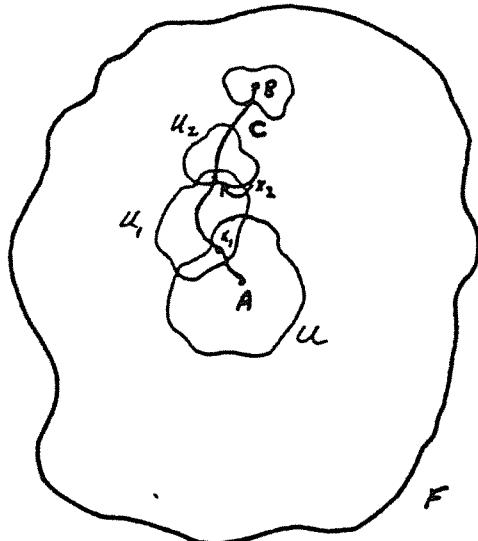
Choose a path C from A to B in F . Cover C by a sequence of "round" charts $U = U_0, U_1, \dots, U_n$ with $\varphi_i : U_i \rightarrow \mathbb{H}^2$. Choose points x_i in C such that $[x_i, x_{i+1}] \subseteq U_i$.

Suppose $\varphi_j = \varphi_{j-1}$ on the component of $U_j \cap U_{j-1}$ containing x_i for $j = 1, \dots, i$. We wish to perform an induction step, so suppose that $\varphi_{i+1} \neq \varphi_i$ on the component of $U_{i+1} \cap U_i$ containing x_{i+1} .

Then $\varphi_{i+1} = g_i \circ \varphi_i$ where g_i is an isometry of \mathbb{H}^2 uniquely determined by the data. Replace φ_{i+1} by $g_i^{-1} \circ \varphi_{i+1} : U_{i+1} \rightarrow \mathbb{H}^2$ and now the chart maps agree on the overlaps.

Set $D(B) = \varphi_n(B)$ and note:

- (1) By refining coordinate covers, it follows that $D(B)$ depends only on C , not on U_i, φ_i, x_i
- (2) If C_1, C_2 are homotopic paths, they define the same value of $D(B)$ as small homotopies don't leave a coordinate cover.
- (3) F simply-connected $\Rightarrow D(B)$ is well-defined.



Observe that $D \circ E = \text{id}_{\mathbb{H}^2}$, $E \circ D = \text{id}$ on a non-empty closed subset of F . But as $E \circ D$ is a retraction, $E^{-1}(E)$ is open by invariance of domain \Rightarrow this subset of F is also open. F connected $\Rightarrow E \circ D = \text{id}_F$

9/17 Casson

Apply this to the universal cover \tilde{F} of a closed hyperbolic surface F . \tilde{F} is isometric to \mathbb{H}^2 .

Further as $F = \tilde{F}/\{\text{deck translations}\} \cong \mathbb{H}^2/\Gamma$ where Γ is a subgroup of $PSL_2(\mathbb{C})$ isomorphic to $\pi_1 F$. As we can lift ϵ -neighborhoods of points to the universal cover, Γ is a discrete subgroup of $PSL_2(\mathbb{C})$. This fact also shows that $\Gamma - \{1\}$ contains no parabolic elements. Indeed, F compact $\Rightarrow \exists$ a uniform ϵ on $F \Rightarrow \exists \epsilon > 0$ such that for every $g \in \Gamma - \{1\}$ the hyperbolic distance between $x, g(x) > \epsilon$.

In the half-plane model, a parabolic element is represented by a horizontal translation thru a Euclidian distance b . But the metric in this model is given by $\frac{ds}{y}$, and so points "high up" on the "Y-axis" are moved an arbitrarily small hyperbolic distance.

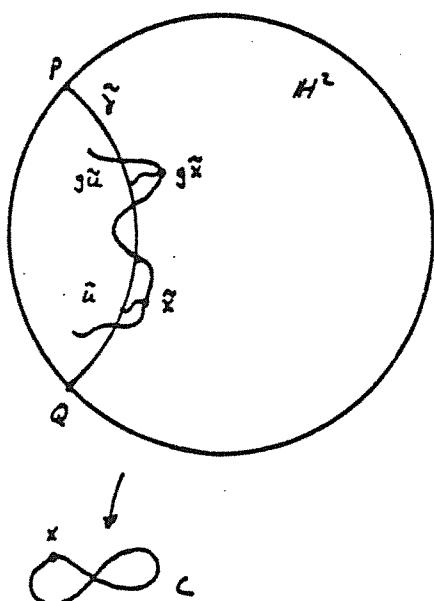
Notice that as the action of $\pi_1 F$ on \mathbb{H}^2 is free, $\Gamma - \{1\}$ has no elliptic elements. We conclude that $\Gamma - \{1\}$ has all hyperbolic elements.

Remark: This rules out $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup of $\pi_1 F \Rightarrow$ torus is not hyperbolic

Lemma 2.3 Every essential closed curve in a closed hyperbolic surface is freely homotopic to a unique closed geodesic.

Proof Let $C \subseteq F$ be an essential closed curve, \tilde{C} some "component" of the preimage in \mathbb{H}^2 . Choose an x on C and an \tilde{x} on \tilde{C} projecting to x . We can find a $g \in \Gamma$ such that the segment $\tilde{x}, g\tilde{x}$ goes once around C .

As g is hyperbolic, g has a geodesic axis $\tilde{\gamma}$. Consider $\gamma = \text{image } \tilde{\gamma} \text{ in } F$, a closed geodesic. Choose any point on $\tilde{\gamma}$ and any path \tilde{u} connecting it to \tilde{x} . The circuit from the point to \tilde{x} , along \tilde{C} to $g\tilde{x}$, along $g\tilde{u}$ to g (point), back to the point on γ bounds a singular "rectangle" in \mathbb{H}^2 , which projects down to a singular annulus in F . This annulus is a free homotopy from C to γ .



Proof (cont) For uniqueness: Let C be an essential closed curve free homotopic to a closed geodesic γ , with free homotopy $f: S^1 \times I \rightarrow F$. As before let \tilde{C} be a "component" of the pre-image of C . Let $\tilde{f}: \mathbb{H}^2 \times I \rightarrow \mathbb{H}^2$ cover f such that $\tilde{f}(\mathbb{H}^2 \times 0) = \tilde{C}$.

Let $\tilde{\gamma} = f(\mathbb{H}^2 \times 1)$ with endpoints P, Q (see pix previous page)

Claim: $\exists d > 0$ such that $\tilde{f}(\mathbb{H}^2 \times I) \subseteq d$ -neighborhood of $\tilde{\gamma}$ (hyperbolic distance).

Pf: $d = 1, u.b. \{f(p \times I)\}$

\Rightarrow In the Euclidian sense $d(\tilde{C}, \tilde{\gamma}) \rightarrow 0$ as we tend to P or Q . So \tilde{C} compactifies to a closed interval with endpoints P, Q .

In particular if C, γ are free homotopic geodesics, they coincide.

II.

Lemma 2.4 Every essential simple closed curve in a closed hyperbolic surface is isotopic to a unique simple closed geodesic.

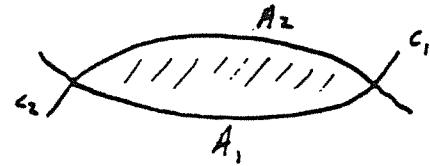
Proof Suppose $C \subseteq F$ is an essential simple closed curve, then the preimage of C in \mathbb{H}^2 is a disjoint union of open arcs. Note that if \tilde{C}, \tilde{C}' are disjoint, then endpoints aren't separated i.e. If P, Q are the endpoints of \tilde{C} ; P', Q' the endpoints of \tilde{C}' , then P, Q don't separate P' and Q' . Hence the geodesic PQ does not meet the geodesic $P'Q'$.

Obtain the geodesic γ as in Lemma 2.3 and note that by the above the preimage of γ in \mathbb{H}^2 is a collection of disjoint geodesics. Hence γ is simple.

It remains to show that C is isotopic to γ .

Exercise: Do there exist simple, non-closed geodesics in a closed hyperbolic surface?

Def Essential s.c.c.'s C_1, C_2 have minimal intersection, if they intersect transversely and if #arcs A_1, A_2 in C_1, C_2 respectively having common endpoints and such that $A_1 \cup A_2$ is the boundary of a disc in F .



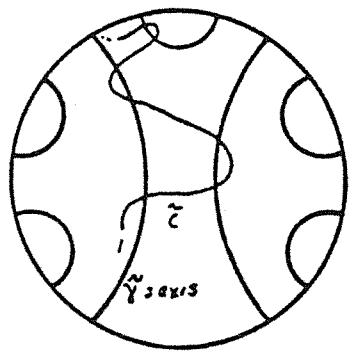
9/22 Casson

Observe that if C_1, C_2 have non-minimal intersection and are transverse, we can find $A_1, A_2 \ni \partial A_1 = \partial A_2$ and $\exists' A_1 \cup A_2$ bounds a disc D with $\text{Int}(D)$ disjoint from $C_1 \cup C_2$. Now by an innermost disc argument, we have that our curve C and geodesic γ have minimal intersection.

Now let $p: \mathbb{H}^2 \rightarrow F$ be the universal covering map. Choose a component \tilde{C} of $p^{-1}(C)$ and examine how it intersects a component of $p^{-1}(\gamma)$.

By an innermost arc argument, $C \cap \gamma$ minimal
 $\Rightarrow \tilde{C}$ meets no component of $p^{-1}(\gamma)$ more than once;
 and as before \tilde{C} has the same endpoints as a component $\tilde{\gamma}$ of $p^{-1}(\gamma)$.

$\therefore \tilde{C}$ can't meet any component of $p^{-1}(\gamma)$ other than $\tilde{\gamma}$.



The deck translation lifting C is an isometry leaving $\tilde{C} \setminus \tilde{\gamma}$ invariant, so if $\tilde{C} \cap \tilde{\gamma} \neq \emptyset$ it contains an orbit of a hyperbolic isometry and would thus be infinite. But $|\tilde{C} \cap \tilde{\gamma}| \leq 1 \Rightarrow \tilde{C} \cap \tilde{\gamma} = \emptyset \Rightarrow C \cap \gamma = \emptyset$

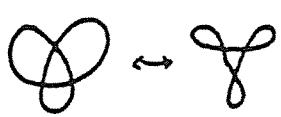
Next $C \cong \gamma \Rightarrow C, \gamma$ homologous; i.e. they cobound part of the surface, call it N . Moreover $C \cong \gamma \Rightarrow N$ has genus 0 $\Rightarrow N$ an annulus $\Rightarrow C, \gamma$ isotopic.

(N.B. This fails for non-orientable surfaces. For example, in the Möbius strip, $C = \text{boundary}, \gamma = \text{central } S^1 \Rightarrow C \cong \gamma^2$)

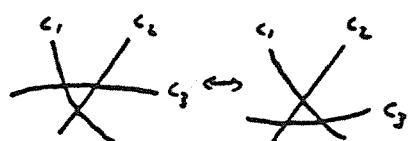
Lemma 2.5 If C_1, C_2 are transverse, essential, simple closed curves on a hyperbolic surface F then C_1, C_2 have minimal intersection iff \exists a homeomorphism $h: F \rightarrow F$ isotopic to τ , $\exists' h(C_1), h(C_2)$ are both closed geodesics.

N.B. Simplicity of C_1, C_2 is essential; as is the number of curves, due to problems with triple points.

$\nabla_3:$



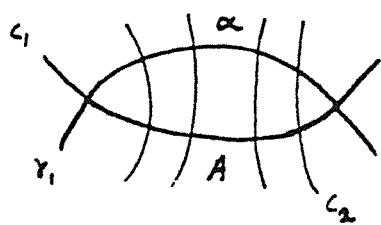
geodesics



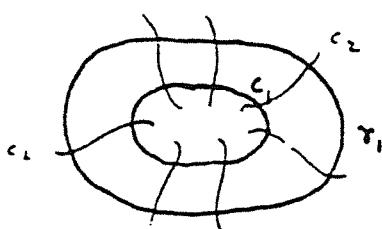
curves

Proof (\Leftarrow) IF both C_1, C_2 geodesic they have minimal intersection because no component of $p^{-1}(C_1)$ meets any component of $p^{-1}(C_2)$ more than once.

(\Rightarrow) Suppose C_1, C_2 have minimal intersection with C_2 geodesic. By isotopy adjust C_1 so that there are no triple points. Now by lemma 2.4 C_1 is isotopic to a simple closed geodesic γ_1 . We must choose this isotopy so that it leaves C_2 invariant.



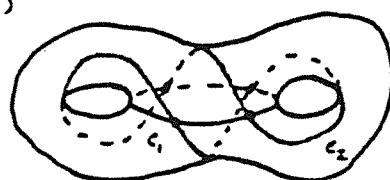
Suppose $A \cup \alpha$ bounds a disc; C_1, C_2 have minimal intersection. γ_1, C_2 have minimal intersection as both are geodesics $\Rightarrow C_2 \cap$ disc is a family of arcs passing "through" the disc from "top to bottom". So we can choose our isotopy to leave C_2 invariant (altho not pointwise fixed) so that $C_1 \cap \gamma_1 = \emptyset$.



Now we can isotope C_1 to γ_1 via an isotopy which is the identity outside a neighborhood of an annulus. As above the pieces of C_2 must go "through" this annulus (maybe they wind around but this can be fixed by homeomorphism) and so essentially the same proof as above works here as well.

II.

Exercise: Define $i(C_1, C_2) = \inf \{ |C'_1 \cap C'_2| : C'_i \cong C_i \}$
 $=$ the geometric intersection number. Prove that
 C_1, C_2 have minimal intersection \Leftrightarrow
 $|C_1 \cap C_2| = i(C_1, C_2)$



$$i(C_1, C_2) = 4$$

Def Essential simple closed curves $C_1, C_2 \subseteq F$ with minimal intersection fill F if every component of $F - (C_1 \cup C_2)$ is a 2-cell.
(See illustration in above exercise)

Exercise: On any surface F , \exists s.c.c.'s C_1, C_2 filling F

9/24 Casson

Thm 2.6 Let $h: F \rightarrow F$ be an orientation preserving automorphism of a closed hyperbolic surface F . If for every essential simple closed curve $C \subseteq F$ $\exists k \geq 1 \ni h^k(C) \cong C$, then $\exists n \ni h^n$ is isotopic to τ .

Proof Choose geodesics C_1, C_2 filling F . There exist integers k_1, k_2 such that $h^{k_i}(C_i) \cong C_i$. So set $k = k_1 \cdot k_2$.
 Then $h^k(C_i) \cong C_i \quad i = 1, 2$.

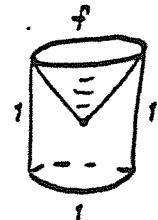
The curves $h^k(C_i)$ have minimal intersection, so by lemma 2.5 there exists $g \in \text{Homeo}(F)$ isotopic to τ such that

$gh^k(C_i) \quad i = 1, 2$ is a closed geodesic

Hence

$$gh^k(C_i) = C_i \Rightarrow gh^k(C_1 \cup C_2) = C_1 \cup C_2$$

Considering $C_1 \cup C_2$ as a graph on F , gh^k permutes the vertices of this graph, so there exists an $l > 0$ such that $(gh^k)^l$ is isotopic to an $f: F \rightarrow F$ which restricts to the identity on $C_1 \cup C_2$. Now, g isotopic to $\tau \Rightarrow h^{kl}$ is isotopic to an f which restricts to the identity on $C_1 \cup C_2$. The Alexander trick shows that f is isotopic to $\tau \Rightarrow h^{kl}$ is isotopic to τ .



Alexander Trick

11.

Goal: Determine the area of a hyperbolic surface

Suppose F is a closed surface made by identifying the geodesic edges of a polygon in pairs. Suppose this polygon has $2e$ edges, then this gives rise to a cell decomposition of F with a single 2-cell, e edges, and some number v of vertices

The Euler characteristic of F , $K_F = v - e + 1$

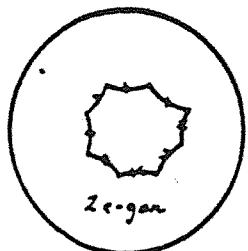
$$\Rightarrow e = v + 1 - K_F \Rightarrow 2e = 2(v + 1 - K_F)$$

The Gauss-Bonnet Theorem says the area of our $2e$ -gon is

$$2(v - K_F)\pi - 2\pi v = -2\pi K_F$$

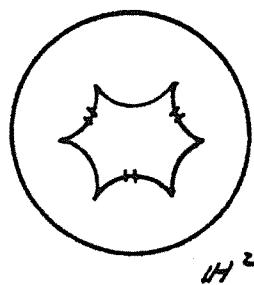
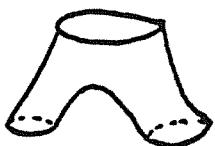
∴

$$\text{Area of closed hyperbolic surface} = -2\pi K_F$$



Lemma 2.7 A compact hyperbolic surface with totally geodesic boundary has area $-2\pi K_F$

Example: Choose a right rectangular geodesic hexagon in \mathbb{H}^2 . Take a second copy and "glue" every other edge. The resulting surface is the hyperbolic "pair of pants"



$$\text{Area} = 2(4\pi - 3\pi) = 2\pi, \quad K_E = -1$$

Proof Double F . ∂F geodesic $\Rightarrow \partial F$ hyperbolic. F compact $\Rightarrow \partial F$ closed and further $K_{\partial F} = K_F + K_E - K_{\partial F}$. But F compact $\Rightarrow K_{\partial F} = 0$ ($K_{\partial F} = \# \text{ components homeomorphic to } \mathbb{H}^2$)
 $\therefore K_{\partial F} = 2K_F$. Area $\partial F = 2 \text{Area } F$ finishes the lemma once we have the following

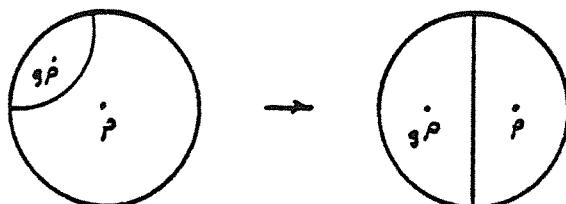
Claim: Every closed hyperbolic surface F is made by glueing the edges of a geodesic polygon in pairs.

Pf: We look for a "fundamental polygon"
As usual $F \cong \mathbb{H}^2/\Gamma$, Γ discrete

Choose $P \in \mathbb{H}^2$ and let $\mathcal{U} = \{x \in \mathbb{H}^2 \mid d(x, P) \leq d(x, gP) \quad \forall g \in \Gamma\}$
note \mathcal{U} is closed, and $\forall x \in \mathbb{H}^2$ there exists a $g \in \Gamma$ such that $gx \in \mathcal{U}$.

We proceed via a series of claims

1. \mathcal{U} is hyperbolically convex, i.e. if $x, y \in \mathcal{U}$ the geodesic segment $[x, y] \subset \mathcal{U}$



Take the perpendicular bisector of the geodesic segment $[P, gP]$ and note that points closer to P are on P side. Move bisector to center to see that \mathcal{U} is the intersection of hyperbolic half planes $\Rightarrow \mathcal{U}$ hyperbolically convex.

9/27 Casson

2. $\text{Fr } \mathcal{U}$ is a locally finite union of geodesic arcs
 By the discreteness of Γ there are only finitely many translates of P at the minimal distance from x . The point x must lie on the geodesic perpendicular bisector of $[P, gP] \Rightarrow x$ is on at most finitely many such geodesics $\Rightarrow \text{Fr } \mathcal{U}$ locally finite.

Further moving x slightly in $\text{Fr } \mathcal{U}$ does not introduce new translates of P at the minimal distance by the discreteness of $\Gamma \Rightarrow x$ is on a geodesic \Rightarrow claim

3. \mathcal{U} is a polygon with geodesic edges
 F compact \Rightarrow There exists a d such that every point of H^2 is within d of some $gP \Rightarrow \mathcal{U}$ bounded $\Rightarrow \text{Fr } \mathcal{U}$ is a finite union of geodesic arcs
 \Rightarrow claim. Call \mathcal{U} a Poincaré polygon for F .

Finally the fact that gP has a Poincaré polygon \Rightarrow edges identified in pairs.

II.

Remark: The above proof is independent of dimension.

Lemma 2.8 An unbounded complete hyperbolic surface F with finite area is homeomorphic to a closed surface less a finite set and has area $-2\pi K_F$

Ex: Ideal Triangle, Area = π

Double an ideal triangle to get a thrice punctured sphere, Area = 2π

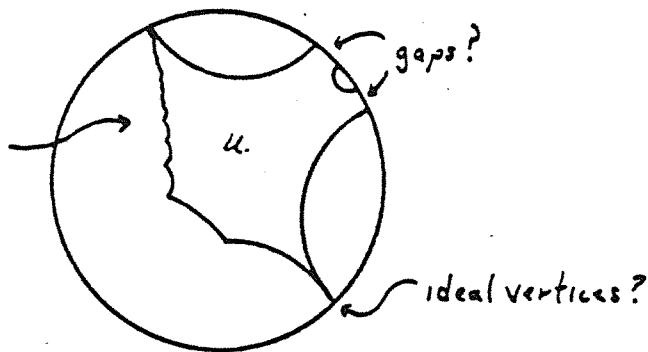


Proof F complete $\Rightarrow F \cong H^2/\Gamma$ with Γ discrete, but possibly non-hyperbolic elements exist in Γ .

As Γ is discrete, we apply the technique of the previous proof again.
 Get $\mathcal{U} = \{k + H^2 / d(x, P) \leq d(x, gP) \quad \forall g \in \Gamma\}$, as before \mathcal{U} is closed, hyperbolically convex, and $\text{Fr } \mathcal{U}$ is a locally finite union of geodesic arcs.

what can go wrong?

infinite decreasing sequence of geodesic edges?



Let α_v be the angle at the vertex v . Vertices occur when v is equidistant from 2 or more distinct translates $\Rightarrow v$ is the intersection of distinct geodesics $\Rightarrow \alpha_v < \pi$

Again we proceed via a series of claims, recall that the area of U is finite

1. The Euclidean closure of U in the Poincaré disc has only finitely many points on the circle at ∞

If n points, as U is hyperbolically convex, U must contain an ideal polygon with these endpoints $\Rightarrow \text{area } U > (n-2)\pi \Rightarrow$ there exists a bound for n

2. Area of $U + 2\pi \geq \sum_{v \text{ in } \partial U} (\pi - \alpha_v)$

Take a finite number of interior vertices and make a finite polygon G contained in U . $\text{Area } G = (n-2)\pi - \text{angle sum}$

$$\Rightarrow n\pi - \text{angle sum} = 2\pi + \text{area } G$$

$$\Rightarrow \sum (\pi - \alpha_{v_i}) = 2\pi + \text{area } G$$

Remark: This shows $\sum (\pi - \alpha_v)$ converges

3. The number of vertices is finite

$$\text{Let } A = \{v \mid \alpha_v \leq 2\pi/3\}; B = \{v \mid \alpha_v > 2\pi/3\}$$

Claim 2 $\Rightarrow A$ finite, now notice that as the edges are identified in pairs, we can divide the vertices into equivalence classes. Each such class has ≤ 2 B -vertices and ≥ 1 A vertices as the angle sum is 2π .

\Rightarrow finitely many equivalence classes

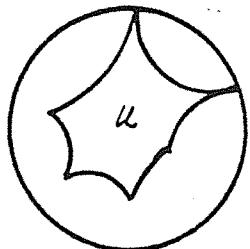
\Rightarrow finitely many vertices

Cor: No gaps or infinite sequences of geodesics.

9/29 Casson

Now take a new polygon $\mathcal{U} \cup \{\text{ideal vertices}\}$. Performing identifications shows that F is made from $\mathcal{U} \cup \{\text{ideal vertices}\}$ by identifying edges in pairs and then deleting the ideal vertices.

$\Rightarrow F \cong \text{closed surface - finite set of points}$



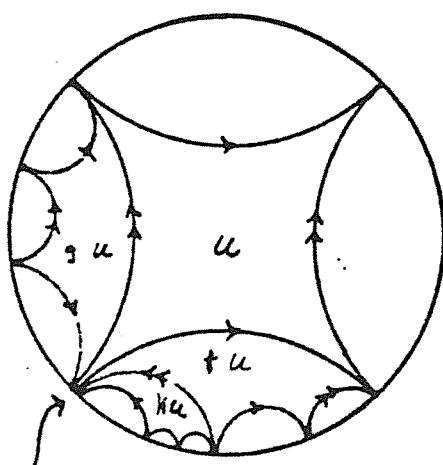
Remark: This proof does not depend on dimension

Now F is a cell-complex with some vertices deleted; one 2-cell, e edges, and v surviving vertices

$$\begin{aligned} \text{area } \mathcal{U} &= (n-2)\pi - \text{angle sum} \\ &= (2e-2)\pi - 2\pi v \quad \text{as the ideal vertices have angle } 0. \\ &= (2e-2-2v)\pi \\ &= -2\pi K_F \end{aligned}$$

Remark: If $F = \frac{n+2}{n}$ is of finite area but non-compact, then P has parabolic elements

Example: A 1-puncture torus - obtained from ideal quadrilateral, area = 2π



composition is parabolic fixing this ideal point.

Theorem 2.9 If F is a complete hyperbolic surface of finite area; totally geodesic boundary, (finite number of β -components), $\text{Area } F = -2\pi \chi_F + \pi \chi_{\partial F}$

Proof Double F ; $\text{Area } \partial F = 2 \text{Area } F$

$$\chi_{\partial F} = \chi_F + \chi_F - \chi_{\partial F}$$

Cor 2.10 The area of any complete hyperbolic surface F with totally geodesic boundary is $n\pi$ ($n \geq 1$). In particular, π is a lower bound for such areas.

§3 Geodesic Laminations

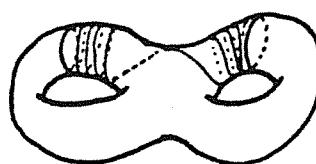
Def A closed hyperbolic surface F ; a geodesic in F is the image of a complete geodesic in $H^2 \cong \mathbb{H}$. A geodesic in F is simple if it has no transverse self intersections.

A (geodesic) lamination on F is a non-empty closed subset L of F which is a disjoint union of simple geodesics in F . We will prove that such an L is a union of geodesics in just one way. The geodesics contained in L are called the leaves of L .

Examples: Finitely many disjoint s.c.c. geodesics form a lamination

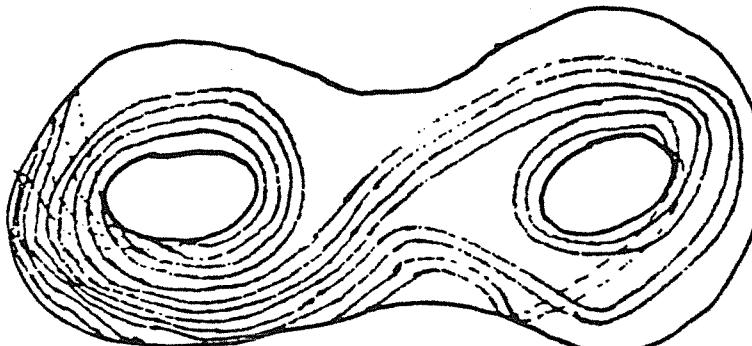


or



or

Take "long" simple closed curve geodesics



Suppose that A, B are closed subsets of a compact metric space. The Hausdorff distance $d(A, B) < \epsilon$, if $A \in N_\epsilon(B)$ and $B \in N_\epsilon(A)$.

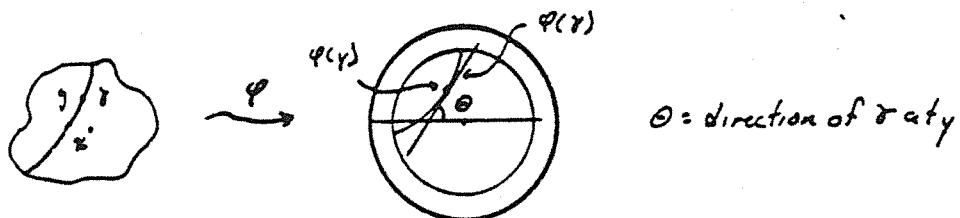
Exercise: Show that $2^X = \{\text{non-empty closed subsets of } X\}$ forms a metric space which is totally bounded & complete, \therefore compact.

We will use convergence in this metric to construct laminations.

10/1

Lemma 3.1 If $L = \bigcup_{x \in L} \gamma_x$ where γ_x is a geodesic thru x if γ_x, γ_y either coincide or are disjoint $\forall x, y \in L$, then the direction the direction of γ_x at w.r.t. any chart varies continuously with x .

Proof Choose a chart for x $\phi: U \hookrightarrow \mathbb{H}^2 \cong \text{Int } \mathbb{D}^2 \subseteq \mathbb{R}^2$. Define the direction of γ at x (w.r.t ϕ) as the angle between $\phi(\gamma)$ and the Euclidean parallel to a fixed line thru $\phi(x)$. (Geodesics here are unoriented)



Now just observe that geodesics will intersect if their directions are far apart. The lemma follows by THINK.

II.

Lemma 3.2 A disjoint union of simple geodesics in a closed (countable) hyperbolic surface F is a proper subset of F , and can be expressed as a union of geodesics in just one way.

Proof F is closed, hyperbolic $\Rightarrow K_F < 0$

$\therefore F$ has no continuously varying line field defined on all of F

\therefore disjoint union of simple geodesics is proper

Next consider $\mathbb{H}^2 \cong \tilde{F}$

$$\begin{array}{c} \downarrow p \\ F \end{array}$$

$L = \bigcup_{x \in L} \gamma_x$. We must show that γ_x is the only geodesic thru x in L . Indeed, we will show no other geodesic arc in L passes thru x .

Suppose \exists an arc α of a geodesic with $x \in \text{Int } \alpha$, $\gamma_x \cap \alpha$ transverse, and $\alpha \subseteq L$. Define $\phi: \alpha \times \mathbb{R} \rightarrow \mathbb{H}^2: (y, t) \mapsto$ point on γ_y a distance t from y (Here we have chosen an orientation on the normal to α , maybe α had to be shrunk).

Lemma 3.1 $\Rightarrow \phi$ continuous

Notice that $\phi(\alpha \times \mathbb{R}) \subseteq p^{-1}(L)$. Given $d > 0$, $\exists U$ on γ_x such that the hyperbolic d -neighborhood of $U \subseteq \phi(\alpha \times \mathbb{R})$. Set $d = \text{diameter of a Poincaré polygon for } F$
 $\Rightarrow p(d\text{-nbhd of } U) = F \Rightarrow L \subseteq F \quad \leftarrow$

H.

Skolem: No open subset of $F \subseteq L \Rightarrow$ laminations are nowhere dense

10/3

Theorem 3.4 Let $\Lambda = \Lambda(F)$ be the set of all geodesic laminations on the closed (orientable) hyperbolic surface F . Hausdorff distance defines a compact metric on Λ .

Before we prove this theorem we need:

Lemma 3.3 The closure of a non-empty disjoint union L of geodesics is a lamination

Proof We must show \bar{L} is a disjoint union of simple geodesics

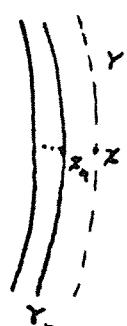
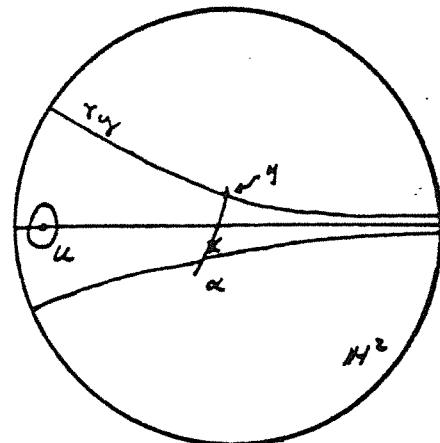
Take $x \in \bar{L}$, $x_n \rightarrow x$, $x_n \in L$, γ_n a leaf of L thru x_n

Observe that the set of directions of geodesics at x is compact, indeed $\cong \mathbb{RP}^1$

$\Rightarrow \exists$ a subsequence of directions $\gamma_n \rightarrow$ direction of γ thru x

so replace $\{\gamma_n\}$ by this subsequence

(directions measured in some chart about x)



10/3 CASSON

Next take y on γ , $\text{dist}(x, y) = d$. Find y_n on γ_n a distance d from x_n measured in the same direction as y . Then $y_n \rightarrow y$

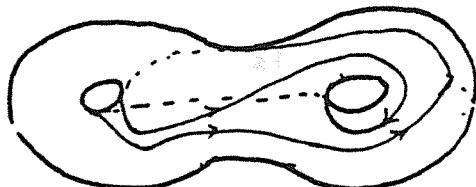
$$\Rightarrow x \in \gamma \subseteq L$$

$\Rightarrow L$ is a union of geodesics

Any two geodesics formed in this way are disjoint, as any geodesic so formed can be approximated by a geodesic in L which also approximates direction. So if two geodesics $\gamma, \gamma' \in L - L$ intersected, we could find intersecting geodesics in L . A similar argument shows γ_x simple.

11.

Remark: The geodesics which are members of a lamination are considered unoriented. This is necessary for if one forms the lamination which "limits" the "sequence" of longer & longer s.c. curve geodesics, it must be unoriented as the example below shows



We are ready to prove

Theorem 3.4 The set of all geodesic laminations on a closed (orientable) hyperbolic surface has a compact metric defined by Hausdorff distance.

Recall: Hausdorff distance = K compact metric, $2^X = \{\text{non-}\emptyset \text{ closed subsets of } X\} = \{\text{non-}\emptyset \text{ compacta in } K\}$

$$d(A, B) \leq \epsilon \Leftrightarrow A \subseteq N_\epsilon(B) \wedge B \subseteq N_\epsilon(A)$$

Note: $d(A, B) = 0 \Leftrightarrow A = B$

Topology on 2^X depends only on the topology of X as any two metrics on X are equivalent

X compact $\Rightarrow 2^X$ compact

Proof $\lim_{n \rightarrow \infty} A_n = A$ if $\forall \epsilon > 0 \exists N \ni n \geq N \Rightarrow A \subseteq N_\epsilon(A_n)$ and $A_n \subseteq N_\epsilon(A)$

$$\text{so } \lim_{n \rightarrow \infty} A_n = \{x \in X \mid \forall \epsilon > 0 \exists N \ni n \geq N \Rightarrow x \in N_\epsilon(A_n)\}$$

Let $L_n \rightarrow L$ in the Hausdorff metric, each L_n a lamination. L is closed, pick $x \in L$. we can find nearby points $y_n \in L_n$. \exists subsequence such that $y_n \rightarrow r$ a geodesic.

But suppose you chose a different subsequence. Then maybe the y you get might not have the same direction as the first. So at this stage the previous lemma must be reproved. Instead, we control the directions of geodesics as well.

Consider the Projective tangent bundle $PT(F) = \{(x, \zeta) \mid x \in F, \zeta \text{ a closed geodesic segment of length } 2, \text{ say, centered on } x\}$
Topologize $PT(F)$ with charts:

$$(x, \zeta) \in PT(F) \supset p^{-1}(U) \approx U \times \mathbb{RP}^1$$

$$\begin{array}{ccc} & \downarrow p & \downarrow p \\ x & \in F & \supset U \xrightarrow{\text{id.}} U \\ & \text{chart} & \end{array}$$

projection

i.e. Take the collection of p^{-1} (chart neighborhoods) as a basis for the topology
 $p: PT(F) \rightarrow F$ is continuous, $PT(F)$ is a compact 3-manifold.
 Moreover $PT(F)$ metrizable $\Rightarrow PT(F)$ compact metric $\Rightarrow 2^{PT(F)}$ compact metrizable

Given γ a geodesic in F , we have the (lifted) geodesic $\hat{\gamma} = \{(x, \zeta) \mid \zeta \in \gamma\}$ in $PT(F)$. Notice that every point in $PT(F)$ lies on a unique lifted geodesic.
 For L a lamination in F , $\hat{L} = \{\hat{\gamma} \mid \gamma \in L\}$, $\hat{A} = \{\hat{L} \mid L \in A(F)\}$

Now:

$$\begin{array}{ccc} \hat{L} & \subset & PT(F) \\ \stackrel{1-1}{\text{onto}} & \downarrow p & \downarrow p \\ L & \subset & F \\ \text{compact} & \nearrow & \end{array}$$

Lemma 3.1 $\Rightarrow p^{-1}$ continuous on $L \Rightarrow p$ a homeomorphism $\Rightarrow \hat{L}$ compact

10/6 Casson

Hence the lifted laminations are in $\hat{\Lambda} \subset 2^{\text{PT}(F)}$

we want $\hat{\Lambda}$ closed because:

2^F Hausdorff $\Rightarrow \hat{\Lambda}$ Hausdorff;
 $2^{\text{PT}(F)}$ compact, so if $\hat{\Lambda}$ closed in $2^{\text{PT}(F)}$
then $\hat{\Lambda}$ compact
 $\Rightarrow p_* : \hat{\Lambda} \rightarrow \Lambda$ a homeomorphism
 $\Rightarrow \Lambda$ compact

$$\begin{array}{ccc} \hat{\Lambda} & \subset & 2^{\text{PT}(F)} \\ \downarrow \text{onto} & & \downarrow p_* \\ \text{continuous } \Lambda & \subset & 2^F \end{array}$$

So suppose $\hat{L}_n \in \hat{\Lambda}$, $\hat{L}_n \rightarrow A \in 2^{\text{PT}(F)}$. To show: $A = \hat{L}$ for some lamination L .
Put $L = p(A)$. Note that A is non- \emptyset and compact $\Rightarrow L$ compact.

(1) L is a union of geodesics

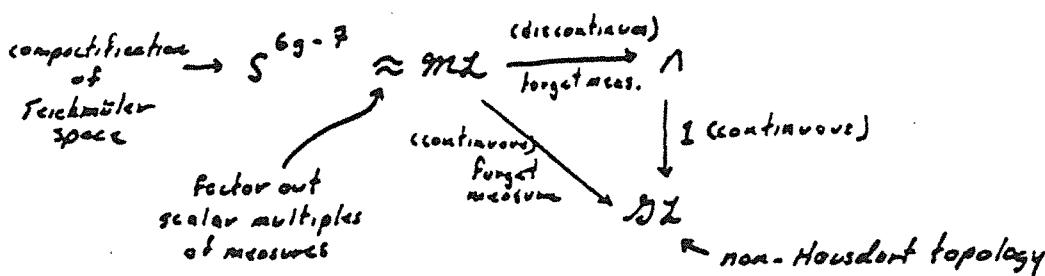
$\exists f \times \epsilon \in L$, $x = p(a)$, $a \in A \subset \text{PT}(F)$

$a = (x, \alpha)$ is a length 2 geodesic segment which we will (sloppily) call a Now $\exists a_n \in \hat{L}_n$ s.t. $a_n \rightarrow a$. Let x_n = center point of $p(a_n)$. Then $x_n \rightarrow x$ and direction $a_n \rightarrow$ direction α . Let γ = prolongation of a , as before $\gamma \in L$
(2) Repeat the argument of Lemma 3.3 to see L a disjoint union of simple geodesics

16.

Remark: Thurston considers two spaces of laminations

\mathcal{ML} = measured laminations, \mathcal{GL} = geometric laminations



A sketch of the above proof is

Lemma 3.5 Hausdorff distance on 2^F and $2^{\text{PT}(F)}$ define the same topology on $\Lambda(F)$

Def A leaf γ of a lamination L is isolated if for each $x \in \gamma$ there is a neighborhood U of x such that $(U, U \cap L)$ is homeomorphic to $(\text{disk}, \text{diameter})$.

Exercise: This holds for every $x \in Y \Leftrightarrow$ it holds for some $y \in Y$

Set $L' = L - \{\text{isolated leaves}\}$. Note that L' is closed, if $L' \neq \emptyset$ we call L' the derived lamination of L .

Lemma 3.6 If $L' = \emptyset$, then L is a finite union of s.c.c's and L is an isolated point in $A(F)$.

Proof If $L' = \emptyset$, all the leaves of L are isolated $\Rightarrow L$ is a 1-dim submanifold of F (notice that not all 1-dim submanifolds are laminations with just isolated leaves however)

$\therefore L$ is a disjoint union of s.c.c. geodesics

$\therefore \exists \epsilon > 0$ such that $\overline{N_\epsilon(L)}$ is a disjoint union of annuli

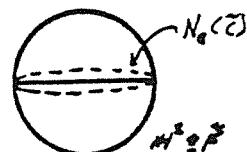
Suppose L_α is such that $d(L_\alpha, L) < \epsilon$. Let γ be a leaf of L_α ,

$\gamma \in N_\epsilon(L) \Rightarrow \exists$ s.c.c. geodesic C in L such that $\gamma \in N_\epsilon(C)$

\Rightarrow Some lift of γ is in $N_\epsilon(\tilde{C}) \subseteq H^2$. But the only geodesic in this neighborhood is \tilde{C}

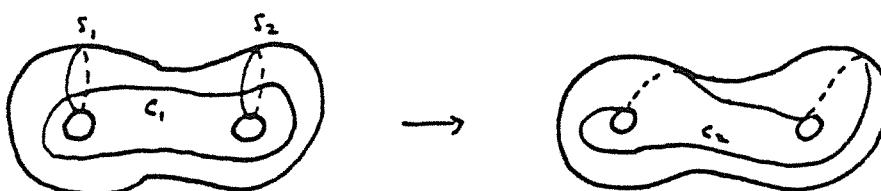
$\therefore \gamma = C \Rightarrow \gamma \in L$

II.



Querry: 1) Is every isolated point of $A(F)$ a 1-dim submanifold?

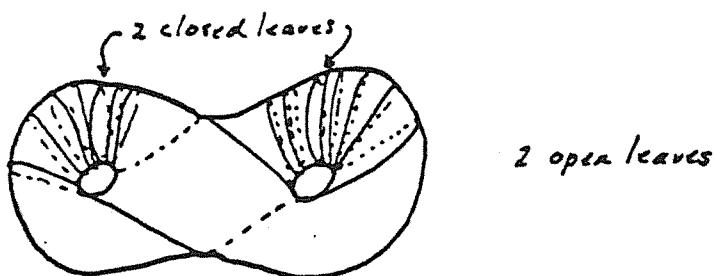
Example: Take a fixed s.c.c. and apply Dehn twists in others



Here C_{new} is the simple geodesic homotopic to $(T_1)^m (T_2)^n (C_1)$

10/8 - 10/10 Casson

Limit:



Limit Lamination in Hausdorff Topology

Delete an open leaf, is the resulting lamination isolated?

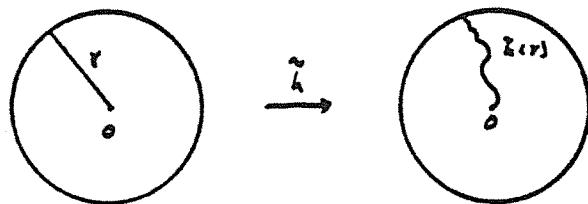
2) Can you find a perfect lamination? i.e. a lamination $L \ni L' \subset L$

Lemma 3.7 Let $h: F_1 \rightarrow F_2$ be a homeomorphism of closed (orientable) hyperbolic surfaces, and let $\tilde{h}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be a lift of h to the universal cover. Then \tilde{h} has a unique continuous extension over $\mathbb{H}^2 \cup S'_\infty$, where S'_∞ : circle at ∞ .

Remark: This theorem is true in higher dimensions as well.

Proof We already know the lemma for isometries as \tilde{h} is uniformly continuous in the Euclidean metric on the Poincaré disc.

We shall show that for a geodesic γ in \mathbb{H}^2 , $\tilde{h}(\gamma)$ converges to a point on S'_∞ . wlog let $\tilde{h}(0) = 0$.



Notice that as $\tilde{h}, \tilde{h}^{-1}$ are lifts of continuous maps of closed hyperbolic surfaces they are uniformly continuous w.r.t. the hyperbolic metric. Hence $\exists \lambda > 0$ such that

$$d(x, y) \leq 1 \Rightarrow d(\tilde{h}x, \tilde{h}y) \leq \lambda \quad (1)$$

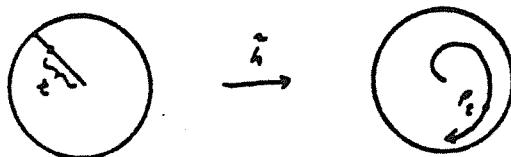
$$\therefore d(\tilde{h}x, \tilde{h}y) \leq 1 \Rightarrow d(x, y) \leq \lambda/2$$

$$\therefore \delta = d(\tilde{h}x, \tilde{h}y) \leq \kappa \Leftrightarrow d = d(x, y) \leq \kappa/2$$

Set $n = [\tilde{d}] + 1 \Rightarrow d \leq \frac{1}{2}(\tilde{d}+1) \Rightarrow \tilde{d} \geq \frac{2d}{\lambda} - 1$
 So if $d(x, y) \geq \lambda$, then $\tilde{d} \geq d/\lambda$, we get the condition

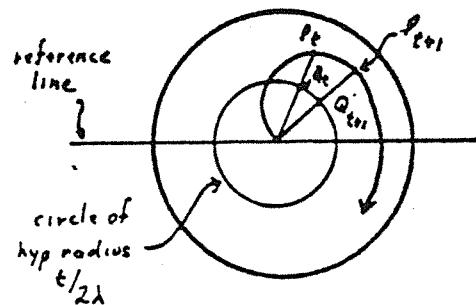
$$d(x, y) \geq \lambda \Rightarrow d(\tilde{x}, \tilde{y}) \geq \frac{d(x, y)}{\lambda} \quad (2)$$

Remark: we want to rule out "spiroaling"



Let γ be an oriented geodesic thru O , set $P_t = \tilde{h}$ (point on γ at distance t from O)
 Pick a fixed reference line thru O , let Θ_t be the angle between OP_t and this
 fixed reference line. We want Θ_t to have a limit as $t \rightarrow \infty$

Every point on the arc P_t, P_{t+1} has distance
 $\geq t/\lambda \geq t/\lambda - \lambda$ from O by (2), by (1)
 $d(P_t, P_{t+1}) \leq \lambda$



∴ Every point on the geodesic arc P_t, P_{t+1}
 has distance $\geq t/\lambda - \lambda$ from O
 $\geq t/2\lambda$ if $t \geq 2\lambda^2$

Now

$$d(P_t, P_{t+1}) \geq \text{arc } Q_t, Q_{t+1} = |\Theta_{t+1} - \Theta_t| \cdot \sinh \frac{t}{2\lambda} \geq \frac{|\Theta_{t+1} - \Theta_t|}{4} \cdot e^{t/2\lambda}$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \geq \frac{1}{2}e^x (x \geq 1)$$

So

$$|\Theta_{t+1} - \Theta_t| \leq 4\lambda e^{-t/2\lambda}, \text{ if } t \geq 2\lambda^2$$

For $u = t \geq 2\lambda^2$

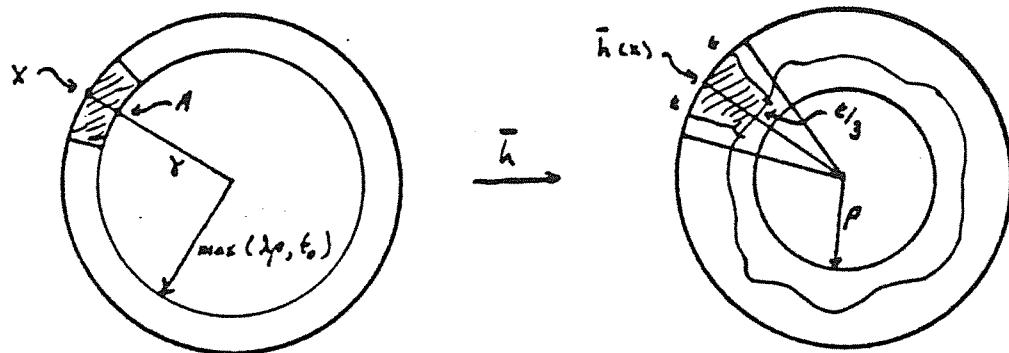
$$|\Theta_u - \Theta_t| \leq \int_{t=1}^u 4\lambda e^{-s/2\lambda} ds = C e^{-t/2\lambda} \text{ where } C \text{ is a function of } \lambda \text{ only}$$

$\Rightarrow \lim_{t \rightarrow \infty} \Theta_t$ exists, call it Θ .

So we can define our extension of \tilde{h} , \bar{h} , by taking the endpoints of geodesics
 to angle $\Theta \cap S^\infty$.

10/10 Casson

Is this continuous? Note that a neighborhood of $\tilde{h}(x)$ can be thought of as a 2ϵ -sector \cap exterior of a circle of hyperbolic radius ρ . To find a neighborhood of x mapped into this set, consider the exterior of a circle



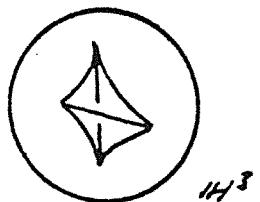
of hyperbolic radius $\max(\lambda\rho, t_0)$ where $t_0 > 2\lambda^2$ and $(e^{-t_0/2\lambda}) < \frac{\epsilon}{3}$
and of width determined by the continuity of \tilde{h} at A (see pix above)
Note: $\lambda\rho$ insures that image of circle is outside of circle of radius ρ .

11.

Remarks:

1. With a bit more work the hypothesis can be weakened to h a homotopy equivalence
2. In higher dimensions this is the first step in the proof of Mostov's Rigidity Theorem. In that argument one concludes that h is homotopic to an isometry.
3. If $h_1, h_2 : F_1 \rightarrow F_2$ are homotopic via a homotopy $H : F_1 \times I \rightarrow F_2$ and \tilde{h}_1, \tilde{h}_2 are the lifts of h_1, h_2 resp. obtained by lifting the homotopy, then
 $\tilde{h}_1|S'_\infty \equiv \tilde{h}_2|S'_\infty$
 (H uniformly continuous \Rightarrow hyperbolic $d(\tilde{h}_1x, \tilde{h}_2x)$ is bounded
 \Rightarrow as $x \rightarrow S'_\infty$, Euclidean $d(\tilde{h}_1x, \tilde{h}_2x) \rightarrow 0$
 $\therefore \tilde{h}_1(y) = \tilde{h}_2(y), \text{ if } y \in S'_\infty$)
 This also holds in higher dimensions
4. Mostov: For $n \geq 3$, use above argument; then show $\tilde{h}|S'_\infty \equiv$ some isometry $|S'_\infty$ (false in dim 2), conclude $h \cong$ isometry

5. In dimension 2, \exists an isometry taking any ideal triangle to any other ideal triangle; but not all the ideal simplices in H^3 have the same volume \Rightarrow maximum volume possible, any two are isometric. To show that the lift of a homotopy equivalence takes (max. vol) \rightarrow (max vol.) and that this characterizes the action on S^2_∞ , one looks at the image of a face, 2 points on S^2_∞ give a max volume simplex, pt & its reflection, so one is led to considering reflections.



Open Question: Does $\text{Aut}(F)$ act "naturally" on F ?

i.e. $\exists \phi: \text{Homeo}_+(F) \rightarrow \pi_0(\text{Homeo}_+(F)) = \text{Aut}(F)$
is there a 1-sided inverse?

i.e. Is there a group homomorphism $\psi: \text{Aut}(F) \rightarrow \text{Homeo}(F)$
 $\exists' \psi \cdot \phi = \text{id}_{\text{Aut}(F)}$

Partial Solutions:

1) If $\pi \subseteq \text{Aut}(F)$ is finite, is there a group homeomorphism $\psi: \pi \rightarrow \text{Homeo}(F)$ such that $\phi \cdot \psi = \text{id}_\pi$?

(A) Nielsen (1930's) Yes if π cyclic, iterated argument gives Yes, π solvable

(B) Kerckhoff (1979) Yes for any finite π , argument uses Thurston's compactification of Teichmüller space.

2) Consider the unit tangent bundle, a double cover of $\text{PT}(F)$

$$\text{UT}(F) = \{ (x, \sigma) \mid \sigma \text{ an oriented geodesic segment of length } z \}$$

$\text{Aut}(F)$ acts naturally on $\text{UT}(F)$:

$$\begin{array}{ccc}
 \text{Homeo}(\text{UT}(F)) & \xrightarrow{\quad} & \text{Out}(\pi, (\text{UT}(F))) \\
 \downarrow \text{No map here} & \uparrow \text{tangent map} & \searrow \exists \hat{\psi} \quad \downarrow \pi_{*(F)} = \pi_1(\text{UT}(F))/\text{center} \\
 \text{Diff}^+(\text{UT}(F)) & \xrightarrow{\quad} & \text{Out}(\pi, (F)) \cong \text{Aut}(F) \\
 & & \text{Neilson}
 \end{array}$$

$\hat{\psi}$ is a group homomorphism which gives a canonical self-homeomorphism of $\text{UT}(F)$ from each automorphism of F (Gromov-Cheeger)

10/15 Casson

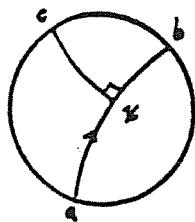
3

Lemma 3.8 If F is a closed complete (oriented) hyperbolic surface, then $UT(F) = Y/\Gamma$ where $F = \mathbb{H}^2/\Gamma$ and $Y = \{(a, b, c) \mid a, b, c \text{ are distinct points on } S_\infty \text{ in counterclockwise order}\}$

Proof: Define $g: Y \rightarrow UT(F): (a, b, c) \mapsto (p(x), ab)$ where x is the foot of the perpendicular geodesic from c to ab .

Note that g is continuous and onto, and induces a homeomorphism $Y/\Gamma \rightarrow UT(F)$

II.



Theorem 3.9 (Gromov, Cheeger) Any orientation preserving homeomorphism $h: F_1 \rightarrow F_2$ of closed hyperbolic surfaces induces a homeomorphism $\tilde{h}: UT(F_1) \rightarrow UT(F_2)$. Moreover, if $h \cong k$, then $\tilde{h} = \tilde{k}$; if $h_i: F_i \rightarrow F_{i+1}$, $i=1, 2$ then $\widehat{h_2} \circ \widehat{h_1} = \widehat{h}_2 \cdot \widehat{h}_1$ (and $\tilde{h} \cong \tilde{k}$). In addition \tilde{h} carries lifted geodesics to lifted geodesics.

Proof $F_i = \mathbb{H}^2/\Gamma_i$; choose $\tilde{h}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ covering $h: F_1 \rightarrow F_2$. If $g_i \in \Gamma_i$, then $\tilde{h}g_1 = g_2\tilde{h}$ for some $g_2 \in \Gamma_2$, $g_2 = h \circ (g_1) \in \Gamma_2$. By Theorem 3.7 we can extend \tilde{h} to $\tilde{h}: S_\infty \rightarrow S'_\infty$; $\tilde{h}g_1 = g_2\tilde{h} \quad (g_2 = h \circ (g_1))$

Notice that

(i) \tilde{h} preserves orientation

(ii) induces $\tilde{h}: Y \rightarrow Y$ satisfying $\tilde{h}g_1 = g_2\tilde{h}$

(iii) induces $\tilde{h}: Y/\Gamma_1 \rightarrow Y/\Gamma_2$, so $\tilde{h}: UT(F_1) \rightarrow UT(F_2)$

(iv) \tilde{h} is independent of the choice of lift of h

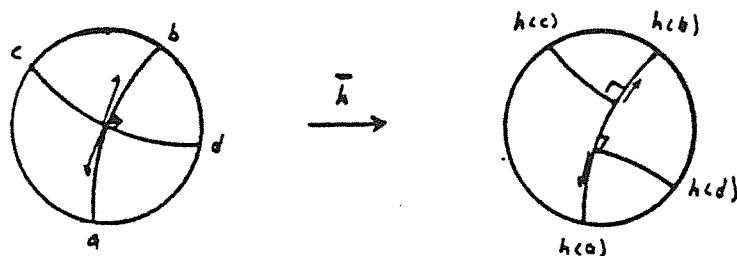
Now $h \cong k \Rightarrow \exists$ lifts \tilde{h}, \tilde{k} such that $\tilde{h} = \tilde{k}$

$$\therefore \tilde{h} = \tilde{k}$$

Moreover, we can choose $\widehat{h_2} \circ \widehat{h_1} = \widehat{h}_2 \cdot \widehat{h}_1$, $\therefore \widehat{h_2} \circ \widehat{h_1} = \widehat{h}_2 \cdot \widehat{h}_1: S_\infty \rightarrow S'_\infty$

$$\therefore \widehat{h_2} \circ \widehat{h_1} = \widehat{h}_2 \cdot \widehat{h}_1$$

Remark: This doesn't work for $PT(F)$:



If $\gamma \in F_1$ is an oriented geodesic, let $\tilde{\gamma}$ be a lift to $M^2 = \tilde{F}_1$, going from a to b on S'_{∞} . The lifted geodesic $\tilde{\gamma}$ in $UT(F)$ is $\{g(a, b, c) | c \in S'_{\infty} - \{a, b\}\}$ such that (a, b, c) is in counterclockwise order}. Then $\hat{h}(\tilde{\gamma})$ is $\{g(\tilde{h}(a), \tilde{h}(b), \tilde{h}(c)) | c \text{ above}\}$ a lifted geodesic.

11.

Note: If γ is an unoriented geodesic in F_1 , with two lifts $\tilde{\gamma}_+, \tilde{\gamma}_-$ in $UT(F)$ then $\hat{h}(\tilde{\gamma}_+), \hat{h}(\tilde{\gamma}_-)$ are lifts of an unoriented geodesic $\hat{h}(\gamma)$ in F_2 .

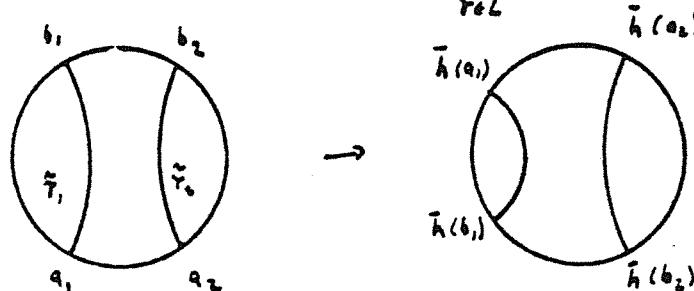
Theorem 3.10 Any orientation preserving homeomorphism $h: F_1 \rightarrow F_2$ of closed hyperbolic surfaces induces a homeomorphism $\hat{h}: \Lambda(F_1) \rightarrow \Lambda(F_2)$. If $h \cong k$, then $\hat{h} = \hat{k}$ and $\widehat{h_2 h_1} = \hat{h}_2 \hat{h}_1$, $i=1$

Proof For each unoriented geodesic $\gamma \in F_1$, \exists geodesic $\hat{h}(\gamma) \subset F_2$; $\hat{h}(\gamma) = p_2 \hat{h}(\tilde{\gamma}_{\pm})$:

$$\begin{array}{ccc} \tilde{\gamma}_{\pm} \in UT(F_1) & \xrightarrow{\hat{h}} & UT(F_2) \quad \hat{h}(\tilde{\gamma}_{\pm}) \\ \downarrow p_1 & & \downarrow p_2 \\ \gamma \in F_1 & & F_2 \quad \hat{h}(\gamma) \end{array}$$

c.e. Take endpoints of γ to endpoints of $\hat{h}(\gamma)$

Let $L \in \Lambda(F_1)$, define $\hat{h}(L) = \bigcup_{\gamma \in L} \hat{h}(\gamma)$



Now: a_1, b_1 do not separate $a_2, b_2 \Rightarrow \hat{h}(a_1), \hat{h}(b_1)$ don't separate $\hat{h}(a_2), \hat{h}(b_2) \Rightarrow$ leaves of $\hat{h}(L)$ are disjoint and simple
 $\& \hat{h}$ preserves order on S'_{∞}

Is \hat{h} continuous?

$$\begin{array}{ccc} \tilde{L} \in UT(F_1) & \xrightarrow{\hat{h}} & UT(F_2) \\ \downarrow & & \downarrow p_1 \\ L \in F_1 & & F_2 \quad \xrightarrow{\hat{h}(L)} \hat{h}(L) = p_2 \hat{h}(\tilde{L}) \end{array}$$

10/18 Casson

\hat{L} compact $\Rightarrow \hat{h}(\hat{L})$ compact, so

$$\begin{array}{ccc} \hat{L}_1 & \xrightarrow{\text{UT}(F_1)} & \hat{L}_2 \\ \approx \theta & \downarrow p_{1\#} & \downarrow p_{2\#} \\ L_1 \subset \hat{L}_1 & \xrightarrow{\hat{h}} & L_2 \subset \hat{L}_2 \end{array}$$

and $\hat{h}\theta = p_{2\#}\hat{h}$ is continuous

II.

Given a lamination $L \subset F$, $F \setminus L$ is open

Theorem 3.11 If L is a geodesic lamination on a closed orientable hyperbolic surface F , then $F \setminus L$ is 'isometric' with $G - \partial G$ for some complete hyperbolic surface G with totally geodesic boundary and finite area.

Proof Recall: Isometric $\Leftrightarrow \exists \Phi : F \setminus L \rightarrow G - \partial G$ which is 1-1, onto, and locally an isometry. Thus $\text{Area}(F)$ finite $\Rightarrow \text{Area } G$ finite.

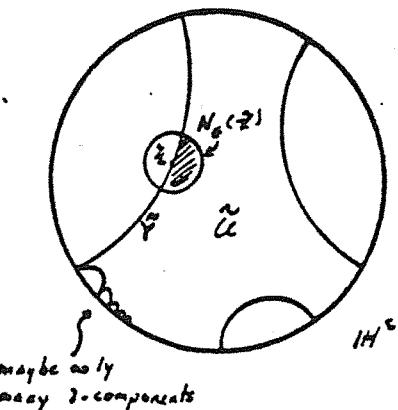
Let U be a component of $F \setminus L$, $\tilde{U} \in H^2$, $p : H^2 \rightarrow F$

Let \tilde{U} be any component of $p^{-1}(U)$. \tilde{U} is a component of $H^2 - \tilde{L}$ where $\tilde{L} = p^{-1}(L)$. Let $\tilde{x} \in \text{Fr}(\tilde{U})$, i.e. $\tilde{x} \in \tilde{Y} \cap \tilde{U}$.

Now \tilde{Y} separates H^2 , \tilde{U} is on one side, i.e. For small ϵ , $N_\epsilon(\tilde{x}) \cap \tilde{U}$ is a single component of $N_\epsilon(\tilde{x}) - \tilde{Y}$

(For otherwise, \exists a frontier point in the half disc on a separating geodesic $\Rightarrow \tilde{Y}$ not in $\text{Fr}(\tilde{U})$)

$\Rightarrow \tilde{U}$ is a surface with totally geodesic boundary.



Similarly \tilde{U} is hyperbolically convex $\Rightarrow \tilde{U}$ contractible
 $\Rightarrow \tilde{U}$ is a universal cover of U .

Set $P_U = \{g \in P \mid g(\tilde{U}) = \tilde{U}\}$. Then $U \cong \tilde{U}/P_U$. The group P_U acts on \tilde{U} , set

$$V_U = \tilde{U}/P_U$$

and note that V_U is a surface with totally geodesic boundary as \tilde{U}/P_U is Hausdorff, and is a subspace of H^2/P_U , a covering surface for F .

Moreover V_U is also complete and hyperbolic; $\text{Ent} V + V - \partial V \cong U$
So set $G = \cup_{\mathbf{U}} V_{\mathbf{U}}$, note that $\text{Area } G = \text{Area}(G - \partial G) = \text{area}(F - L) \leq \text{Area}(F)$

II.

Def A component of $F-L$ is a principal region for L . A boundary leaf of a principal region is a leaf γ of L s.t. $\exists \epsilon > 0 \exists r > 0 \forall x \in N_\epsilon(x) \cap U$ contains at least one component of $N_r(x) \cap Y$. (i.e. γ is isolated from one side) As before condition for some $\gamma \Rightarrow$ condition for all γ)

Remark: Boundary leaves correspond to the boundary components of G , but this correspondence is not always 1-1.

Cor A lamination $L \subset F$ has only finitely many principal regions, each with finitely many boundary leaves.

Proof $\text{Area}(\text{Principal region}) = n\pi$ for some $n > 0$
 \therefore At most $-2K_F(4g-4)$ principal regions

Exercise: Total number of boundary leaves for L is at most $-6K_F(12g-12)$

Lemma 3.12 The union of all boundary leaves of L is dense in L

Proof If $x \in L$, then arbitrarily close to $x \exists u \in F-L$ as L is nowhere dense.
 Let v be the first point on the geodesic arc from u to x which is in L , then v is on a boundary leaf of L .

Theorem 3.13 A geodesic lamination $L \subset F$ a closed (orientable) hyperbolic surface has measure 0. i.e. $\text{Area}(F-L) = \text{Area}(F)$
 i.e. $\text{Area}(G) = \text{Area}(F)$ where G is as in Lemma 3.1
 i.e. $K_G - \frac{1}{2}K_G = K_F$

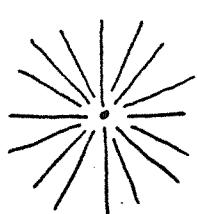
Remark: we only sketch the proof.

Proof: Step 1: There exists a continuous (integrable) line-field \mathcal{S} on F tangent to L everywhere with isolated singularities as illustrated below (and only finitely many such)

Recall: line-field = in neighborhood of a point a non-oriented version of a nowhere zero vector field.

10/20 Casson

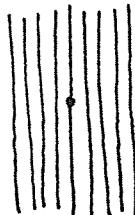
Singularity types:



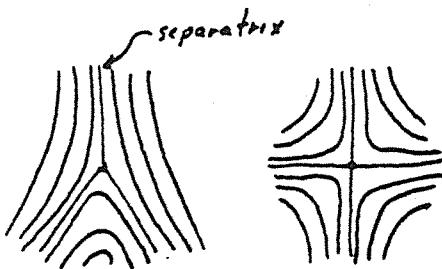
1



$\frac{1}{2}$



0



$-\frac{1}{2}$

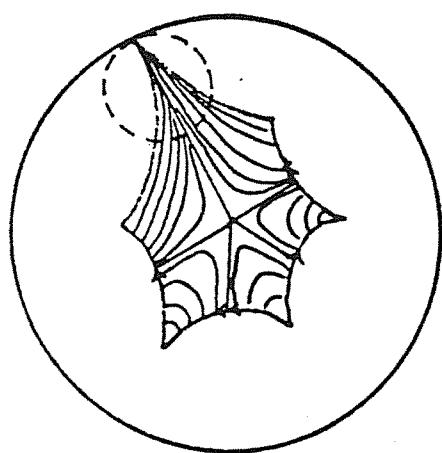
-1

Index of singularity

Def The index of a singularity = $\frac{1}{2\pi}$ (winding number of the line field restricted to the boundary of a disc neighborhood of the singularity). i.e. Orient the line field locally, count the rotations of a tangent vector as you go around the boundary of a disc neighborhood counter-clockwise.

This line field is constructed first on principal regions, indeed on the fundamental domain of such, the line field is an extension of the tangent field to lamination.

Example: Genus 7 non-compact surface with real line boundary. The angles below must add to π so that the boundary is totally geodesic.



index = $1\frac{1}{2}$

Begin in a horoball tangent at an ideal point.
Begin the line field by choosing the geodesics with the ideal point as an endpoint. The line field will be parallel to the unidentified boundary components of the F.R. what's left of the region is compact \Rightarrow line field will be continuous.

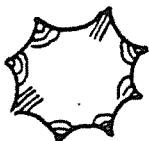
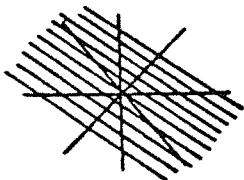
Conc the ideal vertices; centers of finite edges to form the separatrices and extend the line field as illustrated.

Avoid

Step 2: $\sum_s \text{index}(s) = \chi_F$ if F orientable, closed

Regard F as a glued polygon with vertices regular points and assume that the line field is transverse to polygon's edges.

Example: Genus 2 surface



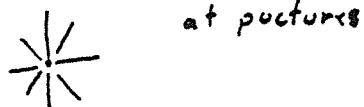
The line field points to two interior angles, all others the field "cuts across". Now count winding number to get total index = χ_F . Observe that if tangency occurs identification cancels it out.

Step 3: a) If F is compact and line field is tangent to boundary

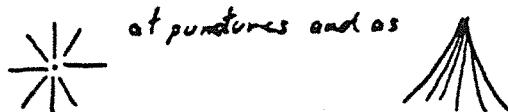
$$\sum_s \text{index}(s) = \chi_F \quad (\text{Doubling Trick})$$

b) F a closed surface - finite set, formula holds if line field is

as at punctures



c) Doubling Trick \Rightarrow formula holds for F a compact surface - finite set if line field is as at punctures and as at δ -punctures.



Here the formula is $\sum_s \text{index}(s) = \chi_F - \frac{1}{2} \chi_{\partial F}$

Step 4: $F - L \approx G - \partial G$; $\chi_F = \sum_s \text{index}(s) = \sum_{s \in F-L} \text{index}(s) + \sum_{s \in \partial F} \text{index}(s) = \chi_G - \frac{1}{2} \chi_{\partial G}$

Now $\text{Area } F = -2\pi \chi_F = -2\pi \chi_G + \pi \chi_{\partial G} = \text{area } G$
 $\Rightarrow \text{measure}(L) = \text{area } F - \text{area } G = 0$

10/22 Casson

Lemma 3.14 If U is a principal region of $L \subset F$, then $\pi_1(U) \rightarrow \pi_1(F)$ is injective

Proof $F = \mathbb{H}^2/\Gamma$, $\Gamma = \pi_1(F)$. Let \tilde{U} be a component of $\rho^{-1}(U)$. \tilde{U} is a hyperbolically convex set $\Rightarrow \tilde{U}$ contractible

$\therefore \pi_1(U) \rightarrow \pi_1(F)$ is injective (indeed $\pi_1(U) \cong \Gamma_u$ the deck-transformations of F which map \tilde{U} into itself)

11.

Lemma 3.15 There is a bound (depending only on F) on the length of strictly monotonic sequences of laminations $F > L_1 > \dots > L_k$

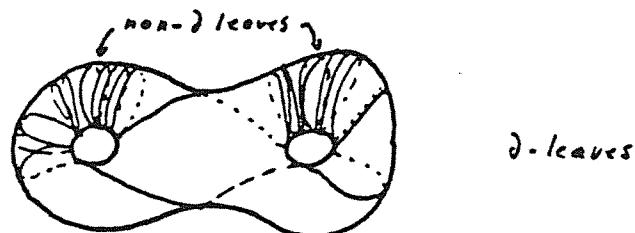
Cor : For any L , either $L^{(n)} = \emptyset$ for some n , or $L^{(n)}$ is perfect.
If L has no closed leaves, then $L^{(n)}$ is perfect for some n .

Proof $F - L_i \subset F - L_{i+1}$, so each principal region of L_i is contained in a unique principal region of L_{i+1} , and every principal region of L_{i+1} contains at least one principal region of L_i

$\therefore \#(\text{principal regions of } L_{i+1}) \leq \#(\text{principal regions of } L_i) \leq -2\%_F$
(as each principal region of L_i has area at least π)

It is enough to bound length when $\#(\text{principal regions})$ is constant.

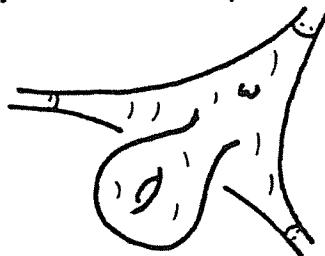
Remark: One might argue that the $\#(\text{boundary leaves})$ is decreasing, but consider



In this example the closed leaves are not boundary leaves as they are not isolated from one-side. Take L' , get only the closed leaves which are boundary leaves of L' .

Returning to the proof: Each principal region U_{i+1} of L_{i+1} contains just one principal region U_i of L_i ; U_i is dense in U_{i+1} .

Digression: $U \cong V - \partial V$ V a complete surface with totally geodesic boundary.
 If $A \subset U$ is compact, we can choose W a compact
 surface such that U is homeomorphic to $W - \partial W$,
 we'll be such that $A \subset \text{Int } W$
 $(W = \alpha - \text{collar (ends)})$



So suppose $U_i \subset U_{i+1} \subset U_{i+2} \dots$ proper inclusions
 $U_i \subset U_{i+1} \subset U_{i+2} \dots$ as dense subsets
 $w_i \subset w_{i+1} \subset w_{i+2} \dots$ U_i homeomorphic to $w_i - \partial w_i$

Observe that $\#(\text{boundary curves of } w_i) \leq \#(\text{boundary leaves of } U_i)$ is bounded depending on F .

Moreover, there exists $i < j$ such that $w_i \rightarrow w_j$ is a homotopy equivalence (w_i' are homotopy equivalent to U_i), and $U_i \rightarrow U_j$ induces an isomorphism on π_1 .

Next let $p: \tilde{M}^3 \rightarrow F$ be the universal cover. Let \tilde{U}_j be a component of $p^{-1}(U_j)$, \tilde{U}_j is a universal cover of U_j . Let $\tilde{U}_i := \tilde{U}_j \cap p^{-1}(U_i)$

Claim: \tilde{U}_i connected

Take two points in \tilde{U}_i , connect them by a path in \tilde{U}_j . Map this path to a loop in U_j , $\pi_1(U_i) \rightarrow \pi_1(U_j)$ is onto, so we get a loop in U_i . Lift this loop to \tilde{U}_i , claim follows.

$\therefore \tilde{U}_i$ is hyperbolically convex, dense in \tilde{U}_j

$\therefore \tilde{U}_i = \tilde{U}_j$

$\therefore U_i = U_j$

Repeat the above for all principal regions \Rightarrow sequence stabilizes
 II.

Remark: Examine the proof carefully and you'll see it works for ascending sequences as well.

Question: What is the best possible bound? (conj: κ_F^2)

10/25 Casson

LEMMA 3.16 If $L \in \Lambda(F)$ and $h: F_1 \rightarrow F_2$ is a homeomorphism, then $\hat{h}(L)$ (isolated leaf) is an isolated leaf and $\hat{h}(\partial\text{-leaf})$ is a ∂ -leaf, hence $\hat{h}(L') = (\hat{h}(L))'$

Proof If a leaf is isolated (from 1-side) its lift to M^2 cannot have both endpoints approximated arbitrarily closely by the endpoints of the lifts of other leaves of L . This is preserved by \hat{h} .

11.

Remark: $\hat{h}(L)$ is actually homeomorphic to L

§4 Automorphisms, after Nielsen (In modern terminology)

Lemma 4.1 If $h: F \rightarrow F$ is a non-periodic automorphism of a closed hyperbolic surface, then $\hat{h}(L) = L$ for some $L \in \Lambda(F)$

Recall: Periodic = $\exists n$ such that h^n is homotopic (\Rightarrow isotopic) to τ_F

Also, h non-periodic, Thm 2.6 $\Rightarrow \exists$ s.c.c. $C \subset F$ such that $h^n(C) \neq C$ for all $n > 0$.

$\therefore h^n(C) \neq h^m(C)$ if $m \neq n$

Proof Let C_n be the geodesic s.c.c. homotopic to $h^n(C)$, $C_{n+1} = \hat{h}(C_n)$, $C_n \in \Lambda(F)$; $\Lambda(F)$ compact $\Rightarrow \{C_n\}$ has a convergent subsequence $C_{n_i} \rightarrow K \in \Lambda(F)$.

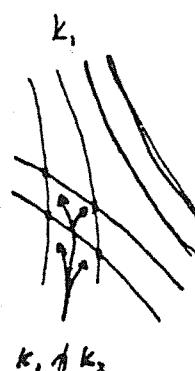
Set $L = \bigcup_{r \in \mathbb{R}} \hat{h}^r(K')$. We claim that L is a lamination, clearly $\hat{h}(L) \in L$

Remark: Taking closure is redundant.

The C_{n_i} are distinct, hence K is not an isolated point of $\Lambda(F)$.
Lemma 3.6 says that $K' \neq \emptyset$.

For laminations $k_1, k_2 \subset F$, let $K, \Delta k_2$ be the set of transverse intersections of k_1, k_2 : $\{x \in F \mid x \circ \gamma_i \in k_i, \gamma_i \notin k_2\}$

We must show $\hat{h}^r(K') \cap \hat{h}^s(K') = \emptyset$ for all $r \neq s$. It is enough to show that $\hat{h}^r(K') \cap K' = \emptyset$ (maybe though $\hat{h}^r(k) \cap k \neq \emptyset$) (Th. 3 shows the leaves of L are distinct, already know they are simple)



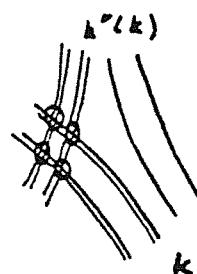
Recall that $C_{n_i} \rightarrow K$, $\hat{h}^r(C_{n_i}) \rightarrow \hat{h}^r(k)$, and that by 3.10 \hat{h} is continuous.

Fix r , $\hat{h}^r(C_{n_i}) = C_{n_i+r} \rightarrow \hat{h}^r(k)$, $C_{n_i} \cong h^{n_i}(c)$, $C_{n_i+r} \cong h^{n_i}h^r(c)$

$$\Rightarrow |C_{r+n_i} \cap C_{n_i}| \leq |h^{n_i}(h^r(c)) \cap h^{n_i}(c)| = |h^r(c) \cap c| = N_r$$

minimal / as curves are geodesics make transverse first

Suppose $|K \cap \hat{h}^r(k)| > N_r$. Take disjoint neighborhoods of the transverse intersections. As $C_{r+n_i} \rightarrow \hat{h}^r(k)$, $C_{n_i} \rightarrow k$ for sufficiently large i ; C_{n_i}, C_{r+n_i} will have at least one transverse intersection in each neighborhood. e.g. $|C_{n_i} \cap C_{r+n_i}| > N_r$



$$\therefore |K \cap \hat{h}^r(k)| \leq N_r < \infty$$

$\therefore K \cap (\hat{h}^r(k))^{\circ} = \emptyset$ as transverse intersection with a non isolated leaf implies infinite intersection with the lamination.

$$\therefore K \cap \hat{h}^r(k') = \emptyset \quad \text{as } (\hat{h}^r(k))^{\circ} = \hat{h}^r(k')$$

and the lemma follows

II.

Remark: The $\bigcup_{r \in \mathbb{Z}}$ in the above proof is just a finite union

Def $h: F \rightarrow F$ is reducible, if there exists a non-empty 1-manifold $C \subset F$ (C = disjoint union of s.c.c's) and a homeomorphism $g \cong h$ such that $g(C) = C$. Otherwise h is irreducible

Lemma 4.2 If $h: F \rightarrow F$ is irreducible and $\hat{h}(L) = L$ for some $L \in \Lambda(F)$, then every principal region of L is contractible (i.e. each is isometric to a finite sided ideal polygon in \mathbb{H}^2) and $L'' \subseteq L'$ (i.e. L' perfect).

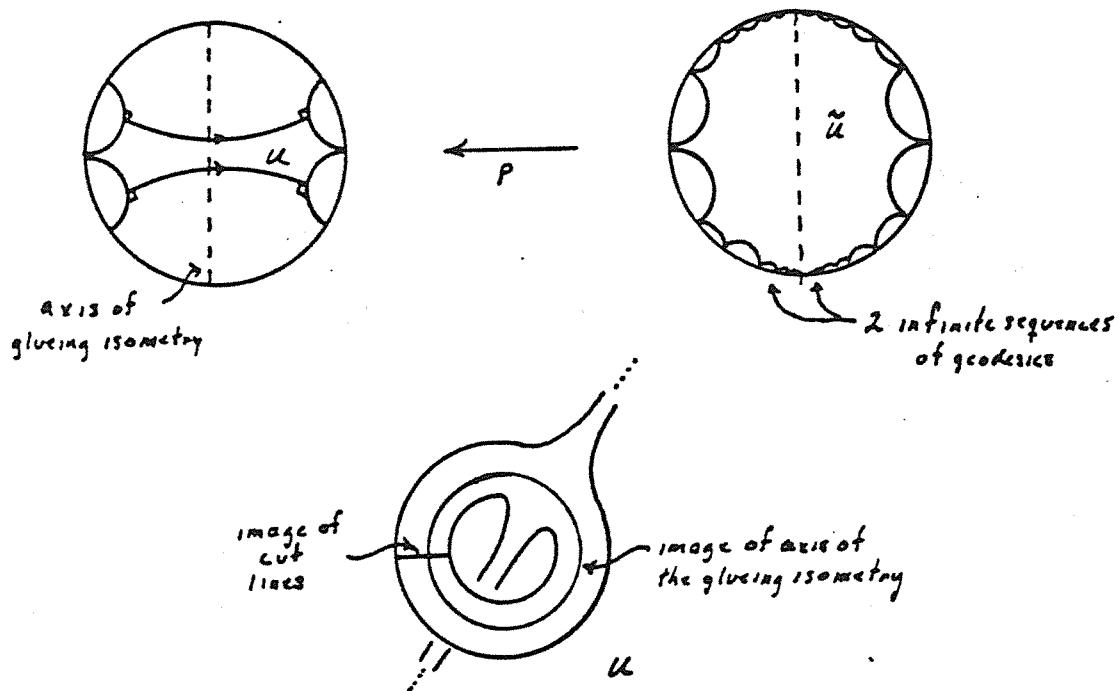
Proof Note that L has no closed leaves, for otherwise the union of all closed leaves of L is a non-empty 1-manifold (notice - a finite number of components) invariant under \hat{h} , implying h reducible.

Let $p: \mathbb{H}^2 \rightarrow F$ be the universal cover, U a principal region of L which is connected but not contractible, \tilde{U} a lift of U to \mathbb{H}^2 .

10/28 Casson

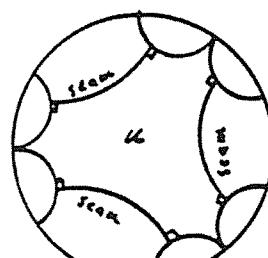
Example: \mathcal{U} = twice boundary punctured annulus

\mathcal{U} is obtained as illustrated below, $\tilde{\mathcal{U}}$ is obtained by replicating the pattern via the axis of the hyperbolic isometry used to "glue". This axis is



mapped to the "core" of \mathcal{U}

A similar phenomena occurs for the boundary punctured "pentalong" = compact core with non-compact boundary.



Front Surface

Returning to the proof: Let $\tilde{h}: \mathbb{H}^2 \cup S'_\infty \rightarrow \mathbb{H}^2 \cup S'_\infty$ be an extension of some lift of $h: F \rightarrow F$. The ideal vertices of $\tilde{\mathcal{U}}$ have the property that if \tilde{V}

is a lift of a leaf of L , then no two vertices of \tilde{C} separate the endpoints of $\tilde{\gamma}$.
 Indeed the vertices of \tilde{C} are a maximal set w.r.t. this property.

$\therefore \tilde{h}$ carries the vertices of \tilde{C} to the vertices of some lift \tilde{C}' of a principal region C' of L .

If C is a simple closed curve in U , essential in U ; then C is essential in F and so homotopic to a geodesic γ in F . If lift \tilde{C} in \tilde{U} , and a lift $\tilde{\gamma}$ of γ with the same endpoints as \tilde{C} . U has no closed boundary leaves, the convexity of \tilde{U} $\Rightarrow \tilde{\gamma} \subset \tilde{C}$; so the homotopy of C to γ stays in U , so $\gamma \subset U$

Next write:

$$U \approx W \cup (\partial W \times [0, \infty)) ; \partial W = C_1 \cup \dots \cup C_k \cong \gamma_1 \cup \dots \cup \gamma_k$$

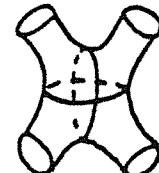
$$U' \approx W' \cup (\partial W' \times [0, \infty)) ; \partial W' = C'_1 \cup \dots \cup C'_k \cong \gamma'_1 \cup \dots \cup \gamma'_k$$

where $\gamma_1, \dots, \gamma_k, \gamma'_1, \dots, \gamma'_k$ are closed geodesics in U, U' resp and W, W' compact.

Claim: \tilde{h} carries $\gamma_1, \dots, \gamma_k$ to $\gamma'_1, \dots, \gamma'_k$ in some order

This follows from the fact that boundary components are the only simple closed curves for which every other s.c.c. may be homotoped off of. Hence $\gamma_1, \dots, \gamma_k$ are the only s.c.c. geodesics in U which are disjoint from (or coincide with) all s.c.c. geodesics in U .

\Rightarrow this property is preserved by $\tilde{h} \Rightarrow k \geq l$ and the claim follows.



Do this for all non-contractible principal regions of L . we obtain a disjoint union Σ of s.c.c. geodesics in F $\exists' \tilde{h}(\Sigma) = \Sigma$

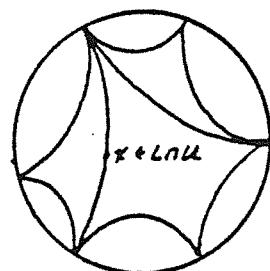
$\therefore h$ reducible



To show L' perfect:

$\tilde{h}(L'') = L''$, so the principal regions of L'' are finite sided ideal polygons. Suppose that $x \in L'$ belongs to an isolated leaf of L' : $x \notin L''$.

$\therefore x \in U$ a principal region of L'' , a finite sided ideal polygon. $L \cap U$ consists of (some of) the finitely many diagonals of $U \Rightarrow x$ is on an isolated leaf of L . (Otherwise, in the universal cover there would be lifts of leaves arbitrarily close to x)



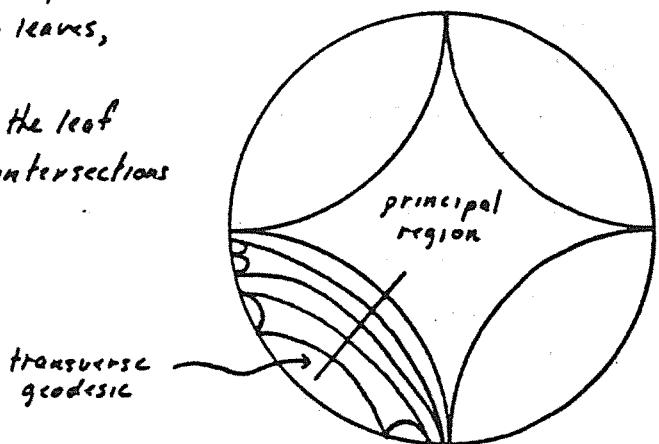
10/30 COSSOR

Remark: These laminations are quite complicated, as we illustrate below.

Example: Consider a point on a boundary leaf.

Arbitrarily close there must be boundary leaves,
so there are lots of principal regions.

Indeed on a transverse geodesic to the leaf
the point lies on there is a countable set of intersections
with other boundary leaves.



Recall: Given a homeomorphism $h: F \rightarrow F$, get $h_*: H_1(F) \rightarrow H_1(F)$. Choose a basis,
get a matrix A . Set $\chi_h(t) = \text{characteristic polynomial of } h_* \circ I - tI$

Lemma 4.3 If $\chi_h(t)$ is irreducible over \mathbb{Z} and the zeros of $\chi_h(t)$ are not
complex n^{th} roots of unity for any n , and $\chi_h(t)$ is not a polynomial in t^n
for any $n > 1$; then h is non-periodic and irreducible.

Proof: If h periodic, then $h^n \equiv 1$, for some n , then all the zeros of $\chi_h(t)$ are
complex n^{th} roots of unity.

If h reducible, then (after isotopy) $h(C) = C$ for some 1-manifold C
whose components may be assumed to be essential, i.e. geodesics.

Case 1: If some component C_i of C is non-separating, then $0 \notin [C_i] \in H_1(F)$.

There exists n such that $h^n(C_i) = C_i$, preserving orientation $\Rightarrow (h_*)^n[C_i] = [C_i]$,
so $(h_*)^n$ has eigenvalue 1 $\Rightarrow h_*$ has an n^{th} root of unity as an eigenvalue.

\Rightarrow zeros of $\chi_h(t)$ are complex roots of unity

Case 2: All components of C separate

we can find a component which is the total boundary of a component
of $F \setminus C$. See pic next page

The exists a component F_0 of $F - C$ such that $F_r F_0 \cong S^1$.
 Let $F_r = h^r F_0$, there exists a least n such that $F_n = F_0$

Now

$$H_1(F) \cong H_1(F_0) \oplus \dots \oplus H_1(F_{n-1}) \oplus H_1(G)$$

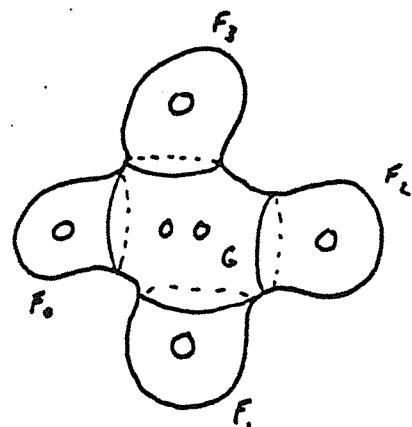
(notice: G might be a punctured sphere)

This splitting is 'h-invariant' i.e. h_g permutes
 the $H_1(F_i)$ cyclically.

Choose a basis for $H_1(F_i)$, then the matrix of h_g ,
 call it A , is ($n=4$):

$$A = \begin{pmatrix} 0 & 0 & 0 & B & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & C \end{pmatrix}$$

Note: C may be 0

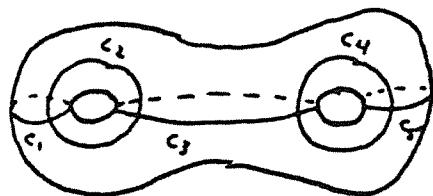


$$\text{Exercise: } \chi_h(t) = |A - tI| = |B - t^n I| \cdot |C - tI|$$

But $|B - t^n I|$ is a polynomial in t^n

II.

Example: Let T_i = Dehn twist in C_i as illustrated below (positive twist to right)



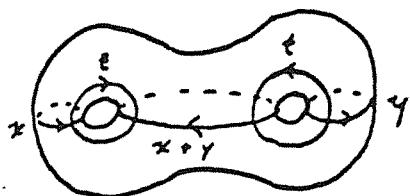
Claim: $T_1 T_3 T_5^{-2} T_2^{-1} T_4^{-1}$ is irreducible; non-periodic

Remark: (More is true - Dehn) Any word where odd indices carry same sign,
 and even indices the opposite sign)

10/30-11/1 Lesson

5.

Use the basis:



The matrix for each Dehn twist is

$$T_1^* = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T_2^* = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T_3^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

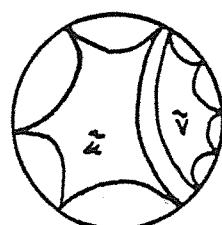
$$T_4^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T_5^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Exercise: Polynomial is $t^4 - 9t^3 + 21t^2 - 9t + 1$ and is irreducible over \mathbb{Q} and has no complex roots of unity as zeros.

Lemma 4.4 Suppose $\tilde{h}(l) = l$, U a contractible component of $F-L$, \tilde{U} a lift of U to the universal covering of F . Then $\exists m > 0$ and a lift \tilde{k} of h^m with extension \bar{k} such that \bar{k} fixes all the vertices of \tilde{U} .

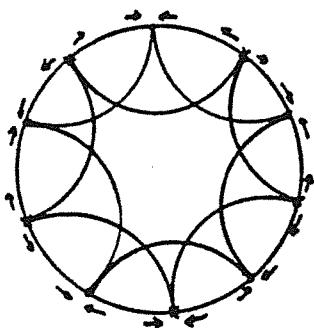
Proof There exists a component V of $F-L$ and a lift \tilde{V} of V such that $\tilde{h}(\{\text{vertices of } \tilde{U}\}) = \{\text{vertices of } \tilde{V}\}$. As $\{\text{vertices of } \tilde{U}\}$ is a finite set joined cyclically by geodesics, $\tilde{h}(\{\text{vertices of } \tilde{U}\})$ is a finite set joined cyclically by geodesics. By the contractibility of U , $F-L$ has only finitely many components, (h permutes the complementary regions), so $\exists m, > 0$ such that $\tilde{h}(\{\text{vertices of } \tilde{U}\}) = \{\text{vertices of } \tilde{U}'\}$ where \tilde{U}' is also a lift of U . Now there exists a deck translation $g \in \pi_1(F-L)$ such that $g(\tilde{U}') = \tilde{U}$.

Put $\tilde{k}_i = g^{-1} \tilde{h}^{m_i}$, then $\tilde{k}_i(\{\text{vertices of } \tilde{U}\}) = \{\text{vertices of } \tilde{U}\}$. Maybe \tilde{k}_i permutes these vertices, but as there are only finitely many \tilde{k}_i , \tilde{k}_i fixes the vertices of \tilde{U} for some i . Setting $m = m_i, g_i = (g^{-1} \tilde{h}^{m_i})^{-1}$ completes the proof.



Goal: h irreducible, non-periodic; \tilde{E} extended lift of h^m . The vertices of \tilde{U} are attracting fixed points of $\tilde{E} \Rightarrow \exists$ repelling fixed points of \tilde{E} .
There are obtained by doing the same procedure to h^m

pix:



Lemma 4.5 Suppose $\tilde{h}^{n_i}(c) \rightarrow k$ in $A(F)$ ($n_i \rightarrow +\infty$), \tilde{k} an extended lift of $k = h^m$ for some $m > 0$ such that \tilde{E} has fixed points on S'_∞ . Let \tilde{c} be a lift of c to the universal cover with endpoints A, B . Then $\lim_{r \rightarrow +\infty} \tilde{E}^r A = A_\infty$,

$\lim_{r \rightarrow +\infty} B = B_\infty$ exist, are fixed points of \tilde{E} , and either $A_\infty = B_\infty$, or
 \nearrow geodesic with these endpoints \searrow considered as $\tilde{E}: S'_\infty \rightarrow S'_\infty$
 $\rho(A_\infty B_\infty)$ is a leaf of $\tilde{h}^q(K)$ for some $0 \leq q \leq m$.

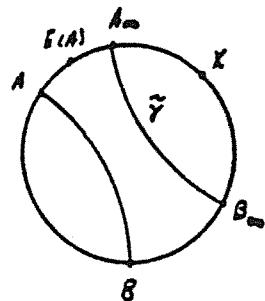
Proof If $\tilde{h}(A) = A$, then $A = A_\infty$. If $\tilde{E}(A) \neq A$, let X be a fixed point of $\tilde{E}|S'_\infty$. Let A_∞ be the first fixed point on the arc $[\tilde{E}(A), A]$ of S'_∞ containing X . Then $\tilde{E}[A, A_\infty] = [\tilde{E}(A), A_\infty]$ and by construction A_∞ is the only fixed point on $[A, A_\infty]$.

$\therefore \tilde{E}^r(A) \rightarrow A_\infty$, similarly $\tilde{E}^r(B) \rightarrow B_\infty$

(note: cannot determine the direction A, B move in)

If $A_\infty \neq B_\infty$ call the geodesic joining them $\tilde{\gamma}$, and set $\gamma = \rho(\tilde{\gamma})$. Now, there exist infinitely many n_i in some residue class mod m . Hence there exists q such that $n_i + q \equiv m \pmod{r_i}$ for infinitely many i . Take a subsequence so that $n_i + q \equiv m r_i$ for all i and renumber:

$$\begin{aligned} \tilde{h}^{n_i+q}(c) &\rightarrow \tilde{h}^q(K) \\ \tilde{h}^{n_i}(c) &\rightarrow \tilde{h}^q(K) \end{aligned}$$



11/3 Casson

Claim: γ is a leaf of $\hat{h}^q(L)$, i.e. $\gamma \in$ Hausdorff limit of $\hat{h}^{ri}(C)$

Any $x \in \gamma$, choose lift \tilde{x} on $\tilde{\gamma}$. Can choose i sufficiently large (depending on ϵ) such that the endpoints $\hat{h}^{ri}(A), \hat{h}^{ri}(B)$ approximate $A_\infty, B_\infty \rightarrow \tilde{\gamma}$ can be approximated arbitrarily closely (ϵ) by a point of a geodesic joining $\hat{h}^{ri}(A), \hat{h}^{ri}(B)$.

Remark: This argument presupposes the existence of the Hausdorff limit as the convergence described above is not uniform. The existence of this Hausdorff limit in this case is guaranteed by other leaves

II.

Lemma 4.6 Suppose $h: F \rightarrow F$ is irreducible and $\hat{h}(L) = L$ for some $L \in \Lambda(F)$.

Let $\tilde{\gamma}$ be a lift of a leaf $\gamma \in L$ and C an essential s.c.c. in F . Then $\exists q \in \mathbb{Z}$ and a lift \tilde{C}_q of C_q where the endpoints separate the endpoints of $\tilde{\gamma}$ on S'_∞

Proof Suppose not and that C is a closed geodesic. Then for all $q \in \mathbb{Z}$ every lift of $\hat{h}^q(C)$ has endpoints not separating the endpoints of $\tilde{\gamma}$. If $q \geq 0$, then C has no transverse intersection with γ ; moreover, C has no transverse intersection with $\hat{h}^{-q}(\gamma)$ for $q \in \mathbb{Z}$.

i.e. $\hat{h}^{-q}(\gamma) \cap C = \emptyset$

Now $L = \hat{h}(L)$, hence $\hat{h}^{-q}(\gamma)$ is a leaf of L . Let $L_i = \overline{\cup_{j \geq 0} \hat{h}^{-q-j}(\gamma)} \subseteq L$. Then $C \cap L_i = \emptyset$ (for otherwise C would meet lots of $\hat{h}^{-q-j}(\gamma)$ s). If $C \cap L_i = \emptyset$, then C is contained in some component of $F \setminus L_i$.

But as $\hat{h}(L_i) = L_i$, and h irreducible, Lemma 4.2 $\Rightarrow F \setminus L$ has contractible components $\Rightarrow C$ inessential

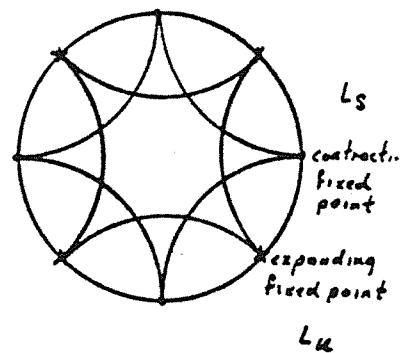
If $C \cap L_i \neq \emptyset$, C is a leaf of L_i ; a lamination with no closed leaves.

II.

Def If $f: S' \rightarrow S'$ has a fixed point A , A is a contracting fixed point if there exists a neighborhood U of A such that $x \in U \Rightarrow f^n(x) \rightarrow A$ as $n \rightarrow +\infty$. A is an expanding fixed point if there exists a neighborhood U of A such that $x \in U \Rightarrow f^n(x) \rightarrow A$ as $n \rightarrow -\infty$.

Remark: We next consider a special case of a main theorem of Nielsen.

Theorem 4.7 Let $h: F \rightarrow F$ be irreducible and non-periodic (h orientation pres.) Then any extended lift of a strictly positive power of h has finitely many fixed points on S^1_∞ , alternately contracting and expanding. Moreover, there is a unique perfect lamination L^S , invariant under \tilde{h} , such that if \tilde{k} is an extended lift of a strictly positive power of h and X, Y are consecutive contracting fixed points of $\tilde{k}|S^1_\infty$, then $p(\text{geodesic joining } X \text{ and } Y)$ is a leaf of L^S .



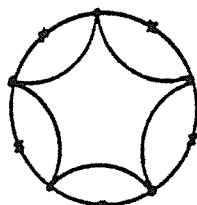
Remark: L^S is called the stable lamination of h . The unstable lamination of h , L^U , is the stable lamination of h^{-1} (and is obtained by substituting expanding for contracting in the above).

Remark: \tilde{h} and $g\tilde{h}$ ($g \in M$) may have differing numbers of fixed points on S^1_∞ .

Strategy of Proof: We'll first construct L^S . If E is an extended lift of a (strictly positive) power of h , we shall consider 3 cases:

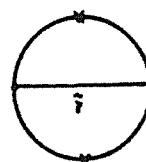
1) \tilde{k} fixes the vertices of a component $\tilde{\pi}$ of $M^2 - p^{-1}(L^S)$

In this case we'll prove that the vertices of $\tilde{\pi}$ are contracting, and that any two vertices which are consecutive are separated by a unique expanding fixed point. Note \tilde{k} has at least 6 fixed points in this case.



2) \tilde{k} fixes some non-boundary leaf $\tilde{\gamma}$ of $p^{-1}(L^S)$

Here we will show that the endpoints of $\tilde{\gamma}$ are contracting and separated by two expanding fixed points (and that no other fixed points exist).



3) \tilde{k} does not fix the endpoints of any leaf of $p^{-1}(L^S)$

Here we'll show that \tilde{k} has exactly 1 expanding; 1 contracting or no fixed points.

Finally we'll prove L^S unique.

11/5 Cesson

We shall need the following addendum to §3

Lemma 3.17 Let L be a geodesic lamination without closed leaves on a closed hyperbolic surface F such that all the components of $F \setminus L$ are contractible. If $Y \in S_\infty$, only finitely many lifts of leaves of L have Y as an endpoint.

Proof Suppose infinitely many, then infinitely many lifts of boundary leaves have endpoint Y , as the boundary leaves are dense. There exists only finitely many oriented boundary leaves in F .

\therefore There exists $g \in \Gamma - id$ carrying a boundary leaf beginning at Y to another such leaf.

$\therefore Y$ is a fixed point of g .
 g hyperbolic $\Rightarrow g$ has an axis \tilde{c} covering an essential closed curve C in F .

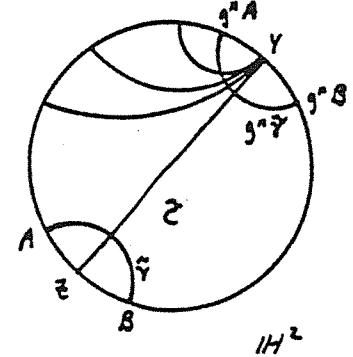
As the components of $F \setminus L$ are contractible and L having no closed leaves it follows that $C \cap L \neq \emptyset$.

$\therefore \tilde{c}$ intersects some leaf $\tilde{\gamma}$ of $p^{-1}(C)$ transversely.

LOG suppose that Y is the contracting fixed point of g . For large n , $g^n \tilde{\gamma}$ has endpoints on either side of Y near Y .

$\therefore g^n \tilde{\gamma}$ (a leaf of $p^{-1}(C)$) intersects leaves of $p^{-1}(C)$ with endpoint Y .

~~✗~~



H.

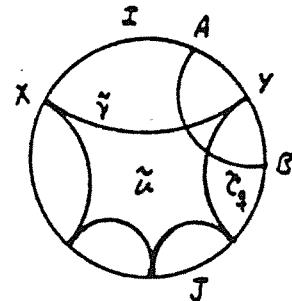
Proof of 4.7 By Lemma 4.1 (h non-periodic) There exists a s.c.c. C such that $\hat{h}^{n_i}(C) \rightarrow K$ as $n_i \rightarrow +\infty$ with $K \neq \emptyset$. Set $L = \bigcup_{k \in E} \hat{h}^k(K)$. Then $\hat{h}(L) = L$.

By Lemma 4.2 (h irreducible) L has no closed leaves, so $L' = L''$. Set $L^S = L'$ which is perfect, $\hat{h}(L^S) = L^S$, the components of $F \setminus L^S$ are contractible, components of $H^2 - p^{-1}(L^S)$ are finite sided ideal polygons.

Case 1: Suppose that \bar{k} , an extended lift of $h^m \circ k$, fixes the vertices of \bar{U} , a component of $H^2 - p^{-1}(L^S)$.

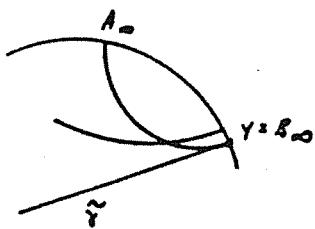
Let $\tilde{\gamma}$ be a boundary leaf of \tilde{A} with endpoints X, Y . Lemma 4.6
 \Rightarrow For some q , there is a lift \tilde{c}_q of the geodesic, c_q , homotopic
to $h^q(c)$ such that the endpoints A, B separate X, Y .

Remark: Call the arc XY of S'_∞ with no endpoints of \tilde{A} the
short arc and denote it by I . Similarly, call the other
arc in S'_∞ the long arc and denote it by J .



Suppose wlog that $A \in I$. By lemma 4.5 $\bar{k}^r(A) \rightarrow A_\infty, \bar{k}^r(B) \rightarrow B_\infty$
as $r \rightarrow +\infty$. Either $A_\infty = B_\infty$ or $p(\text{geodesic forming } A_\infty, B_\infty)$ is a leaf of
 $\bar{h}^q(k)$. From the proof of lemma 4.1 we have $\bar{h}^q(k) \cap L^3 = \emptyset$ (Remark: Actually
showed $\bar{h}^q(k) \cap L = \emptyset$)

$\therefore p(\text{geodesic } A_\infty B_\infty) \cap L^3 = \emptyset$; in particular geodesic $A_\infty B_\infty \cap \tilde{\gamma} = \emptyset$
 $A_\infty \in \bar{I}, B_\infty \in \bar{J}$. If A_∞ is in Int I , then $B_\infty = X$ or Y ; suppose Y .



As $\tilde{\gamma}$ is a boundary leaf, \exists a leaf of $p^{-1}(L^3)$
arbitrarily close to $\tilde{\gamma}$, but not having Y as an
endpoint (Lemma 3.17).

As shown at left, such a leaf would meet
geodesic $A_\infty B_\infty$ transversely



Hence $A_\infty = X$ or Y , say X . It follows that X is a contracting fixed
point from the I -side.

Choose a leaf \tilde{s} of $p^{-1}(L^3)$ so close to $\tilde{\gamma}$ such that

1) Denoting the endpoints of \tilde{s} by u, v ; If $p \in [X, u] \subseteq I$,
then $\bar{k}^r(p) \rightarrow X$ as $r \rightarrow +\infty$

2) u is closer to X , than either of v or $\bar{k}(v)$

This is possible as Y is fixed and \bar{k} continuous.

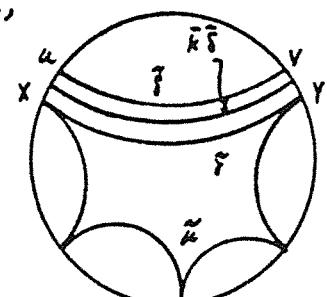
3) $u \neq X, v \neq Y$

Claim: If $Q \in [v, Y] \cap I$, then $\bar{k}^r(Q) \rightarrow Y$ as $r \rightarrow +\infty$

Proof: $\bar{k}(u)$ must go towards X , $\bar{k}\tilde{s} \cap \tilde{s} = \emptyset$

$\Rightarrow \bar{k}(v)$ goes towards $Y \Rightarrow \bar{k}(Q)$ goes towards Y .

$\therefore Y$ is a contracting fixed point of \bar{k} from the I side.

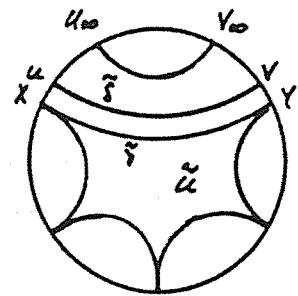


11/8 - 11/10 Casson

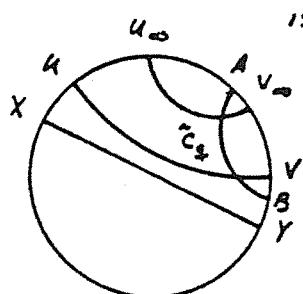
5

Apply the above argument to all the edges of \tilde{U} , then all the vertices of \tilde{U} are contracting fixed points of \tilde{k} .

We have shown that there exists a leaf \tilde{S} of $p^{-1}(L^3)$ with endpoints u, v such that $\tilde{k}^n(u) \rightarrow x, \tilde{k}^n(v) \rightarrow y$ as $n \rightarrow +\infty$.
 $\tilde{k}^{-n}(u) \rightarrow u_\infty, \tilde{k}^{-n}(v) \rightarrow v_\infty$ for some u_∞, v_∞ 'between' u and v . If $u_\infty = v_\infty$, then u_∞ is an expanding fixed point for \tilde{k} and the only fixed point of \tilde{k} 'between' x and y . If $u_\infty \neq v_\infty$, the geodesic $u_\infty v_\infty$ projects to a leaf of L^3 , as L^3 is closed.



In this case, there exists a leaf \tilde{C}_g , a lift of $h^k(c)$ (homotopic to a geodesic c) whose endpoints A and B separate u_∞, v_∞ ; say A is in the 'short' arc. $\tilde{k}^n(A) \rightarrow A_\infty, \tilde{k}^n(B) \rightarrow B_\infty$ as $n \rightarrow +\infty$.



Either $A_\infty = B_\infty$ or $\rho(\text{geodesic } A_\infty B_\infty) \wedge L^3 = \emptyset$
 A_∞ & short closed arc $u_\infty v_\infty \in S'_\infty$
 B_∞ & long " " " "

In fact B_∞ & long closed arc XY in $S'_\infty = J$
 \Rightarrow geodesic $A_\infty B_\infty$ meets the leaf \tilde{S} of $p^{-1}(L^3)$ transversely
 $\Rightarrow \rho(A_\infty B_\infty) \wedge L^3 \neq \emptyset$

★

Hence $u_\infty = v_\infty$ and there is a unique expanding fixed point of \tilde{k} on I .

Case 2 : \tilde{k} fixes the endpoints of a leaf \tilde{Y} of $p^{-1}(L^3)$, but fixes the vertices of no component of $W^2 - p^{-1}(L^3)$

If \tilde{Y} was a boundary leaf, \tilde{k} would 'fix' the isolated side $\Rightarrow \tilde{k}$ fixes the vertices of the whole region, a contradiction.

Thus \tilde{Y} is not a boundary leaf, $\therefore \tilde{Y}$ is not isolated from either side.
 Use the same argument as in case 1 to show that the endpoints of \tilde{Y} are contracting fixed points, separated by a pair of expanding fixed points, with no other fixed points of \tilde{k} .

Case 3: \bar{E} does not fix the endpoints of any leaf of $p^{-1}(L^S)$

Suppose $\bar{E}|S_\infty$ fixes some $x \in S_\infty$ (if not there is nothing to prove)

Notice that X cannot be the endpoint of any leaf of $p^{-1}(L^S)$. For by L3.1F, there are only finitely many such leaves, so one such would have to be fixed.

Let $\tilde{\alpha}$ be any geodesic with one endpoint X . $\tilde{\alpha}$ meets some leaf $\tilde{\gamma}$ of $p^{-1}(L^S)$ transversely (otherwise $\tilde{\alpha}$ is contained in some component of $H^2 - p^{-1}(L^S)$ and therefore a 'diagonal'). This says X is an endpoint of a leaf of $p^{-1}(L^S)$ *)

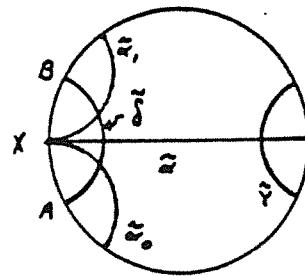
By a compactness argument we obtain for geodesics $\tilde{\alpha}_0, \tilde{\alpha}$, through X that there are finitely many leaves $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ of $p^{-1}(L^S)$ such that every $\tilde{\alpha}_t$ between $\tilde{\alpha}_0$ and $\tilde{\alpha}$, meet some $\tilde{\gamma}_j$. Therefore, there exists a $\tilde{\delta} \in p^{-1}(L^S)$ meeting $\tilde{\alpha}_0$ and $\tilde{\alpha}$, transversely, so there exists $\tilde{\gamma} \in p^{-1}(L^S)$ with endpoints A, B near X and on opposite sides of X .

If close enough $\bar{E}(A)$ and $\bar{E}(B)$ are also close to X ; $\bar{E}(A)$, A on one side and $\bar{E}(B)$, B on one side. It follows that $\lim_{n \rightarrow \infty} \bar{E}^n(A) = \lim_{n \rightarrow \infty} \bar{E}^n(B)$, a

contracting fixed point, and $\lim_{n \rightarrow -\infty} \bar{E}^n(A) = \lim_{n \rightarrow -\infty} \bar{E}^n(B)$ is an expanding fixed

point, as \bar{E} fixes the endpoints of no leaf.

Finally, to see that L^S is unique; consider a perfect lamination containing the geodesics connecting the contracting fixed points. As in the proof of Lemma 4.5, this lamination must contain the boundary leaves of L^S . As these leaves are dense in L^S , our lamination must contain all the leaves of L^S . Any extra leaves lie in principal regions of $p^{-1}(L^S)$. But such a region is a finite sided ideal polygon \Rightarrow extra leaves are 'diagonals' \Rightarrow extra leaves are isolated. But there are no isolated leaves in a perfect lamination.

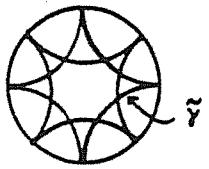


11/15 Cossor

Theorem 4.8 Every leaf of L^S is dense in L^S (similarly for L^U)

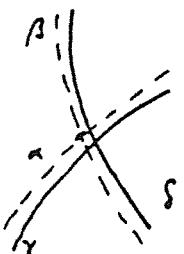
Cor Every leaf of L^S meets every leaf of L^U transversely

Proof of Cor: There exists a leaf γ of L^S and a leaf δ of L^U meeting transversely.
E.g. Take γ a boundary leaf and let $\tilde{\gamma}$ be a lift. Join two
expanding fixed points to get $\tilde{\delta} \in p^{-1}(L^U)$ such that $\tilde{\gamma} \pitchfork \tilde{\delta}$
is a single point.



Now let α, β be any leaves of L^S and L^U respectively.

As α is dense in L^S and β dense in L^U , α
meets β transversely close to any point of
 $\gamma \pitchfork \delta$ proving the corollary.



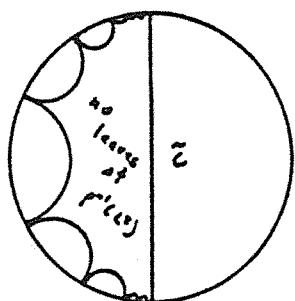
Proof of 4.8 Suppose $L_0 \in L^S$ is a sublamination. If $L^S \neq L_0$ there exists
 $\alpha \in L^S \setminus L_0$, where U is a component of $F \setminus L_0$. Let β be a boundary
leaf of U , $\tilde{\beta}$ a lift of β to H^2 , and $\tilde{\alpha}$ a lift of α .

Let $\tilde{\beta}_n, \tilde{\beta}_{n+1}$ be adjacent boundary leaves. Continue.

Get $\tilde{\beta}_n \cap \tilde{\beta}_{n+1} = \{x\}$. These are all distinct as otherwise
 $\tilde{\beta}$ is a finite sided polygon, with x on a diagonal leaf.
Such a leaf is isolated.

U has only finitely many boundary leaves, so
there is a deck translation $g \in M$ such that $g\tilde{\beta}_n = \tilde{\beta}_n$
for some $n > 0$.

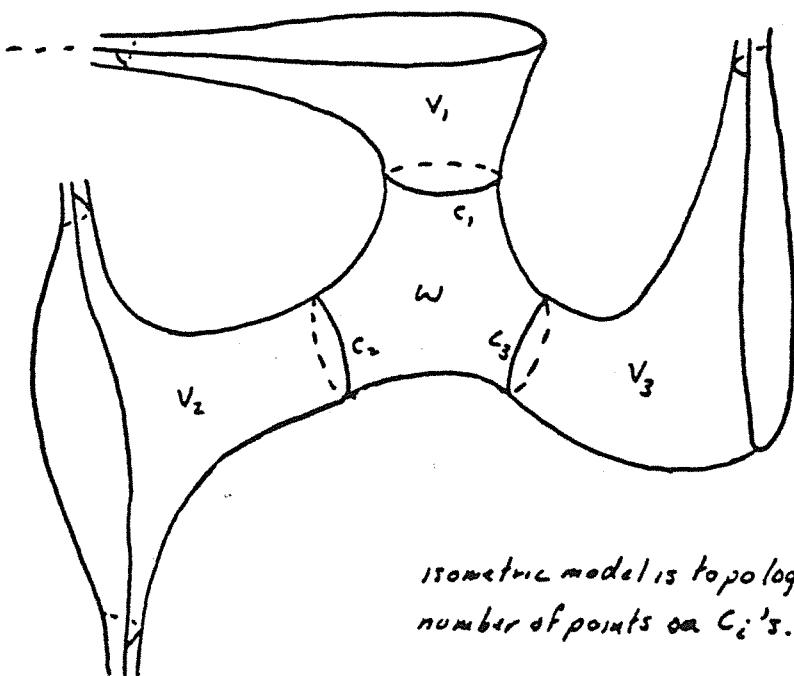
Now, g is hyperbolic, so g has an axis \tilde{C} covering
a closed curve $C \subset F$. Observe that the endpoints of the $\tilde{\beta}_n$'s tend to
the endpoints of \tilde{C} . That is, if B_{n-1}, B_n are the endpoints of $\tilde{\beta}_n$, and
 X, Y are the endpoints of \tilde{C} , $B_n \rightarrow Y$ as $n \rightarrow +\infty$, $B_n \rightarrow X$ as $n \rightarrow -\infty$.



Also note that there are no leaves of $p^{-1}(L^S)$ between the
 $\tilde{\beta}_n$'s and \tilde{C} as such are diagonals and so isolated.

C is also simple: For otherwise, there is a \tilde{C}' with $\tilde{C} \cap \tilde{C}' \neq \emptyset$. \tilde{C}' must be disjoint from the $\tilde{\beta}_n$'s, so an endpoint of \tilde{C}' must also be an endpoint of some $\tilde{\beta}_n$. But \tilde{C}' is the axis of a deck translation $g' \in P$. Applying g' to $\tilde{\beta}_n$ would give an infinite number of leaves of $p^{-1}(L_0)$ with a common endpoint. \star

Do this for all the boundary leaves of U . Get disjoint simple closed geodesics C_1, \dots, C_k inbounding a compact region W , with 'collars' V_1, \dots, V_k .



isometric model is topologically W -finite
number of points on C_i 's.

Now $L^S \cap U \subset W$ for if any leaf was in a collar, then the endpoints of this leaf would coincide with that of a $\tilde{\beta}_n$, again isolating the leaf.

$$\therefore L^S \subset W \cup F \setminus U \text{ (a disjoint union)}$$

As $x \in L^S \cap W$, $L^S \cap W \neq \emptyset$; L^S is a closed subset of a connected surface; so if L^S is disconnected, some component of $F \setminus L^S$ is not simply connected, a contradiction.

$$\therefore L^S \cap F \setminus U = \emptyset \Rightarrow L^S \subset U \subset F \setminus L_0 \Rightarrow L^S \cap L_0 = \emptyset, \text{ but } L_0 \subset L_S$$

\star

Hence L^S has no sublaminations \Rightarrow every leaf is dense

11/15 - 11/17 Lesson

Lemma 4.9 If $h: F \rightarrow F$ is irreducible and non-periodic, and if C is an essential s.c.c. in F , then $h^n(C) \cong h^m(C)$ iff $m = n$.

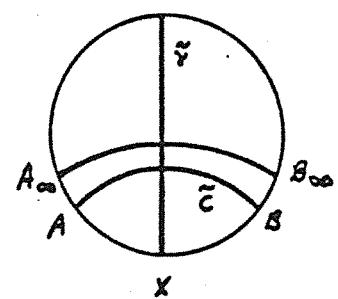
Proof Let C_g be the geodesic homotopic to $h^q(C)$ with $C_0 = C$. If the lemma failed for C , $\bigcup_{g \in \mathbb{Z}} C_g$ is a finite union of geodesics.

Let $\tilde{\gamma}$ be a lift of a leaf γ of L^u with endpoints $X'_f Y$ which are fixed by a lift \tilde{C} of a power of h . $C \cap L^u = \emptyset$, and $\gamma = p(\tilde{\gamma})$ is dense in L^u . Lemma 4.8 $\Rightarrow C \cap \tilde{\gamma} \neq \emptyset$.

\therefore There exists a lift \tilde{C} of C such that $\tilde{C} \cap \tilde{\gamma} \neq \emptyset$.
Let A, B be the endpoints of \tilde{C} . If we iterate \tilde{h} , $A \rightarrow A_\infty, B \rightarrow B_\infty$ contracting fixed points. But $X'_f Y$ are expanding fixed points.

Under our assumption $p^{-1}(L^u(C_g))$ is a closed subset of H^2 . The geodesic $A_\infty B_\infty$ is not in this subset, but it must be in the closure as $\tilde{C}_g \rightarrow$ geodesic $A_\infty B_\infty$ *

(Alternatively, note that $\tilde{h}^n(A), \tilde{h}^n(B) \rightarrow A_\infty, B_\infty \in S_\infty$ as $n \rightarrow \infty$
 $p(A_\infty B_\infty)$ is an infinite geodesic (either a leaf or a diagonal of L^s) and a periodic sequence C_g can't converge to an infinite geodesic)



11.

Theorem 4.10 Let $h: F \rightarrow F$ be irreducible, non-periodic, orientation-preserving automorphism of a closed hyperbolic surface. There exists an $m > 0$ such that for any simple closed geodesic $C \subset F$

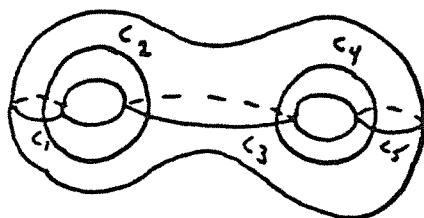
$$\lim_{n \rightarrow \infty} \hat{h}^{mn}(C) = k_C \text{ exists} \quad k_C = \text{stable limit of } C$$

Moreover $k_C \in L^s$ and there are but finitely many possibilities for k_C (depending on C)

Proof Deferred

Remark: From the procedure below one can construct invariant measures on laminations (Thurston)

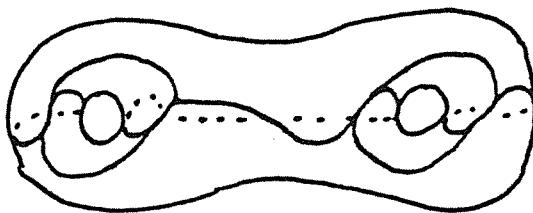
Example: Assume your metric is reasonably flat, so that the pictures below apply. Consider the automorphism $h = T_{C_1} \cdot T_{C_3} \cdot T_{C_5} \cdot T_{C_2}^{-1} \cdot T_{C_4}^{-1}$ in the curves given below:



Let $C = C_3$, we wish to see $h^n(C)$.

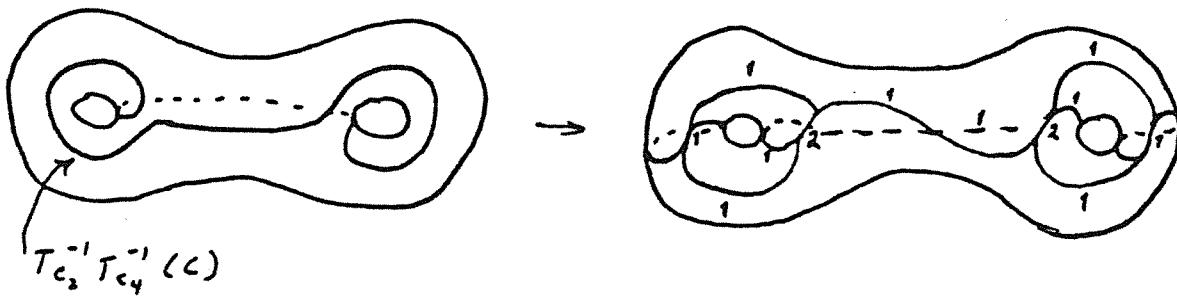
From the C_i 's form a 'train track' as indicated below

Viz:



A train track

Notice that a Dehn twist in one of the C_i 's has no effect on the isotopy class of t.t.
Now consider $T_{C_2}^{-1} T_{C_4}^{-1}(C)$:



The application of $T_{C_2}^{-1} T_{C_4}^{-1}$ to C gives the 'path' of a 'train' on the train track. This path is given by a system of weights on the track. Amalgamating parallel tracks causes the weights to add. Thus the weights will increase according to a linear rule as you apply Dehn twists.

11/17 Casson

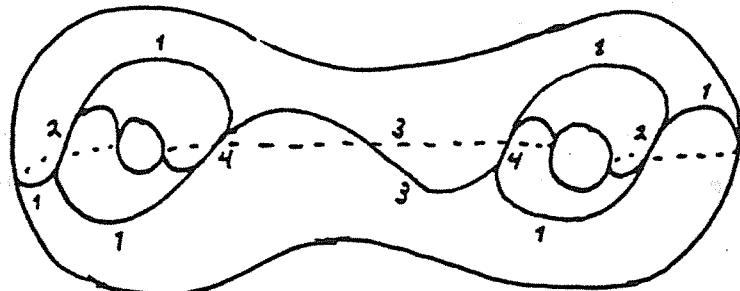
6

- Facts: 1) A given system of weights carries a unique 1-submanifold.
2) Only condition on weights is that they satisfy

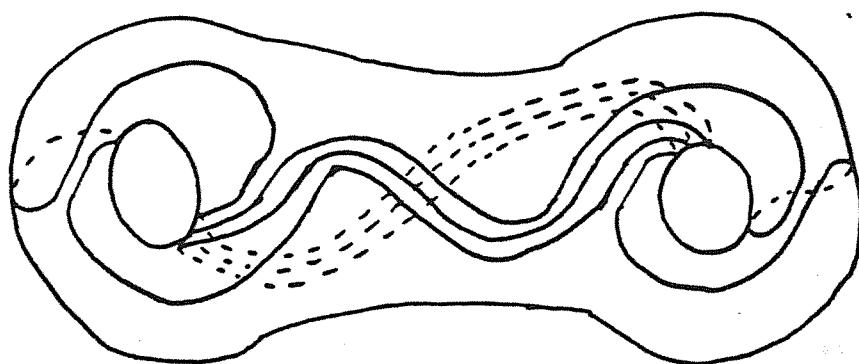
$$\begin{matrix} 9 & \\ & \diagup \\ & 9+b \\ & \diagdown \\ 6 & \end{matrix}$$

3) (Penner) This train track carries the iterates $h^n(c)$ by changing the weights

$$\text{Here is } h(c) = T_1 T_2 T_3 T_2^{-1} T_4^{-1}(c)$$



ρ_{1x} :

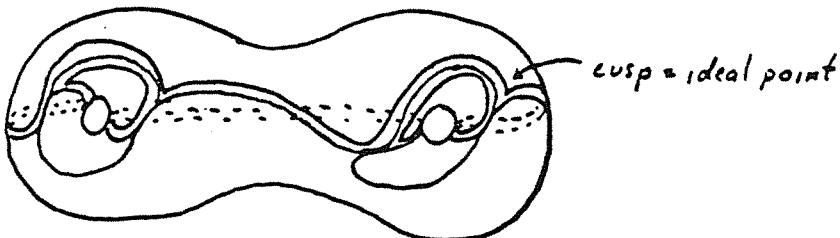


So one can work out the weights for $h^n(c)$. Maybe the limit is not geodesic, but the lamination lies in an ϵ -neighborhood of the train track.
Furthermore one can predict the number of fundamental regions, and the invariant laminations.

11/17 - 11/19

66.

In our example:



Fundamental regions = complementary regions of train track
Here it is two ideal quadrilaterals.

Remark: Penner's method applies to those automorphisms which are Dehn twists in s.c.c.'s which can be divided into two classes, sign of the twist being constant on each class.

Proof of 4.10: By Lemma 4.9 for any essential C the homotopy classes of the $\hat{h}^n(C)$'s are all distinct \Rightarrow the closed geodesics $\hat{h}^n(C)$ are all distinct.

If $\hat{h}^{n_i}(C) \rightarrow K \in \Lambda(F)$, then $K \geq L^S$. (As $L^S = (\cup_{k \in \mathbb{Z}} \hat{h}^k(k))'$
 \Rightarrow some leaf of $L^S \in K$, then $L^S \geq$ all leaves $\in K$ as k is closed.)

There are only finitely many laminations k_1, \dots, k_r containing L^S
(as $k_i = L^S \cup$ diagonal leaves). Let V_1, \dots, V_r be neighborhoods of k_1, \dots, k_r in $\Lambda(F)$.

There exists $n_0 > 0$ such that if $n > n_0$, then $\hat{h}^n(C) \in \bigcup_{j \in I} V_j$

(otherwise \exists a sequence $\hat{h}^{n_i}(C)$ with no subsequence converging to any $K \geq L^S$. \therefore no convergent subsequence, contrary to $\Lambda(F)$ sequentially compact).

Now \hat{h} permutes k_1, \dots, k_r ; so $\exists m > 0$ such that \hat{h}^m fixes k_j for all j .
Choose V_j so that the V_j 's are disjoint and $V_j \cap \hat{h}^m(V_i) = \emptyset$ for $i \neq j$

Then $\exists n_0$ so that $n > n_0 \Rightarrow \hat{h}^{mn}(C) \in V_j$ (say)

Smaller neighborhoods $\Rightarrow \hat{h}^{mn}(C) \rightarrow K$,

11.

11/19 - 11/22 Casson

Remark 1) There is a bound form in terms of genus (F)

2) If all the principal regions of L^S are ideal triangles, then for every C , $h^n(c) \rightarrow L^S$

Exercise: If L^S has principal regions of quadrilaterals, how large need n be? In particular, our example previous.

Theorem 4.11 Let $h: F \rightarrow F$ be irreducible, non-periodic automorphism of a closed hyperbolic surface. Then there exists $\ell(h) \in \mathbb{N}$ (depending on h and the metric) such that for every essential s.c.c. C in F , if we denote by C_g the geodesic homotopic to $h^g(c)$ and if g is sufficiently large, then every arc of C_g of length at least $\ell(h)$ meets every arc of C_{-g} of length at least $\ell(h)$ transversely.

Remark: Theorem does not hold for reducible or periodic h 's

Proof: By Thm 4.10, there is an m such that $C_{mg} \rightarrow k^+ \sharp L^S$ and $C_{-mg} \rightarrow L^- \sharp L^S$ as $g \rightarrow +\infty$.

For the moment, assume $m = 1$.

Note that every leaf γ of k^+ meets every leaf δ in k^- .

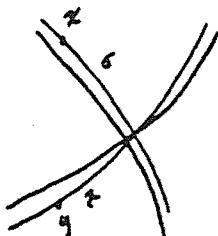
(Consider $\bar{\gamma}$ - a lamination, so a disjoint union of leaves. If $\bar{\gamma} \notin L^S$, then as $\bar{\gamma} \subseteq k^+$, $\bar{\gamma}$ must be a finite union of isolated, infinite leaves, but $\bar{\gamma}$ must also be a lamination, so there must be a closed leaf. A contradiction. So there is a leaf of L^S in $\bar{\gamma}$. But this leaf is dense in $L^S \Rightarrow \bar{\gamma} \supseteq L^S$.)

Similarly, $\bar{\delta} \supseteq L^S \Rightarrow \bar{\gamma} \cap \bar{\delta} \neq \emptyset \Rightarrow \gamma \cap \delta \neq \emptyset$)

Now take $x \in k^+$, $y \in k^-$. Let σ be the open leaf segment of k^+ centered at x of length ℓ , τ the open leaf segment of k^- of length ℓ .

Now if $\ell > \ell(x, y)$ then $\sigma \cap \tau$ have an interior transverse intersection. As illustrated at right, this $\ell(x, y)$ works for nearby pairs (x', y') .

As $k^+ \times k^-$ is compact, there is one ℓ which works for all x', y' ; i.e. there exists one $\ell(x, y)$ such that if σ, τ are



as above, α a geodesic arc of length l within $\epsilon(x, y)$ of σ in the Hausdorff metric, and β a geodesic arc of length l within $\epsilon(x, y)$ of τ . Then $\alpha \cap \beta \neq \emptyset$. By compactness, there is a universal $\epsilon > 0$.

Perform the above argument in the Unit Tangent bundle with the equivalent Hausdorff metric. Point here is that if a point and tangent direction are close to a second point and tangent direction in $UT(F)$, then the length l arcs are close.

So there exists Q such that $q > Q \Rightarrow$ every arc α of C_q of length l is within ϵ' of some arc of k^+ and $q < -Q \Rightarrow$ every arc β of C_q of length l is within ϵ' of some arc of k^- , and hence $\alpha \cap \beta \neq \emptyset$.

The l above depends on k^+ and k^- . But note that there are only finitely many possibilities for either k^+ or k^- . So choose $l(h) = \max\{l(k^+, k^-)\}$
 $k^+ \in L^3, k^- \in L^4\}$.

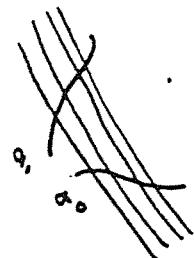
So we don't need $m=1$

11.

§5 Measured Laminations

Let α_0, α_1 be closed arcs transverse to a lamination $L \in \mathcal{L}(F)$.
 Say $\alpha_0 \sim \alpha_1$, if there exists a homotopy $H: I \times I \rightarrow F$ such that
 $H(I \times 0) = \alpha_0$, $H(I \times 1) = \alpha_1$, $H^{-1}(L) = A \times I$ for some $A \subset I$
 where $A \hat{=} \alpha_0 \cap L \oplus \alpha_1 \cap L$.

$H, H^{-1}: \alpha_0 \cap L \rightarrow \alpha_1 \cap L$ is called projection along the leaves.



Def A transverse measure μ on L is a collection of finite, non-negative measures on the transverse arcs to L which is invariant under projection along leaves, consistent with restriction if $\beta \subset \alpha$, and such that $\mu(\alpha \cap L) = 0$.

Example: Let C_1, \dots, C_k be disjoint s.c.c. geodesics. $L = \cup C_i$. If α is transverse to L and $U \cap \alpha$ is open, define $\mu(U) = |U \cap L|$.
 This is the counting measure on L .

More generally, choose $\lambda_1, \dots, \lambda_k \geq 0$. Define $\mu(U) = \sum \lambda_i |U \cap C_i|$
 this is weighted counting measure.

Def A measured lamination is a lamination equipped with a transverse measure

Remark: The set of measured laminations on a surface of genus g can be made into a topological space \mathbb{R}^{6g-6} (Thurston)

Def A leaf γ of L is μ -inessential, if there exists an arc α such that $\gamma \cap \text{Int } \alpha \neq \emptyset$ and $\mu(\alpha) = 0$. The support of $\mu_L = L - \cup$ μ -inessential leaves.

Notice that the set of μ -inessential leaves is open (as projection along leaves leaves μ invariant). Therefore $\text{support}(\mu)$ is closed,
 $\therefore \text{support}(\mu)$ is a lamination, if $\mu \neq 0$.

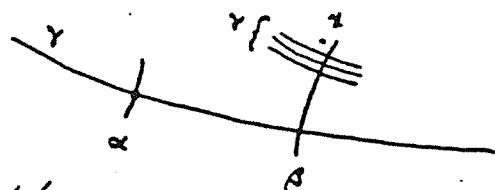
Notice that any isolated leaf in support(μ) is closed.

Suppose otherwise. There is a transverse arc α to γ with $|\text{Int}_{\alpha} \cap \gamma| = z$, $\mu(\alpha) = a > 0$. There exist points in the surface approximated arbitrarily closely by γ . For any transverse β , $\mu(\beta) \geq |\beta \cap \gamma| \cdot a$ as we can make α arbitrarily short and it still has measure a .

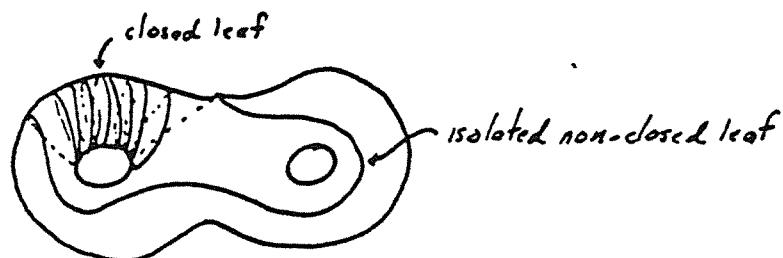
i.e. $\mu(\alpha') = a$ for any $\alpha' \subset \alpha$ such that $\text{Int}_{\alpha'} \cap \gamma \neq \emptyset$.
So for each intersection point of $\beta \cap \gamma$ we get a contributed to $\mu(\beta)$.

\therefore Compact transversals meet γ only finitely often

$\therefore \gamma$ closed



Remark: The lamination given below is not the support of a measure, but is the limit of s.c.c. geodesics. Regarding two measured laminations L_1, L_2 as equivalent if $\mu(L_1) = C \cdot \mu(L_2)$; we get the space of projective measured laminations $\cong S^{6g-3}$ (Thurston). This space is compact, so limits should have measure. In our example below this measure is supported on the closed leaf.



Lemma 5.1 If $L \in \Lambda(F)$, then there exists a non-zero transverse measure μ with $\text{support}(\mu) \subseteq L$.

Proof: If L is a 1-submanifold, L has a transverse measure (the counting measure), so wlog L is not a 1-submanifold. There exist compact arcs $\alpha_1, \dots, \alpha_k$ transverse to L such that every leaf of L meets $\alpha = \alpha_1 \cup \dots \cup \alpha_k$.

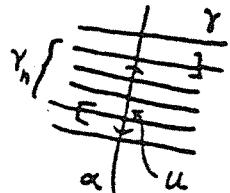
Note that some leaf γ of L meets α infinitely often (Take a non-isolated leaf, consider a transversal. There are only finitely many boundary leaves, so one must meet the transversal infinitely often).

12/6 Casson

Now $\gamma = \bigcup Y_n$ where Y_n is a compact segment of γ and $Y_n < Y_{n+1}$.
Define a measure μ_n on α via

$$\mu_n(u) = \frac{|Y_n \cap u|}{|Y_n \cap \alpha|}$$

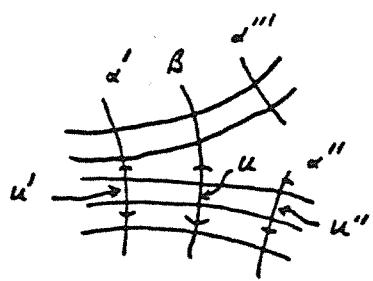
Notice $0 \leq \mu_n(u) \leq 1$ for all $u \subseteq \alpha$.



$$\mu(u) = \frac{3}{4}$$

Now a theorem of Alaoglu (cf: Kelley - General Topology p242) says that there exists a sequence $\{\mu_{n_i}\}$ such that $\mu_{n_i} \rightarrow$ measure μ on α in the sense that $\mu_{n_i}(E) \rightarrow \mu(E)$ for measurable $E \subseteq \alpha$.

Let β be a transverse arc which can be projected along the leaves to a subarc α' in α . Define μ on β by projecting to α' .



Suppose β also projects to $\alpha'' \subseteq \alpha$ and suppose that $u \in \beta$ corresponds to $u' \in \alpha'$ and $u'' \in \alpha''$.

Notice that

$$|u' \cap Y_n| - |u \cap Y_n| \leq 2 \quad \text{and}$$

$$|u'' \cap Y_n| - |u \cap Y_n| \leq 2$$

Hence

$$\frac{|u' \cap Y_{n_i}|}{|\alpha \cap Y_{n_i}|} - \frac{|u'' \cap Y_{n_i}|}{|\alpha \cap Y_{n_i}|} \leq \frac{4}{|\alpha \cap Y_{n_i}|} \rightarrow 0 \text{ as } i \rightarrow \infty$$

$$\therefore \mu(u') = \mu(u'')$$

\therefore definition of μ is consistent with projection along the leaves.

Finally note that μ is non-zero as $\mu(\alpha) = 1$.

11.

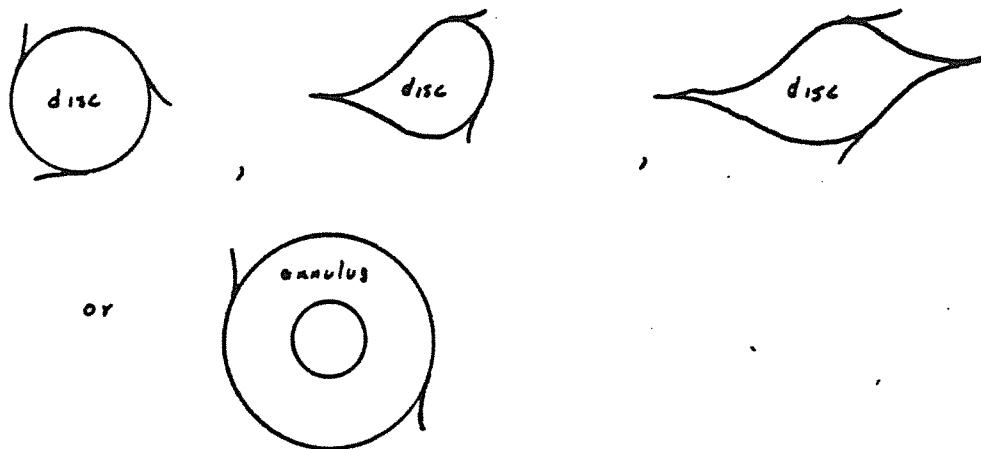
Cor: If $h: F \rightarrow F$ is irreducible and non-periodic, then L^s, L^u support non-zero transverse measures.

Def Let F be a hyperbolic surface. A train track τ in F is a finite graph (i.e. a 1-complex) embedded in F such that

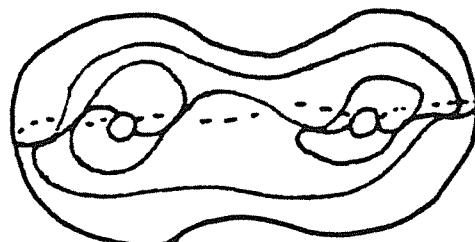
- (1) The edges of the graph (the branches) are C^1 embedded.
- (2) All edges at a vertex (i.e. a switch) are tangent there
- (3) Every vertex is interior to some C^1 arc in τ
- (4) If \bar{V}_R denotes the set of vertices which have an angle θ in a component R of $F - \tau$, then for each such component

$$\chi_R = \frac{1}{2} \chi(\partial R - \bar{V}_R) < 0$$

Remark: Condition (4) rules out the following:



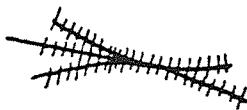
Example: Genus 2 surface



This train track has four triangular complementary regions.

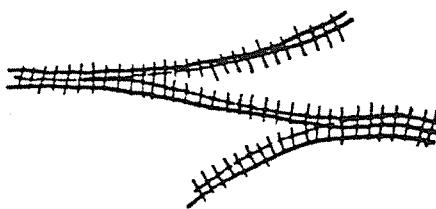
12/8 Casson

Def A standard neighborhood of a train track τ is a neighborhood U of τ which is foliated by arcs (called ties) transverse to τ , such that each transverse arc meets τ in just one point except near the switches (and each branch has at least one such transverse arc). The switches have neighborhoods like:



A lamination is carried by a train track τ if τ has a standard neighborhood U such that $L \subseteq U$ and L is transverse to the ties.

Local Example:



Lemma 5.2 Every geodesic lamination $L \subset F$ is carried by a train track

Proof: $N_\epsilon(L)$ has no more than K_F ($= 12g - 12?$) vertices. The lamination L has zero area

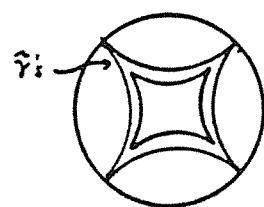
\therefore area $N_\epsilon(L) \rightarrow 0$ as $\epsilon \rightarrow 0$

If ϵ is small, then there is a $\delta > 0$ such that

1) $N_\epsilon(L)$ contains no disc of diameter δ

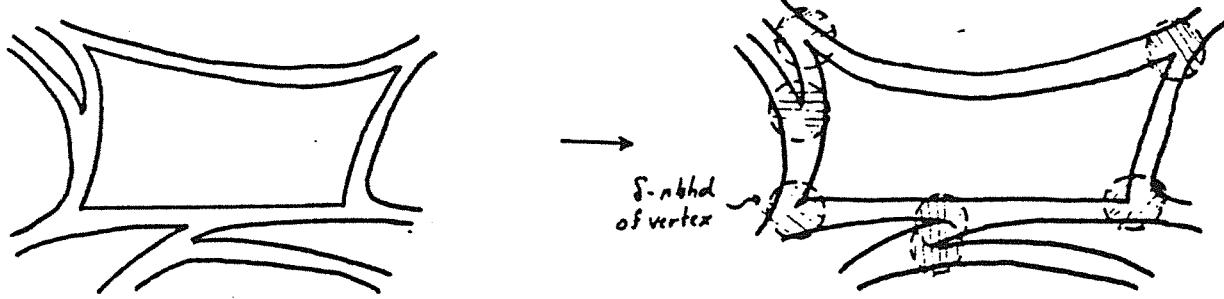
2) Two points of L that are within $K_F \cdot \delta$ of each other have close tangent directions

Remark: One can see the bound on the number of vertices by lifting $N_\epsilon(L)$ to the universal cover:



The interior curves project to the boundary of $N_\epsilon(L)$. They are not geodesics, but are circular arcs thru the vertices. The interior angles are small, but > 0 .

Returning to the proof: In the surface the boundary of $N_\epsilon(L)$ looks like



Foliate the δ -neighborhoods of the vertices by arcs transverse to L . The rest of $N_\epsilon(L)$ is a union of 'strips' of width less than δ . Extend the foliation over them, staying transverse to L . Then \mathcal{E} is the 'core' of $N_\epsilon(L)$.

II.

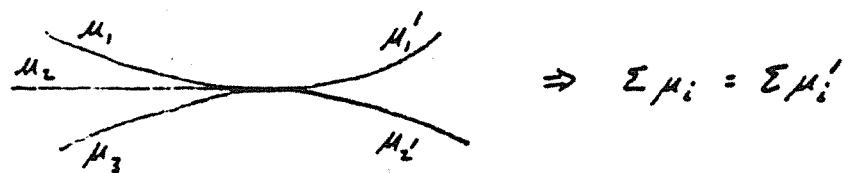
Remark: Thurston passes to universal cover. There he uses horocyclic arcs to foliated neighborhoods of vertices, to begin the foliation.

Remark: If your ϵ is not small enough the from track the above procedure produces may have 'extra' tracks.

Suppose L has a transverse measure μ , L carried by \mathcal{E} in a standard neighborhood U of \mathcal{E} such that L is transverse to the ties. We can assign to each branch b of \mathcal{E} the weight $\mu(b)$ equal to the measure of a tie meeting b in a single point.

Note that the weights do not depend on the choice of standard neighborhood and they satisfy the switch condition i.e. sum of the weights of the entering branches equals the sum of the weights of the exiting branches.

pix:



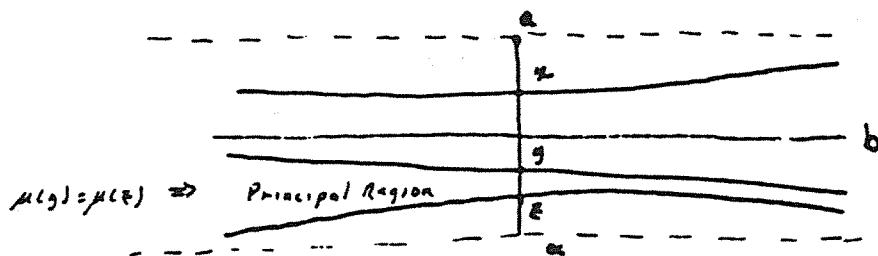
12/10 Casson

Lemma 5.3 There is at most one geodesic lamination L equipped with a transverse measure μ (such that $\text{support}(\mu) = L$) carried by a given train track with given weights.

Proof: Let b be a branch of the train track ε with weight μ_b . Let α be a standard neighborhood with ties transverse to L , α a tie meeting b in a single point.

Let a be an endpoint of α . If $x \in \alpha \cap L$, let $\mu(x) = \mu[a, x]$. Observe that if $\mu(y) > \mu(z)$ then no leaves of L meet α 'between' y and z as $\text{support}(\mu) = L$. Hence g, z lie on boundary leaves, so called 'adjacent' boundary leaves; between which lie a bit of a principal region.

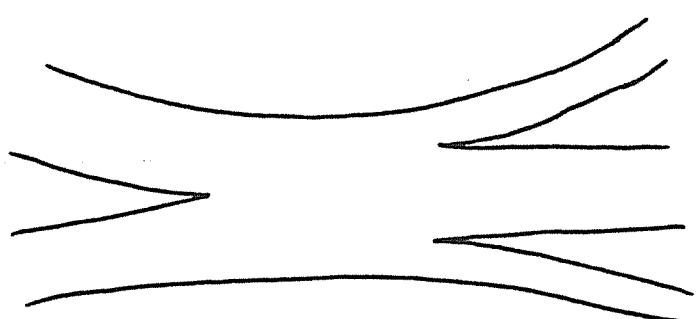
Fix:



So a point x on $\alpha \cap L$ is specified by $\mu(x)$ and (sometimes) a + or - sign distinguishing adjacent boundary leaves.

Make a model of the standard neighborhood as follows; start with ε and the weights. For each branch of ε take a Euclidian strip of width μ_b and splice these strips together at the switches

Example:



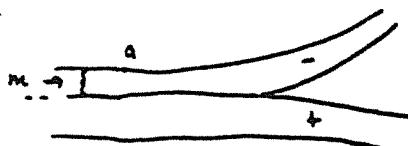
Remark: ε is the 'spine' of the result

12/10 Casson

76

Given $m \in [0, \mu_b]$ and a sign + or -, let λ_m be the immersion of \mathbb{R} into the model passing thru the b strip at a distance m from the α -edge. If necessary, use the + or - to determine the direction at a switch.

pix:

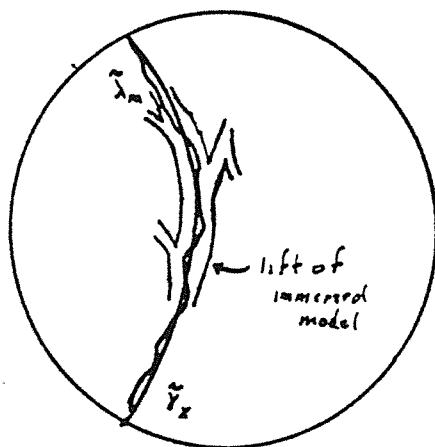


Remark: Make the same choice at each switch as above. This corresponds to the fact that a non-isolated boundary leaf is isolated constantly from one side.

Embed the model smoothly in F making the transverse direction short (c.e. strips thin). Note that the model is not isometrically embedded. If $m = m(x)$ for $x \in \alpha \cap L$, the leaf γ_x of L thru x and the immersion λ_m have the same behavior at the switches.

So lifting the embedded model and L to the universal cover of $F \cong \mathbb{H}^2$ we see that γ_x and λ_m have lifts $\tilde{\gamma}_x$, $\tilde{\lambda}_m$ which remain a bounded distance apart in the \mathbb{H}^2 metric.

Hence the endpoints of $\tilde{\gamma}_x$ are the same as the endpoints of $\tilde{\lambda}_m$, uniquely determining γ_x given $\mu(x) = \mu[\alpha, x]$.



Remark: You might worry that $\mu(x)$ is unattainable, c.e. No open arc of α has measure $\mu(x)$. In this case it follows that γ_x is a closed leaf. In the model there will be a family of closed leaves which lift to a family of curves with common endpoints, those of $\tilde{\gamma}_x$; again uniquely determining the leaf γ_x .

11.

1/19 Casson

Lemma 5.4 If $L \in \Lambda(F)$, the set of transverse measures μ with support $\{\mu\} \subseteq L$ is naturally a non-trivial closed convex cone in a finite dimensional vector space.

Proof Let V be the vector space of signed measures $\mu_1 - \mu_2$ with support $\{\mu_i\}$ contained in L . (μ_i : non-negative and finite on compact arcs)

Let \mathcal{E} be a train track carrying L ; W the set of (not necessarily positive) systems of weights on \mathcal{E} which satisfy the switch condition.

Remark: The switch condition is not necessary for finite dimensionality, but it does lower the dimension.

Notice that there is a linear map $\Phi: V \rightarrow W$ which sends a measured lamination to the vector of associated weights. By Lemma 5.3 this map is injective.

$\therefore V$ is finite dimensional

The positive measures form a convex cone for if x, y are elements of this subset of V and μ, λ are non-negative; then $\lambda x + \mu y$ is also in this subset. Theorem 5.1 shows this cone to be non-trivial, closure is clear.

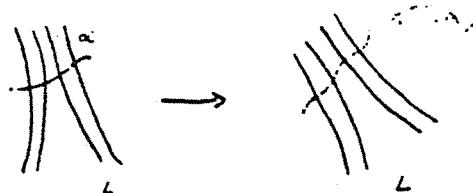
II.

Digression: Suppose $h: F \rightarrow F$ is an automorphism; $L \in \Lambda(F)$. Given a measure μ with support in L ; what is the measure $\hat{h}(\mu)$?

If $\hat{h}(L) = h(L)$, we can define

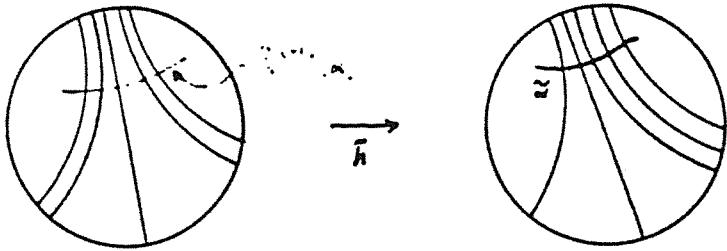
$\hat{h}(\mu)$ via:

α transverse to $h(L)$, $\hat{h}(\mu)(\alpha)$
is just $\mu(h^{-1}(\alpha))$



In general: Lift α to the universal cover, obtain $\tilde{\alpha}$. Let $\tilde{\alpha}' = \hat{h}^{-1}(\alpha)$ ($\tilde{\alpha}'$ geodesic segment with endpoints of the whole geodesic the inverse image under \hat{h} of the endpoints of a prolongation of $\tilde{\alpha}$) be a transversal to \hat{h}^{-1} (leaves meeting $\tilde{\alpha}$). Set $\hat{h}(\mu)(\alpha) = \mu(\hat{h}^{-1}(\alpha))$.

pix:



Theorem 5.5 Let $h: F \rightarrow F$ be non-periodic, irreducible, with invariant laminations L^S, L^U . Then there exists positive measures μ^S, μ^U supported by L^S, L^U and a constant $\lambda > 1$ such that $\hat{h}(\mu^S) = \lambda \mu^S$ and $\hat{h}(\mu^U) = \lambda' \mu^U$. We call λ the stretching factor of h .

Proof: Let $M^S = \{\text{positive measures on } L^S\}; \hat{h}(L^S) = L^S$
 $\therefore \hat{h}$ induces a linear map $\hat{h}: M^S \rightarrow M^S$

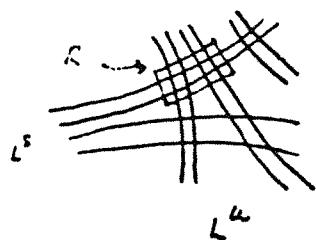
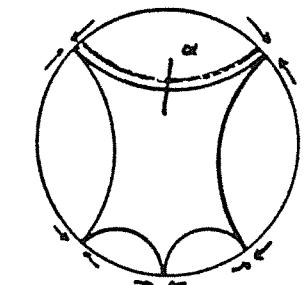
Now M^S/\mathbb{R}^+ is a disc, so by the Brower Fixed Point Theorem there exists a μ^S in M^S such that $\hat{h}(\mu^S) = \lambda \mu^S$ for some positive constant λ .

Now consider the lift to H^2 of a principal region of L^S . Recall that the lift of some power of h leaves the vertices invariant, so the lift of a power takes boundary leaves to themselves. Further, the vertices are contracting fixed points, so we can approximate boundary leaves by leaves which get closer \Rightarrow transverse arcs shrink when we apply $\hat{h} \Rightarrow \hat{h}^{(n)}(\alpha)$ is larger than $\alpha \Rightarrow \mu$ increases
 $\therefore \lambda > 1$

Similarly there exists a measure μ^U on L^U with $\hat{h}(\mu^U) = \lambda' \mu^U$ with $\lambda' < 1$. We sketch the proof that $\lambda \lambda' = 1$.

Define a \mathbb{Z} -parameter measure $\mu^S \cdot \mu^U$ on $L^S \cap L^U$ as follows:

Let R be the topological image of a rectangle in F with 'horizontal' edges disjoint from L^S and 'vertical' edges disjoint from L^U .



Note that $\mu^a('top') = \mu^a('bottom')$ and $\mu^3('left') = \mu^3('right')$ by the invariance of the measures under pushing along the leaves.

So define $\mu^5, \mu^a(R)$ as $\mu^3('vertical edge'), \mu^a('horizontal edge')$ and extend to a measure on $X = L^3 \wedge L^a$.

Remark: This measure is actually defined on the surface but supported on $L^3 \wedge L^a$ and is finite on compact arcs and is non-zero.

Note that $\hat{h}(\mu^3 \mu^a) = \hat{h}(\mu^3) \cdot \hat{h}(\mu^a)$ and, as before defining \hat{h}^{-1} in terms of the endpoint construction, we have:

$$\hat{h}(\mu^3 \mu^a)(x) = \mu^3 \mu^a(\hat{h}^{-1}(x)) = \mu^3 \mu^a(x)$$

"

$$\hat{h}(\mu^3) \cdot \hat{h}(\mu^a)(x) = (\lambda \mu^3)(\lambda' \mu^a)(x) = \lambda \lambda' \mu^3 \mu^a(x)$$

$$\therefore \lambda \lambda' = 1.$$

II.

Goal: $M_L(F)$ = set of all measured laminations on F (with measures positive, finite on compact arcs, and support in lamination) has a natural piecewise linear structure

Note: This set already has the structure of a non-trivial cone, and we can linearly combine any two measures

It follows that $M_L(F)/_{\mathbb{R}^+}$ has a natural piecewise projective structure

which turns out to be S^{6g-7}

Def A weighted train track is a train track \mathcal{T} with a non-negative weight μ_b assigned to each branch b of \mathcal{T} such that the μ_b 's satisfy the switch condition.

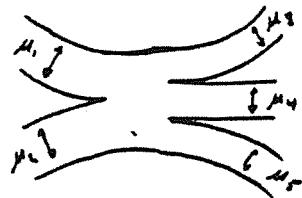
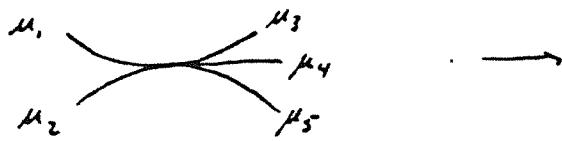
Lemma 5.6 With each weighted train track (τ, μ_b) is associated a unique measured lamination (ℓ, μ) with support $\mu = \ell$. Different weightings of τ are associated with distinct measured laminations.

Cor Let W_τ be the set of weightings of τ (i.e. non-negative μ_b satisfying the switch condition). Let $f_\tau : W_\tau \rightarrow \text{ML}(F)$ be the map defined by lemma 5.6. Then f_τ is injective and $\text{ML}(F) = \bigcup_\tau f_\tau(W_\tau)$

Remark: It follows from the proof that if (ℓ, μ) is carried by (τ, μ_b) then $f_\tau(\mu_b) = (\ell, \mu)$

Proof of 5.6 Given (τ, μ_b) , construct a model for (ℓ, μ) as in the proof of 5.3. Replace each branch of τ by a Euclidean strip of width μ_b . Splice the strips at the switches and imbed the union of the strips in F with τ as spine

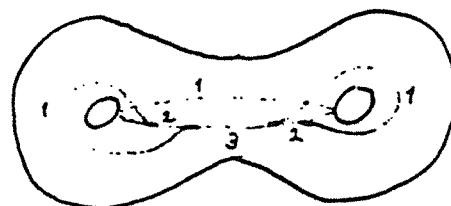
Picture:



A leaf of the model is obtained by following the lines parallel to the edges of the strips

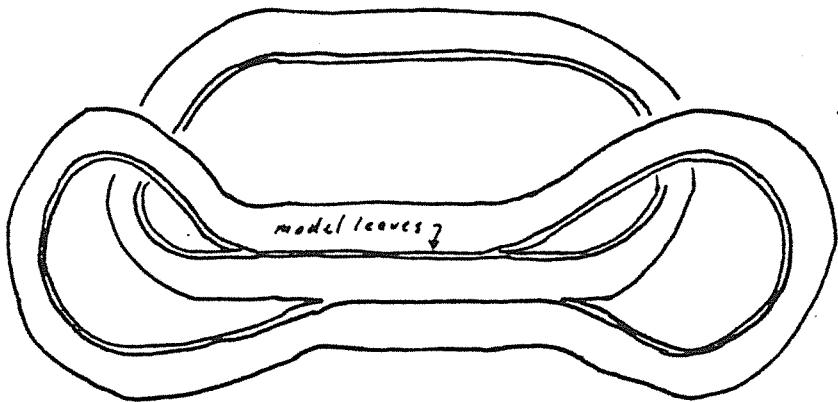
Remark: Should a leaf in the model meet the "vertex" at a switch regard that leaf as a pair of leaves, an "upper" and a "lower" making a consistent choice at each 'vertex' met by the leaf.

Example:



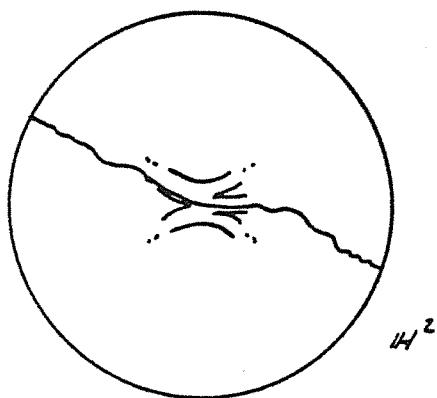
1/24 Cesson

model:



Notice that no ambiguous leaves appear unless $\mu(\text{strip}) = 0$. In this case these leaves have zero measure, and so isolated, and thus not in $\text{support}(\mu)$.

Returning to the proof, lift to the universal cover:



Claim: Each image of a model leaf lifts to a curve in $\tilde{F} \cong H^2$ with well defined and distinct endpoints on S^∞ .

Assuming this claim: Define a leaf of L to be the image in \tilde{F} of the geodesic joining the endpoints of the lift of a model leaf. These leaves are disjoint and form a closed set, and are thus a lamination.

Notice that the transversals to the strips in H^2 have well defined measures, giving a well defined measure on L .

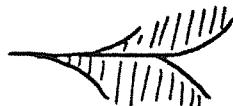
So all that remains is to establish the above claim.

Def A surface with cusps is a C' surface where boundary points are locally modeled by a region bounded by two tangent C' curves.



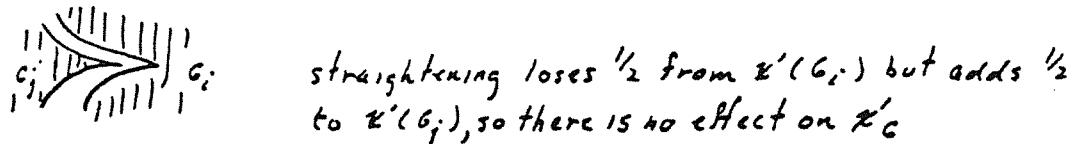
Example: If τ is a C' transverse in F , each component of $F \setminus \tau$ has a compactification which is a surface with cusps, (all outward). Other cusps arise when gluing complementary regions together

Fix:



Lemma 5.7 If G_1, G_2 are compact C' surfaces with cusps such that $G = G_1 \cup G_2$ and $G_i \cap G_j$ is a 1-submanifold of ∂G_i , then $\chi'_G = \chi'_{G_1} + \chi'_{G_2}$ where $\chi'_G = \chi_G - \frac{1}{2} (\# \text{outward cusps}) - (\# \text{inward cusps}) = \chi_G - \frac{1}{2} \text{cusp}_G$

Proof Let $x \in G_1 \cap G_2$. If $x \in \text{Int}(G_1 \cap G_2)$ either x is regular or x is an inward cusp of G_i and an outward cusp of G_j . Fix:



If $x \in \partial(G_1 \cap G_2)$ then either

1) x is an inward cusp of $G_1 \cap G_2$

or

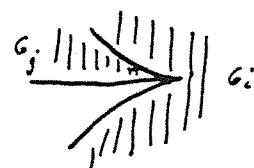
2) x is an outward cusp of G_1 or G_2

or

3) x is an outward cusp of G_1, G_2 , and G

or

4) x is inward for G_i , outward for G_j , and inward for G



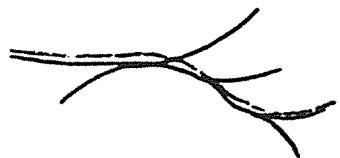
Observe that in all cases: $\text{cusp}(G) = \text{cusp}(G_1) + \text{cusp}(G_2) - 1/2(G_1 \cap G_2)$
 $\Rightarrow 1/2\text{cusp } G = 1/2\text{cusp}(G_1) + 1/2\text{cusp}(G_2) - \chi_{G_1 \cap G_2}$
 But $\chi_G = \chi_{G_1} + \chi_{G_2} - \chi_{G_1 \cap G_2}$; comparing gives the additivity formula.

II.

Def Let τ be a C^1 train track in F . A train route in τ is an equivalence class of C^1 immersions $p: \mathbb{R} \rightarrow F$ such that $p(\mathbb{R}) \subseteq \tau$.

Here equivalence is reparametrization and

we require that $\lim_{t \rightarrow -\infty} p(t)$ and $\lim_{t \rightarrow \infty} p(t)$ do not exist.

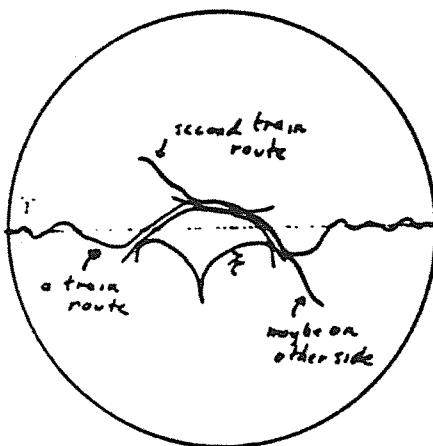


Notice that there exist \pm routes.

Lemma 5.8 Let τ be a train track in F , $\tilde{\tau}$ the preimage of τ in $\tilde{F} \cong \mathbb{H}^2$. Assume $\chi'_G < 0$ for each compactified complementary region G of $F \setminus \tau$. Then every train route in $\tilde{\tau}$ is simple and remains a bounded hyperbolic distance from a uniquely defined geodesic γ in \mathbb{H}^2 .

Remark: Assume every train route has well-defined endpoints on S^1 . Moreover, distinct train routes (oriented or unoriented) in $\tilde{\tau}$ correspond to distinct geodesics. Two train routes in $\tilde{\tau}$ intersect (if at all) tangentially in an arc (possibly a single point).

pix:



Proof If a train route p in $\tilde{\tau}$ is not simple, there exists an arc a

of ρ describing a simple closed curve as below, bounding a disc



G is in H^2 with at most one cusp. Hence $\kappa'_G \geq \frac{1}{2} > 0$. But $G = \cup G_i$ where each $\kappa'_{G_i} < 0$, a contradiction.

Claim: There exists a constant A such that if σ is a segment of a train route from P to Q , then arc length(σ) $\leq A \cdot d(P, Q)$.

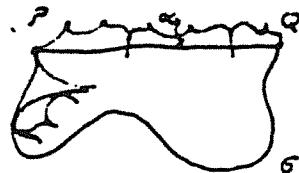
Remark: A can be very large.

Proof of Claim: we can assume that all the components of $F \setminus \tilde{\tau}$ are contractible as we can add extra branches to $\tilde{\tau}$ preserving the $\tilde{\tau}'$ condition. Hence all regions of $H^2 \setminus \tilde{\tau}$ are compact surfaces with cusps, indeed discs with cusps.

Assume first that the hyperbolic distance $d(P, Q)$ is large and that the segment σ meets the geodesic segment α from P to Q only at the endpoints.

Let G be the closure of all the regions of $H^2 \setminus \tilde{\tau}$ meeting the bounded component of $H^2 \setminus (\alpha \cup \sigma)$

Note that σ is the union of segments of boundary components of complementary regions of $\tilde{\tau}$ while α is not. G is a compact surface with cusps.



Exercise: Show G a disc.

By covering α by discs of twice the maximum diameter of the complementary regions of $\tilde{\tau}$ contained in G and noting that each contains only a finite number of cusps, it follows that

$$\# \text{cusps of } G \leq \text{constant} \cdot d(P, Q) \quad (\text{plus a constant, if } d(P, Q) \text{ small})$$

$$\text{So } -\kappa'_G \leq \text{constant} \cdot d(P, Q)$$

Further, the number of regions of $H^2 \setminus \tilde{\tau}$ in G is at least a constant $\cdot l(\sigma)$ as G is bits of branches bounding those regions, a region has but finitely many edges, and there is a lower bound for the length of such a branch.

$$\text{So } -\kappa_G \geq \text{constant} \cdot l(\sigma)$$

$$\text{Hence } l(\sigma) \leq \text{constant} \cdot d(P, Q).$$

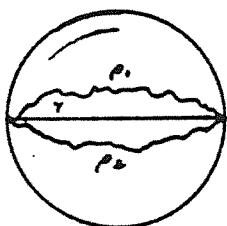
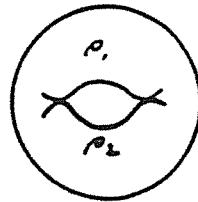
If $d(P, Q)$ is small, then $\ell(\gamma)$ is bounded as it is enough to consider those γ meeting a given fundamental region. Compactness and simplicity of train routes, now gives the constant A for short train routes.

If γ meets α in interior points, decompose γ into subroutes. And the claim follows.

The lemma follows from

Exercise: (cf Nielsen's Thm, Lemma 3.7) A train route ρ in \mathbb{H}^2 has well defined endpoints on S_∞ , and ρ stays a bounded hyperbolic distance from the geodesic joining those endpoints.

Next suppose that ρ_1, ρ_2 have disconnected intersection. Then there exist segments γ_1, γ_2 of ρ_1, ρ_2 cobounding a compact surface (indeed, a disc) G with at most two cusps. Hence $\chi'_G \geq 0$. Impossible as before.



If ρ_1, ρ_2 correspond to the same geodesic, select on $r=1$ and let G be the closure of the union of all the regions of $\mathbb{H}^2 \cdot \mathbb{E}$ which meet $\{\text{region cobounded by } \rho_1, \rho_2\} \cap \text{subdisc of } \mathbb{H}^2$ of Euclidian radius r .

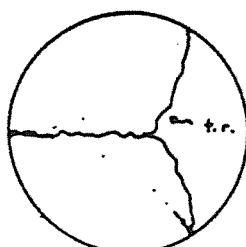
Since ρ_1, ρ_2 are a bounded hyperbolic distance from γ , the number of cusps of G is bounded, so

- χ_G is bounded above for all r . But the number of regions of $G \rightarrow \infty$ as $r \rightarrow 1^-$

$\therefore -\chi_G \rightarrow \infty$, a contradiction.

II.

Remark: Two train routes have a common endpoint exactly when the two train routes are identical "near" the endpoint.

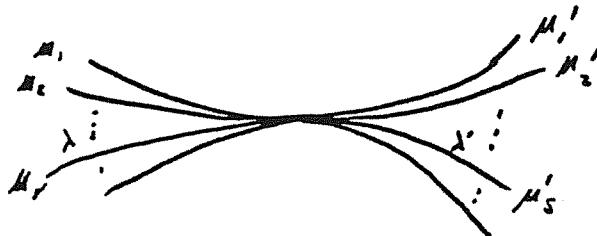


Now we are ready for

Lemma 5.6 Let $\tilde{\tau}$ be a train track in F with $\chi_G' < 0$ for every region G of $F \setminus \tilde{\tau}$ and let $W_{\tilde{\tau}}$ be the set of weightings on $\tilde{\tau}$. Then there exists a naturally defined injection $f_{\tilde{\tau}}: W_{\tilde{\tau}} \rightarrow M_{\tilde{\tau}}(F)$

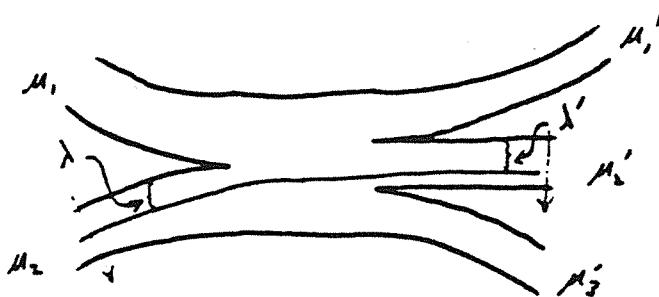
Proof: Let (μ_b) be a weighting for $\tilde{\tau}$. Lift the weights to the branches of $\tilde{\tau} \subset F \times \mathbb{H}^2$. A train route ρ is compatible with (μ_b) , if corresponding to transversely oriented branches \tilde{b} of $\tilde{\tau}$ there are branch numbers $\lambda_{\tilde{b}} \in [0, \mu_b]$ satisfying the switch condition (see below) and such that ρ can be oriented so that no branch number is zero.

Switch condition:



$$\Rightarrow \mu_1 + \dots + \mu_{r-1} + \lambda = \mu'_1 + \dots + \mu'_{s-1} + \lambda'$$

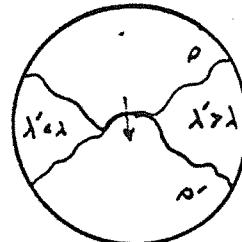
In the model:



Note: The branch numbers are assigned only to branches a leaf "visits"

Next let $L \subset F$ be the set of geodesics corresponding to all train routes compatible with (μ_b) . Two distinct train routes in $\tilde{\tau}$ compatible with

(μ_b) correspond to disjoint geodesics. Suppose that the endpoints of ρ, ρ' separate each other, and ρ, ρ' traverse a common transversely oriented branch. In this situation, the branch numbers must be equal. But now ρ must correspond to a 0 branch number on the left and the full measure on the right, a situation we have ruled out.



\therefore The endpoints of distinct train routes do not separate each other.

Exercise: L is closed

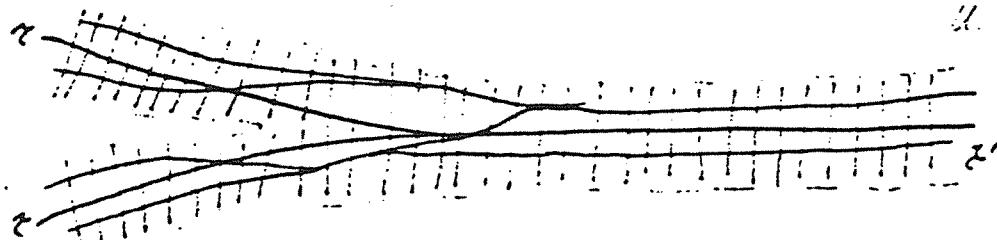
Let \tilde{b} be a branch of \tilde{L} . The branch numbers $\lambda_{\tilde{b}}$ naturally define a transverse measure on the set of leaves of \tilde{L} corresponding to train routes thru \tilde{b} . This defines a transverse measure on sufficiently short transversals to any leaf of \tilde{L} , hence for any compact transversal.

Exercise: Verify invariance under projection along leaves at the train route level.

For injectivity, suppose $f_{\tilde{z}}(\mu_b) = f_{\tilde{z}}(\mu'_b) = (\text{Support}(\mu), \mu)$. Each leaf of L corresponds to a train route ρ compatible with (μ_b) and to a train route ρ' compatible with (μ'_b) . The uniqueness part of Lemma 5.8 implies $\rho = \rho'$ and hence $\mu_b = \mu'_b$.

II.

Def A train track τ' is carried by τ , if there is a standard 'tie' neighborhood U of some τ , ambient isotopic to τ such that $\tau' \subset U$, transverse to the ties.



Remark: In a tie neighborhood of a single branch of τ , the train track τ' must be a tree.

Define $g: W_{\tau'} \rightarrow W_{\tau}$ (writing $g_{\tau, \tau'}$) by assigning to a weighting μ' of τ' a weighting $g(\mu')$ of τ as follows: the weight of a branch b of τ corresponding to the branch b_1 of τ' , is the sum of the weights of the branches of τ' crossing a tie of b_1 . The switch condition guarantees that this weight is independent of the tie chosen.

Lemma 5.9 $g_{\tau, \tau'}: W_{\tau'} \longrightarrow W_{\tau}$ is a linear function of convex cones such that the diagram

$$\begin{array}{ccc} W_{\tau'} & \xrightarrow{g} & W_{\tau} \\ & \searrow & \\ & M_{\tau}(F) & \end{array}$$

commutes.

Further, if τ carries τ' , and τ' carries τ'' , then τ carries τ'' and

$$g_{\tau, \tau''} = g_{\tau, \tau'} \cdot g_{\tau', \tau''}: W_{\tau''} \rightarrow W_{\tau}$$

Proof Assume τ' is in a 'thin' standard neighborhood of τ . Then every train route in τ' projects along the ties to a train route in τ . Notice that this has no effect on the endpoints of the lift in M^2 . Weights μ' on τ' give weights $\mu = g(\mu') \in W_{\tau}$ by projecting along ties. μ' compatible with μ corresponds to a route on τ compatible with μ . Now μ, μ' determine the same lamination L .

Exercise: Check that the measures agree and composition rule holds.

Remark: Even though the image of the g 's may not be open, it is still true that they give linear coordinate charts on $M_{\tau}(F)$

11.

Remark: what follows is primarily due to R. Penner

Def τ is invariant under $h: F \rightarrow F$ (approximated by a C^1 automorphism if necessary), if $h(\tau)$ is carried by τ .

Remark: As 'carried by' is not symmetric, τ invariant under h \neq τ invariant under h^{-1}

2/4 Casson

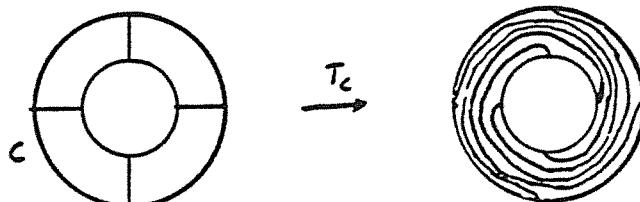
Lemma 5.10 1) If τ is invariant under h , then the diagram

$$\begin{array}{ccccc} W_\tau & \xrightarrow{h} & W_{h(\tau)} & \xrightarrow{g+h(\tau)} & W_\tau \\ f_\tau \downarrow & & \downarrow f_{h(\tau)} & (5.9) & \downarrow f_\tau \\ M_\tau(F) & \xrightarrow{h} & M_{h(\tau)}(F) & \xrightarrow{1} & M_\tau(F) \end{array} \quad \text{commutes, where } h_\mu(h(b)) = \mu(b).$$

2) If τ is invariant under h_1, h_2 ; then τ is invariant under $h_1 \cdot h_2$.

3) If τ has a simple closed C' train route $C \subseteq F$ (i.e. not a monogon) which can be oriented so that all the switches on the right of C' 'diverge forwards' (i.e. ), then τ is invariant under the Dehn twist T_C .

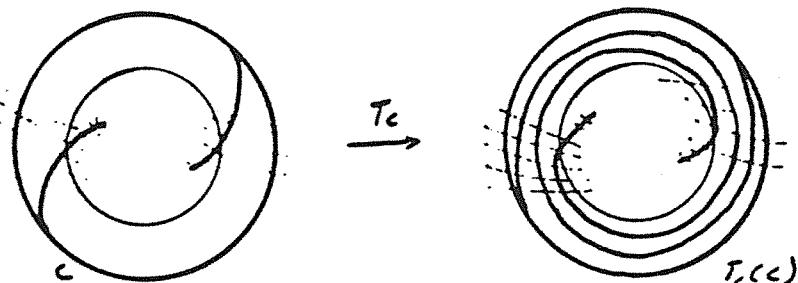
Remark: T_C depends on the orientation of the surface, a positive Dehn twist is illustrated below.



Proof: 1) Exercise

2) $h_2(\tau)$ is carried by τ , $h_1 h_2(\tau)$ is carried by $h_1(\tau)$ and so carried by τ .

3) Let the annulus nbhd of T_C be contained in a tie neighborhood of C . Then this tie neighborhood shows $T_C(\tau)$ carried by τ .

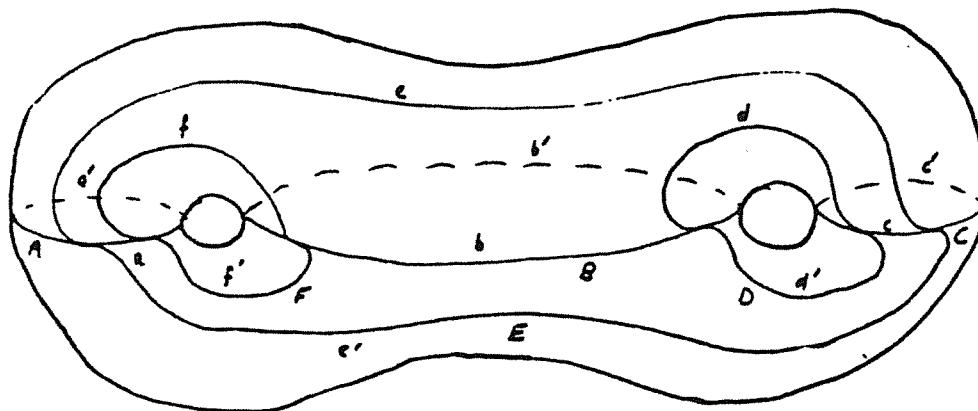


Remark: $T_c^{-1}(\tau)$ is not carried by τ due to failure of transversality of $T_c^{-1}(z)$ to the ties.

We now have the commutative diagram

$$\begin{array}{ccc} W_{\tau} & \xrightarrow{\tilde{h} = \text{old } h \cdot g_{\tau} h(\tau)} & W_{\tau} \\ \downarrow & & \downarrow \\ M_2(F) & \xrightarrow{\tilde{h}} & M_2(F) \end{array}$$

Example: A train track on a genus two surface



The train track τ is invariant under $T_A, T_B, T_C, T_O, T_E, T_F$ and under any member of the semi-group they generate. Weighting the train track as indicated in lower case above, we can give an example of switch calculation:

$$a + a' = a' + b; a + f = a' + f' \Rightarrow a - a' = e - e' = f - f'$$

$$\text{Permuting cyclically} \Rightarrow e - e' = f - f' = d - d' = e' - e$$

$$\text{Hence } a = a', \dots, f = f'$$

Conversely, forgetting primes satisfies the switch condition.

Now τ invariant under $h \Rightarrow W_{\tau}$ is a closed convex cone spanning a vector space V_{τ} . We obtain $\tilde{h}: V_{\tau} \rightarrow V_{\tau}$. Let $\mathbb{Z}_{\tau} \subset V_{\tau}$ be the set of integer weightings. \mathbb{Z}_{τ} is a discrete additive subgroup of V_{τ} , i.e. a lattice, which spans V_{τ} . Hence $\mathbb{Z}_{\tau} \cong \mathbb{Z}^k$ for some integer k . Notice that $\tilde{h}(\mathbb{Z}_{\tau}) \subseteq \mathbb{Z}_{\tau}$, so choosing a basis for \mathbb{Z}_{τ} allows us to represent \tilde{h} by an integer matrix A .

2/4 CASSON

91

The stretching factor is an eigenvalue of this matrix. In our example

$$T_A \sim \left(\begin{array}{c|ccc} I_3 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & & I_3 \end{array} \right); T_B \sim \left(\begin{array}{c|ccc} I_3 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & & I_3 \end{array} \right); T_C \sim \left(\begin{array}{c|ccc} I_3 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & & I_3 \end{array} \right)$$

$$T_D^{-1} \sim \left(\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right); T_E^{-1} \sim \left(\begin{array}{c|c} I_3 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right); T_F^{-1} \sim \left(\begin{array}{c|c} I_3 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right)$$

So let $H = T_A T_B T_C T_D^{-1} T_E^{-1} T_F^{-1}$ (or more generally $(\dots)^a (\dots)^b$ $a, b > 0$)

$$T_A T_B T_C \sim \left(\begin{array}{c|ccc} I_3 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & & I_3 \end{array} \right); T_D^{-1} T_E^{-1} T_F^{-1} \sim \left(\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right)$$

so for convenience let $H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, then $H^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = H + 2I$

Notice that H has minimal polynomial $t^2 - t - 2$

$$\text{Now } H \sim \begin{pmatrix} I & H \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ H & I \end{pmatrix} = \begin{pmatrix} I + H^2 & H \\ H & I \end{pmatrix} = \begin{pmatrix} H + 3I & H \\ H & I \end{pmatrix}$$

We want the eigenvalues

$$\begin{vmatrix} H + (3-\lambda)I & H \\ H & (1-\lambda)I \end{vmatrix} = \begin{vmatrix} H + (3-\lambda)I - \frac{1}{1-\lambda}H^2 & 0 \\ H & (1-\lambda)I \end{vmatrix}$$

$$\begin{aligned}
 &= \begin{vmatrix} (1-\lambda)(H + (3-\lambda)I) - (H+2I) & 0 \\ -\lambda H & I \end{vmatrix} \\
 &= |1-\lambda H + (\lambda^2 - 4\lambda + 1)I| = -\lambda^3 |H - (\lambda + \lambda^{-1} - 4)I|
 \end{aligned}$$

Now H has eigenvalues $2, -1 \Rightarrow A$ has eigenvalues λ such that $\lambda + \lambda^{-1} - 4 = 2$ or $\lambda + \lambda^{-1} - 4 = -1$.

Exercise: The stretching factor is the largest eigenvalue

$$\Rightarrow \lambda + \lambda^{-1} = 6 \Rightarrow \lambda = 3 + 2\sqrt{2}$$

Finally, the eigenvector gives the weights for \mathbb{Z} which give the invariant lamination L^S .

Remark: The eigenvalues λ, λ^{-1} occur in pairs as the stretching factors for L^S, L^U are inverses.

11.

Lemma 5.11 Suppose $h: F \rightarrow F$ leaves a train track τ invariant but leaves no proper subtrack invariant. Then h is non-periodic and irreducible if and only if every component of $F \setminus \tau$ is a disc and, for every h -invariant train track $\tau' \supset \tau$, $\hat{h}: W_{\tau'} \rightarrow W_{\tau'}$ has no non-zero fixed point.

A consequence of this lemma is

Theorem 5.12 If $h: F \rightarrow F$ leaves a train track τ invariant, then it is decidable whether h is non-periodic and irreducible. If it is, the stretching factor and invariant laminations are computable.

Proof of 5.11

1) If h is periodic, $\hat{h}: W_{\tau} \rightarrow W_{\tau}$ must have an eigenvector $w \in W_{\tau} \setminus \{0\}$ (Apply the Brower Fixed Point Theorem to W_{τ}/w^{\perp}). As \hat{h} is also periodic, the corresponding eigenvalue is a real root of 1, hence 1.
 $\therefore \hat{h}w = w$

(Recall: $\hat{h}: V_{\tau} \rightarrow V_{\tau}$ has an eigenvector $w \in W_{\tau}$ with eigenvalue $\lambda > 0$ because $\hat{h}(W_{\tau}) \subseteq W_{\tau}$. But $w_b > 0$ for each branch b of τ for otherwise $Sb: w_b > 0$ is an invariant subtrack of τ)

2) h irreducible \Rightarrow stated conditions:

If some component of $F \setminus \mathcal{F}$ is not a disc, the union of the 'boundary curves' of such components is a 1-submanifold, such that $h(c)$ is isotopic to c . (Exercise, cf proof of 5.12) (Hint: $h(\mathcal{F}) \subset$ standard neighbourhood U of \mathcal{F} transverse to the trees)

$\therefore h$ is reducible

If there is an h -invariant \mathcal{E}' , $\tilde{h}: W_{\mathcal{E}'} \rightarrow W_{\mathcal{E}'}$, and if $\tilde{h}(w) = w$ for some $w \in W_{\mathcal{E}'} \setminus \{o\}$, recall that $W_{\mathcal{E}'} \subset V_{\mathcal{E}'} \supset \mathcal{E}_{\mathcal{E}'}$ and \tilde{h} is represented by an integer matrix A w.r.t. $\mathcal{E}_{\mathcal{E}'}$.

$\therefore \exists$ a rational w w.r.t. $\mathcal{E}_{\mathcal{E}'}$ in $W_{\mathcal{E}'} \setminus \{o\}$ such that $\tilde{h}(w) = w$

$\therefore \exists$ an integer n " " " " " " "

so $f_{\mathcal{E}'}(w)$ is a 1-submanifold C (with counting measure and possibly parallel curves) $\subset F$, invariant up to isotopy under h .

$\therefore h$ is reducible.

3) h reducible \Rightarrow stated conditions fail

h reducible $\Rightarrow h(c) \equiv c$ for some geodesic 1-submanifold of F . Let $(L, \mu) = f_{\mathcal{E}'}(w)$ where $\tilde{h}(L, \mu) = (L, \lambda\mu)$ and $L = \text{support } \mu$.

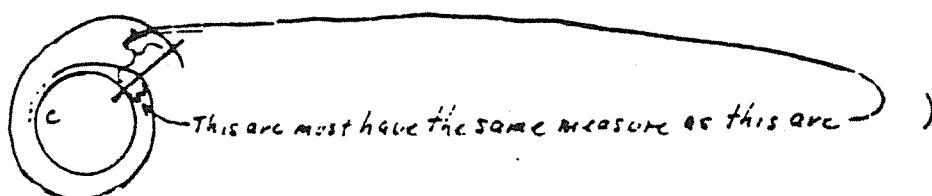
Now $\lambda\mu(c) = \tilde{h}\mu(c) = \tilde{h}\mu(\tilde{h}(c)) = \mu(c)$

\Rightarrow either $\lambda = 1$ or $\mu(c) = 0$

If $\lambda = 1$, $\tilde{h}(W_{\mathcal{E}'}) \cap W_{\mathcal{E}'}$ has a non-zero fixed point

If $\lambda \neq 1$ & $\mu(c) = 0$, then c has no transverse intersection with L . Hence C is a closed leaf of L , or is contained in a complementary region of L .

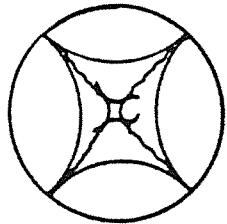
(Recall: $L = \text{support } \mu \Rightarrow$ every closed leaf is isolated, for otherwise we have the following:



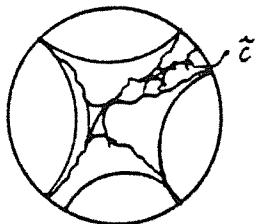
\therefore Some component of $F \setminus L$ is not simply connected.

Next, to each component of $F \setminus \mathcal{Z}$ corresponds a component of $F \setminus L$. This can be seen by taking the train routes corresponding to full branch numbers. Those associated to a fixed 'corner' can either have a common endpoint or they can diverge.

(Case 1) If at each 'corner' the train routes have common endpoints, these endpoints determine the region of $F \setminus L$ corresponding to your component of $F \setminus \mathcal{Z}$ by taking the geodesics joining them.



(Case 2) Maybe the train routes diverge. In this case more than one region of $F \setminus \mathcal{Z}$ corresponds to a region of $F \setminus L$.



Suppose that all the complementary regions of $F \setminus \mathcal{Z}$ are discs. In case 1) there is a simple essential closed curve $C \subset F \setminus L$ carried by \mathcal{Z} . In case 2) there is an h -invariant train track $\tilde{\mathcal{Z}}$, consisting of \mathcal{Z} plus some diagonals, carrying an essential simple closed curve C . In either case $\hat{h}: W_{\mathcal{Z}} \rightarrow W_{\tilde{\mathcal{Z}}}$ has a fixed point.

Remarks: In case 1, $\mathcal{Z}' \supset \mathcal{Z}$; in case 2 we needed to cross regions of \mathcal{Z} to get C which is why we pass to $\tilde{\mathcal{Z}}$. The measure on \mathcal{Z}' is induced by that on \mathcal{Z} and counting measure on C .

II.

Proof of 5.12 Consider $\hat{h}: W_{\mathcal{Z}} \rightarrow W_{\mathcal{Z}}$. Leave out a branch b if $\hat{h}(w_{\mathcal{Z}}, b) \in (w_{\mathcal{Z}}, b) = \{w \in W_{\mathcal{Z}} : |w(b)| = 0\}$

\therefore can assume that \mathcal{Z} has no h -invariant subtracks.

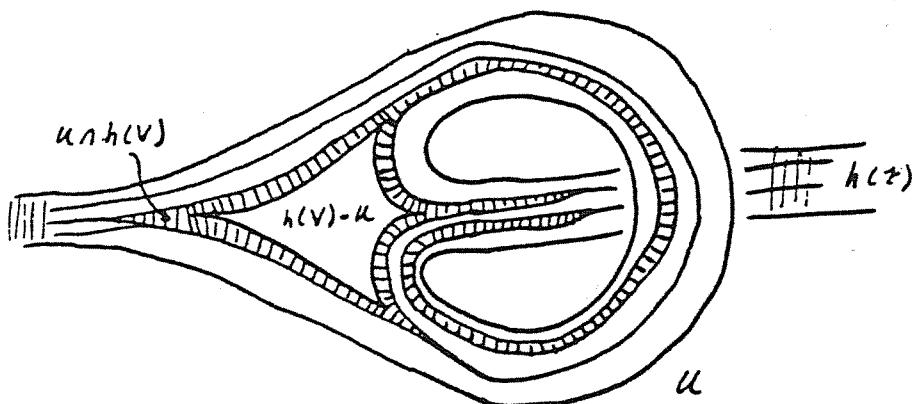
If $F \setminus \mathcal{Z}$ has disc components, we must find all the h -invariant train tracks $\mathcal{Z}' \supset \mathcal{Z}$. From the proof of 5.11 these are just \mathcal{Z} plus some diagonals.

Deciding whether there exists a fixed point for $\hat{h}: W_{\mathcal{Z}} \rightarrow W_{\mathcal{Z}}$ is a problem in linear algebra / linear programming as we have the following action of h on the components of $F \setminus \mathcal{Z}$.

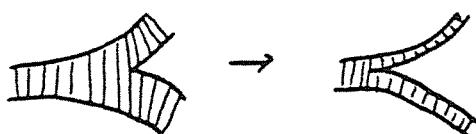
Action of h on components of $F \setminus \tau$

$h(\tau) \subset$ standard neighborhood U of τ . Let V be a component of $F \setminus \tau$, $h(V)$ is a component of $F \setminus h(\tau)$, $F_r(h(V)) \subset U$, $U \cap h(V)$ is a neighborhood of $F_r(h(V))$ in $h(V)$ consisting of ties of U transverse to branches of $h(\tau)$ (except maybe at switches)

Example:



Shrink these ties toward $F_r(h(V))$, elongating the 'horns' toward the switches of $h(\tau)$ as shown below



This gives an isotopy carrying $h(V) - U$ to $h(V)$. Notice $h(V) - U$ is a component of $F \setminus U$ and that τ is a spine of U .

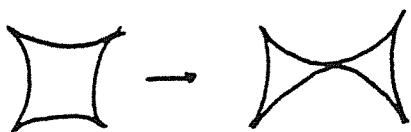
\therefore This component of $F \setminus U$ corresponds to a component $\tilde{h}(V)$ of $F \setminus \tau$
Notice that: branches of $h(V) \rightarrow$ branches of $\tilde{h}(V)$

cusps - " \rightarrow cusps of $\tilde{h}(V)$

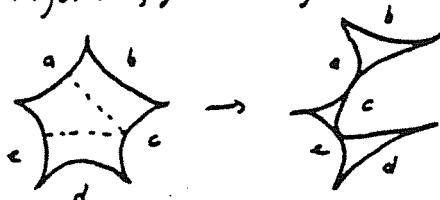
And since adding diagonals to regions of τ gives an h -invariant train track, if and only if the added diagonals are h invariant
diagonals \rightarrow diagonals

Exercise: If h is irreducible and non-periodic and τ is h invariant without h invariant subtracks, then the stretching factor λ is the largest real positive eigenvalue of $\hat{h}: V_\tau \rightarrow V_\tau$

Improvement: (Harer) Instead of adding diagonals, pinch regions



Quadrilaterals

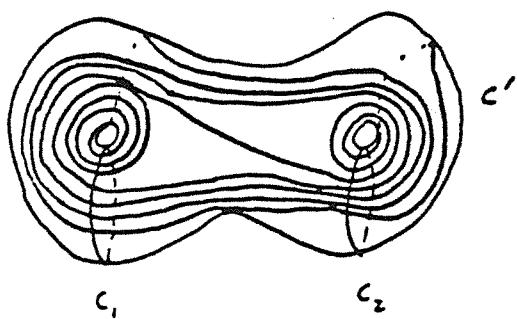


Pentagonals

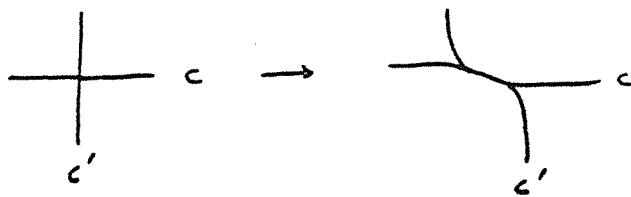
This has the effect of simultaneously adding all diagonal routes. The resulting train track is invariant under some power of h . Deciding irreducibility and periodicity for this power of h also decides these questions for h .

Theorem 5.13 (Penner) Let c, c' be geodesic 1-submanifolds of F having no component in common. Then there exists a train track τ which is invariant under all the products of the T_{c_i} and $T_{c'_j}$ (where $c = \cup c_i$, $c' = \cup c'_j$)

Example :



Proof: To build τ start with c and replace arcs of $c' \setminus c$ by branches permitting a right turn from c onto c'



2/14 Casson

If there are curves in C (c') not meeting $C' (c)$ we kill off the annular components which arise by adding extra branches. In the case of parallel branches, we get forbidden disc regions:

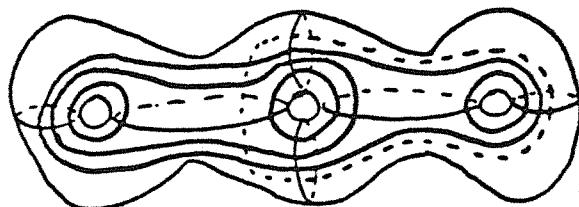


Notice that at this stage our train track is invariant under the appropriate Dehn twists in the curves of C and c' . We amalgamate parallel branches as shown above. Any train route carried by the illegal train track is carried by the new one, but the routes may no longer be simple.

After amalgamating our train track is still invariant as it carries our illegal one and after the appropriate Dehn twists on the curves in C and c' the image of our new track is carried by the illegal one.

11.

Another Example: Splitting a surface into 'pantographs'



Question: (Penner) Do all non-periodic and irreducible automorphisms of F arise in this way?

Example: The Torus $\text{Aut}(\mathbb{T}^2) \hookrightarrow \text{Aut}(\mathbb{Z}^2) \hookrightarrow \text{SL}_2(\mathbb{Q})$

$$C \quad \begin{matrix} T_C \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} ; \quad T_{C'}^{-1} \rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{matrix}$$

Every hyperbolic (non-periodic, irreducible) element of $SL_2(\mathbb{Z})$ is conjugate (i.e. maybe we must rechoose c, c') to

$$\pm \begin{pmatrix} 1 & m_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & m_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n_k & 1 \end{pmatrix}$$

with $m_i, n_i > 0$, unique up to cyclic permutation. The $m_1, n_1, \dots, m_k, n_k$ are the partial quotients of the periodic continued fraction expansion for the algebraic irrational associated to h via stretching factors.

Thm 5.14 If $h: F \rightarrow F$ is any automorphism, then for some $m > 0$, h^m has an invariant train track.

Remark: Notice that there is an upper bound on m depending on genus (F)

Proof Case 1) h periodic. Raise h to period, then every train track is invariant.

Case 2) h reducible $h(C) \subseteq C$ for some 1-submanifold C . C is already an invariant train track

Case 3) h non-periodic and irreducible. This case follows from a theorem we could have proved long ago.

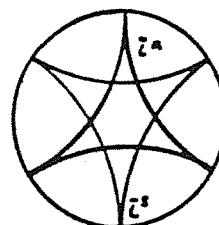
Thm 4.12 If $h: F \rightarrow F$ is irreducible, non-periodic, then h is isotopic to a homeomorphism $h_\# : F \rightarrow F$ such that $h_\#(L^S) = L^S$, $h_\#(L^U) = L^U$ (and $h_\#(\mu^S) = \mu^S$; $h_\#(\mu^U) = \mu^U$)

Remark: $h_\#$ is not necessarily a diffeomorphism

Proof of 4.12 Let \tilde{L}_S, \tilde{L}_U be preimages in $1H^2 \cong \tilde{F}$, $\tilde{h}: \tilde{F} \rightarrow \tilde{F}$ a lift of h , extend to $\tilde{h}: 1H^2 \cup S_\infty \rightarrow 1H^2 \cup S_\infty$. \tilde{h} induces an action on {leaves of \tilde{L}^S } and {leaves of \tilde{L}^U } (endpoints).
 $\therefore \tilde{h}$ induces an action on $\tilde{L}^S \cap \tilde{L}^U$.

Exercise: This action is continuous

(Hint: Angle of intersection between leaves of \tilde{L}^S and \tilde{L}^U is bounded away from 0 and π)

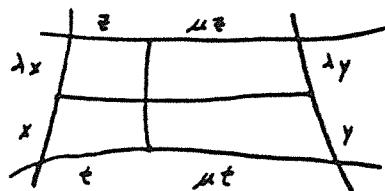


So we get an action on a compact countor set in F . This action can be extended over $\tilde{L}^3 \cup \tilde{L}^4$ by extending the action linearly w.r.t. hyperbolic distance over $L^3 \cup L^4$ and then lifting. One checks that this action is continuous (lower bound on angle of intersection), and hence uniformly so.

Notice that $F \setminus (L^3 \cup L^4)$ has regions whose closures are compact $2k$ -gons. There are only finitely many with $k \geq 3$ as there is only one for each complementary region of L^3 in F and only finitely many of those; but there are infinitely many rectangles. We can extend the \tilde{h} action over the non-rectangles by any $\pi_1(F)$ -equivariant choice of homeomorphisms.

To extend \tilde{h} over rectangles we set up a correspondence between 'opposite sides' using hyperbolic distance, setting up a 'grid' on the rectangle.

Viz:



Now extend the \tilde{h} -action over rectangles linearly via the grid.

Exercise: Extended action is continuous
we get $h_\# : \mathbb{H}^2 \rightarrow \mathbb{H}^2$

Exercise: $h_\#$, $h_\#^{-1}$ are continuous
so we get a homeomorphism $h_\# : F \rightarrow F$; notice that \tilde{h} , $\tilde{h}_\#$ induce the same action on S_∞' and are thus homotopic, and so therefore are $h_\#$ and h .

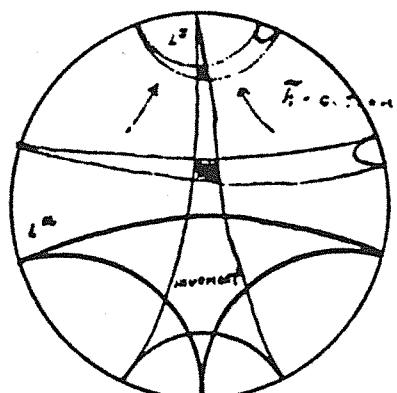
$\therefore h_\#$ is isotopic to h

II.

Returning to the proof of 5.14:

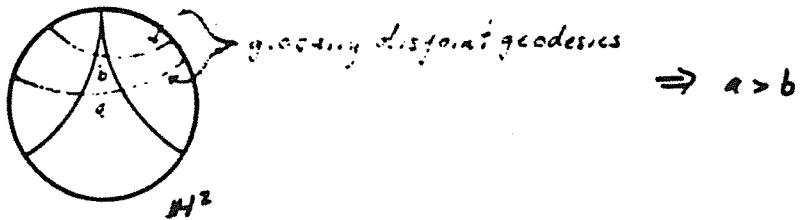
Assume $h = h_\#$, so that $h_\#(L^3) = L^3$ and $h_\#(L^4) = L^4$. Further assume h has been raised to an appropriate power, so that $\tilde{h}_\#$ is a lift fixing the complementary regions indicated in the universal cover. We'll describe a regular neighborhood of the invariant train track.

$F \setminus (L^3 \cup L^4)$ is the union of infinitely many rectangles and finitely many $2k$ -gons ($k \geq 3$).



Choose $\epsilon > 0$ small. Call a rectangle thin, if the unstable edges have length $< \epsilon$. Let $U = L^S \cup (\text{closures of thin rectangles}) \subset F$. Notice that $U \subset N_\epsilon(L^S)$. If ϵ is small, U is the standard neighborhood of a train track, the ties approximately parallel (as in the grid of Thm 4.12) to the leaves of L^u .

Exercise:



Cor: $h_\nu(\text{thin rectangle})$ is thin

Hence $h_\nu(U) \subset U$, and so U is the neighborhood of an invariant train track.

II.

Query: What is F/\sim , where $x \sim y$ if

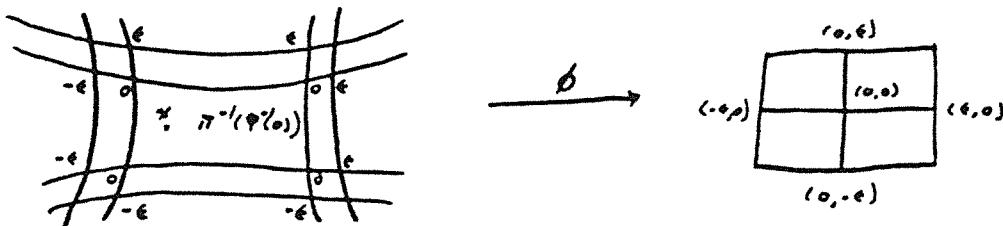
- (i) x, y are in the closure of the same component of $F \setminus (L^S \cup L^u)$
- (ii) x, y are in the closure of the same component of $L^S \setminus L^u$
- (iii) " " - " " " " " " " " $L^u \setminus L^S$
- (iv) $x = y$

Exercise: F/\sim is Hausdorff

Answer: F/\sim is a closed surface

Let $\pi: F \rightarrow F/\sim$, we describe neighborhoods of $\pi(x)$

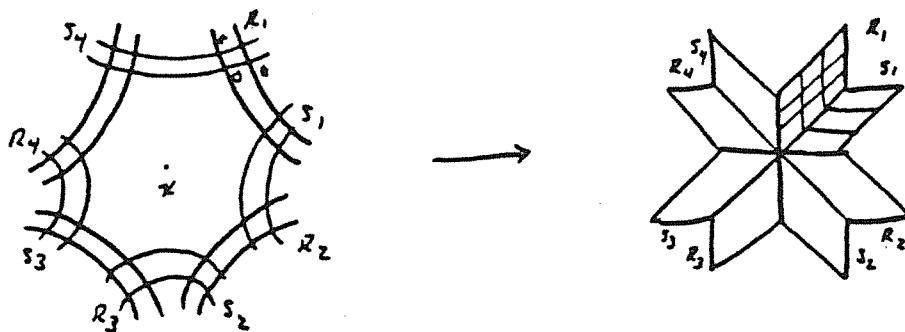
Suppose x is rectangle component of $F \setminus (L^S \cup L^u)$. Use the transverse measures to L^S and L^u to 'epsilon-expand' your rectangle, so that the 'edges' are non-isolated from the inside. From this we get a chart ϕ to $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$



2/18 - 2/21 Casson

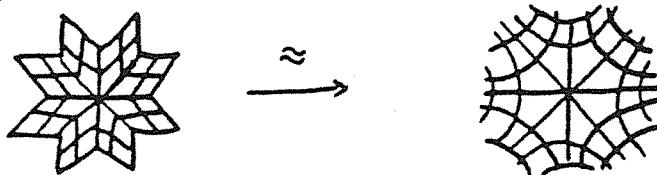
10

Next suppose $x \in 2k\text{-gon}$ for $k \geq 3$. Choose ϵ small. Take any vertex, ϵ expand along the unstable leaves using the stable measure and vice versa. Do this all around, so 'outer edges' are non-isolated from the inside.



Label the resulting $2k$ rectangles as illustrated, and notice they saturate. Make a model out of $2k$ parallelograms and use the coordinates given by the transverse measures to get a non-injective map $S: R_i \rightarrow R_{i,\text{model}}$. This map factors thru \sim to give a homeomorphism $\#$ when the lamination is the support of the measure. The model gives coordinates with 'corners', which are smooth.

Pix:



Exercise: $F/\sim \cong F$ and π is approximable by a homeomorphism. (Alternatively, show π a homotopy equivalence.)

Theorem 5.15 Every irreducible and non-periodic automorphism of a closed orientable surface F is isotopic to a pseudo-Anosov automorphism, $h_\#$. That is, $h_\#$ has a pair of transverse measured singular foliations (\bar{F}^s, μ^s) , (\bar{F}^u, μ^u) such that $h_\#(\bar{F}^s, \mu^s) = (\bar{F}^s, \lambda \mu^s)$ and $h_\#(\bar{F}^u, \mu^u) = (\bar{F}^u, \lambda^{-1} \mu^u)$ where $\lambda > 1$ is the stretching factor of h .

Proof h is isotopic to $h_*: F \rightarrow F$ preserving L^S, L^U . We get

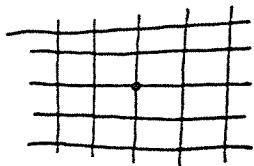
$$\begin{array}{c} h_*: F \rightarrow F \\ \pi \downarrow \quad \downarrow \pi \\ h_*: F/\sim \rightarrow F/\sim \end{array}$$

This lower map is the h_* of the theorem.

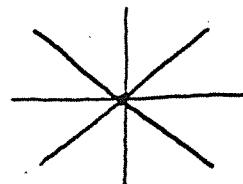
Remark: h_* is not necessarily a diffeomorphism

Proceeding as above, F/\sim has finitely many 'singular' points corresponding to the $2k$ -gons, $k \geq 3$

Pix:

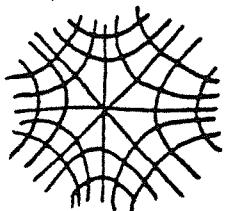


Regular

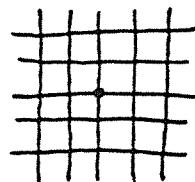


Singular

In our example above, taking double branch covers gives a model for the image of the measured laminations near a singular point which also displays the transverse measure



$$\xrightarrow{z \rightarrow z^2}$$

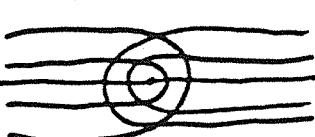
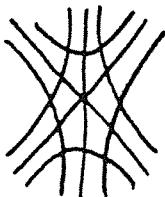


More generally for k even, use $z \rightarrow z^{k/2}$; follow by $z \rightarrow z^2$



Remark: This arises from

For k odd use $z \rightarrow z^k$



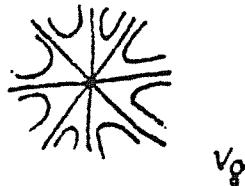
V_3

Def A singular foliation \mathcal{F} on a surface F is a decomposition of F as a disjoint union of leaves. Any point $x \in F$ (with finitely many exceptions) (i.e. any $x \in F \setminus S$ where S is finite) has a chart $\phi: U \rightarrow \mathbb{R}^2$ carrying the components of a leaf to horizontal intervals

$P(x)$:



For $x \in S$, x has a chart $\phi: U \rightarrow \mathbb{R}^2$ carrying $\mathcal{F} \cap U \rightarrow V_k$ for some k



We say x is a 'singularity with k separatrices' or a ' k -pronged singularity'. $\mathcal{F}^s, \mathcal{F}^u$ are transverse if they have the same singular set and at regular points the leaves are transverse. Further, we require the standard V_k model at singular points. A transverse measure is defined as with laminations.

Now if $(\mathcal{F}_1, \mu_1), (\mathcal{F}_2, \mu_2)$ are transverse measured foliations with singular set S , then every point $x \notin S$ has a 'canonical' chart $\phi: U \rightarrow \mathbb{R}^2$ such that

$$\begin{array}{ll} \mathcal{F}_1 \text{ leaves} \rightarrow \text{horizontals} & \mathcal{F}_2 \text{ leaves} \rightarrow \text{verticals} \\ \mu_1 \rightarrow \text{vertical distance} & \mu_2 \rightarrow \text{horizontal distance} \end{array}$$

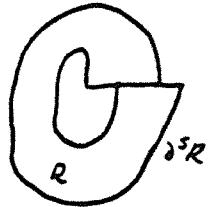
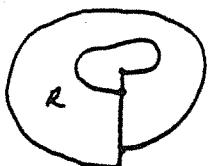
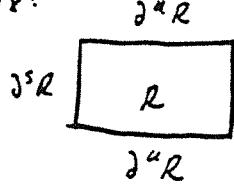
The map ϕ is unique up to translation in \mathbb{R}^2 and π -rotation. So the overlap functions lie in the pseudo-group generated by translations, π -rotation. This gives a branched flat structure on F . (\mathcal{F}_1, μ_1) and (\mathcal{F}_2, μ_2) determine a C^∞ structure on F , so $h_\#$ can be smoothed away from S .

Notice that w.r.t. smooth structure determined as above, $h_\#$ is a diffeomorphism away from singular set. i.e. Anosov away from the singular set.

Exercise: $h_\#$ cannot be smoothed to a diffeomorphism near the singular points.

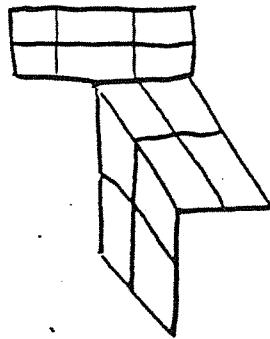
Def A rectangle is a map $\rho: I^2 \rightarrow F$ such that $\rho| \text{Int } I^2$ is an embedding and $\rho(\rho t \times I)$ is unstable leaf and $\rho(I \times \rho t)$ is stable leaf.

$\rho|_I:$



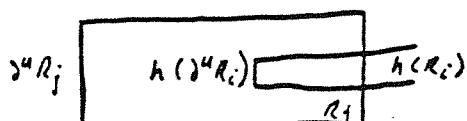
Lemma 5.16 If $h: F \rightarrow F$ is pseudo-Anosov, there exists a decomposition of F as $\bigcup_{i=1}^n R_i$, where each R_i is a rectangle and

- 1) $\text{Int } R_i \cap \text{Int } R_j = \emptyset$ if $i \neq j$
- 2) $h\left(\bigcup_{i=1}^n d^u R_i\right) \subseteq \bigcup_{i=1}^n d^u R_i$
- 3) $h\left(\bigcup_{i=1}^n d^s R_i\right) \subseteq \bigcup_{i=1}^n d^s R_i$

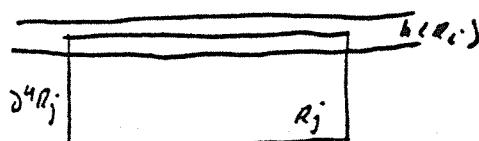


Remarks:

Consider $h(R_i) \cap R_j$



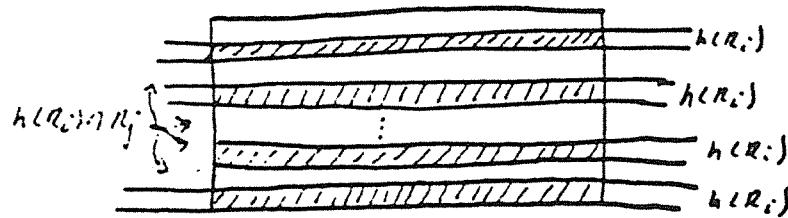
This case is ruled out as $h(\cup d^u) \not\subseteq \cup d^u$



This case is ruled out as $h^{-1}(d^s) \not\subseteq \cup d^s$ as there is an arc of $d^s R_j$ in $\text{Int } h(R_i)$

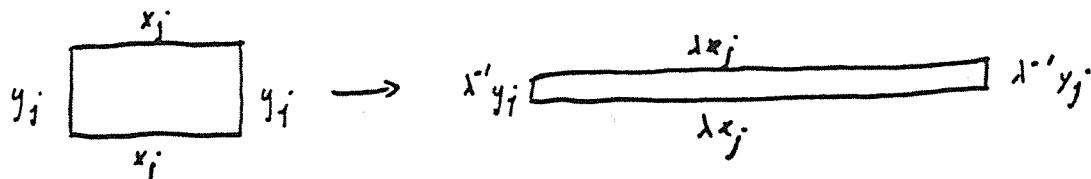
However it may happen that one of $d^u(h(R_i)) \subset d^u R_j$ or that one of $d^s(R_j) \subset d^s(h(R_i))$.

pix:

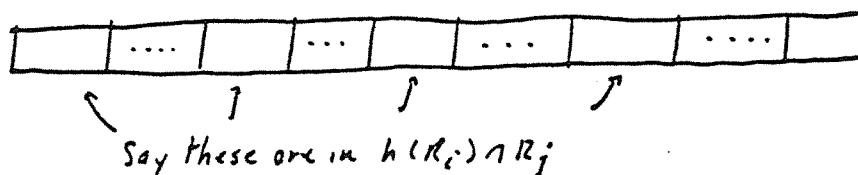


Therefore $h(R_i) \cap R_j$ consists of a finite collection of rectangles with stable dimension μ^u (δ^s component of R_j) i.e. the stable dimension of R_j and unstable dimension μ^s (δ^u component of $h(R_i)$), i.e. the unstable dimension of $h(R_i)$.

Let $A_{ij} = \#$ of components of $\text{Int } h(R_i) \cap \text{Int } R_j$. Let $x_j = \mu^u$ (either stable edge of R_j) = $\mu^u(p_j \in \Sigma \text{ int point})$; $y_j = \mu^s$ (either unstable edge of R_j) = $\mu^s(p_j \in \text{int point} \in \Sigma)$. Then under h



Thus each 'component' of $h(R_i) \cap R_j$ has dimensions $x_j, \lambda^s y_j$
(consider $h(R_i)$)



Then

$$\lambda x_i = \text{stable length of } h(R_i) = \sum_{j=1}^n A_{ij} x_j \text{ and } y_j = \text{unstable length}$$

$$\text{of } R_j = \sum_{i=1}^n A_{ij} \lambda^s y_i$$

Thus $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a positive eigenvector of A with eigenvalue λ

and $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is a positive eigenvector of A^T with eigenvalue λ

Exercise: Show that there exists an n such that every entry of A^n is strictly positive. Deduce that λ is real and has the largest modulus of the eigenvalues of A and is of multiplicity 1. Hence the eigenvector x is (essentially) unique, and so the h -invariant measures are unique. Thus we have a parallel technique to train tracks.

Proof of 5.16 First assume that h fixes the singular points and separatrices.

Let R_1, \dots, R_n be rectangles such that $\text{Int } R_i \cap \text{Int } R_j = \emptyset$ and each component of $\cup_{i=1}^n R_i$ and $\cup_{i=1}^n R_i^c$ contains a singularity.

Remark: This insures that 2) and 3) hold for R_1, \dots, R_n

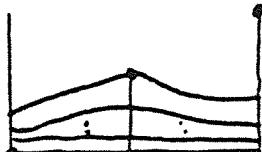
Such systems exist e.g. a rectangle with opposite vertices singular. A rectangle of this type is easily found:

Begin with a singularity and a stable and unstable leaf intersecting there. Choose a second singularity. An unstable leaf through this second singularity is dense and so meets the given stable leaf



Start here

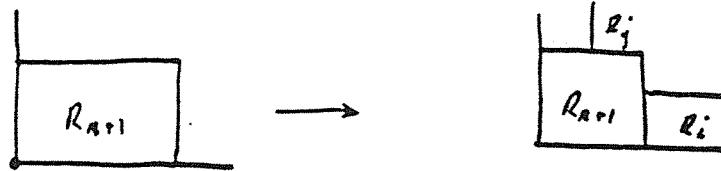
Next examine nearby stable leaves; one hits some singularity, use the unstable leaf of this singularity. Note that it must meet the original stable leaf by transversality. Now you have a rectangle with opposite vertices singular.



rectangle
as required.

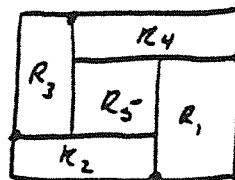
So suppose that $\cup_{i=1}^n R_i \neq F$. Then, if there exists a singularity x on the closure of $F \setminus \cup_{i=1}^n R_i$, adjoin a new rectangle with x as vertex, call it R_{n+1} .

Push the edges of R_{i+1} as far as possible, i.e. until you meet a singularity



or some R_i , in which case extend the other side until you hit another rectangle or singularity. Notice that this preserves the invariance property. If all the singular points are encircled by $\cup R_i$, then $F \setminus \cup R_i$ is already a disjoint union of rectangles. Adjoin those rectangles.

Example: $R_1 - R_4$ have opposite vertices singular, R_5 has no singular vertices.



Exercise: Complete the proof for the cases when h permutes the separatrices and singularities.

Remark: The decomposition of F into rectangles is a Markov partition of F .

11.

Chapter 6 Teichmuller Space

A closed orientable surface F with negative Euler characteristic has hyperbolic structures: each is specified by a hyperbolic metric ρ .

If $h: F \rightarrow F$ is a homeomorphism, define $h^*\rho$ via

$$h^*\rho(x, y) = \rho(h(x), h(y))$$

and define $h_*\rho$ via

$$h_*\rho(x, y) = \rho(h^{-1}(x), h^{-1}(y))$$

Notice $h^{**} = h^{-1}_*$

Define hyperbolic metrics ρ_1, ρ_2 on F to be equivalent if $\rho_2 = h_*\rho_1$ for some homeomorphism $h: F \rightarrow F$ isotopic (equivalently homotopic) to 1_F .

Def The Teichmuller space $J(F)$ of F is the set of equivalence classes of hyperbolic structures on F with the topology obtained as the quotient of a subspace of all metrics on F . If F is compact and $\partial F \neq \emptyset$, we require that the hyperbolic structures make ∂F totally geodesic.

Let ρ be a hyperbolic structure. Let $l_\rho(c)$ be the ρ -length of the unique closed geodesic homotopic to c . i.e. $l_\rho(c) = \inf \{ \rho\text{-length of } c' \mid c' \simeq c \}$.

Lemma 6.1 For a fixed essential closed curve $c \subset F$, the map $\rho \mapsto l_\rho(c)$ induces a map $\ell(c): J(F) \rightarrow \mathbb{R}$

Proof Suppose $\rho_1 \sim \rho_2$ i.e. $\rho_2 = h_*\rho_1$ for $h \simeq 1_F$. Then

ρ_2 -length of $c = \rho_1$ -length of $h(c)$. Note that $h(c) \simeq c$, so taking inf over all c' homotopic to c , we get

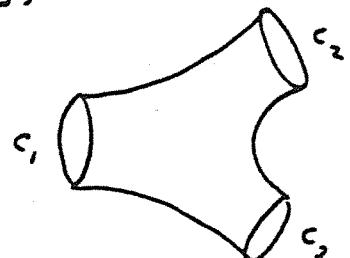
$$l_{\rho_2}(c) = l_{\rho_1}(c).$$

Exercise: $\ell(c): J(F) \rightarrow (0, \infty)$ is continuous.

Lemma 6.2 If P is a 'pair of pants' with $\partial P = C_1 \cup C_2 \cup C_3$, then the map

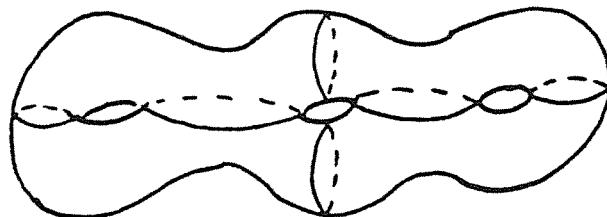
$\rho \rightarrow (\ell_\rho(C_1), \ell_\rho(C_2), \ell_\rho(C_3))$
induces a homeomorphism

$$\mathcal{J}(P) \rightarrow (0, \infty)^3 = \mathbb{R}_+^3$$



Remark: Armed with this lemma we will analyse the hyperbolic structures on closed orientable surfaces by decomposing into pairs of pants.

Ex:



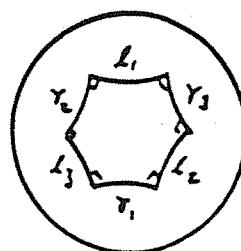
Proof we must show

- 1) Any positive $\ell(C_i)$'s can occur and be realized
- 2) The hyperbolic structure is determined by $\ell(C_i)$ up to orientation preserving isometry preserving boundary curves
- 3) Bicontinuity

We leave 3) as an exercise.

For 1): Given $l_1, l_2, l_3 > 0$, up to an isometry which does not permute edges, there exists a unique hexagon in \mathbb{H}^2 with all angles $\pi/2$ and even edges of length l_1, l_2, l_3 .

Because: Let r_i be the edge opposite the l_i -edge. The l_i -edge is contained in the unique geodesic perpendicular to r_j, r_k and l_i is the perpendicular distance between the geodesics r_j, r_k .



Up to isometry there is a unique pair of geodesics r_2, r_3 with perpendicular distance l_1 .

Having chosen γ_2 and γ_3 , note that the distance l_3 determines a Euclidian circle meeting the endpoints of γ_2 . Define γ_2 . Take γ_1 as the unique geodesic tangent to both these circles.

Now given l_1, l_2, l_3 make a hexagon H with edges $l_{1/2}, l_{2/2}, l_{3/2}$.

Double along the other edges to get P with ∂P totally geodesic of lengths l_1, l_2, l_3 . Establishing 1)

For 2) Given a hyperbolic structure on P , there exists a unique shortest arc joining c_i and c_j , geodesic and perpendicular to c_i and c_j .

$\therefore P$ can be decomposed into the union of two hexagons H, H' such that H, H' are hexagons with right angles and even lengths equal. Hence H and H' are isometric. Thus all hyperbolic structures on P arise by doubling right angled hexagons.

II.

Remark: This generalizes to

$$\gamma(F) \xrightarrow{\text{onto}} \mathbb{R}_+^{3g-3} \quad (\text{take lengths of curves on each pair of pants.})$$

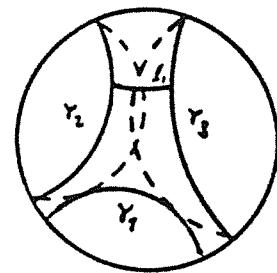
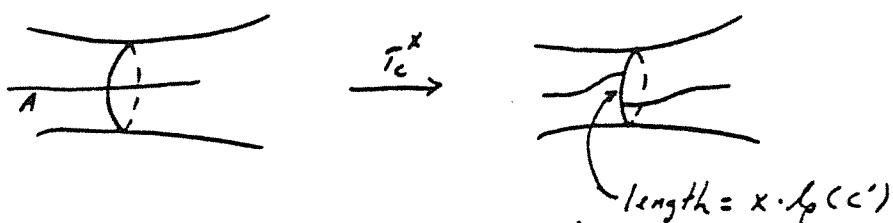
For closed F :

$$\gamma(F)/\mathbb{R}^{3g-3} \simeq \mathbb{R}_+^{3g-3}.$$

Def If C is a simple closed curve, ρ a hyperbolic structure on F , deform C to a closed geodesic, cut F along C' and reglue after a twist through an angle of $2\pi x$. The result is $T_C^x(\rho)$. We get

$$T_C^x : \gamma(F) \rightarrow \gamma(F).$$

Fix:

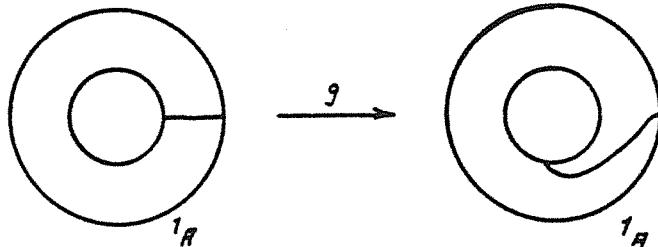


3/4 Lesson

Exercise: $T_c^n(p) = (T_c^n)_* p$ where right hand T_c^n is the Dehn-twist in C .

More formally, choose a collar neighborhood N of C and a homeomorphism $\phi: N \rightarrow \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ such that $\phi(C) = \text{unit circle}$. Define

$$g(r, \theta) = (r, \theta + 2\pi \times (r-2))$$



Now define $f: F \setminus C \rightarrow F \setminus C$ via $\begin{cases} \phi g \phi^{-1} & \text{on } N \\ f & \text{off } N \end{cases}$

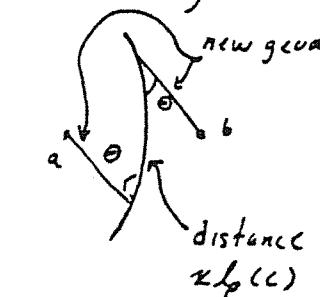
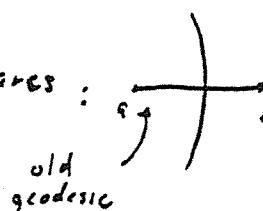
Next define $T_c^x(p)$ by charts. Away from C a chart is ψf^{-1} where ψ is a p -chart. Note $\psi f^{-1}: U \rightarrow \mathbb{H}^2$.

On C define your chart in two pieces, glued by an appropriate isometry of \mathbb{H}^2

Picture:



, For short transverse arcs :
in old metric



If C is a simple closed curve on F ; $T_c^x: \mathcal{J}(F) \rightarrow \mathcal{J}(F)$ is well-defined and depends only on the isotopy class of C . If p_1, p_2 are equivalent hyperbolic structures via h , the curve C is homotopic to closed p_i -geodesics C_i . Note $C_2 = h(C_1)$. Choosing compatible collar neighborhoods shows that h induces an isometry $T_{C_2}^x(p_2) \sim T_{C_1}^x(p_1)$.

Lemma 6.3 Let ρ be a hyperbolic structure on F and let

$G_\rho = \{h: F \rightarrow F \mid h_*\rho = \rho \text{ as metrics}\}$, the isometry group of ρ and let

$G_{[\rho]} = \{h \in \text{Aut}(F) \mid h_*[\rho] = [\rho]\}$, the isotopy group of $[\rho]$
(where $[\rho]$ = class of ρ in $\mathcal{Y}(F)$)

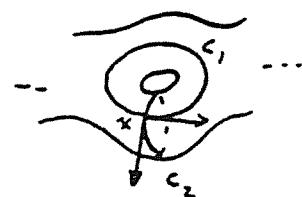
Then the natural inclusion $i: G_\rho \rightarrow G_{[\rho]}$ is an isomorphism and both groups are finite.

Remark: $\text{Aut}(F) = \{\text{orientation preserving homeomorphisms } F \rightarrow F\} / \text{isotopy}$
= mapping class group, acts on $\mathcal{Y}(F)$ via $(h, \rho) \mapsto h_*\rho$
The quotient space $\mathcal{Y}(F) / \text{Aut}(F)$ is called the moduli space of F

Proof i is onto: Given $h: F \rightarrow F$ such that $h_*[\rho] = [\rho]$, then $h_*\rho = g_*\rho$ for some $g: F \rightarrow F$ homotopic to 1_F . Set $\tilde{h} = g^{-1}h \simeq h$
Then $\tilde{h}_*\rho = g_*^{-1}h_*\rho = \rho$
 $\therefore \tilde{h} \in G_\rho$
 $\therefore i$ is onto

i is 1-1: Suppose $h \in G_\rho$ i.e. h is an isometry of ρ . We must show that if $h \simeq 1_F$, then $h = 1_F$

Let C_1, C_2 be simple closed ρ -geodesics with a single common point $x = C_1 \cap C_2$. h an isometry $h(C_i) = C_i$, indeed preserving orientations $\Rightarrow h(x) = x$. Further h induces a map on $T_x F$ which fixes all the tangent vectors. $\Rightarrow h$ fixes a neighborhood of x . A connectedness argument shows $h = 1$ on F .

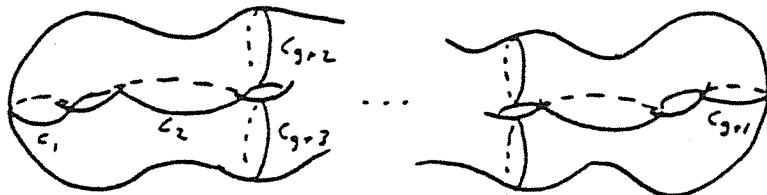


Exercise: If h is an isometry of ρ , \mathbb{Z}_p -homologous to 1_F for some prime $p \Rightarrow h = 1$

Finally G_ρ is finite as it is discrete and bounded (hence compact) in the metric topology.

Theorem 6.4 Let F be a closed orientable surface with $\chi_F < 0$. Then $\gamma(F) \cong \mathbb{R}^{3g-6}$

Proof Dissect F into pairs of pants by disjoint simple closed curves. There



are $(g+1) + 2(g-2) = 3g-3$ such. Call the geodesics homotopic to these curves c_1, \dots, c_{3g-3} .

Define $l_c : \gamma(F) \rightarrow (\sigma, \infty)^{3g-3} : \rho \mapsto (l_{c_1}(\rho), \dots, l_{c_{3g-3}}(\rho))$

This map is onto: The lengths of c_1, \dots, c_{3g-3} determine hyperbolic structures on the $2g-2$ pairs of pants. Any gluing of these pairs of pants determines $\rho \in \gamma(F)$ such that $l_c(\rho) = (l_1, \dots, l_{3g-3})$.

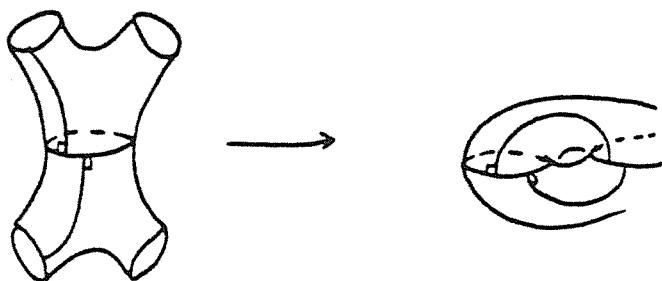
Now notice that the abelian group \mathbb{R}^{3g-3} acts on $\gamma(F)$: $x = (x_1, \dots, x_{3g-3}) \mapsto T_c^x = T_{c_1}^{x_1} \cdots T_{c_{3g-3}}^{x_{3g-3}}$. Note $T_{c_i}^{x_i}$ does not change $l(c_i)$.

Further: ρ, ρ' in $\gamma(F)$ are in the same orbit $\Leftrightarrow l_c(\rho) = l_c(\rho')$

Finally, we claim \mathbb{R}^{3g-3} acts without fixed points.

For if $T_c^x([\rho]) = [\rho]$, then x_1, \dots, x_{3g-3} are integers as a non-integer twist changes the distance between the feet of the unique perpendiculars from c_j 's to c_i .

Fix:



So if T_c^X has a fixed point, then T_c^X is induced by an automorphism $h: F \rightarrow F$
 $h = T_{c_1}^{n_1} \cdots T_{c_{3g-3}}^{n_{3g-3}}$

If $h_*[\rho] = [\rho]$, then h is an element of the isotropy group of $[\rho]$ which
is finite by Lemma 6.3. But if any $n_i \neq 0$, h has infinite order \Rightarrow all $n_i = 0$

$$\Rightarrow h = 1_F$$

$\therefore \mathbb{R}^{3g-3}$ acts without fixed points and $\mathcal{G}(F)/\mathbb{R}^{3g-3} \cong \mathbb{R}^{3g-3}$

i.e. $\mathcal{G}(F)$ is an \mathbb{R}^{3g-3} bundle over \mathbb{R}^{3g-3}

Exercise: This bundle is locally trivial

$$\therefore \mathcal{G}(F) \cong \mathbb{R}^{3g-3} \times \mathbb{R}^{3g-3} \cong \mathbb{R}^{3g-6} = \mathbb{R}^{-3} \mathcal{L}_F$$

II.

Theorem 6.5 Let F be a closed, orientable, hyperbolic surface and let $h: F \rightarrow F$
satisfy $h^n \approx 1_F$ for some n . Then $h \approx g$ where $g^n \approx 1_F$.

Proof We'll prove $h_*: \mathcal{G}(F) \rightarrow \mathcal{G}(F)$ has a fixed point $[\rho]$. Then h is
in the isotropy group of $[\rho]$. By lemma 6.3 $h \approx$ unique g in the isometry group of ρ .
Then g^n is (the unique) isometry of ρ homotopic to $h^n \approx 1_F$, so by the proof
of lemma 6.3, $g^n \approx 1_F$.

First suppose $n=p$ a prime. Then $h_*: \mathcal{G}(F) \rightarrow \mathcal{G}(F)$; $h^n \approx 1 \Rightarrow$
 $(h_*)^n = 1_{\mathcal{G}(F)}$. If h_n had no fixed point in $\mathcal{G}(F)$, h_n would generate a fixed
point free action of \mathbb{Z}_p on $\mathcal{G}(F) \cong \mathbb{R}^{6g-6}$.

Then $\mathcal{G}(F)/\mathbb{Z}_p$ would be a finite dimensional $K(\mathbb{Z}_p, 1)$. But $H^*(K(\mathbb{Z}_p, 1))$
to in infinitely many dimensions \Rightarrow No such space exists

$\therefore h_*$ has a fixed point

Exercise: Complete the proof for the general case.

Hint: Argument above applies to $(h^n/p)^{p_1} \approx 1_F$. Induct.

II.

Remarks: Given a finite group G , a corollary to the following is

$$\begin{array}{ccc} \text{3 lift} & \xrightarrow{\quad} & \text{Homeo}_+(F) \\ & \downarrow & \\ G & \hookrightarrow & \text{Aut}_+(F) \end{array} \quad \text{For general } G \text{ unknown.}$$

Theorem 6.6 (Kerckhoff) Let C_1, \dots, C_k be essential closed curves in F that fill F (note: C_i 's not necessarily simple). Then there exists a unique $[\rho] \in \mathcal{Y}(F)$ minimizing $\sum_{i=1}^n l_p(C_i)$.

Remarks: Theorem is false if the C_i 's don't fill F . Also, observe that $[\rho] \in \mathcal{Y}(F)$ depends only on the homotopy classes of C_1, \dots, C_k because $l_p(C_i)$ depends only on $[C_i]$ in $\mathcal{Y}(F)$ and the homotopy class of the C_i . Recall C_1, \dots, C_k fills \Leftrightarrow For all s.c.c.'s C , there exists j such that $i(C, C_j) > 0$.

Proof Deferred

Cor If G is a finite subgroup of $\text{Aut}(F)$ then there exists $[\rho] \in \mathcal{Y}(F)$ such that $G \subseteq$ isotropy group of $[\rho] \cong$ isometry group of ρ .

Proof of Cor: There exists a finite G -invariant set S of homotopy classes of simple closed curves filling F . e.g. If C_1, \dots, C_k fill F , take $S = \{[gC_i] \mid g \in G, 1 \leq i \leq k\}$

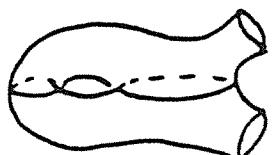
By Theorem 6.6, there exists a unique $[\rho]$ minimizing $\sum_{[C] \in S} l_p(C)$

By uniqueness, $g_*[\rho] = [\rho]$ for all $g \in G \Rightarrow G \subseteq$ isotropy group of $[\rho]$

11.

Lemma 6.7 If F is compact orientable connected with $\chi_F < 0$, then $\mathcal{Y}(F) \approx \mathbb{H}^{3\chi_F}$
(we require boundary components to be geodesics)

Proof Decompose F into pairs of pants. $\chi(P) = -1$, number of pairs of pants $= -\chi_F$. $\dim \mathcal{Y}(F) = 2(\# \text{ internal curves}) + (\# \partial\text{-curves}) = 3(\# \text{ pants}) = -3\chi_F$



11.

Lemma 6.8 If F is a closed orientable connected surface with $\chi_F < 0$ then
 $\gamma(F) \approx \{\text{discrete, faithful representations } \pi_1(F) \rightarrow PSL_2(\mathbb{R})\} / \text{conjugation in } PSL_2(\mathbb{R})$

Proof Identify $\pi_1(F) \cong \pi \cong \text{deck transformation group of } \tilde{F}$. If ρ is a hyperbolic structure on F it lifts in a unique way to $\tilde{\rho}$ on \tilde{F} . There exists a unique isometry $D: \tilde{F}, \tilde{\rho} \rightarrow \mathbb{H}^2$, the developing map. This map is unique up to composition with an element of $\text{Iso}^+(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$.

The holonomy map $H: \pi \rightarrow \text{Iso}^+(\mathbb{H}^2)$ is defined via $Dg = H(g)D$ i.e. $H(g)$ is g conjugated by D ($g \in \pi$ a deck translation). H depends only on D . Replacing D by hD where h is an isometry of \mathbb{H}^2 , we see that $(hD)_g = hH(g)h^{-1}(hD)$, so H is replaced by hHh^{-1} .

Thus, up to conjugacy in $PSL_2(\mathbb{R})$, ρ determines a unique representation $H: \pi \rightarrow \text{Iso}^+(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$.

Exercise: g acts without fixed points $\Rightarrow H$ is faithful, discrete

Next, replacing ρ by $f_*\rho$ where $f \cong f_F$ only changes H by conjugation so we get a well-defined map

$$\gamma(F) \longrightarrow \{\text{discrete, faithful representations } \pi \rightarrow PSL_2(\mathbb{R})\} / \text{conjugation in } PSL_2(\mathbb{R})$$

We must check that this map is 1-1, onto.

For $\pi = \pi_1(F)$, F closed gives $\phi: \pi \rightarrow \text{Iso}^+(\mathbb{H}^2)$ discrete, faithful. Set $M = \phi(\pi) \cong \pi \cong \pi_1(F)$. Then $G = \mathbb{H}^2/M$ is a hyperbolic surface with $\pi_1(G) = \Gamma = \pi = \pi_1(F)$. Theorem follows from the fact that any isomorphism between $\pi_1(G)$ and $\pi_1(F)$ is induced by a homeomorphism.

II.

Remarks: The above shows that we have the right dimension for $\gamma(F)$

For topologically $PSL_2(\mathbb{R})$ is the 3-manifold $S^1 \times \mathbb{R}$; $\pi_1(F) = \langle x_1, y_1, \dots, x_g, y_g : [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$, so to choose a representation $\phi: \pi_1(F) \rightarrow PSL_2(\mathbb{R})$ we need $2g$ points in $S^1 \times \mathbb{R}$: $\phi(x_i) = X_i$, $\phi(y_i) = Y_i$

$\Rightarrow 6g$ dimensional space of representations from the free group $\langle x_1, y_1, \dots, x_g, y_g \rangle$ to $PSL_2(\mathbb{R})$. We want those representations satisfying

3/11 Lesson

$[X_1, Y_1] \cdots [X_g, Y_g] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ a matrix equation
 So we expect a $6g-3$ dimensional space of representations $\pi_1(F) \rightarrow PSL_2(\mathbb{R})$
 Finally, as $\dim PSL_2(\mathbb{R}) = 3$ we expect conjugation by elements of $PSL_2(\mathbb{R})$
 to drop the dimension to $6g-6$.

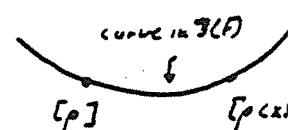
Lemma 6.9 Let F be a closed orientable surface with hyperbolic structure ρ .

Let $C \subset F$ be an essential s.c.c., let C' be an essential closed curve in F ,

let $\rho(x) = T_C^x(\rho)$. If $i(C', C) > 0$ then $f(x) = l_{\rho(x)}(C')$

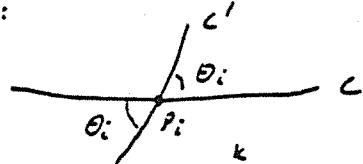
is strictly convex and $f(x) \rightarrow +\infty$ as $x \rightarrow \pm\infty$

$\rho(x)$ -geodesic $\approx C'$



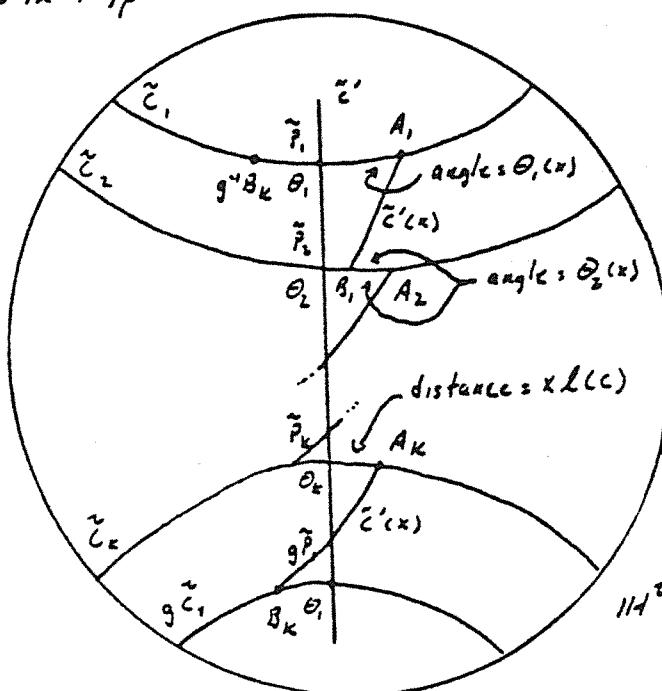
Proof Begin by finding $f'(0)$. Deform C, C' to ρ -geodesics, let
 $C \cap C' = \{p_1, \dots, p_k\}$, and let θ_i be the angle between C, C' at p_i ; $0 < \theta_i < \pi$.

$\rho(x)$:



Claim: $f'(0) = \sum_{i=1}^k l(C) \cos \theta_i$

Proof of Claim: Let $\tilde{C}'(x)$ be the $\rho(x)$ geodesic homotopic to C' . $\tilde{C}'(x)$ will appear
 as k segments in $\tilde{F}, \tilde{\rho}$:



Note: x is small, C' represents g in deck-translations

Note: length between $\tilde{p}_i, g\tilde{p}_i$ is $l_{\rho}(C')$

Angles must match up so that geodesics form.

Also: ρ -distance (A_{i+1}, B_i)
 $= \rho(x)$ -distance (A_{i+1}, B_i)
 $= x l(C)$

$$\text{Hd}^2 \tilde{F}, \tilde{\rho}$$

We claim that for small x , $f(x) = f(0) + x l_p(c) \sum_{i=1}^k \cos \theta_i + o(x)$

Exercise: $d(A_i, B_i) - l_p$ (orthogonal projection on C') is $o(x)$.

$$\therefore f(x) = \sum_{i=1}^k p\text{-distance } (A_i, B_i) = \sum_{i=1}^k p\text{-length (orthogonal projection)}$$

of A_i, B_i on C') + $o(x) = l_p(C') + \sum k p\text{-length (orthogonal projection of}$
 A_i, B_i on $C')$ + $o(x)$. Note: An implicit sign convention here

$$= f(0) + \sum_{i=1}^k x l_p(c) \cos \theta_i + o(x)$$

$$\Rightarrow f'(0) = l_p(c) \sum_{i=1}^k \cos \theta_i \quad \text{and our claim follows.}$$

$$\text{Similarly } f'(x) = l_p(c) \sum_{i=1}^k \cos \theta_i \cdot cx$$

We wish to see that $f'(x)$ is strictly increasing, we'll prove that $\theta_i(x)$ is a strictly decreasing function of x . It is enough to show that if $x > 0$ then $\theta_i(x) < \theta_i(\theta) = \theta_i$.

Choose a transverse orientation for \tilde{C}' so that A_{i+1} is 'right' of B_i . Suppose A_i is to the right of P_1 (if not just reverse the orientation). If $\theta_i(x) \geq \theta_i$ for some $x > 0$, then B_i (and hence A_{i+1}) is to the 'right' of P_2 .

As our geometry is hyperbolic:

$$(\pi - \theta_i) + \theta_2 + \theta_i(x) + (\pi - \theta_2) < 2\pi$$

(note: here, it is essential that the segment of $\tilde{C}'(x)$ is to the 'right' of \tilde{C}') $\therefore 0 \leq \theta(x) - \theta_i < \theta_2(x) - \theta_2$

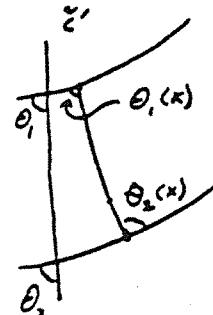
Repeating the argument we eventually get $\theta_i(x) - \theta_i > \dots > \theta_1(x) - \theta_1$

$\therefore \theta_i(x)$ is strictly decreasing

$\therefore f'(x)$ is strictly increasing

$\therefore f(x)$ is convex

Exercise: $f(x) \rightarrow \infty$ as $x \rightarrow \pm \infty$



3/25 Casson

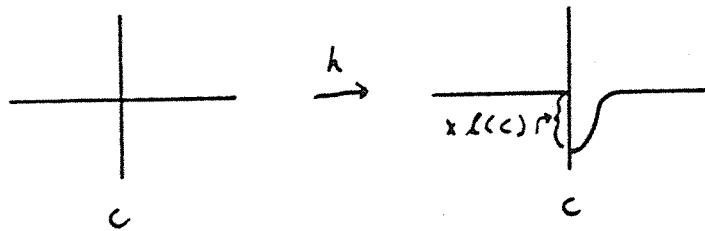
11

Remark: The above lemma suggests Kerchoff's theorem, but as there are only countably many closed geodesics, we need a more general notion than twisting in a closed geodesic.

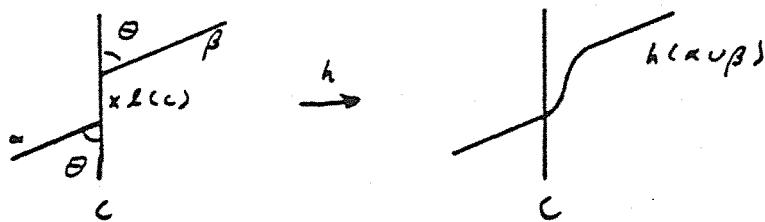
Earthquakes

$T_c^x(p)$ is a special case of an earthquake where our lamination is C and $x\ell(C)$ represents counting measure.

$T_c^x(p) = p(x)$ is a homeomorphism $h: F \setminus C \rightarrow F \setminus C$; F off a neighborhood of C with a discontinuity $x\ell(C)$ across C . We choose h to be a homeomorphism with C a boundary curve of the neighborhood.



If ϕ is a p -chart away from C , then ϕh^{-1} is a $p(x)$ -chart. If α, β are p -geodesic segments as illustrated below, then $h(\alpha \cup \beta)$ is a $p(x)$ -geodesic:



There is also the holonomy $H_p: \pi_1(F) \rightarrow \text{Isom } \mathbb{H}^2$; $(F, p) = \mathbb{H}^2 / H_p(\pi_1(F))$ and developing maps

$$D: (\tilde{F}, p) \rightarrow \mathbb{H}^2 \quad \& \quad D_x: (\tilde{F}, p(x)) \rightarrow \mathbb{H}^2$$

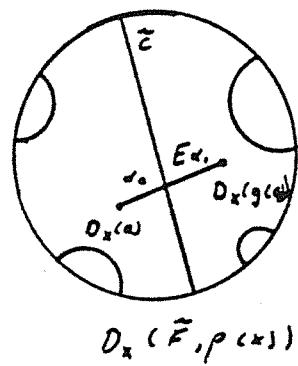
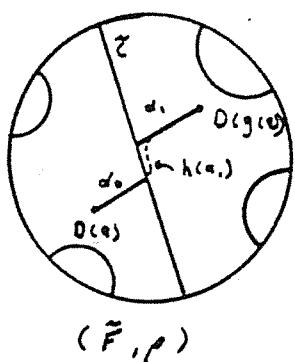
Now let a be a base point in \tilde{F} . We can arrange that $a \notin p^{-1}(C)$ and that $D(a) = D_x(a)$ and preserves tangent directions. So let U_0 be the component of $\mathbb{H}^2 \setminus$ preimage of C (under D) containing Da . By the above there is an isometric component of $\mathbb{H}^2 \setminus$ preimage of C (under D_x) containing $D_x a$, call that U_0 as well. For convenience set $s = x\ell(C)$.

3/25 Casson

120

Consider $g \in \pi_1 F$. Suppose for the moment that $Dg(a)$ is in an adjacent component U_1 . Where is $D_x(g(a))$?

Call α the p -geodesic segment in U_0 of the p -geodesic forming



$D(a)$ to $D(g(a))$; α , the p -geodesic segment in U_1 of this geodesic. Now the $p(x)$ -geodesic joining $D_x(a)$ to $D_x(g(a))$ is made up of the segment α_0 and a segment $E\alpha_1$, where $E\alpha_1$ is the image of α_1 under the action of a hyperbolic isometry with axis the common boundary component of U_0 and U_1 , shifting a distance s .

We build the earthquake function $E : \mathbb{H}^2 \setminus \pi^{-1}(C) \rightarrow \mathbb{H}^2 \setminus \pi_x^{-1}(C)$ (where $\pi = p D^{-1}$, $\pi_x = p D_x^{-1}$, $p : \tilde{F} \rightarrow F$) inductively. For example, if $D(a)$ and $D(g(a))$ are separated by two components of \tilde{C} , shifts in the two axes, the outer one first.

Then 1) E maps each component of $\mathbb{H}^2 \setminus \pi^{-1}(C)$ to a component of $\mathbb{H}^2 \setminus \pi_x^{-1}(C)$

2) If U_1, U_2 are components of $\mathbb{H}^2 \setminus \pi^{-1}(C)$ such that $\bar{U}_1 \cap \bar{U}_2 = Y$, then the isometries extending $E|_{U_1}$ satisfy $E|_{U_2} = (E|_{U_1}) R_Y$ on Y where R_Y is the translation with axis Y shifting a distance s to the right as viewed from U_1 .

3) $E|_{U_0} = 1_{U_0}$ for some component U_0 of $\mathbb{H}^2 \setminus \pi^{-1}(C)$

4) There is a unique such E .

Note: Picture $\Rightarrow H_{p(x)}(g) = E H_p(g) E^{-1}$ locally, away from $\pi^{-1}(C)$

Lemma 6.10 Let (F, ρ) be a closed hyperbolic surface with developing map $D: \tilde{F} \rightarrow \mathbb{H}^2$. Let (L, μ) be a measured geodesic lamination on (F, ρ) and let $\tilde{L} = D(\rho^{-1}(L)) \subseteq \mathbb{H}^2$. Then there exists a right earthquake map $E: \mathbb{H}^2 \setminus \tilde{L} \rightarrow \mathbb{H}^2$ such that

(1) $E|_{U_i}$ is an isometry for each component U_i of $\mathbb{H}^2 \setminus \tilde{L}$

(2) If y is a leaf of \tilde{L} and $x, y \in \mathbb{H}^2 \setminus \tilde{L}$ are close and separated by y , then $d(Ey), Ex(Ry)) = d(x, y)$ where $Ex: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is the isometry extending $E|_x$ -component of $\mathbb{H}^2 \setminus \tilde{L}$ and R is the hyperbolic isometry with axis y and translation length $\mu[x, y]$ to the right when viewed from x to y .

Remarks: Conditions (1), (2) determine E up to composition with an isometry of \mathbb{H}^2 . Further $E(y) = Ex(Ry)$ for simple closed curves.

Let Γ be the holonomy of $\pi, F \in \Sigma$ on \mathbb{H}^2 . Then $(F, \rho) \circ \mathbb{H}^2/\Gamma$. Also EgE^{-1} is an isometry a.e. (i.e. off \tilde{L}) for $g \in \Gamma$. If L has no isolated leaves or leaves with atomic measure and $L = \text{support}(\mu)$, then E is continuous, so $E\Gamma E^{-1}$ is a group of isometries, discrete, fixed point free, so $\mathbb{H}^2/E\Gamma E^{-1}$ is a hyperbolic surface and thus defines a point $E_{(L, \mu)}(\rho)$ in $\gamma(F)$ obtained from ρ by a right earthquake in (L, μ) .

Proof Choose a component U_0 of $\mathbb{H}^2 \setminus \tilde{L}$ and define $E|_{U_0} = 1_{U_0}$.

Pick a base point $a \in U_0$ and let $x \in \mathbb{H}^2 \setminus \tilde{L}$.

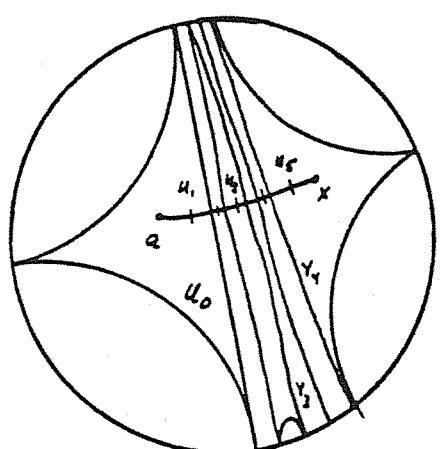
Dissect the geodesic segment $[a, x]$ via

$a = u_0, u_1, \dots, u_k = x$ such that $u_i \notin \tilde{L}$

Let y_i be a leaf of \tilde{L} separating u_{i+1} from u_i ; if there is one, y_i arbitrary otherwise.

Let R_i be translation with axis y_i thru length $\mu[u_{i+1}, u_i]$ to the right as viewed from a to x .

Set $E'(x) = R_1 R_2 \cdots R_k(x)$.



3/30 Casson

Claim: $E'(x) \rightarrow$ unique limit $E(x)$ as $\max(d(u_{i-1}, u_i)) \rightarrow 0$

Suppose this is so, we must check (1) and (2).

For (1), suppose y is in the same component of $\mathbb{H}^2 \setminus \tilde{L}$ as x . Then the geodesic segment $[a, y]$ also meets the leaves separating a and x , so for every dissection of $[a, x]$ there is a corresponding dissection of $[a, y]$ with identical measures as the measure is invariant under projection along leaves. We can even choose the same separating geodesics. So if $R_1, \dots, R_k(x)$ is an approximation to $E(x)$ w.r.t. a dissection of $[a, x]$, then $R_1, \dots, R_k(y)$ is an approximation to $E(y)$ w.r.t. corresponding dissection of $[a, y]$. In particular $d(E(x), E(y)) = d(x, y)$.

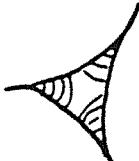
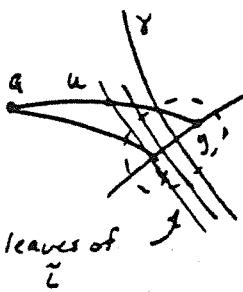
$\therefore E$ is an isometry on each component of $\mathbb{H}^2 \setminus \tilde{L}$

For (2), notice that for a geodesic triangle, the leaves of \tilde{L} can span any two sides.

We note that in the case that x is close to y , then

no leaves of \tilde{L} meets $[x, y]$ and $[a, x]$

i.e. If x close the geodesic through a and x meets y .



Now we can find a u on $[a, y]$ that is in the same component of $\mathbb{H}^2 \setminus \tilde{L}$ as x , and as the measure is invariant under projection along leaves, $E'(u) = E'(x)$. So wlog a, x, y are on the same geodesic and (2) follows.

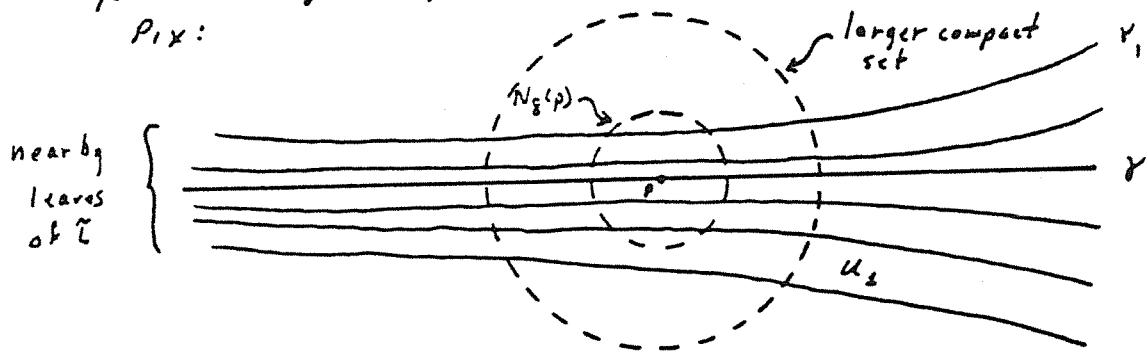
Proof of claim: Let $V = \{x \in \mathbb{H}^2 \setminus \tilde{L} \mid \text{construction for } E(x) \text{ converges}\}$. Note $V \neq \emptyset$. Indeed, V is a union of components of $\mathbb{H}^2 \setminus \tilde{L}$. Moreover, if the construction converges for x , it converges for the geodesic connecting a to x . Hence, ignoring the points on \tilde{L} , V is a hyperbolically convex set.

Let γ be a leaf of \tilde{L} such that γ is a frontier leaf of V and let $p \in \gamma$. We claim that there exists a $\delta > 0$ such that if U_δ is a component of $\mathbb{H}^2 \setminus \tilde{L}$ meeting $N_\delta(p)$, then there exists a map F defined on all the components of $\mathbb{H}^2 \setminus \tilde{L}$ meeting $N_\delta(p)$ such that $F|_{U_\delta} = \gamma|_{U_\delta}$ and such that F satisfies conditions (1) & (2).

Notice that this proves the claim as we can choose U_1 to be on the 'a-side' of γ , so that $E|_{U_1}$ is already defined. We can then extend E across γ via $E(x) = E_{U_1} \cdot F(x)$ for $x \in \text{domain}(F)$.

Construction of F : Given p we need a larger compact set such that if we apply the construction of E to any point in $N_\delta(p)$ by any direction, then $E(\text{point})$ stays in this larger compact set.

Fix:



Fix $\mu > 0$, $s \in (0, 1]$ so that $0 \leq s \leq \mu$. Then there exists a $\delta > 0$ such that, if γ is a leaf of $\tilde{\mathcal{L}}$ meeting $N_\delta(p)$ and if $x \in N_{4\mu}(p)$ then $d(R_y^s(x), R_y^s(x)) \leq \varepsilon s \leq s \leq \mu$ and $d(R_y^s(x), x) \leq 2s \leq 2\mu$ for those x between γ and some leaf meeting $N_\delta(p)$. Also note that δ does not depend on γ or p .

It follows that, if $s = s_1 + \dots + s_k \leq \mu$, $s_i \geq 0$; $x \in N_{4\mu}(p)$ and between γ and some leaf of $\tilde{\mathcal{L}}$ meeting $N_\delta(p)$; and γ_i a leaf meeting $N_\delta(p)$, then

$$d(R_{\gamma_1}^{s_1} \cdot R_{\gamma_2}^{s_2} \cdots R_{\gamma_k}^{s_k}(x), R_y^s(x)) \leq \varepsilon s \leq \mu$$

Because setting $x_i = R_{\gamma_i}^{s_i} \cdots R_{\gamma_k}^{s_k}(x)$ we can prove by induction that the x_i 's lie in $N_{4\mu}(p)$.

For this assume $x_{i+1} \in N_{4\mu}(p)$ and note that $x_i = R_{\gamma_i}^{s_i}(x_{i+1})$

$$\text{Then } d(x_i, R_y^{s_i}(x_{i+1})) \leq \varepsilon s_i \Rightarrow d(R_y^{s_i} x_{i+1}, R_y^{s_i+s_{i+1}}(x_{i+1})) \leq \varepsilon s_{i+1}$$

$$\dots \Rightarrow \dots \Rightarrow d(x_i, R_y^{s_i+s_{i+1}+\dots+s_k}(x)) \leq \varepsilon(s_i + \dots + s_k) \leq \varepsilon s s/\mu$$

$$\Rightarrow d(x, R_y^s(x)) \leq 2\mu$$

$$\Rightarrow x \in N_{4\mu}(p).$$

4/1 - 4/4 COSSON

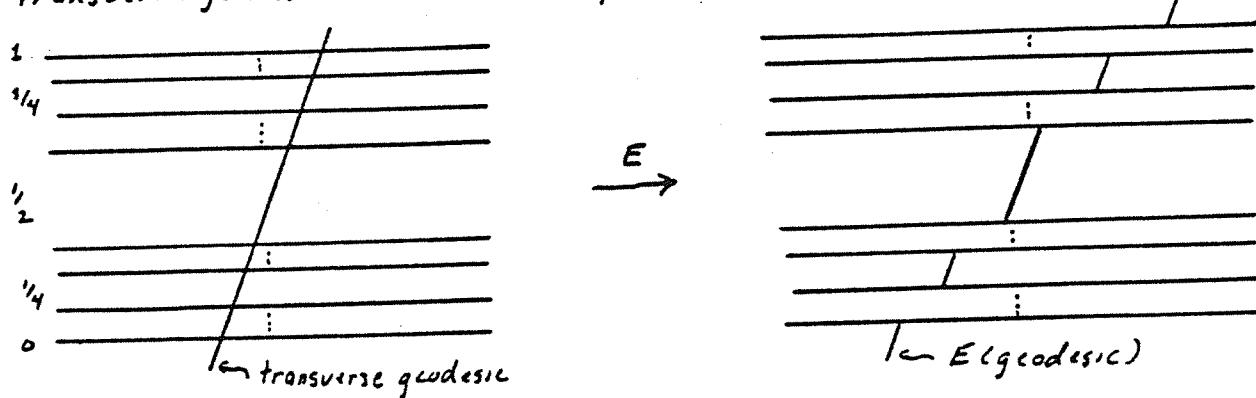
Finally a uniform continuity argument shows the convergence.

Exercise: Show this is unique up to composition with an isometry of \mathbb{H}^2

II.

Remark: A local picture

Recall that the transverse structure to a lamination is a Cantor set. Illustrating this as a classic 'middle thirds' set we can see what happens to a transverse geodesic under the earthquake map.



The picture is essentially the graph of a Cantor function, i.e. a function which is increasing and differentiable almost everywhere but not equal to the integral of its derivative.

Moreover, $E(x)$ is continuous except at atoms of the measure. In particular if L has no isolated leaves, then E is continuous.

Now given $(L, \mu) \in (F, \rho)$ we want to define the right earthquake $E_{(L, \mu)}[\rho]$ in $\mathcal{T}(F)$.

Regard F as \mathbb{H}^2/Γ where Γ is the image of the holonomy of the developing map. For $g \in \Gamma$, $g\tilde{\ell} = \tilde{\ell}$; so $Eg = g'E$ for some $g' \in \text{Iso}(\mathbb{H}^2)$. Hence $EgE^{-1} = g'$ for some $g' \in \text{Iso}(\mathbb{H}^2)$ for $g \in \Gamma$.

We obtain

Cor If $(F, \rho) = \mathbb{H}^2/\Gamma$ and $(L, \mu) \in \mathcal{ML}(F)$ and $E: \mathbb{H}^2 \setminus \tilde{\ell} \rightarrow \mathbb{H}^2$ is the associated right earthquake map, then for each $g \in \Gamma$, EgE^{-1} agrees with an isometry g_1 of \mathbb{H}^2 away from $\tilde{\ell}$. Set $\Gamma_1 = \{g_1 \mid g_1 = EgE^{-1} \text{ off } \tilde{\ell} \text{ for some } g \in \Gamma\}$ and set $(F_1, \rho_1) = \mathbb{H}^2/\Gamma_1$. The explicit isomorphism

$g \mapsto g_1$ is induced by a homomorphism $h: F \rightarrow F$ which is unique up to isotopy. Thus (F, ρ) represents a well-defined point $E_{(L, \mu)}[\rho] \in \mathcal{Y}(F)$.

Exercise: Check Γ , discrete, fixed point free.

Hint: M , not discrete $\Rightarrow \Gamma$, must bring a arbitrarily close to a .

We make the convention that all negative (non-positive) measures on a lamination is a left earthquake; i.e. $E_{(L, -\mu)}$ represents the left earthquake in (L, μ) .

Note: A left earthquake is not the inverse of a right earthquake in general as a right earthquake is given by $R_1 \cdots R_k$ and a left earthquake by $R_1^{-1} R_2^{-1} \cdots R_k^{-1}$ and the group of translations is not abelian.

Further note: If μ has no atoms, then $E: \mathbb{H}^2 \setminus \tilde{\Gamma} \rightarrow \mathbb{H}^2$ is continuous and equivariant w.r.t. Γ and Γ' , thus inducing an explicit homeomorphism $\hat{E}: F \rightarrow F$, which is an isometry a.e.

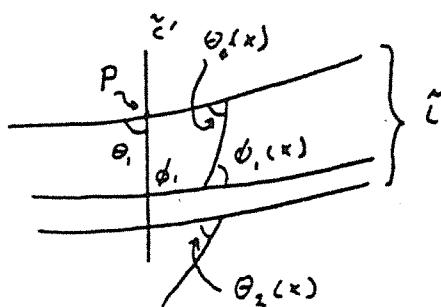
Theorem 6.11 (Kerckhoff) Let (F, ρ) be a hyperbolic surface, $(L, \mu) \in \mathcal{M}_2(F, \rho)$, and let $\rho(x) = E_{(L, x\mu)}[\rho]$. If c' is a closed ρ -geodesic

such that $\mu(c') > 0$, then $f(x) = L_{\rho(x)}(c')$ is strictly convex and $f(x) \rightarrow \infty$ as $x \rightarrow \pm \infty$

Proof (Adapt 6.9) As before, we begin by showing $F'(0) = \int_{c'} \cos \theta \, \mu$
(a Stieltjes integral)

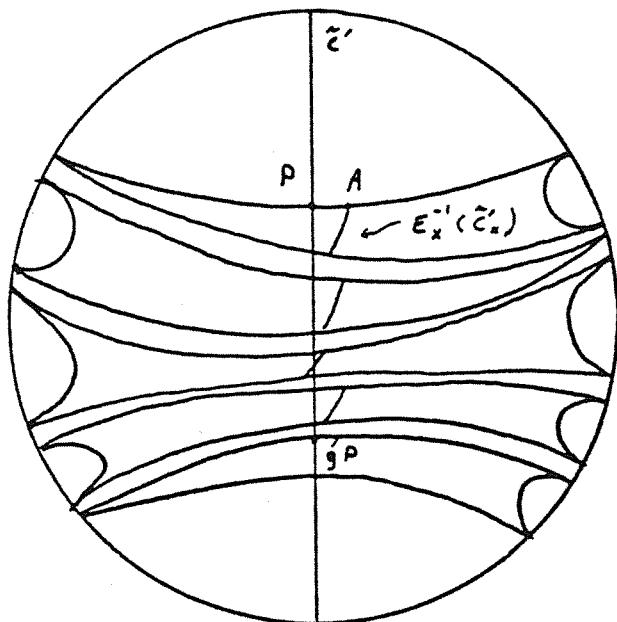
For $p \in \tilde{\Gamma}' \cap \tilde{\Gamma}$, let c'_x be the $\rho(x)$ -geodesic homotopic to c' , $E_x = E_{(L, x\mu)}$. We want to examine $E_x^{-1}(c'_x) \subseteq \mathbb{H}^2$. Its singularities are on $\tilde{\Gamma}$ and in complementary regions, it will appear as geodesic segments as a 'Cantor function'.

Local pix:



Note that the Lebesgue measure of $[p, gP] \cap \tilde{L}$ ($g \in P$) is 0, so we can use finitely many complementary regions of \tilde{L} , thereby determining finitely many intervals of $[p, gP]$ which contain all but ϵ of $[p, gP] \setminus \tilde{L}$.

Global pix:



Now as the leaves of \tilde{L} are close, the translations in these leaves are close to commuting. By means of intersecting with a finite number of components of $H^2 \cdot \tilde{L}$ which almost fill the length of C' , we can in analogous fashion to the proof of Lemma 6.9 proceed with the analysis of overlaps and underlaps under orthogonal projection. In Lemma 6.9, $d_i(x) = \theta_2(x)$ etc., but here they made $o(x)$ close, as we can make the gaps small.

$$\therefore f'(x) = \int_{C'} \cos \theta \mu$$

In general: $f'(x) = \int_{C'} \cos \theta_x \mu_x$ where (L_x, μ_x) is the lamination corresponding to (L, μ) .

Remark: In what sense might the formula $E_{(L, \mu)} \cdot E_{(L, \gamma\mu)} = E_{(L, (\gamma + \eta)\mu)}$ hold?

Note $E_{(L, \gamma\mu)} \cong H^2 / \mu \cong (F, \rho_\gamma)$. So we can take the canonical homeomorphism $F \rightarrow F$, and pull back the metric ρ_γ , and L remains geodesic. So the formula holds but with L "shifted" by $\gamma\mu$.

4/8 Casson



Cone structure via right earthquakes

Def Let $\underline{\mathcal{PL}(F)} = \underline{M_2(F)}/(L, \mu) \sim (L, t\mu)$ $t \in \mathbb{R}^+$, the projective lamination space

Lemma 6.13 $\underline{\mathcal{PL}(F)}$ is a closed manifold of dimension $6g - 7$

Remark: Recall that given a train track τ we get an injective map $f_\tau: W_\tau \rightarrow M_2(F) = \bigcup_\tau f_\tau(W_\tau)$. The topology is the extreme topology which makes all the f_τ 's continuous.

Proof of 6.13 Let's describe a neighborhood of (L, μ) , as usual assume that $L = \text{support } (\mu)$.

If $\alpha_1, \dots, \alpha_k$ are transverse geodesic arcs with endpoints off L , and $\varepsilon > 0$

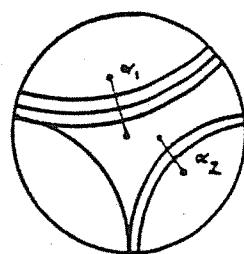
$$N_\varepsilon(L, \mu) = \{(L, \mu') \mid \mu_i(\alpha_i) - \mu'(\alpha_i) < \varepsilon\}$$

Now $f_\tau: W_\tau \hookrightarrow M_2(F)$ induces the map

$$f_\tau: PW_\tau \hookrightarrow \underline{\mathcal{PL}(F)}$$

Moreover PW_τ is topologically a disc (including the boundary) and is thus compact. So to prove that $\underline{\mathcal{PL}(F)}$ is compact, it is enough to show that there are finitely many train tracks τ_1, \dots, τ_k such that $\underline{\mathcal{PL}(F)} = \bigcup_{i=1}^k f_{\tau_i}(PW_{\tau_i})$

For this, first observe that 1-manifolds with (multiples of) counting measure are dense in $\underline{\mathcal{PL}(F)}$ as they are dense in each W_τ . (we can approximate by a train track with integer weights)



Returning to the proof, we want to show $\Theta(x) \in (0, \pi)$ is strictly decreasing

Claim: $\Theta(x) < \Theta(0)$ for $x > 0$ (similarly for $x > g$, $\Theta(x) < \Theta(g)$)

For this choose a transverse orientation of \tilde{C}' , and wlog A is to the right of p . Suppose $\Theta_1(x) \geq \Theta_1$, then

$$\phi_2(x) - \phi_2 < \Theta_1(x) - \Theta_1$$

But $\Theta_2(x) - \Theta_2 \approx \phi_2(x) - \phi_2$, and utilizing the fact that E_x is defined in $\Theta(x)$ terms, we can proceed as in Lemma 6.9 and conclude

$$\Theta_1(x) - \Theta_1 > \Theta_k(x) - \Theta_k > \dots > \Theta_1(x) - \Theta_1$$

a contradiction

$\therefore \Theta(x)$ is strictly decreasing

$\therefore f(x)$ is convex

Finally to insure that $f(x) \rightarrow \infty$, note that there exists a leaf of \tilde{C} entirely to the right of \tilde{C}' , so $l_{\rho(x)}(c') \rightarrow \infty$.

II.

Addendum: Note that the above techniques show that $f(x) \rightarrow \infty$ locally uniformly as $x \rightarrow \pm\infty$. That is, given K , there exists a cone neighborhood W of (L, μ) in $MX(F)$ and a compact set $A \subseteq MX(F)$ such that if (L_i, μ_i) is in $W \setminus A$, then $l_{\rho_i}(c') > K$ where $\rho_i = E_{(L_i, \mu_i)}(p)$.

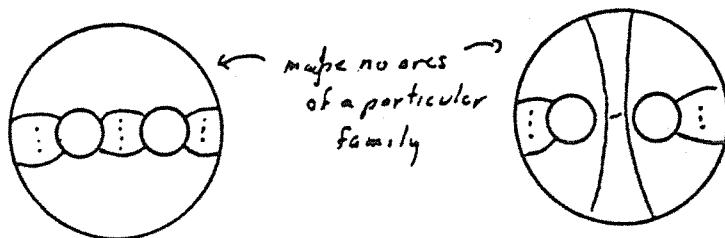
This will be needed when we show that if c_1, \dots, c_k $\parallel F$, then $\sum_{i=1}^k l_{\rho_i}(c_i)$ has a unique minimum.

Theorem 6.12 (Thurston) If $\rho \in \mathcal{Y}(F)$, then $(L, \mu) \mapsto E_{(L, \mu)}[\rho]$ defines a homeomorphism $E[\rho]: MX(F) \rightarrow \mathcal{Y}(F)$

Cor: If $\rho_1, \rho_2 \in \mathcal{Y}(F)$, then there exists a unique right earthquake carrying ρ_1 to ρ_2

Remark: For the torus, $\mathcal{Y}(\text{torus}) \cong \mathbb{H}^2$. Earthquake lines are horocycles with the counter clock-wise orientation. This gives us a picture of the cone structure given by right earthquakes.

So it suffices to find τ_1, \dots, τ_k such that every 1-manifold is carried by some τ_i . Let C_1, \dots, C_{3g-3} dissect F into pairs of pants and let C be any 1-manifold having minimal intersection with the C_i 's and all components geodesic. Suppose for a moment that no C_i is a component of C . Then if P is a pair of pants $C \cap P$ has no circles as no C_i is a component of C , so up to permutation of boundary components, there are essentially two possibilities:

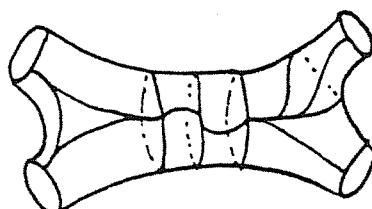


These are carried by (up to permutation of boundary components)



so we only need finitely many for each pair of pants. Now to include the C_i 's, when you form pairs of pants decide if you are going to twist left or right and place a $\overleftarrow{\text{---}}$ or a $\overrightarrow{\text{---}}$ in an annular neighborhood of C_i .

$P_1 \times :$



Note: 4 choices on each P , 2 for each C_i , so a large power of 2 train tracks carries all the 1-manifolds
 $\therefore P_2(F)$ is compact.

$\mathcal{PL}(F)$ a manifold: Observe that every (ℓ, μ) in $\mathcal{PL}(F)$ is carried by a minimal train track, i.e. one with strictly positive weights. Hence any $(\ell, \mu) \in f_\varepsilon(\text{Int } \mathcal{PW}_\varepsilon)$ for some ε . Moreover, ε can be assumed to be complete, i.e. all the regions of $F \setminus \varepsilon$ triangles as we can pinch non-triangular regions as in the remarks following Theorem 5.12 without inducing branches with zero weights.

Now, if ε is complete we assert that $f_\varepsilon(\text{Int } \mathcal{PW}_\varepsilon)$ is open in $\mathcal{PL}(F)$. This follows by using the train track definition of the $\mathcal{PL}(F)$ topology, i.e. a set in $\mathcal{PL}(F)$ is open if and only if its preimage under the f_ε 's is open. Note that $f_\varepsilon(\text{Int } \mathcal{PW}_\varepsilon)$ may not in general be open.

Exercise: $\cup_{\substack{\varepsilon \in \mathcal{E} \\ \varepsilon \text{ minimal}}} f_\varepsilon(\text{Int } \mathcal{PW}_\varepsilon)$ is an open set in $\mathcal{PL}(F)$

Now ε complete implies that any minimal ε containing ε is equal ε . The theorem now follows from

Exercise: If ε is a complete train track, $\mathcal{PW}_\varepsilon \cong D^{6g-7}$ provided there exists a set of strictly positive weights on ε

11.

The next step in the proof of Theorem 6.12 is

Lemma 6.14 $E[\rho]: M_2(F) \rightarrow \mathcal{Y}(F)$ is a proper map, i.e. the preimage of a compact set is compact.

Proof of 6.14 Let c_1, \dots, c_k fill F , and let $\gamma_r(F) = \{\rho \in \mathcal{Y}(F) \mid$

$\sum_{i=1}^k \lambda_{\rho_i}(c_i) < r\}$. This set is open as the λ_{ρ_i} 's are continuous and so we get an open covering of $\mathcal{Y}(F)$.

\therefore Any compact subset of $\mathcal{Y}(F)$ is contained in some $\gamma_r(F)$

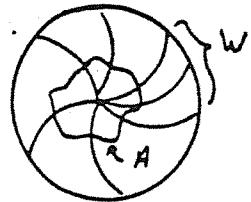
Notice that $M_2(F)$ is naturally the open cone on $\mathcal{PL}(F)$.

So we need only show that the preimage of $\gamma_r(F)$ is contained in a compact set.

4/10 - 4/13 Casson

13

For this, let (L, μ) be in $M_2(F) \setminus \{\phi\}$ with $L = \text{support}(\mu)$. Since c_1, \dots, c_k fill F , $\mu(c_i) > 0$ for some i . The addendum to theorem 6.11 says that given $r > 0$ there exists a cone neighborhood W of (L, μ) in $M_2(F)$ and compact subset A of $M_2(F)$ such that if $(L_1, \mu_1) \in W \setminus A$, then $\rho_{p_i}(c_i) \geq r$ where $p_i = E_{(L_i, \mu_i)}[\rho]$.



$$\therefore \sum_i \rho_{p_i}(c_i) \geq r \Rightarrow E[\rho](w \setminus A) \text{ is outside } \gamma_r(F)$$

The cone neighborhoods of (L, μ) in $P_2(F)$ satisfying this condition form an open cover of $P_2(F)$. Thus there exists a finite subcover w_1, \dots, w_n with associated compact sets A_1, \dots, A_n such that $E[\rho](w_i \setminus A_i) \cap \gamma_r(F) = \emptyset$. Setting $A = A_1 \cup \dots \cup A_n$, then $E[\rho](M_2(F) \setminus A) \cap \gamma_r(F) = \emptyset$

$$\therefore E[\rho]^{-1}(\gamma_r(F)) \subseteq A$$

II.

Lemma 6.15 $E[\rho]: M_2(F) \rightarrow \gamma(F)$ is injective

Remark: Recall that $\rho_f: PW_f \rightarrow P_2(F)$ is injective

Proof of 6.15: Suppose that $E_{(L_1, \mu_1)}[\rho] = E_{(L_2, \mu_2)}[\rho]$ in $\gamma(F)$.

Let $(F, \rho) = \mathbb{H}^2/\mu$, $(\tilde{L}_i, \tilde{\mu}_i)$ ($i=1, 2$) be the preimage of (L_i, μ_i) in \mathbb{H}^2 , and let $E_i: \mathbb{H}^2 \setminus \tilde{L}_i \rightarrow \mathbb{H}^2$ be the earthquake map corresponding to (L_i, μ_i) , well-defined up to composition with an isometry of \mathbb{H}^2 . In $\gamma(F)$, $E_{(L_1, \mu_1)}[\rho]$ is represented by \mathbb{H}^2/μ'_1 where $\mu'_1 = E_1 \circ E_1^{-1}$.

$E_{(L_1, \mu_1)}[\rho] = E_{(L_2, \mu_2)}[\rho] \Rightarrow$ there exists an isometry $h: \mathbb{H}^2/\mu'_1 \rightarrow \mathbb{H}^2/\mu'_2$

inducing the isomorphism $\Gamma_1 \rightarrow \Gamma_2: E_1 g E_1^{-1} \rightarrow E_2 g E_2^{-1}$ for all $g \in \Gamma$. Let \tilde{h} be a lift of h , and note that up to conjugacy

$$\tilde{h} E_1 g E_1^{-1} = E_2 g E_2^{-1} \cdot \tilde{h} \quad \text{for all } g \in \Gamma$$

$$\therefore (E_2^{-1} \tilde{h} E_1) g (E_1^{-1} \tilde{h} E_2) = g \quad \text{for all } g \in \Gamma.$$

So replace E_i by \tilde{E}_i , and then $E_2^{-1}E_i$ commutes with all the g 's in M .
 Easy exercise: Earthquake maps have a unique extension to S'_∞ and
 this extension is always continuous on S'_∞ .

Consider $E_2^{-1}E_i|_{S'_\infty}$. We claim it is $\tau_{S'_\infty}$. Recall that all the g 's $\neq 1$
 in M are hyperbolic and the set of contracting fixed points is dense in S'_∞ .
 $E_2^{-1}E_i$ must fix these contracting fixed points and so $E_2^{-1}E_i = \tau_{S'_\infty}$ on S'_∞ .

Next suppose that $L_1 \cap L_2 \neq \emptyset$, i.e. suppose $\gamma_1 \in L_1, \gamma_2 \in L_2$ intersect
 transversely, wlog γ_1, γ_2 are boundary leaves as the boundary leaves are dense.
 Compose the E_i 's with isometries if necessary, so that E_i is the identity on a
 component of $H^2 \setminus \bar{\Gamma}_i$ with γ_i as a boundary leaf.

We still have $kE_i = E_2$ for some isometry k
 of H^2 . But in our situation, the k -action on S'_∞
 must be 'clockwise' near the endpoints of γ_1 and
 'anti-clockwise' near the endpoints of γ_2 . This
 behavior is inconsistent with k an isometry (k
 has too many fixed points), indeed we cannot even
 have $k = 1$.

$$\therefore L_1 \cap L_2 = \emptyset.$$

$\therefore L_1 \cup L_2$ contains the support of μ_1, μ_2

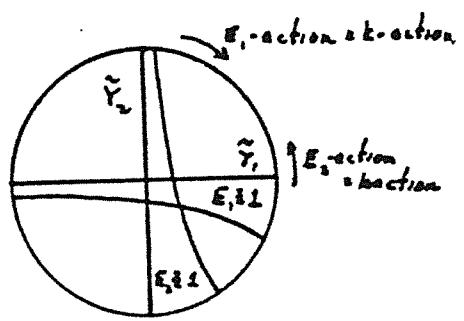
Now normalize E_i, \tilde{E}_2 so both are the identity on the same component of
 $H^2 \setminus \bar{\Gamma}$, where $\bar{\Gamma} = L_1 \cup L_2$. Then $E_2^{-1} \cdot E_i|_{S'_\infty}$ is an isometry fixing at least
 three points

$\therefore E_2 = E_i$ on S'_∞ and each is an isometry on each component of $H^2 \setminus \bar{\Gamma}$

$$\therefore E_i = E_2 \text{ on } H^2$$

Finally, refer to the definition of \tilde{E}_i as $E(L_i, \mu_i)$ to see that $\mu_1 = \mu_2$
 as measures on L

II.



Proof of Theorem 6.12 : Observe that it suffices to show that the map $E[\rho] : M_2(F) \rightarrow \mathcal{G}(F)$ is onto as the continuity of the inverse map follows from the properness of $E[\rho]$.

Set $E = E[\rho]$ and consider $E(M_2(F) \setminus \{\rho\})$. It is open by Invariance of Domain and the injectivity of E . If $c \in \overline{E(M_2(F))}$, then (with respect to any metric in $\mathcal{G}(F)$) $E^{-1}(\overline{N_\delta(c)})$ is compact by Lemma 6.14 and non-empty, thus there exists u in $\bigcap_{\delta > 0} E^{-1}(\overline{N_\delta(c)})$

$$\therefore E(u) = c$$

$\therefore E(M_2(F))$ is closed in $\mathcal{G}(F)$

$\therefore E(M_2(F) \setminus \{\rho\})$ is open and closed in $\mathcal{G}(F) \setminus \{\rho\}$

$$\therefore E(M_2(F)) = \mathcal{G}(F)$$

II.

Cor (Proof of Theorem 6.6) IF C_1, \dots, C_k fill F , then there exists a unique ρ minimizing $f(\rho) = \sum_{i=1}^k \lambda_i \rho(C_i)$

Exercise: Directly prove that f attains some minimum value as $f : \mathcal{G}(F) \rightarrow \mathbb{R}$ is a proper map.

By the convexity theorem (6.11); this minimum is unique.

II.

Theorem 6.16 (Thurston) $\tilde{\mathcal{G}}(F) = \mathcal{G}(F) \cup \partial \mathcal{G}(F)$ can be topologized functorially such that

1) $\tilde{\mathcal{G}}(F) \approx D^{6g+6}$ with $\partial \mathcal{G}(F)$ corresponding to ∂D , $\mathcal{G}(F)$ corresponding to $\text{Int } D$

2) A homeomorphism $h : F_i \rightarrow F_j$ induces a homeomorphism $h_* : \tilde{\mathcal{G}}(F_i) \rightarrow \tilde{\mathcal{G}}(F_j)$ agreeing with the previously defined h_* 's on $\mathcal{G}(F_i), \partial \mathcal{G}(F_i)$.

In particular: The action of $\text{Aut}(F)$ on $\mathcal{G}(F)$ extends to an action on $\tilde{\mathcal{G}}(F)$.

Proof Take $\bar{G}(F) = \text{cone}(\mathcal{P}_X(F))$. Choose a g and a base point ρ so that we can identify $\gamma(F)$ with the open cone $(\mathcal{P}_X(F))^\circ$

$$\text{V.i.3: } \text{open cone}(\mathcal{P}_X(F)) \xrightarrow[g]{\cong} \mathcal{M}_X(F) \xrightarrow{\mathcal{E}[\rho]} \gamma(F)$$

Exercise: Show that the topology on $\bar{G}(F)$ inherited from $\text{cone}(\mathcal{P}_X(F))$ via $\mathcal{E}[\rho] \cdot g$ does not depend on the choice of g or ρ .

Remark: At this stage \bar{G} is a functor from (surfaces, homeomorphisms) to (sets, maps) as the cone structure is only present to give a topology.

Note also that

$$\begin{aligned} h_x : \bar{G}(F_1) &\longrightarrow \bar{G}(F_2) \\ &\quad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \text{is continuous.} \\ \gamma(F_1) \amalg \mathcal{P}_X(F_1) &\qquad \qquad \qquad \gamma(F_2) \amalg \mathcal{P}_X(F_2) \end{aligned}$$

Now let $\mathcal{C} = \{ \text{isotopy classes of essential simple closed curves in } F \};$
 $\mathbb{R}_+^{\mathcal{C}} = [0, \infty); \mathbb{R}_+^{\mathcal{C}} = \{ \text{functions } \phi : \mathcal{C} \rightarrow \mathbb{R}_+ \}$ with the product topology;

$\ell : \gamma(F) \rightarrow \mathcal{M}_+^{\mathcal{C}} : [\rho] \mapsto (\langle \cdot \rangle \mapsto \ell_\rho(\langle \cdot \rangle));$ and let

$$m : \mathcal{M}_X(F) \rightarrow \mathbb{R}_+^{\mathcal{C}} : (L, \mu) \mapsto (\langle \cdot \rangle \mapsto \inf_{c' \in \mathcal{C}} \mu(c'))$$

Projectivize by letting $\text{PR}_+^{\mathcal{C}} = \frac{\mathbb{R}_+^{\mathcal{C}} - \{0\}}{\phi \sim \lambda \phi} \quad \text{for all } \lambda \in (0, \infty)$
and giving it the quotient topology. Now consider ℓ, m as having range $\text{PR}_+^{\mathcal{C}}$
and note we can also projectivize the domain of m to get a new map

$$n : \mathcal{P}_X(F) \rightarrow \text{PR}_+^{\mathcal{C}}$$

So we get $\ell \amalg m : \bar{G}(F) = \gamma(F) \amalg \mathcal{P}_X(F) \longrightarrow \text{PR}_+^{\mathcal{C}}$

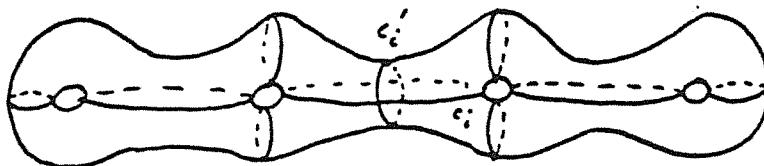
If we can show that $\ell \amalg m$ is injective, the theorem will follow by taking the subspace topology of $\text{PR}_+^{\mathcal{C}}$ on $\bar{G}(F) \amalg \mathcal{P}_X(F)$, i.e. we make $\ell \amalg m$ an embedding.

Lemma 6.17 $\ell: \gamma(F) \rightarrow P\mathbb{R}_+^G$ is injective

Proof 1) $\ell: \gamma(F) \rightarrow P\mathbb{R}_+^G$ is injective

Let c_1, \dots, c_{3g-3} dissect F into pairs of pants, and assume that the closures of the pairs of pants are imbedded. Observe that since there are no similarities of hyperbolic polygons, projectivization does not interfere with these coordinates.

Now assume that there are two hyperbolic metrics such that the lengths of the c_i 's are identical, we must show that the 'twist' parameters are equal. So let c'_1, \dots, c'_{3g-3} be such that $|c_i \cap c'_j| = 2s_{ij}$, and let c''_1, \dots, c''_{3g-3}

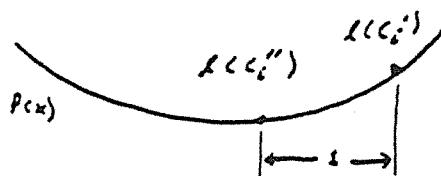


be such that $c''_i = T_{c_i}(c'_i)$. Note that we have used an implicit sign convention and that a large number of twists in the c_i sends $\ell(c_i') \rightarrow \infty$.

Claim: $\ell: \gamma(F) \rightarrow \mathbb{H}^{9g-9}_{++}$ is injective

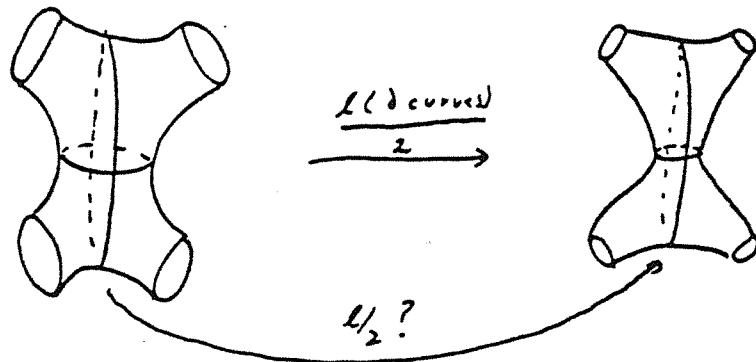
Claim follows from: If $\ell_{p_1}(c_i) = \ell_{p_2}(c_i)$ for all i , then p_2 is obtained from p_1 by twisting in c_1, \dots, c_{3g-3} . If in addition $\ell_{p_1}(c'_i) = \ell_{p_2}(c''_i)$ and $\ell_{p_1}(c''_i) = \ell_{p_2}(c''_i)$, Theorem 6.9 asserts that the i^{th} twist parameter is zero.

Recall: From 6.9 we are sampling two points on the convex $P\mathbb{R}^k$



If this happens for all i , then $p_1 = p_2$

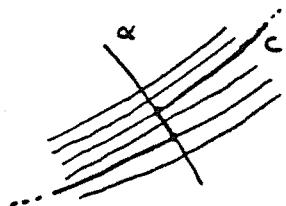
Exercise: Projectivization question, if the length of the boundary curves of adjacent pairs of pants is divided by two. Do the lengths of the c_i' , c_i'' curves above also divide by two?



$$2) \quad l(\gamma(F)) \cap m(\partial\mathbb{Z}(F)) = \emptyset$$

If $[\rho] \in \gamma(F)$, then $\{l_p(c), c \in G\}$ is bounded away from zero as every point in F has a p -neighborhood homeomorphic to a disc for some $p > 0$; F compact \Rightarrow there is a uniform δ , and hence no essential simple closed curve has length less than δ . Note this is invariant under projectivization.

On the other hand, if $(L, \mu) \in M_2(F)$ and α is a transverse arc to L , by following a leaf we can find a simple closed curve C in G with arbitrarily small $\mu(C)$.



$$3) \quad m: \partial\mathbb{Z}(F) \rightarrow \mathbb{R}_+^G \text{ injects}$$

Note that it suffices to show that $m: M_2(F) \rightarrow \mathbb{R}_+^{G^2}$ is injective as m preserves the cone structure.

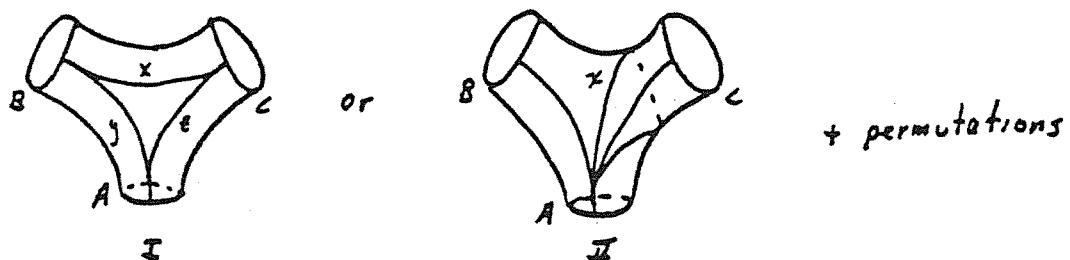
We show:

$$m: M_2(F) \rightarrow \mathbb{R}_+^G \xrightarrow{\cong} \mathbb{R}_+^{q_g q} \text{ injects where the } q_g q$$

coordinates are given by $c_i, c_i', T_{c_i}^{-1}(c_i')$

Suppose $m(L_1, \mu_1) = m(L_2, \mu_2)$ where $L_r = \text{support } \mu_r$. Now as $\mu_1(C_i) = \mu_2(C_i)$ for all i , it follows that (L_r, μ_r) $r=1,2$ is carried by a weighted train track τ_r such that the τ_r 's with weights agree away from annulus neighborhoods of C_1, \dots, C_{3g-3}

This is because on a single pair of pants there are only finitely many possibilities:



And if $a = \mu(A)$, $b = \mu(B)$, $c = \mu(C)$; we can decide which configuration occurs from a, b , and c :

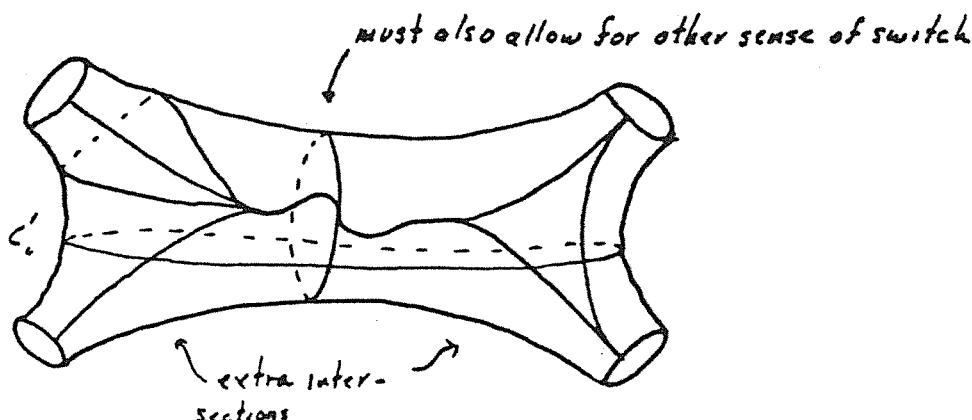
$$\begin{aligned} 0 &\leq 2x = b + c - a \\ 0 &\leq 2y = c + a - b \\ 0 &\leq 2z = a + b - c \end{aligned}$$

x, y, z thus determined by a, b, c
which satisfy a strict triangle
inequality.

Here x may be zero, but
 a, b, c do not satisfy a
strict triangle inequality

The annulus neighborhoods of the C_i cause problems as we can't necessarily put in C'_i in such a way that guarantees the correct calculation of $\mu(C')$ as we might have extra intersections, giving essential loops formed by C'_i and τ_r .

Fix:



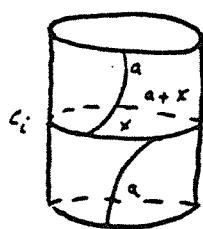
Notice, however, that any simple train route carried by



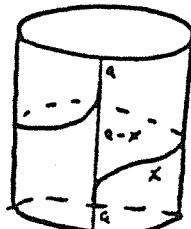
is also carried by



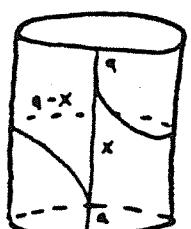
So in an annulus neighborhood of C_i there are 4 possibilities of behavior



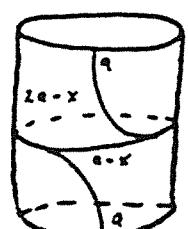
I



II



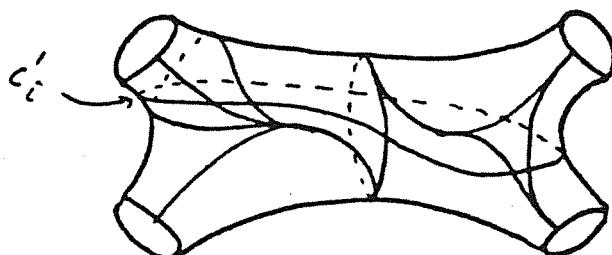
III



IV

Suppose you knew which type of behavior occurred, I say. Then you can accurately calculate $\mu(C'_i)$ as the loops formed by C'_i and train track are essential.

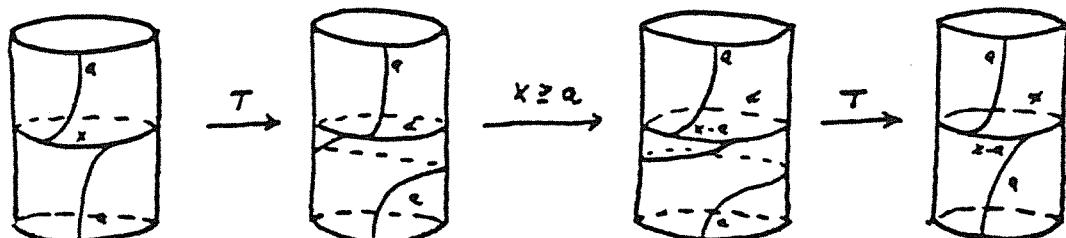
Pic:



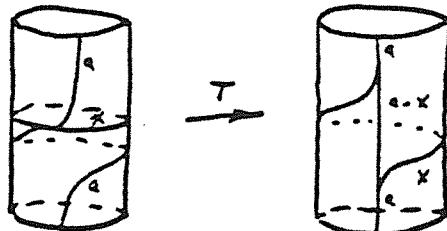
$$\mu(C'_i) = 2x + f(\mu(C_1), \dots, \mu(C_{3g-3}))$$

We can thus calculate x and fill in all the weights, implying $(L_1, \mu_1) = (L_2, \mu_2)$, but we can't be sure that type I behavior occurs. We need to investigate the effect of a left-handed Dehn twist in C_i which we denote by T .

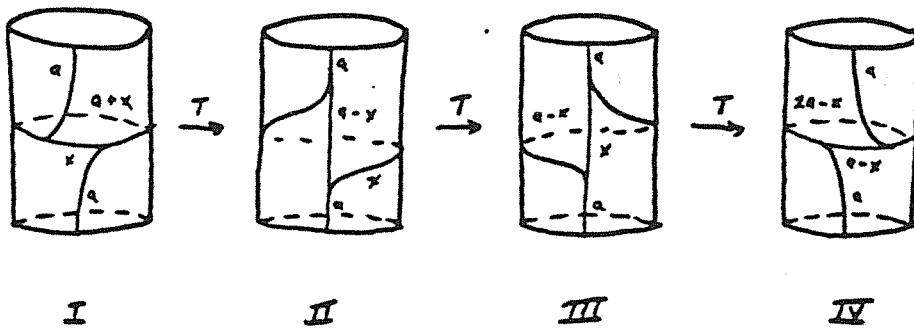
Suppose first that $x \geq a$ in a type I situation



So wlog we have $x \leq a$, but if $a \leq x < q$



So for $a \leq x \leq q$, we get



Notice also that we obtain the correct value of $\mu(C_i')$ if we have type **II** behavior.
Here $\mu(C_i') = 2(a-x) + g(\mu(c_1), \dots, \mu(c_{3g-3}))$

Our situation is thus: we have the map

$$m: \text{MFLS} \rightarrow \mathbb{R}_+^6 \rightarrow \mathbb{R}_+^{9g-9}: (L, \mu) \mapsto (C \rightarrow \begin{matrix} \text{if } \mu(C') \\ \text{csc} \end{matrix})$$

and laminations $(L_1, \mu_1), (L_2, \mu_2)$ such that

$m(L_1, \mu_1) = m(L_2, \mu_2)$ and $(L_1, \mu_1), (L_2, \mu_2)$ are carried by weighted train tracks t_1, t_2 which agree away from annular neighborhoods of the C_i 's.
Now fix i . If $a=0$ we have



so we just ignore I-IV and can find weighted train tracks t_1, t_2 which also agree near C_i as $\mu(C_i') = \mu_2(C_i')$

Now if $a > 0$ and if (L_i, μ_i) is not of type I, then $T^2(L_i, \mu_i)$ and $T^3(L_i, \mu_i)$ are both of type **IV**. Notice that

$\mu(C_i') = 3a - 2x + \text{constant}$ w.r.t. Dehn twist in an annular neighborhood of C_i
and $\mu(T^3 C_i') = 5a - 2x + \text{same constant}$

Therefore $\mu_1(T^{-3}c'_i) > \mu_1(c'_i)$

$$\therefore \mu_2(T^{-3}c'_i) > \mu_2(c'_i)$$

$\therefore T^3(L_2, \mu_2)$ must be of type IV, for otherwise $\mu_2(c'_i) > \mu_2(T^3c'_i)$

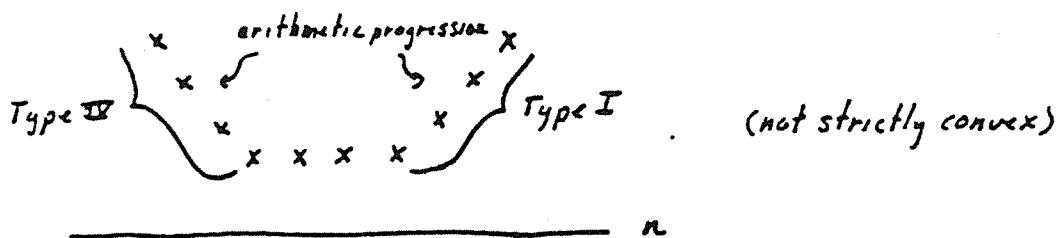
\therefore weights agree near c_i

Finally, if $T^3(L_1, \mu_1)$ is not of type IV near c_i , both are type I and the same argument applies.

$\therefore m$ is injective

II.

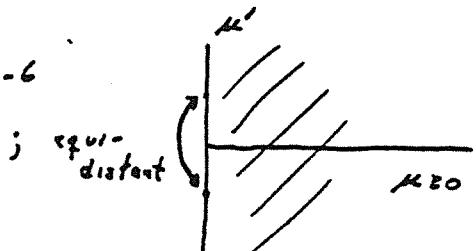
Remark: Coordinates of $T^{-n}(c'_i)$



Lemma 6.18 There is a homeomorphism

$$m_Z(F) \longrightarrow \left(\frac{m^2}{\pm 1}\right)^{3g-3} \approx m^{\frac{6g-6}{2}}$$

i.e. mod out by antipodal map
 $\approx m^2$



This homeomorphism is defined by $(L, \mu) \rightarrow (\mu_1, \mu'_1, \mu_2, \mu'_2, \dots, \mu_{3g-3}, \mu'_{3g-3})$
 where $\mu'_i = \inf_{c'_i \in c_i} \mu(c'_i)$ and μ'_i is defined in the cases above as

Case	I	II	III	IV
μ'_i	$a+x$	x	$x-a$	$x-2a$

Remark: Note that in case I we distinguish the 'long' branch and take its weight, in case IV we find the 'long' branch and take - its weight. Same idea in II & III
 This is analogous to the situation in $\mathcal{Y}(F)$ where there is no canonical choice of twist parameter.

Remarks: In the case of type I behavior

$$\mu(C_i') = 2x + a + f(\mu(C_1), \dots, \mu(C_{3g-3})) \quad (\text{recall } a = \mu(C_i))$$

To determine $f(\mu(C_1), \dots, \mu(C_{3g-3}))$ it requires the knowledge of the triangle inequalities which occur in the adjoining pairs of pants, i.e. knowledge of the adjoining train tracks. One considers

$$\mu(T^n(C_i')) = 2na + 2x + a + f(\mu(C_1), \dots, \mu(C_{3g-3}))$$

which gives

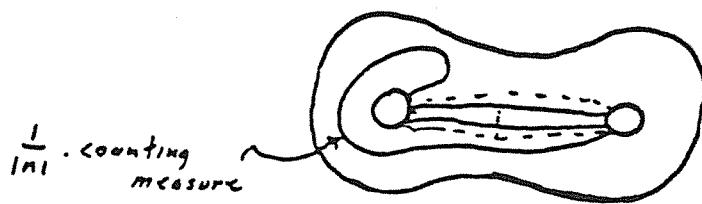
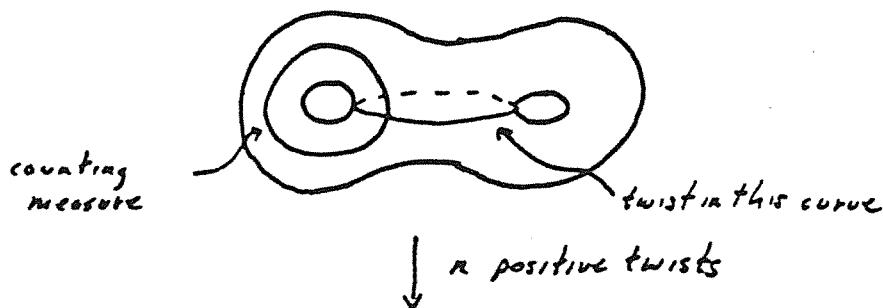
$$2x = \mu(T^n(C_i')) - (2n+1)a - f(\mu(C_1), \dots, \mu(C_{3g-3}))$$

Moreover, if n is large enough, one can use this formula to find μ' if you don't have type I behavior.

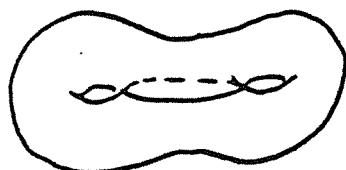
$\therefore \mu'_i$ is defined by the train track carrying (C_i, μ) in a neighborhood of C_i

Note that this allows μ'_i to be either positive or negative, so a point (μ_i, μ'_i) is in the right half plane. If $\mu_i = 0$, then in the above equations $a = 0$ and types II, III don't occur. Moreover, types I, IV reduce to a single circuit of weight κ . If $a = 0$, then x has no well-defined sign, as it just represents the number of parallel curves to the single circuit, so we must identify $+x$ to $-x$ as illustrated in the statement of the lemma.

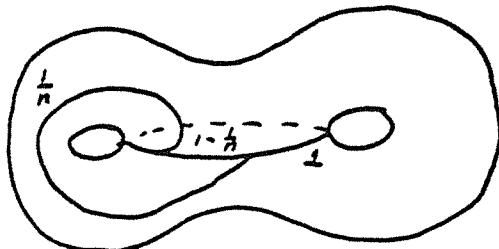
Example: Begin with counting measure on the simple closed curve illustrated below in a genus 2 surface, and examine the effect of twisting on another s.c.c.



we get a sequence of measured laminations and as $n \rightarrow \pm\infty$ the sequence tends to counting measure on



In the n^{th} lamination μ (above curve) is $\frac{1}{n}$, so it tends to zero as $n \rightarrow \pm\infty$.
The train track carrying the n^{th} lamination is



(reverse switches for negative n)

The limit is the same as $n \rightarrow \pm\infty$, μ coordinate $\rightarrow 0$ but as $n \rightarrow +\infty$, $\mu' \rightarrow 1$ and as $n \rightarrow -\infty$, $\mu' \rightarrow -1$.

Proof of 6.18 Onto: If each μ' 's corresponding μ , draw the train tracks in each pair of pants as to the triangle inequality, next draw appropriate annular train track. Other case is an exercise.

1-1: Apply Dehn twists until you have type I behavior and apply previous argument

II.

Remark: The coordinates given by Lemma 6.18 preserve the cone structure and thus goes to the projective lamination space and give it a piecewise projective structure giving the following

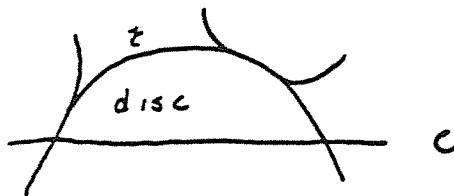
Cor $\partial X(F) \cong S^{6g-7}$

9/25 Casson

14.

Lemma 6.19 If τ is a train track on F and C is an essential s.c.c. in F with $m_C : \text{ML}(F) \rightarrow \mathbb{R}_+$ defined via $m_C(L, \mu) = \inf_{\tau' \sim \tau} \mu(C')$ then $m_C \cdot f_\tau : W_\tau \rightarrow [0, \infty)$ is piecewise linear. In fact: There exists finitely many train tracks τ_1, \dots, τ_k ; all carried by τ such that $f_\tau(W_\tau) = \bigcup_{i=1}^k f_{\tau_i}(W_{\tau_i})$ and $m_C \cdot f_{\tau_i} : W_{\tau_i} \rightarrow [0, \infty)$ is linear.

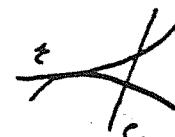
Proof Set $w_i = w_{\tau_i}$. Our aim is to find τ_1, \dots, τ_k such that $\bigcup_{i=1}^k f_{\tau_i}(w_i) = f_\tau(w_\tau)$ and such that for all i , τ_i has minimal intersection with C c.e.



Does not occur

If τ_i does have minimal intersection with C and $(L, \mu) = f_{\tau_i}(w_i)$ then $m_C(L, \mu)$ is the sum of the weights of the branches meeting C .
 $\therefore \tau_i$ minimal $\Rightarrow m_C \cdot f_{\tau_i}$ is linear

Note: Discs with cusps are allowed



Observe that 'sliding' won't do:



Instead, we use a 'subdivision operation', replacing a train track

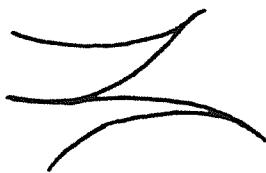
by a finite number of train tracks. For example replace



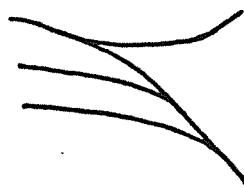
By one of



or

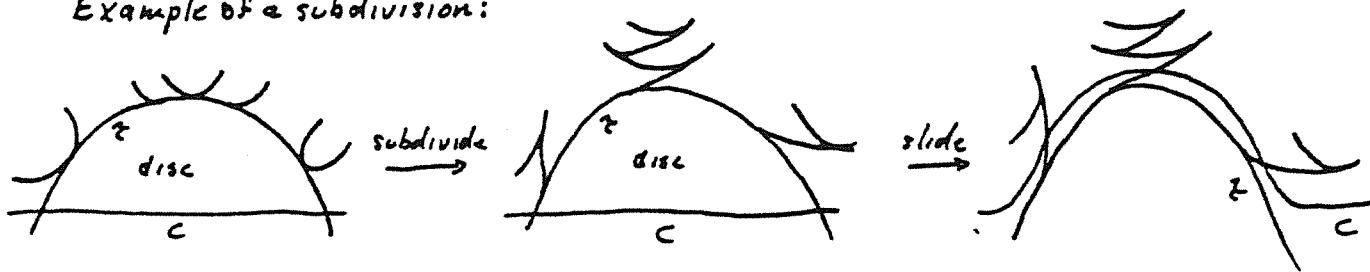


or

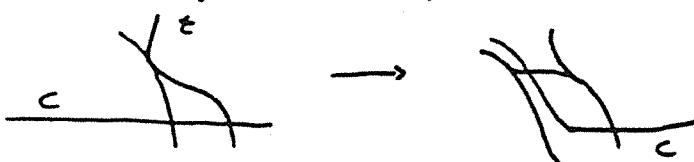


You can do this more generally if you have more branches at each switch

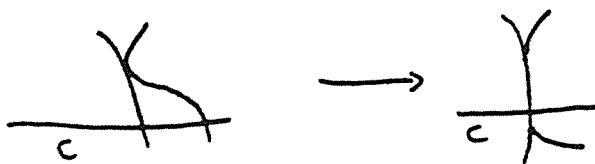
Example of a subdivision:



At this stage we haven't guaranteed that there are no bad discs:



To fix this, perform the operation below first



Finally, do lexicographic induction on $\max_{1 \leq i \leq k} (\# \text{bad discs of } z_i, c, 1 \cap z_i)$

increasing the number of train tracks until there are no bad discs.

II.

Theorem 6.20 (Thurston) There is a unique piecewise linear structure on $\mathcal{M}_2(F)$ such that $f_\tau: W_\tau \rightarrow \mathcal{M}_2(F)$ is PL for all τ and $m_c: \mathcal{M}_2(F) \rightarrow [0, \infty)$ is PL for all essential s.c.c.'s C .

Remark: Unique in the strong sense, it does not refer to a homeomorphism class of PL structures, but to a single structure.

Proof 1) Uniqueness is clear as the m_c 's embed $\mathcal{M}_2(F)$ as a non-compact polyhedron in \mathbb{R}^{qg-q}

2) Existence: There exists essential s.c.c.'s C_1, \dots, C_k such that $m: \mathcal{M}_2(F) \rightarrow \mathbb{R}^k: (L, \mu) \mapsto (m_{C_1}(L, \mu), \dots, m_{C_k}(L, \mu))$ is injective and there exists train tracks τ_1, \dots, τ_p such that

$$\mathcal{M}_2(F) = \bigcup_{i=1}^p f_{\tau_i}(W_{\tau_i})$$

Use lemma 6.19 to replace τ_1, \dots, τ_p by train tracks τ'_1, \dots, τ'_m ; drop the primes from the notation, and now $\{\tau_1, \dots, \tau_m\}$ is such that

$$\mathcal{M}_2(F) = \bigcup_{i=1}^m f_{\tau_i}(W_{\tau_i})$$

and

$m_{c_i} f_{\tau_j}: W_{\tau_j} \rightarrow \mathbb{R}$ is linear for all i, j

$\therefore m \circ f_{\tau_j}: W_{\tau_j} \rightarrow \mathbb{R}^k$ is linear and injective

So $m(\mathcal{M}_2(F)) \subset \mathbb{R}^k$ is a union of finitely many non-compact convex 'cells' $m(f_{\tau_j}(W_{\tau_j}))$ $j=1, \dots, m$ giving $\mathcal{M}_2(F)$ a PL structure

By lemma 6.19 $f_\tau: W_\tau \rightarrow \mathcal{M}_2(F)$ and $m_c: \mathcal{M}_2(F) \rightarrow \mathbb{R}$ are all PL

II.

Remarks: Suppose we are given an automorphism $h: F \rightarrow F$. Here is an algorithmic method for determining whether h is periodic, reducible, or pseudo-Anosov. Recall that our old method did not explicitly give a method of finding the invariant train tracks. We want a general procedure such that given h we can calculate explicitly $h_0: MZ(F) \rightarrow MZ(F)$, and, in particular, find the fixed points of $h_0: PZ(F) \rightarrow PZ(F)$.

For this, select c_1, \dots, c_k in G , essential s.c.c.'s such that $m: MZ(F) \rightarrow \mathbb{R}^k$ is injective. Apply Lemma 6.19 to the curves c_1, \dots, c_k

$c_1, \dots, c_k, h^{-1}(c_1), \dots, h^{-1}(c_k)$ to find train tracks τ_1, \dots, τ_k such

that $\bigcup_{i=1}^k f_{\tau_i}(W_{\tau_i}) = MZ(F)$; and $f_{\tau_i}: W_{\tau_i} \xrightarrow{\sim} \mathbb{R}^k$ and

$m \cdot h_0 \cdot f_{\tau_i}: W_{\tau_i} \rightarrow \mathbb{R}^k$ are all linear; and such that $m \cdot f_{\tau_i}$ and $h^{-1}(c_1), \dots, h^{-1}(c_k)$

$m \cdot h_0 \cdot f_{\tau_i}$ are represented by integer matrices A_i, B_i respectively

If $h_0(L, \mu) = \lambda(L, \mu)$ and $(L, \mu) \in f_{\tau_i}(w)$ with w in W_{τ_i} , then $m \cdot h_0 \cdot f_{\tau_i}(w) = \lambda^k f_{\tau_i}(w)$

$$\therefore B_i w = \lambda A_i w \text{ with } w \text{ in } W_{\tau_i}.$$

Conversely, if $B_i w = \lambda A_i w$ with w in W_{τ_i} , then $(L, \mu) \in f_{\tau_i}(w)$ satisfies $h_0(L, \mu) = \lambda(L, \mu)$

You can decide using Linear Programming whether there exists a w in W_{τ_i} with $\lambda = 1$. This happens if and only if h is reducible for a reducing submanifold corresponds via f_{τ_i} to an integer solution w to $B_i w = A_i w$ and conversely.

You can also decide by Numerical Analysis if there exists a solution to $B_i w = \lambda A_i w$, although you may not be able to determine it. For $\lambda > 1$ this corresponds to a contracting fixed point of $h_0: PZ(F) \rightarrow PZ(F)$.

Suppose there is such a point. Then there exists a neighborhood defined by rational inequalities, carried into itself by B . The existence of such a neighborhood can be determined by a finite calculation, deciding if h is pseudo-Anosov.

If there are no solutions to $B_i \cdot w = \lambda A_i \cdot w$ for any i and λ , then $h_\alpha : P\mathbb{X}(F) \rightarrow P\mathbb{X}(F)$ has no fixed points. The Brouwer Fixed Point Theorem then implies that $h_\alpha : \mathcal{Y}(F) \rightarrow \mathcal{Y}(F)$ has a fixed point and hence h is periodic.

Exercise: h periodic \Rightarrow period $\leq 84(g-1)$

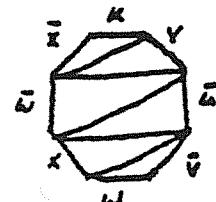
Hints: Can assume that h is periodic on the nose, not just up to isotopy. $F/\langle h \rangle$ is a surface, quotient map is a branched covering. Now use the Euler characteristic and the fact that 84 is the minimal hyperbolic area of a hyperbolic triangle which is a fundamental domain to finish.

So set up two computers - one looking for a contracting neighborhood, the other for periodicity. One must finish.

Remark: Recall that $M\mathbb{X}(F) \cong (\mathbb{H}^2/\pm_1)^{3g-3}$. This parametrizes G via $C \mapsto (c_i(c), c_j)$, twist γ . This parametrization was known to Dehn and Thurston gave the completion. Things are actually simpler if you puncture the surface once.

Let $G = G(F \setminus a) = \frac{\{\text{closed } 1\text{-submanifolds of } F \setminus a, \text{ all components essential}\}}{\text{isotopy in } F \setminus a}$

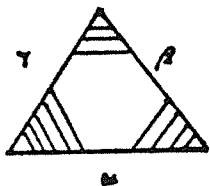
Observe that there exists a 'triangulation' k of F with just one vertex, etc
 (Take a $4g$ -gon \tilde{f} identifications. Add diagonals so that the result is simplicial)



$$\begin{aligned} \text{If } e \text{ edges, } f \text{ faces: } 2e &= 3f \Rightarrow f = \frac{2}{3}e \\ \Rightarrow \chi_F &= 1 - e + \frac{2}{3}e = 2 - 2g \\ \Rightarrow -\frac{1}{3}e &= 1 - 2g \\ \Rightarrow 2 &= 6g - 3 = -3\chi_{F \setminus a} \end{aligned}$$

LEMMA 6.21 $G(F \setminus a) \cong$ Set of non-negative integer weightings of the edges of K such that the weights of the boundary edges of any 2-simplex satisfy the weak triangle inequality and have even sum.

Proof Given weights w_a for edges a of K satisfying the conditions, there exists a 1-submanifold $C \subset F \setminus a$ such that $w_a \geq |C \cap a|$ for all edges a and such that each component of $C \cap \Delta$ is a linear segment in the 2-simplex Δ :



i.e. No circles or components which enter and leave on the same edge of Δ .
Such C is unique up to isotopy

If Δ is a 2-simplex with edges α, β, γ set

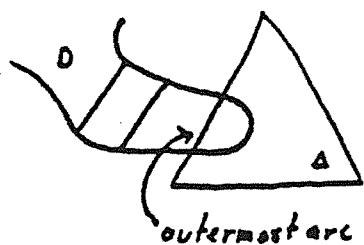
$$r = \frac{1}{2} (w_\beta + w_\gamma - w_\alpha)$$

$$s = \frac{1}{2} (w_\gamma + w_\alpha - w_\beta) \quad \Rightarrow r, s, t \text{ are positive integers}$$

$$t = \frac{1}{2} (w_\alpha + w_\beta - w_\gamma) \quad (\text{possibly just non-negative})$$

\therefore we can take $\left\{ \begin{array}{l} r \text{ disjoint linear arcs joining } \beta, \gamma \\ s \text{ " " " " } \gamma, \alpha \\ t \text{ " " " " } \alpha, \beta \end{array} \right\}$ with endpoints

evenly spaced on α, β , and γ . Their union is a 1-submanifold of $F \setminus a$ missing the vertex. Call this 1-submanifold C . If a component of C bounded a disc D in $F \setminus a$, an outermost arc of $D \cap K^3$ would give rise to:



a situation we have forbidden (cf. Haben)
(other intersections of C and D possible)

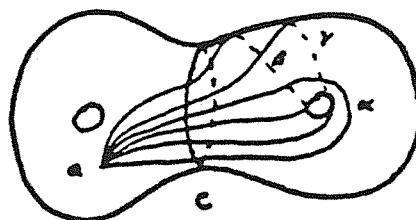
Given $C \in \mathcal{C}(F \setminus a)$ we can isotope C so that all the components of $C \cap (\alpha \text{ two simplex})$ are linear segments. (Assume the intersection is transverse and misses the vertex. If a situation as above occurs, just push C off the 2-simplex. Each move reduces $|C \cap k^2|$ and so must have an end to the pushing process. Now each component of $C \cap \text{simplex}$ is a spanning arc, isotope to a linear segment.)

\therefore we get a system of weights satisfying the conditions

$\therefore \{\text{weights, etc}\} \rightarrow \mathcal{C}(F \setminus a)$ is onto

We need to check that the above correspondence is 1-1. One would like to set $w_a = 1_{\alpha \cap C}$, but the following example shows that one must be careful, as if we moved a we could get rid of $C \cap \alpha$, $C \cap \beta$, $C \cap \gamma$

Example:

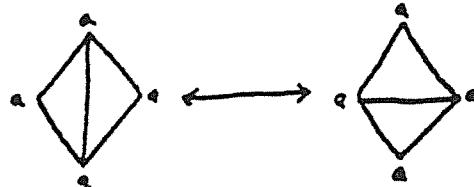


Notice that there exists a hyperbolic metric such that C is geodesic, replacing the edge a of K by a geodesic, we can set $w_a = 1_{\alpha \cap C}$. This requires

Exercise: Generalize Lemma 2.4 to pairs of curves C, α where C is a simple closed curve and α is a simple loop based at $a \notin C$, i.e. C and α are simultaneously isotopic to a geodesic and a geodesic with an angle at a .

11.

Exercise: If k_1, k_2 are 'triangulations' of F with a single vertex at a , then k_1, k_2 are related by isotopy a moves of form

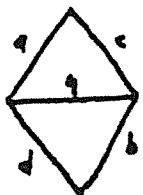


Hmt: Single vertex \Rightarrow can't subdivide

LEMMA 6.22 If



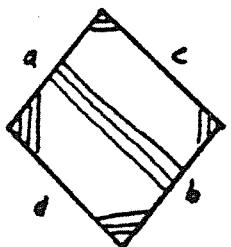
and



are weights

arising from $c \in G(F \setminus \text{base point})$ then $x+y = \max(a+b, c+d)$

Proof Exercise from the following picture



← maybe other diagonal - simplicity says
only one can occur.

Hint: Diagonal edges count in $x+y$, so consider all the arcs in picture II.

Cor Any $h: F \rightarrow F$ preserving the base point (or of punctured surface) is a chain of these operations, so the effect on $G(F \setminus a)$ can be calculated directly (by hand calculator).

Remark: This method is due to Mosher

X