Deformation spaces of 3-dimensional affine space forms

William M. Goldman

Department of Mathematics University of Maryland

Hyperbolicity in Geometry, Topology and Dynamics
A workshop and celebration of Caroline Series' 60th birthday
University of Warwick
26 July, 2011

Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
  - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
  - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's
- Closely related to hyperbolic geometry on surfaces

Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
  - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
  - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's
- Closely related to hyperbolic geometry on surfaces

Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
  - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
  - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's
- Closely related to hyperbolic geometry on surfaces

Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
  - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
  - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's
- Closely related to hyperbolic geometry on surfaces

Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
  - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
  - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's
- Closely related to hyperbolic geometry on surfaces

Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
  - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
  - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's
- Closely related to hyperbolic geometry on surfaces

Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
  - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
  - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's
- Closely related to hyperbolic geometry on surfaces

Complete affine 3-manifolds

- A *complete affine manifold* M^n is a quotient $M = \mathbb{R}^n/\Gamma$ where Γ is a discrete group of affine transformations acting properly and freely.
- Which kind of groups Γ can occur?
- Two types when $n = 3$:
    - Γ is solvable: M^3 is finitely covered by an iterated fibration of circles and cells.
    - Γ is free: M^3 is (conjecturally) an open solid handlbody with complete flat Lorentzian structure.
- First examples discovered by Margulis in early 1980's
- Closely related to hyperbolic geometry on surfaces

Euclidean manifolds

- If M compact, then Γ finite extension of a subgroup of *translations* $\Gamma \cap \mathbb{R}^n = \Lambda \cong \mathbb{Z}^n$ (Bieberbach 1912);
- M finitely covered by flat torus $\mathbb{R}^n/\Lambda$ (where $\Lambda \subset \mathbb{R}^n$ lattice).

Euclidean manifolds

- If M compact, then Γ finite extension of a subgroup of *translations* $\Gamma \cap \mathbb{R}^n = \Lambda \cong \mathbb{Z}^n$ (Bieberbach 1912);
- M finitely covered by flat torus $\mathbb{R}^n/\Lambda$ (where $\Lambda \subset \mathbb{R}^n$ lattice).

Euclidean manifolds

- If M compact, then Γ finite extension of a subgroup of *translations* $\Gamma \cap \mathbb{R}^n = \Lambda \cong \mathbb{Z}^n$ (Bieberbach 1912);
- M finitely covered by flat torus $\mathbb{R}^n/\Lambda$ (where $\Lambda \subset \mathbb{R}^n$ lattice).

Euclidean manifolds

- If M compact, then Γ finite extension of a subgroup of *translations* $\Gamma \cap \mathbb{R}^n = \Lambda \cong \mathbb{Z}^n$ (Bieberbach 1912);
- M finitely covered by flat torus $\mathbb{R}^n/\Lambda$ (where $\Lambda \subset \mathbb{R}^n$ lattice).

Consequences of Bieberbach theorems

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.
- $\pi_1(M)$ is finitely generated.
- $\pi_1(M)$ is finitely presented.
- $\chi(M) = 0$.

Consequences of Bieberbach theorems

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.
- $\pi_1(M)$ is finitely generated.
- $\pi_1(M)$ is finitely presented.
- $\chi(M) = 0$.

Consequences of Bieberbach theorems

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.
- $\pi_1(M)$ is finitely generated.
- $\pi_1(M)$ is finitely presented.
- $\chi(M) = 0$.

Consequences of Bieberbach theorems

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.
- $\pi_1(M)$ is finitely generated.
- $\pi_1(M)$ is finitely presented.
- $\chi(M) = 0$.

Consequences of Bieberbach theorems

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.
- $\pi_1(M)$ is finitely generated.
- $\pi_1(M)$ is finitely presented.
- $\chi(M) = 0$.

Consequences of Bieberbach theorems

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.
- $\pi_1(M)$ is finitely generated.
- $\pi_1(M)$ is finitely presented.
- $\chi(M) = 0$.

Example: Hyperbolic torus bundles

- Mapping torus M^3 of automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ induced by hyperbolic $A \in \mathrm{SL}(2,\mathbb{Z})$ inherits a complete affine structure.
    - Flat Lorentz metric (A-invariant quadratic form).
- Extend $\mathbb{Z}^2$ to $\mathbb{R}^2$ and A to one-parameter subgroup $\exp\big(t\log(A)\big)$ to get solvable Lie group $G \cong \mathbb{R}^2 \rtimes \mathbb{R}$ acting simply transitively on E.
- $M^3 \cong \Gamma\backslash H$ is a *complete affine solvmanifold.*

Example: Hyperbolic torus bundles

- Mapping torus M^3 of automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ induced by hyperbolic $A \in \mathrm{SL}(2,\mathbb{Z})$ inherits a complete affine structure.
  - Flat Lorentz metric (A-invariant quadratic form).
- Extend $\mathbb{Z}^2$ to $\mathbb{R}^2$ and A to one-parameter subgroup $\exp\big(t\log(A)\big)$ to get solvable Lie group $G \cong \mathbb{R}^2 \rtimes \mathbb{R}$ acting simply transitively on E.
- $M^3 \cong \Gamma\backslash H$ is a *complete affine solvmanifold.*

Example: Hyperbolic torus bundles

- Mapping torus M^3 of automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ induced by hyperbolic $A \in \mathsf{SL}(2,\mathbb{Z})$ inherits a complete affine structure.
  - Flat Lorentz metric (A-invariant quadratic form).
- Extend $\mathbb{Z}^2$ to $\mathbb{R}^2$ and A to one-parameter subgroup $\exp\big(t\log(A)\big)$ to get solvable Lie group $G \cong \mathbb{R}^2 \rtimes \mathbb{R}$ acting simply transitively on E.
- $M^3 \cong \Gamma\backslash H$ is a *complete affine solvmanifold.*

Example: Hyperbolic torus bundles

- Mapping torus M^3 of automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ induced by hyperbolic $A \in \mathrm{SL}(2,\mathbb{Z})$ inherits a complete affine structure.
  - Flat Lorentz metric (A-invariant quadratic form).
- Extend $\mathbb{Z}^2$ to $\mathbb{R}^2$ and A to one-parameter subgroup $\exp\big(t\log(A)\big)$ to get solvable Lie group $G \cong \mathbb{R}^2 \rtimes \mathbb{R}$ acting simply transitively on E.
- $M^3 \cong \Gamma\backslash H$ is a *complete affine solvmanifold.*

Example: Hyperbolic torus bundles

- Mapping torus M^3 of automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ induced by hyperbolic $A \in \mathrm{SL}(2,\mathbb{Z})$ inherits a complete affine structure.
    - Flat Lorentz metric (A-invariant quadratic form).
- Extend $\mathbb{Z}^2$ to $\mathbb{R}^2$ and A to one-parameter subgroup $\exp\big(t\log(A)\big)$ to get solvable Lie group $G \cong \mathbb{R}^2 \rtimes \mathbb{R}$ acting simply transitively on E.
- $M^3 \cong \Gamma\backslash H$ is a *complete affine solvmanifold.*

Proper affine actions

- Suppose $M = \mathbb{R}^n/G$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \Longrightarrow$ go to ∞ in every orbit Gx).
- Discreteness does not imply properness.

Proper affine actions

- Suppose $M = \mathbb{R}^n / G$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \Longrightarrow$ go to ∞ in every orbit Gx).

    More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

    is a proper map (preimages of compacta are compact).
- Discreteness does not imply properness.

Proper affine actions

- Suppose $M = \mathbb{R}^n/G$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

is a proper map (preimages of compacta are compact).

- Discreteness does not imply properness.

Proper affine actions

- Suppose $M = \mathbb{R}^n/G$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \text{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
- Discreteness does not imply properness.

Proper affine actions

- Suppose $M = \mathbb{R}^n/G$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
- Discreteness does not imply properness.

Proper affine actions

- Suppose $M = \mathbb{R}^n/G$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

    is a proper map (preimages of compacta are compact).
- Discreteness does not imply properness.

Proper affine actions

- Suppose $M = \mathbb{R}^n/G$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \Longrightarrow$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
- Discreteness does not imply properness.

Proper affine actions

- Suppose $M = \mathbb{R}^n/G$ is a complete affine manifold:
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \text{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

    is a proper map (preimages of compacta are compact).
- Discreteness does not imply properness.

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    $L(G) \subset O(2,1)$ is isomorphic to G.
    M^3 noncompact complete flat Lorentz 3-manifold
- Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
- Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $L(G) \subset O(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
- Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
- Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathsf{L}(G) \subset \mathsf{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
- Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
- Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathsf{L}(G) \subset \mathsf{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
- Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
- Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathsf{L}(G) \subset \mathsf{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
- Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
- Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathsf{L}(G) \subset \mathsf{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
- Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
- Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathsf{L}(G) \subset \mathsf{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
- Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
- Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycycic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated S^1-fibration.
  - Dimension 3: M^3 compact $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983).

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycyclic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated S^1-fibration
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycycic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated S^1-fibration
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycycic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated S^1-fibration
  - Dimension 3: M^3 *compact* $\implies$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),

Evidence?

Milnor offers the following results as possible "evidence" for a negative answer to this question.

- *Connected Lie group G admits a proper affine action $\iff$ G is amenable (compact-by-solvable).*
- *Every virtually polycyclic group admits a proper affine action.*

Evidence?

Milnor offers the following results as possible “evidence” for a negative answer to this question.

- *Connected Lie group G admits a proper affine action $\iff$ G is amenable (compact-by-solvable).*
- *Every virtually polycyclic group admits a proper affine action.*

Evidence?

Milnor offers the following results as possible "evidence" for a negative answer to this question.

- *Connected Lie group G admits a proper affine action $\iff G$ is amenable (compact-by-solvable).*
- *Every virtually polycyclic group admits a proper affine action.*

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
  - These actions are *not* affine.
- Milnor suggests:

  Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.
  However it seems difficult to decide whether the resulting group action is properly discontinuous.

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
  - These actions are *not* affine.
- Milnor suggests:

> Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.
>
> However it seems difficult to decide whether the resulting group action is properly discontinuous.

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
  - These actions are *not* affine.
- Milnor suggests:

  Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.
  However it seems difficult to decide whether the resulting group action is properly discontinuous.

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
  - These actions are *not* affine.
- Milnor suggests:

  Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely. However it seems difficult to decide whether the resulting group action is properly discontinuous.

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1x_2 + y_1y_2 - z_1z_2$$

  and Minkowski space $\mathsf{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathsf{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathrm{O}(2,1)$.
- The stabilizer of the origin is the group $\mathrm{O}(2,1)$ which preserves the hyperbolic plane

$$\mathsf{H}^2 := \{v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0\}.$$

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1x_2 + y_1y_2 - z_1z_2$$

  and Minkowski space $\mathsf{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathsf{Isom}(\mathsf{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathsf{O}(2,1)$.
- The stabilizer of the origin is the group $\mathsf{O}(2,1)$ which preserves the hyperbolic plane

$$\mathsf{H}^2 := \{v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0\}.$$

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1 x_2 + y_1 y_2 - z_1 z_2$$

  and Minkowski space $\mathsf{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathsf{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathrm{O}(2,1)$.
- The stabilizer of the origin is the group $\mathrm{O}(2,1)$ which preserves the hyperbolic plane

$$\mathsf{H}^2 := \{ v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0 \}.$$

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1x_2 + y_1y_2 - z_1z_2$$

  and Minkowski space $\mathsf{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathsf{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathrm{O}(2,1)$.
- The stabilizer of the origin is the group $\mathrm{O}(2,1)$ which preserves the hyperbolic plane

$$\mathsf{H}^2 := \{v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0\}.$$

Lorentzian and Hyperbolic Geometry

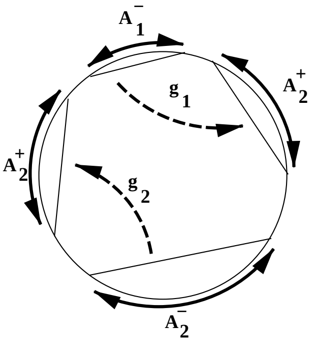
- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1 x_2 + y_1 y_2 - z_1 z_2$$

  and Minkowski space $\mathsf{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathsf{Isom}(\mathsf{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathsf{O}(2,1)$.
- The stabilizer of the origin is the group $\mathsf{O}(2,1)$ which preserves the hyperbolic plane

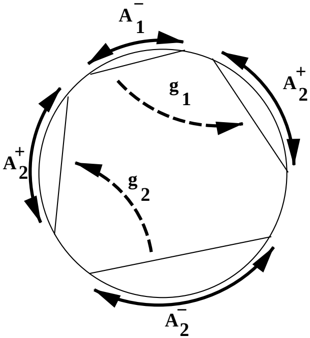
$$\mathsf{H}^2 := \{v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0\}.$$

A Schottky group



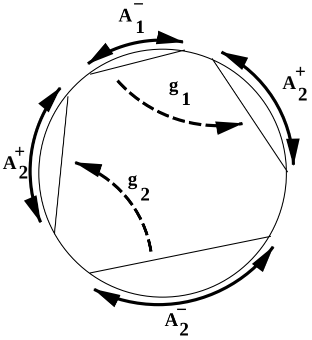
- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathrm{H}^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $\mathrm{H}^2 \setminus \bigcup A_i^\pm$.

A Schottky group



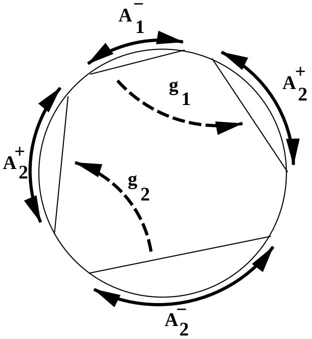
- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathsf{H}^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $\mathsf{H}^2 \setminus \bigcup A_i^\pm$.

A Schottky group



- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathsf{H}^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $\mathsf{H}^2 \setminus \bigcup A_i^\pm$.

A Schottky group



- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathrm{H}^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $\mathrm{H}^2 \setminus \bigcup A_i^\pm$.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathrm{GL}(3, \mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathrm{O}(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

where Σ complete hyperbolic surface.

  - Mess (1990): Σ *not compact*.
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(3, \mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2, 1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

where Σ complete hyperbolic surface.

  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

where Σ complete hyperbolic surface.

  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

where Σ complete hyperbolic surface.

  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

where Σ complete hyperbolic surface.

  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

where Σ complete hyperbolic surface.

  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is *not solvable.*

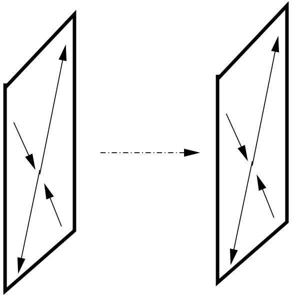
- (Fried-G 1983): Let $\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathsf{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - L injective.
- Homotopy equivalence

$$M^3 := \mathsf{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathsf{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples in dimension three.

Cyclic groups

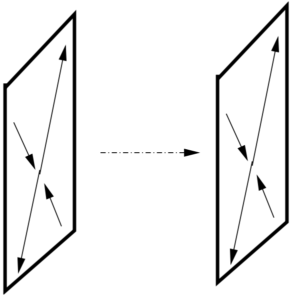
- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1)$. A fundamental domain is the *slab* bounded by two parallel planes.



A boost identifying two parallel planes.

Cyclic groups

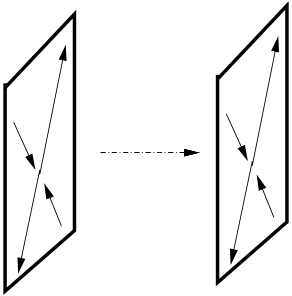
- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1)$. A fundamental domain is the *slab* bounded by two parallel planes.



A boost identifying two parallel planes

Cyclic groups

- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1)$. A fundamental domain is the *slab* bounded by two parallel planes.



A boost identifying two parallel planes

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $E^{2,1}/\Gamma$ is a *closed geodesic*. Like hyperbolic surfaces, most loops are freely homotopic to (unique) closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

  - $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ in Σ^2
  - $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ in M^3.

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathsf{E}^{2,1}/\Gamma$ is a *closed geodesic*. Like hyperbolic surfaces, most loops are freely homotopic to (unique) closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

- $\ell(\gamma) \in \mathbb{R}^{+}$: *geodesic length* of γ in Σ^2
- $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ in M^3.

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathsf{E}^{2,1}/\Gamma$ is a *closed geodesic.* Like hyperbolic surfaces, most loops are freely homotopic to (unique) closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

  - $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ in Σ^2
  - $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ in M^3.

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $E^{2,1}/\Gamma$ is a *closed geodesic.* Like hyperbolic surfaces, most loops are freely homotopic to (unique) closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

  - $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ in Σ^2
  - $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ in M^3.

Geodesics on Σ

- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric.
  - γ translates along C_γ by $\alpha(\gamma)$.
- Closed geodesics on $\Sigma \longleftrightarrow$ closed *spacelike* geodesics on M^3.
- Orbit equivalence: Recurrent orbits of geodesic flow on $U\Sigma$ $\longleftrightarrow$ Recurrent *spacelike* geodesics on M^3. (G-Labourie 2011)

Geodesics on Σ

- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric.
  - γ translates along C_γ by $\alpha(\gamma)$.
- Closed geodesics on $\Sigma \longleftrightarrow$ closed *spacelike* geodesics on M^3.
- Orbit equivalence: Recurrent orbits of geodesic flow on $U\Sigma$ $\longleftrightarrow$ Recurrent *spacelike* geodesics on M^3. (G-Labourie 2011)

Geodesics on Σ

- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric.
  - γ translates along C_γ by $\alpha(\gamma)$.
- Closed geodesics on $\Sigma \longleftrightarrow$ closed *spacelike* geodesics on M^3.
- Orbit equivalence: Recurrent orbits of geodesic flow on $U\Sigma$ $\longleftrightarrow$ Recurrent *spacelike* geodesics on M^3. (G-Labourie 2011)

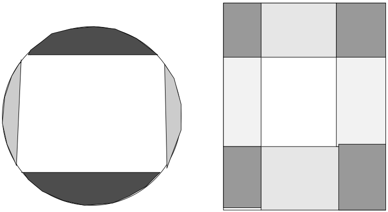
Geodesics on Σ

- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric.
  - γ translates along C_γ by $\alpha(\gamma)$.
- Closed geodesics on $\Sigma \longleftrightarrow$ closed *spacelike* geodesics on M^3.
- Orbit equivalence: Recurrent orbits of geodesic flow on $U\Sigma$ $\longleftrightarrow$ Recurrent *spacelike* geodesics on M^3. (G-Labourie 2011)

Geodesics on Σ

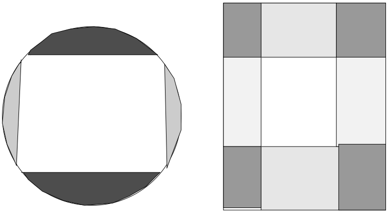
- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric.
  - γ translates along C_γ by $\alpha(\gamma)$.
- Closed geodesics on $\Sigma \longleftrightarrow$ closed *spacelike* geodesics on M^3.
- Orbit equivalence: Recurrent orbits of geodesic flow on $U\Sigma$ $\longleftrightarrow$ Recurrent *spacelike* geodesics on M^3. (G-Labourie 2011)

Slabs don't work!



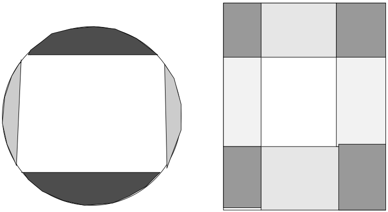
- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Slabs don't work!



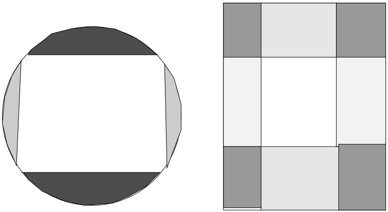
- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Slabs don't work!



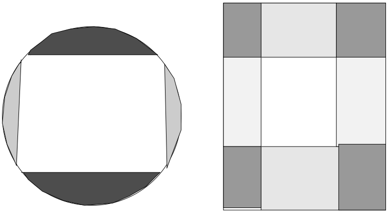
- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Slabs don't work!



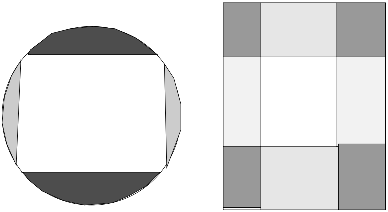
- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Slabs don't work!



- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

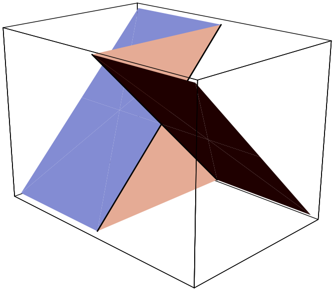
Slabs don't work!

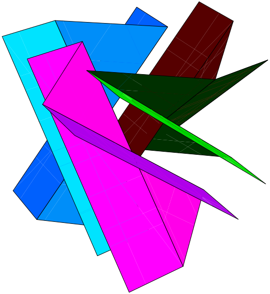


- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

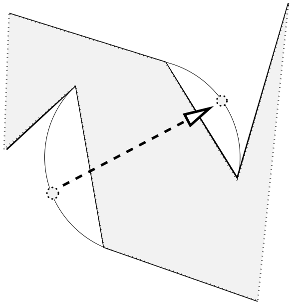
Drumm's Schottky groups

The classical construction of Schottky groups fails using affine half-spaces and slabs. Drumm's geometric construction uses *crooked planes*, PL hypersurfaces adapted to the Lorentz geometry which bound fundamental polyhedra for Schottky groups.



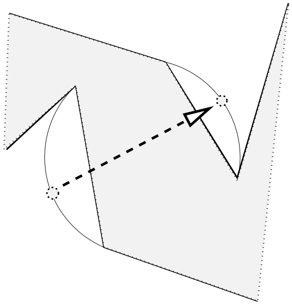


Crooked polyhedron for a boost



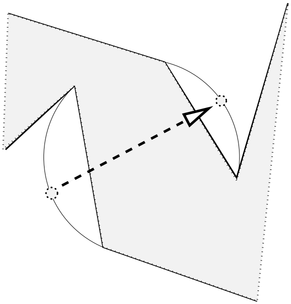
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $E^{2,1}$;
- Extend outside light cone in $E^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



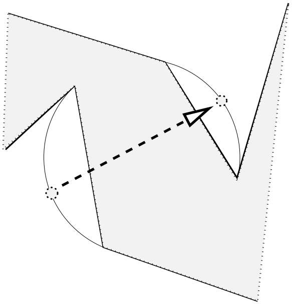
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $E^{2,1}$;
- Extend outside light cone in $E^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



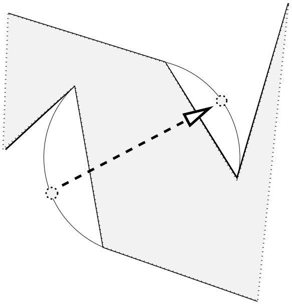
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $E^{2,1}$;
- Extend outside light cone in $E^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



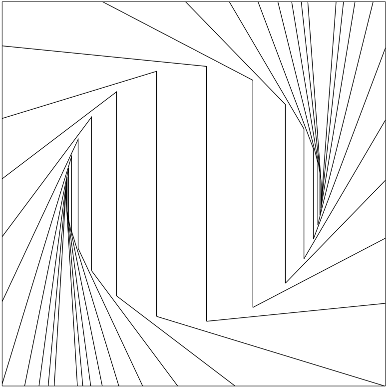
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $E^{2,1}$;
- Extend outside light cone in $E^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



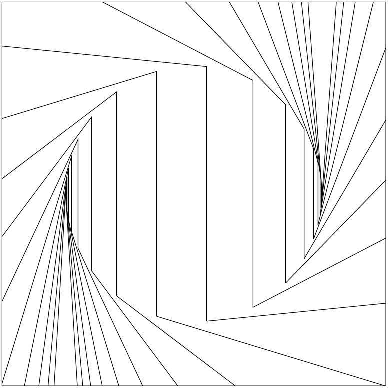
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $E^{2,1}$;
- Extend outside light cone in $E^{2,1}$;
- Action proper except at the origin and two null half-planes.

Images of crooked planes under a linear cyclic group



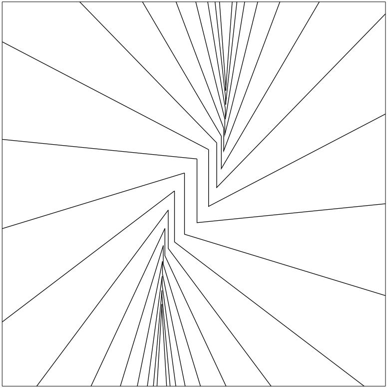
The resulting tessellation for a linear boost.

Images of crooked planes under a linear cyclic group



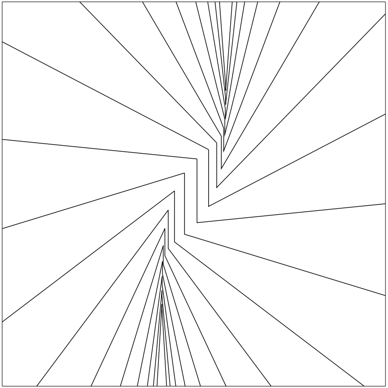
The resulting tessellation for a linear boost.

Images of crooked planes under an affine deformation



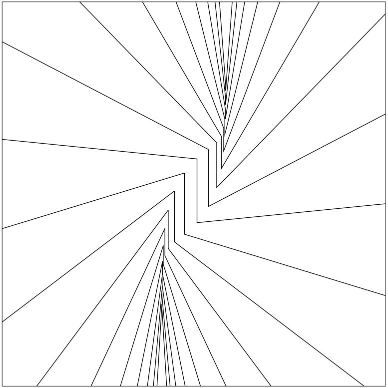
- Adding translations frees up the action
- — which is now proper on *all* of $E^{2,1}$.

Images of crooked planes under an affine deformation



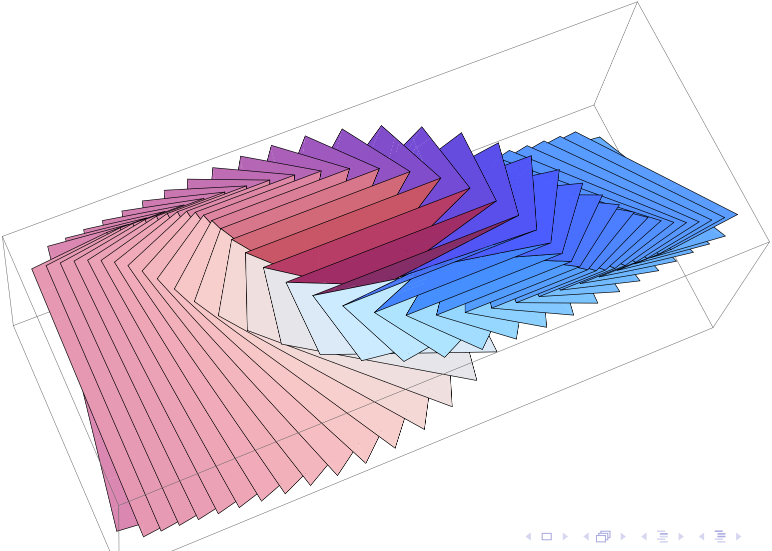
- Adding translations frees up the action
- — which is now proper on *all* of $E^{2,1}$.

Images of crooked planes under an affine deformation

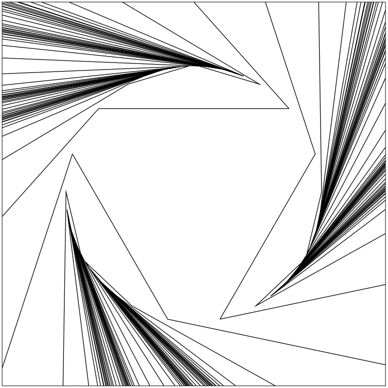


- Adding translations frees up the action
- — which is now proper on *all* of $E^{2,1}$.

A foliation by crooked planes

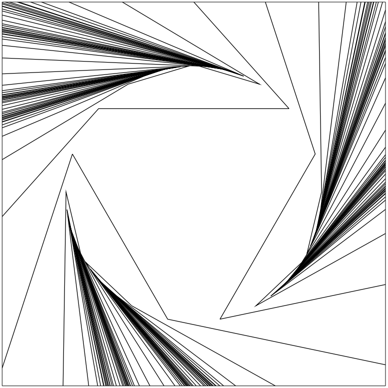


Linear action of Schottky group



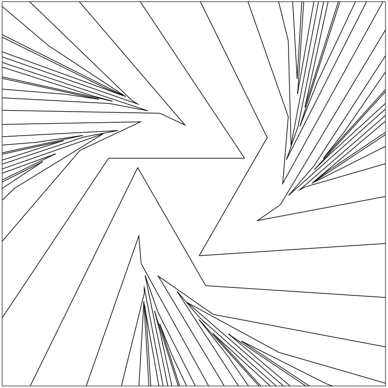
Crooked polyhedra tile H^2 for subgroup of $O(2,1)$.

Linear action of Schottky group



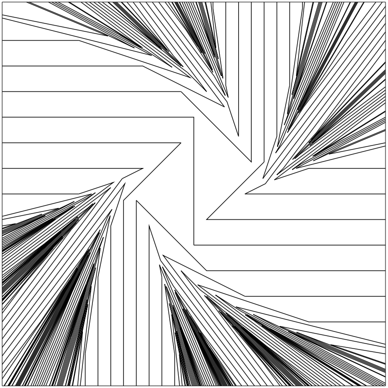
Crooked polyhedra tile H^2 for subgroup of $O(2,1)$.

Affine action of Schottky group



Carefully chosen affine deformation acts properly on $E^{2,1}$.

Affine action of level 2 congruence subgroup of $\mathrm{GL}(2, \mathbb{Z})$



Proper affine deformations exist even for *lattices* (Drumm).

An arithmetic example

- Minkowski space compactifies into the space of Lagrangian 2-planes in a 4-dimensional symplectic $\mathbb{R}$-vector space (V, ω).
- Choose two transverse Lagrangian 2-planes L_0 and L_∞.
- Minkowski $2+1$-space $\mathsf{E}^{2,1}$ is the space of Lagrangian 2-planes $L \subset V$ transverse to L_∞.
  - Graphs of *symmetric* maps $L_0 \xrightarrow{f} L_\infty$.
  - Lorentzian inner product defined by $f \longmapsto \mathrm{Det}(f)$
- $\mathbb{R}^{2,1} \longleftrightarrow \{\ 2 \times 2 \text{ symmetric matrices }\}$.

An arithmetic example

- Minkowski space compactifies into the space of Lagrangian 2-planes in a 4-dimensional symplectic $\mathbb{R}$-vector space (V, ω).
- Choose two transverse Lagrangian 2-planes L_0 and L_∞.
- Minkowski $2+1$-space $\mathsf{E}^{2,1}$ is the space of Lagrangian 2-planes $L \subset V$ transverse to L_∞.
  - Graphs of *symmetric* maps $L_0 \xrightarrow{f} L_\infty$.
  - Lorentzian inner product defined by $f \longmapsto \mathrm{Det}(f)$
- $\mathbb{R}^{2,1} \longleftrightarrow \{\ 2 \times 2$ symmetric matrices $\}$.

An arithmetic example

- Minkowski space compactifies into the space of Lagrangian 2-planes in a 4-dimensional symplectic $\mathbb{R}$-vector space (V, ω).
- Choose two transverse Lagrangian 2-planes L_0 and L_∞.
- Minkowski $2+1$-space $\mathsf{E}^{2,1}$ is the space of Lagrangian 2-planes $L \subset V$ transverse to L_∞.
  - Graphs of *symmetric* maps $L_0 \xrightarrow{f} L_\infty$.
  - Lorentzian inner product defined by $f \longmapsto \mathrm{Det}(f)$
- $\mathbb{R}^{2,1} \longleftrightarrow \{\ 2 \times 2 \text{ symmetric matrices } \}$.

An arithmetic example

- Minkowski space compactifies into the space of Lagrangian 2-planes in a 4-dimensional symplectic $\mathbb{R}$-vector space (V, ω).
- Choose two transverse Lagrangian 2-planes L_0 and L_∞.
- Minkowski $2+1$-space $\mathsf{E}^{2,1}$ is the space of Lagrangian 2-planes $L \subset V$ transverse to L_∞.
  - Graphs of *symmetric* maps $L_0 \xrightarrow{f} L_\infty$.
  - Lorentzian inner product defined by $f \mapsto \mathrm{Det}(f)$
- $\mathbb{R}^{2,1} \longleftrightarrow \{ \ 2 \times 2 \text{ symmetric matrices } \}$.

An arithmetic example

- Minkowski space compactifies into the space of Lagrangian 2-planes in a 4-dimensional symplectic $\mathbb{R}$-vector space (V, ω).
- Choose two transverse Lagrangian 2-planes L_0 and L_∞.
- Minkowski $2+1$-space $\mathsf{E}^{2,1}$ is the space of Lagrangian 2-planes $L \subset V$ transverse to L_∞.
  - Graphs of *symmetric* maps $L_0 \xrightarrow{f} L_\infty$.
  - Lorentzian inner product defined by $f \longmapsto \mathrm{Det}(f)$
- $\mathbb{R}^{2,1} \longleftrightarrow \{\ 2 \times 2 \text{ symmetric matrices }\}$.

An arithmetic example

- Minkowski space compactifies into the space of Lagrangian 2-planes in a 4-dimensional symplectic $\mathbb{R}$-vector space (V,ω).
- Choose two transverse Lagrangian 2-planes L_0 and L_∞.
- Minkowski $2+1$-space $\mathsf{E}^{2,1}$ is the space of Lagrangian 2-planes $L \subset V$ transverse to L_∞.
  - Graphs of *symmetric* maps $L_0 \xrightarrow{f} L_\infty$.
  - Lorentzian inner product defined by $f \mapsto \mathrm{Det}(f)$
- $\mathbb{R}^{2,1} \longleftrightarrow \{\ 2 \times 2 \text{ symmetric matrices }\}$.

An arithmetic example

- Minkowski space compactifies into the space of Lagrangian 2-planes in a 4-dimensional symplectic $\mathbb{R}$-vector space (V, ω).
- Choose two transverse Lagrangian 2-planes L_0 and L_∞.
- Minkowski $2+1$-space $\mathsf{E}^{2,1}$ is the space of Lagrangian 2-planes $L \subset V$ transverse to L_∞.
  - Graphs of *symmetric* maps $L_0 \xrightarrow{f} L_\infty$.
  - Lorentzian inner product defined by $f \mapsto \mathrm{Det}(f)$
- $\mathbb{R}^{2,1} \longleftrightarrow \{\ 2 \times 2 \text{ symmetric matrices } \}$.

Minkowski space inside $\mathrm{Sp}(4, \mathbb{R})$

- L_0 and L_∞ dual under symplectic form $L_0 \times L_\infty \xrightarrow{\omega} \mathbb{R}$
- $g \in \mathrm{GL}(L_\infty)$ induces *linear symplectomorphism* of $V = L_\infty \oplus L_0$, represented as block upper-triangular matrices:

$$g \oplus (g^\dagger)^{-1} = \begin{bmatrix} g & 0 \\ 0 & (g^\dagger)^{-1} \end{bmatrix}$$

- Translations of Minkowski space correspond to *shears:* (fixing L_∞ and L/L_∞):

$$\begin{bmatrix} I_2 & f \\ 0 & I_2 \end{bmatrix}$$

where $L_0 \xrightarrow{f} L_\infty$ is a symmetric linear map.

Minkowski space inside $\mathrm{Sp}(4, \mathbb{R})$

- L_0 and L_∞ dual under symplectic form $L_0 \times L_\infty \xrightarrow{\omega} \mathbb{R}$
- $g \in \mathrm{GL}(L_\infty)$ induces *linear symplectomorphism* of $V = L_\infty \oplus L_0$, represented as block upper-triangular matrices:

$$g \oplus (g^\dagger)^{-1} = \begin{bmatrix} g & 0 \\ 0 & (g^\dagger)^{-1} \end{bmatrix}$$

- Translations of Minkowski space correspond to *shears:* (fixing L_∞ and L/L_∞):

$$\begin{bmatrix} I_2 & f \\ 0 & I_2 \end{bmatrix}$$

where $L_0 \xrightarrow{f} L_\infty$ is a symmetric linear map.

Minkowski space inside $\mathrm{Sp}(4, \mathbb{R})$

- L_0 and L_∞ dual under symplectic form $L_0 \times L_\infty \xrightarrow{\omega} \mathbb{R}$
- $g \in \mathrm{GL}(L_\infty)$ induces *linear symplectomorphism* of $V = L_\infty \oplus L_0$, represented as block upper-triangular matrices:

$$g \oplus (g^\dagger)^{-1} = \begin{bmatrix} g & 0 \\ 0 & (g^\dagger)^{-1} \end{bmatrix}$$

- Translations of Minkowski space correspond to *shears:* (fixing L_∞ and L/L_∞):

$$\begin{bmatrix} I_2 & f \\ 0 & I_2 \end{bmatrix}$$

where $L_0 \xrightarrow{f} L_\infty$ is a symmetric linear map.

Minkowski space inside $\mathrm{Sp}(4, \mathbb{R})$

- L_0 and L_∞ dual under symplectic form $L_0 \times L_\infty \xrightarrow{\omega} \mathbb{R}$
- $g \in \mathrm{GL}(L_\infty)$ induces *linear symplectomorphism* of $V = L_\infty \oplus L_0$, represented as block upper-triangular matrices:

$$g \oplus (g^\dagger)^{-1} = \begin{bmatrix} g & 0 \\ 0 & (g^\dagger)^{-1} \end{bmatrix}$$

- Translations of Minkowski space correspond to *shears:* (fixing L_∞ and L/L_∞):

$$\begin{bmatrix} I_2 & f \\ 0 & I_2 \end{bmatrix}$$

where $L_0 \xrightarrow{f} L_\infty$ is a symmetric linear map.

Affine deformation of $\mathrm{SL}(2, \mathbb{Z})$

- For $i = 1, 2, 3$ choose three positive integers μ_1, μ_2, μ_3. Then the subgroup Γ of $\mathrm{Sp}(4, \mathbb{Z})$ generated by

$$\begin{bmatrix} -1 & -2 & \mu_1 + \mu_2 - \mu_3 & 0 \\ 0 & -1 & 2\mu_1 & -\mu_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -\mu_2 & -2\mu_2 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is a proper affine deformation of a rank two free group.
  - M^3 genus two handlebody and Σ^2 triply–punctured sphere.
  - Depicted example is $\mu_1 = \mu_2 = \mu_3 = 1$.

Affine deformation of $\mathrm{SL}(2,\mathbb{Z})$

- For $i = 1, 2, 3$ choose three positive integers μ_1, μ_2, μ_3. Then the subgroup Γ of $\mathrm{Sp}(4,\mathbb{Z})$ generated by

$$\begin{bmatrix} -1 & -2 & \mu_1+\mu_2-\mu_3 & 0 \\ 0 & -1 & 2\mu_1 & -\mu_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -\mu_2 & -2\mu_2 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

  is a proper affine deformation of a rank two free group.

  - M^3 genus two handlebody and Σ^2 triply–punctured sphere.
  - Depicted example is $\mu_1 = \mu_2 = \mu_3 = 1$.

Affine deformation of $\mathrm{SL}(2,\mathbb{Z})$

- For $i = 1,2,3$ choose three positive integers μ_1, μ_2, μ_3. Then the subgroup Γ of $\mathrm{Sp}(4,\mathbb{Z})$ generated by

$$\begin{bmatrix} -1 & -2 & \mu_1+\mu_2-\mu_3 & 0 \\ 0 & -1 & 2\mu_1 & -\mu_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -\mu_2 & -2\mu_2 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

  is a proper affine deformation of a rank two free group.
  - M^3 genus two handlebody and Σ^2 triply–punctured sphere.
  - Depicted example is $\mu_1 = \mu_2 = \mu_3 = 1$.

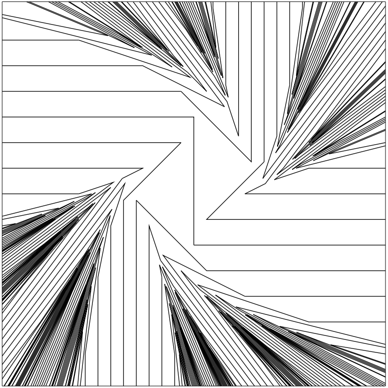
Affine deformation of SL$(2, \mathbb{Z})$

- For $i = 1, 2, 3$ choose three positive integers μ_1, μ_2, μ_3. Then the subgroup Γ of Sp$(4, \mathbb{Z})$ generated by

$$\begin{bmatrix} -1 & -2 & \mu_1 + \mu_2 - \mu_3 & 0 \\ 0 & -1 & 2\mu_1 & -\mu_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -\mu_2 & -2\mu_2 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

  is a proper affine deformation of a rank two free group.
  - M^3 genus two handlebody and Σ^2 triply–punctured sphere.
  - Depicted example is $\mu_1 = \mu_2 = \mu_3 = 1$.

Affine action of level 2 congruence subgroup of $GL(2,\mathbb{Z})$



Symmetrical example: $\mu_1 = \mu_2 = \mu_3 = 1.$

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Every *noncompact* complete hyperbolic surface Σ (with $\pi_1(\Sigma)$ finitely generated) admits a *proper* affine deformation.
- M^3 homeomorphic to solid handlebody.

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Every *noncompact* complete hyperbolic surface Σ (with $\pi_1(\Sigma)$ finitely generated) admits a *proper* affine deformation.
- M^3 homeomorphic to solid handlebody.

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Every *noncompact* complete hyperbolic surface Σ (with $\pi_1(\Sigma)$ finitely generated) admits a *proper* affine deformation.
- M^3 homeomorphic to solid handlebody.

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Every *noncompact* complete hyperbolic surface Σ (with $\pi_1(\Sigma)$ finitely generated) admits a *proper* affine deformation.
- M^3 homeomorphic to solid handlebody.

Marked Signed Lorentzian Length Spectrum

- For every affine deformation $\Gamma \xrightarrow{\rho=(\mathsf{L},u)} \mathsf{Isom}(\mathsf{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathsf{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- For every affine deformation $\Gamma \xrightarrow{\rho=(\mathsf{L},u)} \mathsf{Isom}(\mathsf{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathsf{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- For every affine deformation $\Gamma \xrightarrow{\rho=(\mathsf{L},u)} \mathsf{Isom}(\mathsf{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathsf{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- For every affine deformation $\Gamma \xrightarrow{\rho=(\mathsf{L},u)} \mathsf{Isom}(\mathsf{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathsf{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- For every affine deformation $\Gamma \xrightarrow{\rho=(\mathsf{L},u)} \mathsf{Isom}(\mathsf{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathsf{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Opposite Sign Lemma

(Margulis 1983) Let ρ be a *proper* affine deformation.

- $\alpha_u(\gamma) > 0\ \forall\gamma \neq 1$, or
- $\alpha_u(\gamma) < 0\ \forall\gamma \neq 1$.

Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset O(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$

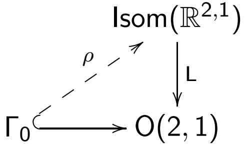
determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\iff$ adding a coboundary to u.
- *Translational conjugacy classes* of affine deformations of Γ_0 form the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset \mathrm{O}(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$



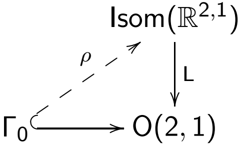
determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\Longleftrightarrow$ adding a coboundary to u.
- *Translational conjugacy classes* of affine deformations of Γ_0 form the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset O(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$



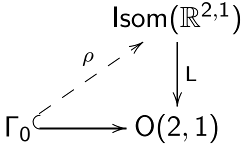
  determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\iff$ adding a coboundary to u.
- *Translational conjugacy classes* of affine deformations of Γ_0 form the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset \mathsf{O}(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$



determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\Longleftrightarrow$ adding a coboundary to u.
- *Translational conjugacy classes* of affine deformations of Γ_0 form the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{R})) \cong H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{R})) \cong H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{R})) \cong H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\Big|_{t=0} \ell_t(\gamma)$$

    (G-Margulis 2000).
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- Converse: When Σ is homeomorphic to a three-holed sphere or two-holed $\mathbb{R}P^2$.

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\left.\frac{d}{dt}\right|_{t=0} \ell_t(\gamma)$$

    (G-Margulis 2000).
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- Converse: When Σ is homeomorphic to a three-holed sphere or two-holed $\mathbb{R}P^2$.

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\left.\frac{d}{dt}\right|_{t=0} \ell_t(\gamma)$$

    (G-Margulis 2000).
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- Converse: When Σ is homeomorphic to a three-holed sphere or two-holed $\mathbb{R}P^2$.

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
    - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
    - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\Big|_{t=0} \ell_t(\gamma)$$

(G-Margulis 2000).

- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- Converse: When Σ is homeomorphic to a three-holed sphere or two-holed $\mathbb{R}P^2$.

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\Big|_{t=0} \ell_t(\gamma)$$

    (G-Margulis 2000).
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- Converse: When Σ is homeomorphic to a three-holed sphere or two-holed $\mathbb{R}P^2$.

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

    $$\frac{d}{dt}\bigg|_{t=0} \ell_t(\gamma)$$

    (G-Margulis 2000).
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- Converse: When Σ is homeomorphic to a three-holed sphere or two-holed $\mathbb{R}P^2$.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
    - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
    - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
    - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- $\mathcal{C}(\Sigma)$ connected $\Longrightarrow$ Either $\Psi_u(\mu)$ are all *positive* or all *negative*.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- $\mathcal{C}(\Sigma)$ connected $\Longrightarrow$ Either $\Psi_u(\mu)$ are all *positive* or all *negative*.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- $\mathcal{C}(\Sigma)$ connected $\Longrightarrow$ Either $\Psi_u(\mu)$ are all *positive* or all *negative*.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\iff$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- $\mathcal{C}(\Sigma)$ connected $\implies$ Either $\Psi_u(\mu)$ are all *positive* or all *negative*.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- $\mathcal{C}(\Sigma)$ connected $\Longrightarrow$ Either $\Psi_u(\mu)$ are all *positive* or all *negative*.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\iff$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- $\mathcal{C}(\Sigma)$ connected $\implies$ Either $\Psi_u(\mu)$ are all *positive* or all *negative*.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- $\mathcal{C}(\Sigma)$ connected $\Longrightarrow$ Either $\Psi_u(\mu)$ are all *positive* or all *negative*.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathsf{L}(\gamma)$ given *decorations* of the cusps (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - Margulis invariant extends to continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2010)
- When $\mathsf{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- $\mathcal{C}(\Sigma)$ connected $\Longrightarrow$ Either $\Psi_u(\mu)$ are all *positive* or all *negative*.

The Deformation Space

- Deformation space of marked Margulis space-times corresponding to surface S fibers over space of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
- Fiber is subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (*all* affine deformations) consisting of *proper* affine deformations Σ.
  - Nonempty (Drumm 1989).
- (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by generalized Margulis functionals of measured geodesic laminations on Σ.

The Deformation Space

- Deformation space of marked Margulis space-times corresponding to surface S fibers over space of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
- Fiber is subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (*all* affine deformations) consisting of *proper* affine deformations Σ.
  - Nonempty (Drumm 1989).
- (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by generalized Margulis functionals of measured geodesic laminations on Σ.

The Deformation Space

- Deformation space of marked Margulis space-times corresponding to surface S fibers over space of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
- Fiber is subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (*all* affine deformations) consisting of *proper* affine deformations Σ.
  - Nonempty (Drumm 1989).
- (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by generalized Margulis functionals of measured geodesic laminations on Σ.

The Deformation Space

- Deformation space of marked Margulis space-times corresponding to surface S fibers over space of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
- Fiber is subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (*all* affine deformations) consisting of *proper* affine deformations Σ.
  - Nonempty (Drumm 1989).
- (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by generalized Margulis functionals of measured geodesic laminations on Σ.

The Deformation Space

- Deformation space of marked Margulis space-times corresponding to surface S fibers over space of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
- Fiber is subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (*all* affine deformations) consisting of *proper* affine deformations Σ.
  - Nonempty (Drumm 1989).
- (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by generalized Margulis functionals of measured geodesic laminations on Σ.

The Crooked Plane Conjecture

- Conjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \approx [0, \infty)^b \times (0, \infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \approx [0, \infty)^b \times (0, \infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \;\approx\; [0,\infty)^b \times (0,\infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \;\approx\; [0,\infty)^b \times (0,\infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \approx [0,\infty)^b \times (0,\infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \approx [0,\infty)^b \times (0,\infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \approx [0,\infty)^b \times (0,\infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

$$\mathfrak{F}(\Sigma) \approx [0,\infty)^b \times (0,\infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

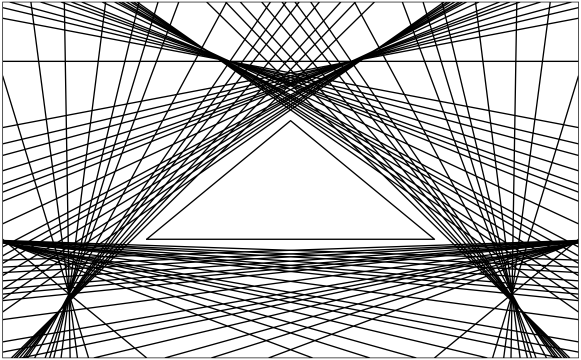
$$\mathfrak{F}(\Sigma) \approx [0, \infty)^b \times (0, \infty)^{3-b}.$$

The Crooked Plane Conjecture

- **C**onjecture: Every Margulis spacetime M^3 admits a fundamental polyhedron bounded by disjoint crooked planes.
  - Corollary: (Tameness) $M^3 \approx$ open solid handlebody.
- Proved when $\chi(\Sigma) = -1$ (that is, $rank(\pi_1(\Sigma)) = 2$). (Charette-Drumm-G 2010)
- Four possible topologies for Σ:
  - Three-holed sphere;
  - Two-holed cross-surface (projective plane);
  - One-holed Klein bottle;
  - One-holed torus.
- If $\partial\Sigma$ has b components, then the Fricke space

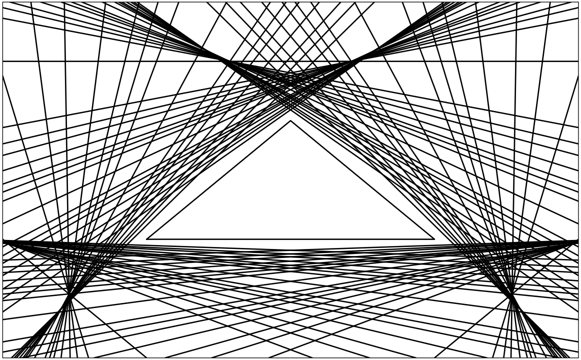
$$\mathfrak{F}(\Sigma) \ \approx \ [0,\infty)^b \times (0,\infty)^{3-b}.$$

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ three-holed sphere



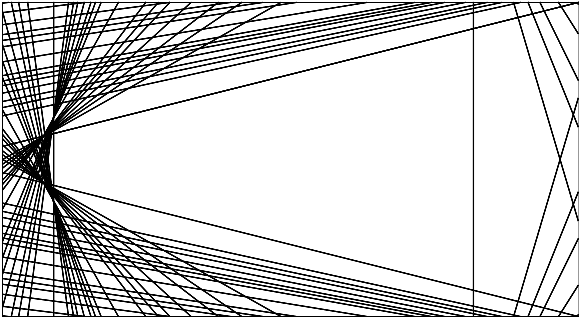
Charette-Drumm-Margulis functionals of $\partial\Sigma$ completely describe deformation space as $(0, \infty)^3$.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ three-holed sphere



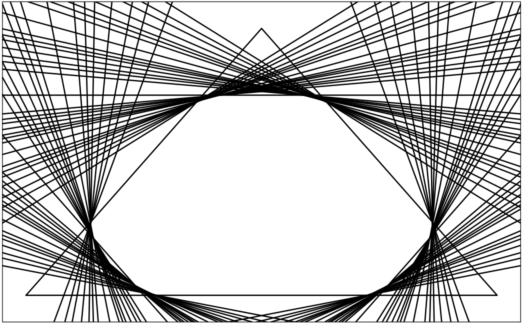
Charette-Drumm-Margulis functionals of $\partial\Sigma$ completely describe deformation space as $(0, \infty)^3$.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ two-holed $\mathbb{R}P^2$.



Deformation space is quadrilateral bounded by the four lines defined by CDM-functionals of $\partial\Sigma$ and the two orientation-reversing interior simple loops.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ one-holed torus



Properness region bounded by infinitely many intervals, each corresponding to simple loop.

Structure of the boundary

- ∂-points lie on intervals or are points of strict convexity (irrational laminations) (G-Margulis-Minsky).
- Birman-Series argument $\Longrightarrow$ For 1-holed torus, these points of strict convexity have Hausdorff dimension zero.

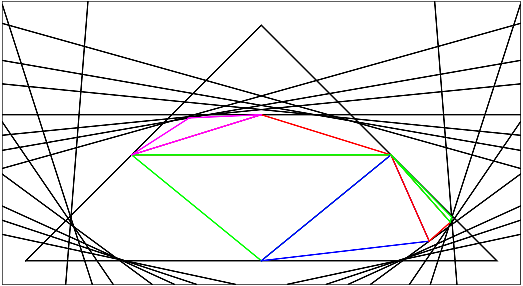
Structure of the boundary

- ∂-points lie on intervals or are points of strict convexity (irrational laminations) (G-Margulis-Minsky).
- Birman-Series argument $\Longrightarrow$ For 1-holed torus, these points of strict convexity have Hausdorff dimension zero.

Structure of the boundary

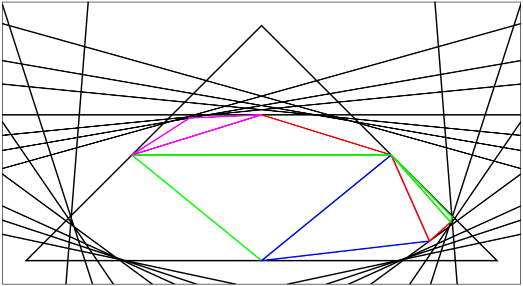
- ∂-points lie on intervals or are points of strict convexity (irrational laminations) (G-Margulis-Minsky).
- Birman-Series argument $\Longrightarrow$ For 1-holed torus, these points of strict convexity have Hausdorff dimension zero.

Realizing an ideal triangulation by crooked planes



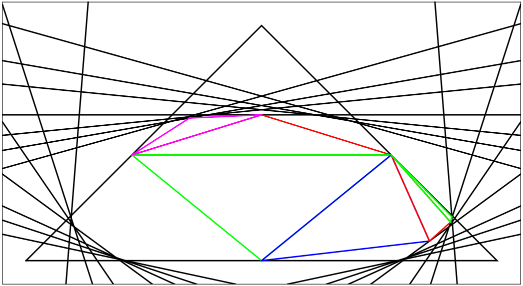
- Properness region tiled by triangles.
- Triangles $\longleftrightarrow$ ideal triangulations of Σ.
- Flip of ideal triangulation $\longleftrightarrow$ moving to adjacent triangle.

Realizing an ideal triangulation by crooked planes



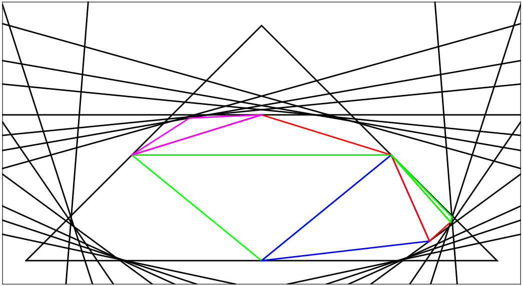
- Properness region tiled by triangles.
- Triangles $\longleftrightarrow$ ideal triangulations of Σ.
- Flip of ideal triangulation $\longleftrightarrow$ moving to adjacent triangle.

Realizing an ideal triangulation by crooked planes



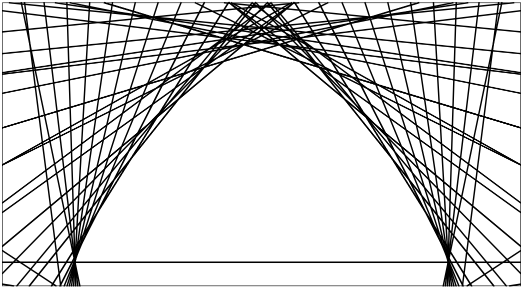
- Properness region tiled by triangles.
- Triangles $\longleftrightarrow$ ideal triangulations of Σ.
- Flip of ideal triangulation $\longleftrightarrow$ moving to adjacent triangle.

Realizing an ideal triangulation by crooked planes



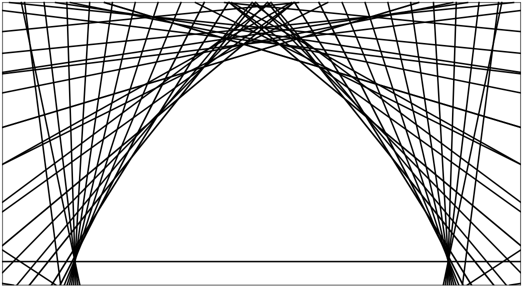
- Properness region tiled by triangles.
- Triangles $\longleftrightarrow$ ideal triangulations of Σ.
- Flip of ideal triangulation $\longleftrightarrow$ moving to adjacent triangle.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ one-holed Klein bottle



Properness region bounded by infinitely many intervals, each defined by CDM-invariants of simple orientation-reversing loops, arranged cyclically, and the one orientation-preserving interior simple loop.

Functionals $\alpha(\gamma)$ when $\Sigma \approx$ one-holed Klein bottle



Properness region bounded by infinitely many intervals, each defined by CDM-invariants of simple orientation-reversing loops, arranged cyclically, and the one orientation-preserving interior simple loop.

Happy Birthday, Caroline!!!