Three-Dimensional Manifolds

Michaelmas Term 1999

Prerequisites

Basic general topology (eg. compactness, quotient topology)
Basic algebraic topology (homotopy, fundamental group, homology)

Relevant books

Armstrong, *Basic Topology* (background material on algebraic topology)
Hempel, *Three-manifolds* (main book on the course)
Stillwell, *Classical topology and combinatorial group theory* (background material, and some 3-manifold theory)

§1. Introduction

Definition. A *(topological) n-manifold* M is a Hausdorff topological space with a countable basis of open sets, such that each point of M lies in an open set homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. The *boundary* ∂M of M is the set of points not having neighbourhoods homeomorphic to $\mathbb{R}^n$. The set $M - \partial M$ is the *interior* of M, denoted $\mathrm{int}(M)$. If M is compact and $\partial M = \emptyset$, then M is *closed*.

In this course, we will be focusing on 3-manifolds. Why this dimension? Because 1-manifolds and 2-manifolds are largely understood, and a full 'classification' of n-manifolds is generally believed to be impossible for $n \geq 4$. The theory of 3-manifolds is heavily dependent on understanding 2-manifolds (surfaces). We first give an infinite list of closed surfaces.

Construction. Start with a 2-sphere S^2. Remove the interiors of g disjoint closed discs. The result is a compact 2-manifold with non-empty boundary. Attach to each boundary component a 'handle' (which is defined to be a copy of the 2-torus T^2 with the interior of a closed disc removed) via a homeomorphism between the boundary circles. The result is a closed 2-manifold F_g of *genus* g. The surface F_0 is defined to be the 2-sphere S^2.

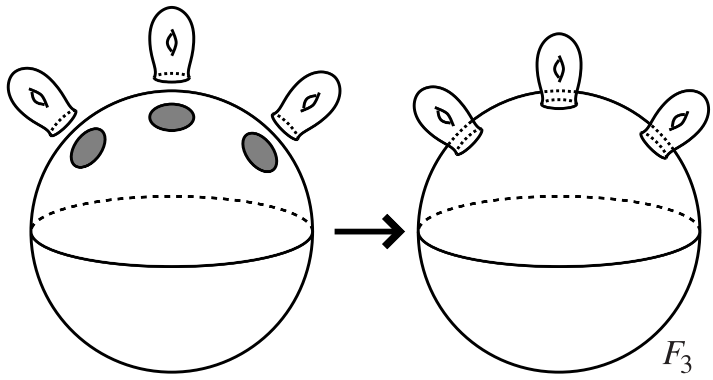

Figure 1.

Construction. Start with a 2-sphere S^2. Remove the interiors of h disjoint closed discs ($h \geq 1$). Attach to each boundary component a Möbius band via homeomorphisms of the boundary circles. The result is a closed 2-manifold N_h.

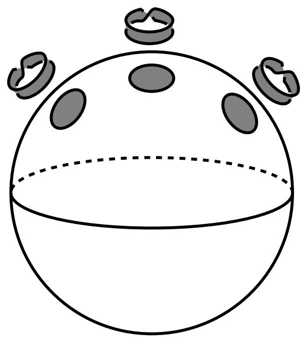

Figure 2.

Exercise. N_1 is homeomorphic to the real projective plane P^2.

Theorem 1.1. (Classification of closed 2-manifolds) *Each closed 2-manifold is homeomorphic to precisely one F_g for some $g \geq 0$, or one N_h for some $h \geq 1$.*

This is an impressive result. There is a similar result for compact 2-manifolds with boundary.

Theorem 1.2. (Classification of compact 2-manifolds) *Each compact 2-manifold is homeomorphic to precisely one of $F_{g,b}$ or $N_{h,b}$, where $g \geq 0$, $b \geq 0$ and $h \geq 1$, and $F_{g,b}$ (resp. $N_{h,b}$) is homeomorphic to F_g (resp. N_h) with the interiors of b disjoint closed discs removed.*

The surface $F_{0,1}$ is a disc D^2, $F_{0,2}$ is an *annulus* and $F_{0,3}$ is a *pair of pants*;

the surfaces $F_{0,i}$ ($i \geq 1$) are the *compact planar surfaces*.

There is in fact a classification of non-compact 2-manifolds, but the situation is significantly more complicated than in the compact case. In dimensions more than two, it is usual to concentrate on compact manifolds (which are usually hard enough). Below are some examples of non-compact 2-manifolds (without boundary) that exhibit a wide range of behaviour.

Examples. (i) $\mathbb{R}^2$.

(ii) The complement of a finite set of points in a closed 2-manifold.

(iii) $\mathbb{R}^2 - (\mathbb{Z} \times \{0\})$.

(iv) Glue a countable collection of copies of $F_{1,2}$ 'end-to-end'.

(v) Start with an annulus. Glue to each boundary component a pair of pants. The resulting 2-manifold has four boundary components. Glue to each of these another pair of pants. Repeat indefinitely.

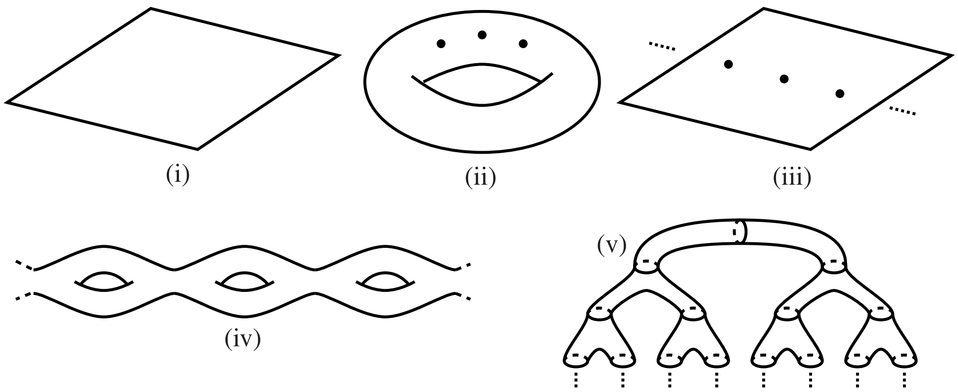

Figure 3.

It is quite possible that there is some sort of classification of compact 3-manifolds similar to the 2-dimensional case, but inevitably much more complicated. The simplest closed 3-manifold is the 3-sphere, which is most easily visualised as $\mathbb{R}^3$ 'with a point at infinity'.

Exercise. Prove that, for any point $x \in S^3$, $S^3 - \{x\}$ is homeomorphic to $\mathbb{R}^3$.

Construction. Let X be a subset of S^3 homeomorphic to the solid torus $S^1 \times D^2$. Then $S^3 - \text{int}(X)$ is a compact 3-manifold, with boundary a torus. Note that there are many possible such X in S^3 (one is given in Figure 4), and hence there are many such 3-manifolds.

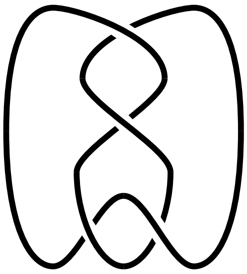

Figure 4.

Despite the large number of different 3-manifolds, they have a well-developed theory.

Definition. Let M_1 and M_2 be two oriented 3-manifolds. (The definition of an oriented manifold will be given in the next section.) Pick subsets B_1 and B_2 homeomorphic to closed 3-balls in the interiors of M_1 and M_2. Let $M_1 \# M_2$ be the manifold obtained from $M_1 - \text{int}(B_1)$ and $M_2 - \text{int}(B_2)$ by gluing ∂B_1 and ∂B_2 via an orientation-reversing homeomorphism. Then $M_1 \# M_2$ is the *connected sum* of M_1 and M_2.

The resulting 3-manifold $M_1 \# M_2$ is in fact independent of the choice of B_1, B_2 and orientation-reversing homeomorphism $\partial B_1 \to \partial B_2$. The 3-sphere is the union of two 3-balls glued along their boundaries. When one is forming $M \# S^3$ for any 3-manifold M, we may assume that one of these 3-balls is used in the definition of connected sum. Hence, $M \# S^3$ is obtained from M by removing a 3-ball and then gluing another back in. Hence, $M \# S^3$ is homeomorphic to M. A 3-manifold M is *composite* if it is homeomorphic to $M_1 \# M_2$, for neither M_1 nor M_2 homeomorphic to S^3; otherwise it is *prime*.

Here is an example of a theorem in this course.

Theorem 1.3. (Topological rigidity) *Let M_1 and M_2 be closed orientable prime 3-manifolds which are homotopy equivalent. Suppose that $H_1(M_1)$ and $H_1(M_2)$ are infinite. Then M_1 and M_2 are homeomorphic.*

The theorem can be false:

- if M_1 and M_2 are not prime,
- if $H_1(M_1)$ and $H_1(M_2)$ are finite,
- if M_1 and M_2 have non-empty boundary, or
- if M_1 and M_2 are non-compact.

Example. The following is a construction of two compact orientable prime 3-manifolds M_1 and M_2, with non-empty boundary, that are homotopy equivalent but not homeomorphic. Pick two disjoint simple closed curves in a torus T^2, bounding disjoint discs in T^2. Attach to each curve a copy of $F_{1,1}$ along the boundary curve of $F_{1,1}$. The resulting space X will be homotopy equivalent to both M_1 and M_2.

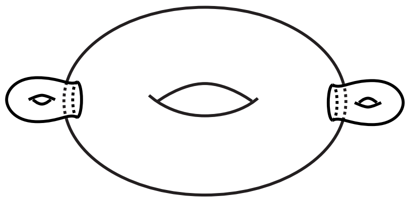

Figure 5.

We construct M_1 and M_2 by 'thickening' T^2 and the two copies of $F_{1,1}$ to $T^2 \times [0,1]$ and two copies of $F_{1,1} \times [0,1]$. We build M_1 by gluing the two copies of $\partial F_{1,1} \times [0,1]$ to disjoint annuli in $T^2 \times \{0\}$ (the annuli separating off disjoint discs in $T^2 \times \{0\}$). Note that M_1 is a 3-manifold with ∂M_1 being three tori and a copy of F_3. We construct M_2 similarly, except we attach one of the two copies of $\partial F_{1,1} \times [0,1]$ to $T^2 \times \{0\}$ and one to $T^2 \times \{1\}$. The resulting manifold M_2 has ∂M_2 being two tori and two copies of F_2. Hence, M_1 and M_2 are not homeomorphic, but they are both homotopy equivalent to X. (We cannot at this stage prove that they are prime, but this is in fact true.)

However, it is widely believed that (in a sense that can be made precise) 'almost all' homotopy equivalent closed 3-manifolds are in fact homeomorphic. A special case of this is the following, which is one of the most famous unsolved conjectures in topology.

Poincaré Conjecture. *A 3-manifold homotopy equivalent to S^3 is homeomorphic to S^3.*

§2. Which category?

In manifold theory, it is very important to specify precisely which 'category' one is working in. For example, one can deal not only with topological manifolds, but also smooth manifolds (which we will not define) and piecewise-linear (pl) manifolds, which are defined below. It turns out that 3-manifold theory often takes place in the pl setting.

Definition. The *n-simplex* is the set

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \ldots + x_{n+1} = 1 \text{ and } x_i \geq 0 \text{ for all } i\}.$$

The *dimension* of Δ^n is n. A *face* of an n-simplex Δ^n is a subset of Δ^n in which some co-ordinates are set to zero. A face of dimension zero is a *vertex*.

Definition. A *simplicial complex* is the space K obtained from a collection of simplices by gluing their faces together via linear homeomorphisms, such that any point of K has a neighbourhood intersecting only finitely many simplices.

Remark. This definition is more general than the usual definition of a simplicial complex, where one insists that each collection of points forms the vertices of at most one simplex.

Note. The underlying space of a simplicial complex is compact if and only if it has finitely many simplices.

Definition. A *triangulation* of a space M is a homeomorphism from M to some simplicial complex.

Example. The space obtained from two copies of Δ^n by identifying their boundaries using the identity map is a simplicial complex. It forms a triangulation of the n-sphere.

Definition. A *subdivision* of a simplicial complex K is another simplicial complex L with the same (i.e. homeomorphic) underlying space as K, where each simplex of L lies in some simplex of K in such a way that the inclusion map is affine.

Definition. A map $f\colon K \to L$ between simplicial complexes is *pl* if there exists subdivisions K' and L' of K and L so that f sends vertices of K' to vertices of L', and sends each simplex of K' linearly (but not necessarily homeomorphically) onto a simplex of L'.

Thus, by definition, there exists a pl homeomorphism between two simplicial complexes if and only if they have a common subdivision.

Exercise. The composition of two pl maps is again pl. Hence, simplicial complexes and pl maps form a category.

Definition. A *pl n-manifold* is a simplicial complex in which each point has a neighbourhood pl homeomorphic to the n-ball

$$D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq 1 \text{ for each } i\}$$

(with a standard triangulation).

An important fact that simplifies much of 3-manifold theory is the following theorem, due to Moise.

Theorem 2.1. *A topological 3-manifold possesses precisely one smooth structure (up to diffeomorphism) and precisely one pl structure (up to pl homeomorphism).*

This theorem is false in dimensions greater than three. When studying 3-manifold theory, however, it does not matter which category one pursues it from. For simplicity, we will now work entirely in the pl category without explicitly stating this. **Thus, all manifolds will be pl, and all maps will be pl.**

We now introduce a couple of concepts that are probably familiar, in a pl setting.

Orientability

Definition. An *orientation* on an n-simplex is an equivalence class of orderings on its vertices, where we treat distinct orderings as specifying the same orientation if and only if the orderings differ by an even permutation. If the vertices are ordered as $v_0, \ldots, v_n$ (say), then we write $[v_0, \ldots, v_n]$ for this orientation. We write $-[v_0, \ldots, v_n]$ for the other orientation. The orientation $[v_0, \ldots, v_n]$ *induces* the orientation $(-1)^i[v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$ on the face opposite v_i.

Definition. An *orientation* on an n-manifold M is a choice of orientation on each n-simplex of M, such that, if σ is any $(n-1)$-simplex adjacent to two n-simplices, then the orientations that σ inherits from these simplices disagree. The manifold is then *oriented*. If a triangulation of a manifold does not admit an orientation, then the manifold is *non-orientable*.

Note. A compact n-manifold M is orientable if and only if $H_n(M, \partial M) = \mathbb{Z}$. In this case, an orientation is a choice of generator for $H_n(M, \partial M)$. Hence, orientability is independent of the choice of triangulation for compact manifolds (and in fact for all manifolds).

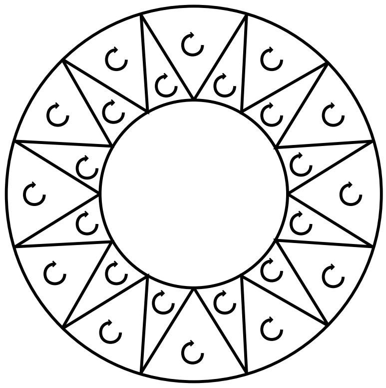

Figure 6.

Examples. The Möbius band M is non-orientable, whereas the annulus A is orientable. See Figure 6, where the arrows on each 2-simplex specify an orientation on that 2-simplex in the obvious way. Note that M and A are homotopy equivalent.

SUBMANIFOLDS

Note that D^k sits inside D^n for $k \leq n$, by setting the co-ordinates $x_{k+1}, \dots, x_n$ to zero.

Definition. A *submanifold* X of a pl manifold M is a subset which is simplicial in some subdivision of M, such that each point of X has a neighbourhood N and a pl homeomorphism $(N, N \cap X) \to (D^n, D^k)$. Note that this implies that $\partial X = X \cap \partial M$.

Definition. A map $X \to M$ between simplicial complexes is an *embedding* if it is a pl homeomorphism onto its image. It is a *proper embedding* if M is a manifold and the image of X is a submanifold of M.

Example. A 1-dimensional submanifold of S^3 is a *link*. If it is connected, it is a *knot*. If K is a knot in S^3 that does not bound a disc and we 'cone' the pair

(S^3, K), the result is a 2-disc embedded in the 4-ball, but not properly embedded.

Exercise. Show that if S is a surface embedded in a 3-manifold M such that $S \cap \partial M = \partial S$, then S is properly embedded. (You will need to know that any circle embedded in S^2 is 'standard'.)

We will see that studying submanifolds of M will shed considerable light on the properties of M.

We will prove the following result in §6.

Proposition 2.2. *Let X be an orientable codimension one submanifold of an orientable manifold. Then X has a neighbourhood homeomorphic to $X \times [-1, 1]$, where $X \times \{0\}$ is identified with X, and where $(X \times [-1, 1]) \cap \partial M) = \partial X \times [-1, 1]$.*

Isotopies

Let M be a simplicial complex.

Definition. Two homeomorphisms $h_0 \colon M \to M$ and $h_1 \colon M \to M$ are *isotopic* if there is a homeomorphism $H \colon M \times [0, 1] \to M \times [0, 1]$ such that, for all i, $H|_{M \times \{i\}}$ is a homeomorphism onto $M \times \{i\}$, and so that $H|_{M \times \{0\}} = h_0$ and $H|_{M \times \{1\}} = h_1$.

Remark. It is possible to impose a topology on the set $\mathrm{Homeo}(M, M)$ of all (pl) homeomorphisms $M \to M$, such that the path-components of $\mathrm{Homeo}(M, M)$ are precisely the isotopy classes.

Definition. Let K_0 and K_1 be subsets of M. They are *ambient isotopic* if there is a homeomorphism $h \colon M \to M$ that is isotopic to the identity and that takes K_0 to K_1.

Subsets of M that are ambient isotopic are, for almost all topological purposes, 'the same' and we will feel free to perform ambient isotopies as necessary.