

THE COARSE GEOMETRY OF THE TEICHMÜLLER METRIC

BRIAN H. BOWDITCH

ABSTRACT. We study the coarse geometry of the Teichmüller space of a compact orientable surface in the Teichmüller metric. We show that this admits a ternary operation, natural up to bounded distance, which endows the space with the structure of a coarse median space whose rank is equal to the complexity of the surface. We deduce that Teichmüller space satisfies a coarse quadratic isoperimetric inequality. We describe when it admits a quasi-isometric embedding of a euclidean space, or a euclidean half-space. We give a weak form of quasi-isometric rigidity for Teichmüller space, and deduce that, apart from some well known coincidences, the Teichmüller spaces are quasi-isometrically distinct. We show that any asymptotic cone is bilipschitz equivalent to a CAT(0) space, and so in particular, is contractible.

1. INTRODUCTION

In this paper we explore various properties of the large scale geometry of the Teichmüller space in the Teichmüller metric of a compact orientable surface. Our starting point is a combinatorial model of the Teichmüller space [R, D1], which we use to show that Teichmüller space admits a coarse median structure in the sense of [Bo1]. From this, a number of facts follow immediately, though others require additional work. As we will note, some of these results have been obtained in some form before, while others seem to be new. The paper makes use of constructions in [Bo4], which studies the geometry of the mapping class group metric from a similar perspective. The idea of using medians (or “centroids”) in the mapping class group originates in [BeM]. We remark that Teichmüller space in the Weil-Petersson metric also admits a coarse median [Bo5], which also has consequences for the large-scale geometry of that space.

Let Σ be a compact orientable surface of genus g with p boundary components. Let $\xi = \xi(\Sigma) = 3g + p - 3$ be the *complexity* of Σ . Unless

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otherwise stated, we will assume in this introduction that $\xi(\Sigma) \geq 2$. We will sometimes use $S_{g,p}$ to denote a generic surface of this type.

We write $\mathbb{T}(\Sigma)$ for the Teichmüller space of Σ , that is, the space of marked finite-area hyperbolic structures on the interior of Σ . We give $\mathbb{T}(\Sigma)$ the Teichmüller metric. This endows it with the structure of a complete Finsler manifold, diffeomorphic to $\mathbb{R}^{2\xi(\Sigma)}$. Note that the mapping class group, $\text{Map}(\Sigma)$, acts properly discontinuously on $\mathbb{T}(\Sigma)$. The properties we are mainly interested in here are quasi-isometry invariant, so for most of the paper we will be referring instead to the “decorated marking graph”, $\mathcal{R}(\Sigma)$, which is a slight variation on the “augmented marking graph” of [D1]. Both these spaces are equivariantly quasi-isometric to $\mathbb{T}(\Sigma)$.

The central observation of this paper is that Teichmüller space admits an equivariant median, with similar properties to that of the mapping class group [BeM, Bo1]. More specifically, we show:

Theorem 1.1. *There is a ternary operation, $\mu : \mathbb{T}(\Sigma)^3 \rightarrow \mathbb{T}(\Sigma)$, which is canonical up to bounded distance, and which endows $\mathcal{T}(\Sigma)$ with the structure of a coarse median space of rank $\xi(\Sigma)$.*

A more precise formulation of this (for $\mathcal{R}(\Sigma)$) is given as Theorem 4.1. Roughly speaking, this means that $\mathcal{R}(\Sigma)$ behaves like a median algebra of rank $\xi(\Sigma)$ up to bounded distance. In fact, any finite subset of $\mathbb{T}(\Sigma)$ lies inside a larger finite subset of $\mathbb{T}(\Sigma)$ which can be identified with the vertex set of a finite CAT(0) cube complex of dimension at most $\xi(\Sigma)$, in such a way that the median operation in $\mathbb{T}(\Sigma)$ agrees up to bounded distance with the usual median operation in a cube complex (see Section 2). As with the mapping class group, or the Weil-Petersson metric, the median can be characterised in terms of subsurface projection maps. Moreover, it is $\text{Map}(\Sigma)$ -equivariant up to bounded distance.

Theorem 1.1 is used here to prove various facts about the quasi-isometry type of $\mathbb{T}(\Sigma)$.

For example:

Theorem 1.2. *For any compact surface Σ , $\mathbb{T}(\Sigma)$ satisfies a coarse quadratic isoperimetric inequality.*

Here the term “coarse quadratic isoperimetric inequality” refers to the standard quasi-isometric invariant formulation of the quadratic isoperimetric inequality, and will be made more precise in Section 5.

We note that Theorem 1.2 is also a consequence of the fact that Teichmüller space admits a bicombing, see [KR].

We also have the following result, which has been proven by different methods in [EMR1]. An argument using asymptotic cones has been found independently by Durham [D2] (see the remark after Theorem 1.8).

Theorem 1.3. *There is a quasi-isometric embedding of a euclidean n -dimensional half-space into $\mathbb{T}(\Sigma)$ if and only if $n \leq \xi(\Sigma)$.*

The question of quasi-isometric embeddings of $\mathbb{R}^{\xi(\Sigma)}$ requires additional work. we will show:

Theorem 1.4. *There is a quasi-isometric embedding of the euclidean space, $\mathbb{R}^{\xi(\Sigma)}$, into $\mathbb{T}(\Sigma)$ if and only if Σ has genus at most 1, or is a closed surface of genus 2.*

Theorem 1.4 turns out to be equivalent to finding an embedded $(\xi(\Sigma) - 1)$ -sphere in the curve complex of Σ , which we will see happens precisely in the above cases (see Proposition 5.6).

Note that Theorem 1.3 immediately implies that if Σ and Σ' are both compact orientable surfaces and $\mathbb{T}(\Sigma)$ is quasi-isometric to $\mathbb{T}(\Sigma')$, then $\xi(\Sigma) = \xi(\Sigma')$. (One can distinguish some further cases by bringing Theorem 1.4 into play.) In fact we have:

Theorem 1.5. *If $\mathbb{T}(\Sigma)$ is quasi-isometric to $\mathbb{T}(\Sigma')$ then $\xi(\Sigma) = \xi(\Sigma')$, and if $\xi(\Sigma) = \xi(\Sigma') \geq 4$, then Σ and Σ' are homeomorphic. Moreover, if one of Σ or Σ' is homeomorphic to $S_{1,3}$, then they both are.*

This statement can be paraphrased by saying that if two Teichmüller spaces are quasi-isometric then they are isometric, since it is well known each of the pairs $\{S_{2,0}, S_{0,6}\}$, $\{S_{1,2}, S_{0,5}\}$ and $\{S_{1,1}, S_{0,4}\}$ have isometric Teichmüller spaces. They are all different apart from the above coincidences. Theorem 1.5 is a simple consequence of Theorem 1.6 below, together with the corresponding statement for the mapping class group.

Recall that we have a *thick part*, $\mathbb{T}_T(\Sigma)$, of $\mathbb{T}(\Sigma)$. Up to bounded Hausdorff distance, it can be defined in a number of equivalent ways. For example, given $\epsilon > 0$, it can be defined as the set those finite area hyperbolic structures with systole (i.e. length of the shortest closed geodesic) at least ϵ . For this, we need to chose ϵ sufficiently small so that $\mathbb{T}_T(\Sigma)$ is connected. Note that $\text{Map}(\Sigma)$ acts cocompactly on $\mathbb{T}_T(\Sigma)$. So, for example, $\mathbb{T}_T(\Sigma)$ is a bounded Hausdorff distance from any $\text{Map}(\Sigma)$ -orbit.

Theorem 1.6. *If $\phi : \mathbb{T}(\Sigma) \rightarrow \mathbb{T}(\Sigma)$ is any quasi-isometry, then the Hausdorff distance from $\phi(\mathbb{T}_T(\Sigma))$ and $\mathbb{T}_T(\Sigma)$ is finite, and bounded above in terms of $\xi(\Sigma)$ and the quasi-isometric parameters of ϕ .*

Our proof will use asymptotic cones. However, it has been pointed out to me by Kasra Rafi that it is also a consequence of the results of Mosher [Mo] and Minsky [Mi1, Mi2] which imply that a quasigeodesic in Teichmüller space is stable if and only if it lies a bounded distance from the thick part. Thus, up to bounded distance, the thick part can be characterised as the union of all stable quasigeodesics.

Now, up to bounded Hausdorff distance, $\mathbb{T}_T(\Sigma)$, can be viewed as uniformly embedded copy of (any Cayley graph of) $\text{Map}(\Sigma)$. It follows that ϕ gives rise to a quasi-isometry of $\text{Map}(\Sigma)$ to $\mathbb{T}_T(\Sigma)$. By quasi-isometric rigidity of $\text{Map}(\Sigma)$, [BeKMM, Ham] (see also [Bo4]), it follows that $\phi|_{\mathbb{T}_T(\Sigma)}$ agrees up to bounded distance with the map induced by an element of $\text{Map}(\Sigma)$.

With some more work, one can get:

Theorem 1.7. *There is a function $l : [0, \infty) \rightarrow [0, \infty)$ depending on $\xi(\Sigma)$ and quasi-isometry parameters, with $l(t)/t \rightarrow 0$ as $t \rightarrow \infty$, such that if $\phi : \mathbb{T}(\Sigma) \rightarrow \mathbb{T}(\Sigma)$ is a quasi-isometry, then $\rho(x, \phi x) \leq l(\phi(x, \mathbb{T}_T(\Sigma)))$ for all $x \in \mathbb{T}(\Sigma)$.*

This is a weak form of quasi-isometric rigidity. In fact, I am informed that full quasi-isometric rigidity (replacing the sublinear bound above with a constant one) has been obtained independently by other means, by Eskin, Masur and Rafi [EMR2]. From this, Theorems 1.5 and 1.6 follow.

The proofs of Theorems 1.3 to 1.7 involve studying an asymptotic cone, $\mathbb{T}^\infty(\Sigma)$, of $\mathbb{T}(\Sigma)$ (see [G, VaW]). As a consequence of Theorem 1.1, we know that $\mathbb{T}^\infty(\Sigma)$ is a topological median algebra of rank $\xi(\Sigma)$, and can be bilipschitz embedded in a finite product of \mathbb{R} -trees [Bo1, Bo2]. In particular, it is bilipschitz equivalent to a median metric space. We also derive the following facts:

Theorem 1.8. *Any asymptotic cone, $\mathbb{T}^\infty(\Sigma)$, of $\mathbb{T}(\Sigma)$ has locally compact dimension $\xi(\Sigma)$. It is bilipschitz equivalent to a $CAT(0)$ space (and so, in particular, is contractible).*

Here the *locally compact dimension* of a topological space is the maximal dimension of any locally compact subset.

Another proof that the asymptotic cone has locally compact dimension at most $\xi(\Sigma)$ has been found independently by Durham [D2]. From this, the “only if” part of Theorem 1.3 follows.

In fact, analysing the structure of $\mathbb{T}^\infty(\Sigma)$ will be a significant part of the work of this paper.

We remark that there are various other spaces naturally associated to a compact surface. Of particular note are the Weil-Petersson metric

on Teichmüller space, the (Cayley graph of the) mapping class group, and the curve complex. Some discussion of the quasi-isometry types of these and other related spaces can be found in [Y]. Various results regarding rank and rigidity of such spaces can be found, for example, in [BeKMM, Ham, EMR1, Bo1, Bo4], and references therein.

Theorem 1.1 of this paper is proven in Section 4 (modulo the lower bound on rank, which will be a consequence of Theorem 1.3). Theorems 1.2, 1.3, 1.8 and half of 1.4 are proven in Section 5. Theorems 1.5, 1.6 and 1.7 are proven in Section 7. The remaining half of Theorem 1.4 is proven in Section 8.

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2. BACKGROUND

In this section, we review a few items of background material.

2.1. Conventions and terminology. If $x, y \in \mathbb{R}$, we often use the notation $x \sim y$ to mean that $|x - y|$ is bounded above by some additive constant. The factors determining the relevant constant at any given moment, if not specified, should be clear from context. Ultimately, it should only depend on the parameters of the hypotheses, namely the complexity of the surface, or quasi-isometry constants etc. If a, b are points in a metric space, we write $a \sim b$ to mean that they are bounded distance apart.

If $x, y > 0$, we will similarly write $x \asymp y$ to mean that y is bounded above and below by increasing linear functions of x . Again, the factors determining these functions should be clear from context.

We will generally behave as though the relations \sim and \asymp were transitive, though clearly each application of the transitive law will implicitly entail a change in the defining constants.

Except when we are working in the asymptotic cone, maps between our various metric spaces are generally defined up to bounded distance. It will generally be assumed that maps between graphs send vertices to vertices.

In the context of asymptotic cones, we will be using ultraproducts of various sets or graphs associated to a surface, for example curves, multicurves and markings. In this case, we will understand a “curve” to mean an element of the ultraproduct of the set of curves, and a “standard curve” to mean an element of the original set, that is, a curve

in the traditional sense. We apply similar terminology to multicurves and subsurfaces etc. We will elaborate on this later.

Let $\phi : (M, \rho) \rightarrow (M', \rho')$ is a map (not necessarily continuous) between metric spaces. We say that ϕ is *coarsely lipschitz* if for all $x, y \in M$, $\rho'(\phi x, \phi y)$ is bounded above by a linear function of $\rho(x, y)$. We say it is a *quasi-isometric embedding* if, in addition, $\rho(x, y)$ is bounded above by a linear function of $\rho'(\phi x, \phi y)$. A *quasi-isometry* is a quasi-isometric map with cobounded image (i.e., every point of M' is a bounded distance from $\phi(M)$).

2.2. Marking graphs. Let Σ be a compact orientable surface of genus g with p boundary components. Let $\xi = \xi(\Sigma) = 3g + p - 3$ be the *complexity* of Σ . We will write $S_{g,p}$ to denote the topological type of Σ .

As usual, a *curve* in Σ will mean a homotopy class of essential non-peripheral simple closed curves (except when it refers to an element of the ultraproduct of such, as mentioned above). We write $\iota(\alpha, \beta)$ for the geometric intersection number of two curves, α, β . We write $\text{Map}(\Sigma)$ for the mapping class group of Σ . A *multicurve* in Σ is a set of pairwise disjoint curves. Here we will generally allow empty multicurves. We write $\mathcal{S} = \mathcal{S}(\Sigma)$ for the set of multicurves on Σ . A multicurve is *complete* if it cuts Σ into $S_{0,3}$'s. A complete multicurve has exactly $\xi(\Sigma)$ components.

Central to our discussion will be the notion of a “marking graph”. One version of this was given in [MaM2], though there are many variations. We first describe the essential properties we require of it.

A *marking graph* is a connected graph, \mathcal{M} , whose vertices are “markings” of Σ . By a “marking”, we mean a finite set of curves in Σ which fill Σ . We require that the set, \mathcal{M}^0 , of vertices, is $\text{Map}(\Sigma)$ -invariant (under the natural action on markings induced by the action on curves). We also require that this action extends to an action on \mathcal{M} , and that the action is cofinite. Note, in particular, that if $a, b \in \mathcal{M}^0$ are equal or adjacent, and $\alpha \in a$, $\beta \in b$, then $\iota(\alpha, \beta)$ is bounded. (We should therefore think of a marking as having bounded self-intersection.) For many purposes, the above would be sufficient. Examples of such are the marking graph described in [MaM2]. Alternatively, we could choose any sufficiently large integers, $q \geq p > 0$ and set \mathcal{M}^0 to be the set of markings, a , with $\iota(\alpha, \beta) \leq p$ for all $\alpha, \beta \in a$, and deem markings a, b to be adjacent if $\iota(\alpha, \beta) \leq q$ for all $\alpha \in a$ and $\beta \in b$. (In fact, we could take $p = q = 4$.) This was the definition used in [Bo1].

Here we will require a bit more, namely that some marking should contain a complete multicurve. Note that it follows that if $a \in \mathcal{M}^0$

contains a multicurve, τ , then a is a bounded distance from some $b \in \mathcal{M}^0$, where b contains a complete multicurve containing τ . We also require the following. Moreover, if $a, b \in \mathcal{M}^0$ both contain a multicurve τ , then they can be connected by a path in \mathcal{M} , whose vertices all contain τ , and whose length is bounded above in terms of the distance between a and b in \mathcal{M} . This is easy to arrange. For example, it is true of the marking graph described in [MaM2].

(We note that, in [MaM2] and its application in [D1], the complete multicurve is viewed as part of the structure of the marking, though that is not essential here.)

In any case, the notion is quite robust. For some applications, it is convenient to suppose that the self-intersection bound on markings is sufficiently large.

We will fix one marking graph, and denote it by $\mathcal{M}(\Sigma)$.

2.3. Subsurfaces. By a *subsurface* in Σ , we mean a subsurface X of Σ , defined up to homotopy, such that the intrinsic boundary, ∂X , of X is essential in Σ , and such that X is not a three-holed sphere. We write \mathcal{X} for the set of subsurfaces. We can partition \mathcal{X} as $\mathcal{X}_A \sqcup \mathcal{X}_N$ into annular and non-annular subsurfaces. Given a curve, γ , in Σ , we write $X(\gamma) \in \mathcal{X}_A$, for the regular neighbourhood of γ . Given $X \in \mathcal{X}$, write $\partial_\Sigma X$ for the relative boundary of X in Σ , thought of as a multicurve in Σ .

Given $X, Y \in \mathcal{X}$, we have the following pentachotomy:

$X = Y$.

$X \prec Y$: $X \neq Y$, and X can be homotoped into Y but not into ∂Y .

$Y \prec X$: $Y \neq X$, and Y can be homotoped into X but not into ∂X .

$X \wedge Y$: $X \neq Y$ and X, Y can be homotoped to be disjoint.

$X \pitchfork Y$: none of the above.

We will be using subsurface projections to curve graphs and marking complexes.

We can associate to each $X \in \mathcal{X}$ the *curve graph*, $\mathcal{G}(X)$, in the usual way. Thus, if $X \in \mathcal{X}_N$, then the vertex set, $\mathcal{G}^0(X)$, is the set of curves in X , where two curves are adjacent if they have minimal possible intersection number. If $\xi(\Sigma) \geq 2$, this is 0. We write σ_X^0 for the combinatorial metric on $\mathcal{G}(X)$. (The caret subscript will be explained later.) It is shown in [MaM1] that $\mathcal{G}(X)$ is Gromov hyperbolic.

We will need to deal with annular surfaces differently. We begin with a general discussion.

Suppose that A is a compact topological annulus. Let $\mathcal{G}(A)$ be the graph whose vertex, $\mathcal{G}^0(A)$, consists of arcs connecting the two boundary components of A , defined up to homotopy fixing their endpoints, and where two such arcs are deemed adjacent in $\mathcal{G}(A)$ if they can be realised so that they meet at most at their endpoints. We write σ_A^\wedge for the combinatorial metric on $\mathcal{G}(A)$. It is easily seen that $\mathcal{G}(A)$ is quasi-isometric to the real line. In fact, we will choose points x, y in the different boundary components, and let $\mathcal{G}_0(A)$ be the full subgraph of $\mathcal{G}(A)$, whose vertex set, $\mathcal{G}_0^0(A)$, consists of those arcs with endpoints at x and y . Now $\mathcal{G}_0(A)$ can be identified with real line, \mathbb{R} , with vertex set \mathbb{Z} . In fact, if t is the Dehn twist in A , and $\delta \in \mathcal{G}_0^0(A)$ is any fixed arc, then the map $[r \mapsto t^r \delta]$ gives an identification of \mathbb{Z} with the vertex set. One can also check that the inclusion of $\mathcal{G}_0(A)$ into $\mathcal{G}(A)$ is an isometric embedding. Moreover, $\mathcal{G}(A)$ is the 1-neighbourhood of its image. It follows that $|\sigma_A^\wedge(\delta, t^r \delta) - |r|| \leq 1$, for all $r \in \mathbb{Z}$. In fact, since x, y and δ can be chosen arbitrarily, this holds for all $\delta \in \mathcal{G}^0(A)$. (The fact that we have only an additive error here is important for later discussion.)

Now given $X \in \mathcal{X}$, we have a well defined ‘‘subsurface projection’’ map: $\theta_X^\wedge : \mathcal{M}(\Sigma) \rightarrow \mathcal{G}(X)$, well defined up to bounded distance (see [MaM2]). (Here we are using the notation ‘‘ θ_X^\wedge ’’ to remind us that we are dealing with marking graphs and curve graphs. In [Bo1], this was simply denoted θ_X . However, we use that notation here for projection between ‘‘decorated’’ graphs, which will play an equivalent role in this paper. This also explains the notation σ_X^\wedge .)

We also have a projection map $\psi_X^\wedge : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(X)$. In fact, these maps can be defined intrinsically to subsurfaces. In this way, if $Y \preceq X$, then $\theta_Y^\wedge \circ \psi_X^\wedge = \theta_Y^\wedge : \mathcal{M}(\Sigma) \rightarrow \mathcal{G}(Y)$ and $\psi_Y^\wedge \circ \psi_X^\wedge = \psi_Y^\wedge : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(Y)$. Moreover, if $\gamma \in a \in \mathcal{M}(\Sigma)$ with $\gamma \prec X$, we may always assume that $\gamma \in \psi_X^\wedge a$.

If $Y \preceq X$ or $Y \pitchfork X$, then we also have a projection, $\theta_X^\wedge Y \in \mathcal{G}(X)$.

The distance formula of [MaM2] relates distances in $\mathcal{M}(\Sigma)$ to subsurface projection distances. In particular, they show:

Lemma 2.1. *There is some $r_0 \geq 0$ depending only on $\xi(\Sigma)$, such that if $r \geq r_0$ $a, b \in \mathcal{M}(\Sigma)$ then the set $\mathcal{A}^\wedge(a, b; r) = \{X \in \mathcal{X} \mid \sigma_X^\wedge(\theta_X^\wedge a, \theta_X^\wedge b) \geq r\}$ is finite. Moreover, $\rho(a, b) \asymp \sum_{X \in \mathcal{A}^\wedge(a, b)} \sigma_X^\wedge(\theta_X^\wedge a, \theta_X^\wedge b)$.*

Here the linear bounds implicit in \asymp depend only on $\xi(\Sigma)$ and r . (A similar formula for Teichmüller space was given in [R]. It is given as Proposition 4.8 here.)

One immediate consequence is the following:

Lemma 2.2. *Given $r \geq 0$, there is some $r' \geq 0$, depending only on $\xi(\Sigma)$ and r such that if $a, b \in \mathcal{M}(\Sigma)$ and $\sigma_X^\wedge(\theta_X^\wedge a, \theta_X^\wedge b) \leq r$ for all $X \in \mathcal{X}$, then $\rho(a, b) \leq r$.*

Another important ingredient is the following lemma of Behrstock [Be] (given as Lemma 11.3 of [Bo1]).

Lemma 2.3. *There is a constant, l , depending only on $\xi(\Sigma)$ with the following property. Suppose that $X, Y \in \mathcal{X}$ with $X \pitchfork Y$, and that $a \in \mathcal{M}^0$. Then $\min\{\sigma_X^\wedge(\theta_X^\wedge a, \theta_X^\wedge Y), \sigma_Y^\wedge(\theta_Y^\wedge a, \theta_Y^\wedge X)\} \leq l$.*

2.4. Median algebras. Let (M, μ) be a median algebra; that is, a set, M , equipped with a ternary operation, $\mu : M^3 \rightarrow M$, such that $\mu(a, b, c) = \mu(b, c, a) = \mu(c, a, b)$, $\mu(a, a, b) = a$ and $\mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e)$, for all $a, b, c, d, e \in M$. (For an exposition, see for example [BaH].) Given $a, b \in M$, write $[a, b] = \{x \in M \mid \mu(a, b, x) = x\}$, for the *median interval* from a to b . A subset, A , of M is a *subalgebra* if $\mu(a, b, c) \in A$ for all $a, b, c \in A$. It is *convex* if $[a, b] \subseteq A$ for all $a, b \in A$. One defines homomorphisms and isomorphisms between median algebras in the obvious way. If $a, b \in M$, then $[a, b]$ admits a partial order, \leq , defined by $x \leq y$ if $x \in [a, y]$ (or equivalently $y \in [b, x]$). If $[a, b]$ has intrinsic rank 1, then this is a total order.

An n -cube in M is a subalgebra isomorphic to the direct product on n two-point median algebras: $\{-1, 1\}^n$. The *rank* of M is the maximal n such that M contains an n -cube (deemed infinite, if there is no such bound). In [Bo1], we also defined “ n -colourability” for a median algebra, though that will only be referred to indirectly here, so we will not repeat the definition.

We will refer to a 2-cube as a *square*. We will generally denote a square by cyclically listing its points as a_1, a_2, a_3, a_4 , so that $\{a_i, a_{i+1}\}$ is a side for all i . Note that $a_i \in [a_{i-1}, a_{i+1}]$ for all i (which one can check is, in fact, an equivalent way of characterising a square, provided we assume the a_i to be pairwise distinct).

Two ordered pairs, x, y and x', y' of elements of M are said to be *parallel* if $(x = y$ and $x' = y')$ or $(x = x'$ and $y = y')$ or x, y, y', x' form a square.

Suppose $C \subseteq M$ is any subset. A map $\omega : M \rightarrow C$ is a *gate map* or *quasiprojection* if $\omega(x) \in [c, x]$ for all $x \in M$ and $c \in C$. If such a map exists, then it is unique, and $\omega|_C$ is the identity. Moreover, C must be convex (since if $a, b \in C$ and $d \in [a, b]$, then $\omega(d) \in [a, d] \cap [d, b] = \{d\}$, so $d \in C$).

We will also use the notion of a topological median algebra. This consists of a hausdorff topological space, M , and a continuous ternary operation $\mu : M^3 \rightarrow M$ such that (M, μ) is a median algebra. We say

that M is *locally convex* if every point has a base of convex neighbourhoods. We say that M is *weakly locally convex* if, given any open set $U \subseteq M$ and any $x \in U$, there is another open set, $V \subseteq U$ containing x such that if $y \in V$, then $[x, y] \subseteq U$. (In fact, finite rank together with weakly locally convex implies locally convex, see Lemma 7.1 of [Bo1].) All the topological median algebras that arise here will be locally convex. Examples of topological median algebras arise as median metric spaces. Again, we only refer to this latter notion in passing here. More detailed discussion can be found in [Ve, Bo3].

2.5. Coarse median spaces. Let (Λ, ρ) be a geodesic metric space. The following definition was given in [Bo1]:

Definition. We say that (Λ, ρ, μ) is a coarse median space if it satisfies:

(C1): There are constants, $k, h(0)$, such that for all $a, b, c, a', b', c' \in \Lambda$,

$$\rho(\mu(a, b, c), \mu(a', b', c')) \leq k(\rho(a, a') + \rho(b, b') + \rho(c, c')) + h(0).$$

(C2): There is a function $h : \mathbb{N} \rightarrow [0, \infty)$ such that $1 \leq |A| \leq p < \infty$, then there is a finite median algebra (Π, μ_Π) and a map $\lambda : \Pi \rightarrow \Lambda$ such that

$$\rho(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) \leq h(p)$$

for all $x, y, z \in \Pi$ and such that $\rho(a, \lambda\pi a) \leq h(p)$ for all $a \in A$.

We say that (Λ, ρ, μ) has rank at most n if Π can always be chosen to have rank at most n .

We say that (Λ, ρ, μ) is *n-colourable* if we can always choose Π to be *n-colourable*.

Given $a, b \in \Lambda$, write $[a, b] = \{\mu(a, b, x) \mid x \in \Lambda\}$, for the *coarse interval* from a to b . If $c \in [a, b]$, then one can check that $\mu(a, b, c) \sim c$.

Definition. We say that $C \subseteq \Lambda$ is *r-quasiconvex* if $[a, b] \subseteq N(C; r)$ for all $a, b \in C$. We say that C is *quasiconvex* if it is *r-convex* for some $r \geq 0$.

Note that a quasiconvex set, C , is always quasi-isometrically embedded, or more precisely, there is some s depending only on r and the parameters of Λ , such that the inclusion of $N(C; s)$ into Λ is a quasi-isometric embedding.

We can define a *coarse gate map* to be a map $\omega : \Lambda \rightarrow C$ such that $\mu(x, \omega(x), c) \sim \omega(x)$ for all $x \in \Lambda$ and $c \in C$. Similarly as with gate maps in a median algebra, the existence of such a map implies that C is quasiconvex.

We note the following:

Lemma 2.4. *Suppose that $a, b, c \in \Lambda$ and $r \geq 0$, with $\rho(\mu(a, b, c), c) \leq r$. Then $\rho(a, c) + \rho(c, b) \leq k_1\rho(a, b) + k_2$, where k_1, k_2 are constants depending only on r and the parameters of Λ .*

Proof. This is equivalent to saying that $\rho(a, c)$ is bounded above in terms of $\rho(a, b)$. This, in turn follows from the fact that the map $[x \mapsto \mu(a, b, x)]$ is coarsely lipschitz, which is an immediate consequence of hypothesis (C1) of the definition of coarse median. \square

In other words, if c lies “between” a and b in the coarse median sense, then $\rho(a, c) + \rho(c, b)$ agrees with $\rho(a, b)$ up to linear bounds.

A map, $\phi : (\Lambda, \rho, \mu) \rightarrow (\Lambda', \rho', \mu')$ between two coarse median spaces is said to be a *h-quasimorphism* if $\rho'(\phi\mu(x, y, z), \mu(\phi x, \phi y, \phi z))$ for all $x, y, z \in \Lambda$. We abbreviate this to *quasimorphism* if the constant is understood from context.

A particular example of a coarse median space is a Gromov hyperbolic space. In this case, the median is the usual centroid of three points (a bounded distance from geodesic triangle connecting any two of these points). Such a coarse median space has rank at most 1. (In fact, any rank-1 coarse median space arises in this way.)

2.6. Asymptotic cones. Let \mathcal{Z} be a countable set equipped with a non-principal ultrafilter. Given a \mathcal{Z} -sequence of sets, $\vec{A} = (A_\zeta)_\zeta$, we write $\mathcal{U}\vec{A}$ for its ultraproduct, that is, $\mathcal{U}\vec{A} = \left(\prod_{\zeta \in \mathcal{Z}} A_\zeta\right) / \approx$, where $(a_\zeta)_\zeta \approx (b_\zeta)_\zeta$ if $a_\zeta = b_\zeta$ almost always. If $A_\zeta = A$ is constant, we write $\mathcal{U}A = \mathcal{U}\vec{A}$. We can identify A as a subset of $\mathcal{U}A$ via constant sequences. We refer to an element of A in $\mathcal{U}A$ as being *standard*.

Note that the ultraproduct of the reals $\mathcal{U}\mathbb{R}$ is an ordered field. We say that $x \in \mathcal{U}\mathbb{R}$ is *infinitesimal* if $|x| < y$ for all $y \in \mathbb{R}$ with $y > 0$. We say that $x \in \mathcal{U}\mathbb{R}$ is *limited* if $|x| < y$ for some $y \in \mathbb{R}$. If we quotient $\mathcal{U}\mathbb{R}$ by the infinitesimals (i.e. two numbers are equivalent if they differ by an infinitesimal), we get the “extended reals”, \mathbb{R}^* , which is an ordered abelian group, an ordered field, containing \mathbb{R} as the convex subgroup of *limited* extended reals.

Suppose that $((\Lambda_\zeta, \rho_\zeta))_\zeta$ is a \mathcal{Z} -sequence of metric spaces. Then $(\mathcal{U}\vec{A}, \mathcal{U}\rho)$ is a $(\mathcal{U}\mathbb{R})$ -metric space. After identifying points an infinitesimal distance apart, we get a quotient (Λ^*, ρ^*) , which is an \mathbb{R}^* -metric space. We say that two points of Λ^* lie in the same *component* if they are a limited distance apart. Thus, each component of Λ is a metric space in the usual sense (that is, the metric takes real values). In fact, one can show that such a component is a complete metric space.

In particular, suppose that (Λ, ρ) is a fixed metric space, and that $\mathfrak{t} \in \mathcal{UR}$ is a positive infinitesimal. Let $(\Lambda_\zeta, \rho_\zeta) = (\Lambda, t_\zeta \rho)$. In this case, we write (Λ^*, ρ^*) for the resulting \mathbb{R}^* -metric space (where the scaling factors, t_ζ , are implicitly understood). We will use Λ^∞ to denote a general component of Λ^* , and ρ^∞ for the restriction of ρ^* . Thus, $(\Lambda^\infty, \rho^\infty)$ is a complete metric space in the usual sense. This is called an *asymptotic cone* of Λ and we refer to Λ^* as the *extended asymptotic cone*. Given a \mathcal{Z} -sequence of points $x_\zeta \in \Lambda_\zeta$, and $x \in \Lambda^*$, write $x_\zeta \rightarrow x$ to mean that x is the class corresponding to $(x_\zeta)_\zeta$. If Λ is a geodesic space, so is Λ^∞ . We put the metric topology on each component of Λ^* , and topologise Λ^* as the disjoint union of its components. Note that Λ^* has a preferred basepoint, namely that corresponding to any constant sequence in Λ . The component of Λ^* containing this basepoint is sometimes referred to as *the* asymptotic cone of Λ . (Again, the choice of scaling factors is implicitly assumed.)

If a group Γ acts by isometry on Λ , we get an action of its ultraproduct $\mathcal{U}\Gamma$ on Λ^* .

If (Λ, ρ) and (Λ', ρ') are metric spaces, and $f : \Lambda \rightarrow \Lambda'$ is a coarsely lipschitz map, we get an induced map $f^* : \Lambda^* \rightarrow (\Lambda')^*$, which is lipschitz (in the sense that the multiplicative bound is real). This restricts to a lipschitz map $f : \Lambda^\infty \rightarrow (\Lambda')^\infty$. If f is a quasi-isometric embedding, then f^∞ is bilipschitz onto its range. In particular, if f is a quasi-isometry then, f^∞ is a bilipschitz homeomorphism.

A similar discussion applies if we have a \mathcal{Z} -sequence of uniformly coarsely lipschitz maps, $f_\zeta : \Lambda \rightarrow \Lambda'$.

If (Λ, ρ, μ) is a coarse median space, then we get a ternary operation $\mu^* : (\Lambda^*)^3 \rightarrow \Lambda^*$ which restricts to $\mu^\infty : (\Lambda^\infty)^3 \rightarrow \Lambda^\infty$. One can check that (Λ^*, μ^*) is a median algebra, with $(\Lambda^\infty, \mu^\infty)$ as a subalgebra. Note that $(\Lambda^\infty, \mu^\infty)$ is a topological median algebra, in the sense that the median is continuous with respect to the topology induced by ρ^∞ .

A standard example is that of a Gromov hyperbolic space, Λ , in which case, Λ^* is an \mathbb{R}^* -tree. Each component, Λ^∞ , is an \mathbb{R} -tree. For example, if Λ is the hyperbolic plane, then Λ^∞ is the unique complete 2^{\aleph_0} -regular tree. If Λ is a horodisc, then Λ^∞ is a closed subtree thereof. In both cases, Λ^∞ is “almost furry” which means that it is non-trivial, and no point has valence 2.

3. A COMBINATORIAL MODEL

In this section, we describe a combinatorial model, $\mathcal{R} = \mathcal{R}(\Sigma)$, for $\mathbb{T}(\Sigma)$. It is a slight variation on the “augmented marking complex” described in [R] and in [D1]. In order to distinguish it, we will refer

to the model described here as the “decorated marking complex”. The model $\mathcal{R}(\Sigma)$ contains $\mathcal{M}(\Sigma)$ as a subgraph. We define $\mathcal{R}(\Sigma)$ as follows.

A vertex, a , of \mathcal{R} consists of a marking, $\bar{a} \in \mathcal{M}^0$, together with a map $\eta = \eta_a : \bar{a} \rightarrow \mathbb{N}$ such that if $\alpha, \beta \in \bar{a}$, with $\eta(\alpha) > 0$ and $\eta(\beta) > 0$, then $\iota(\alpha, \beta) = 0$. Thus, $\hat{a} = \{\alpha \in \bar{a} \mid \eta(\alpha) > 0\}$ is a (possibly empty) multicurve in Σ . We refer to such an a as a *decorated marking*, and to $\eta_a(\alpha)$ as the *decoration* on α . Two decorated markings, $a, b \in \mathcal{R}^0$ are deemed adjacent in \mathcal{R} if one of the following three conditions hold:

(E1): $\bar{a} = \bar{b}$ and $|\eta(\alpha) - \eta(\beta)| \leq 1$ for all $\alpha \in \bar{a}$.

(E2a): $\hat{a} = \hat{b}$, $\eta_a|_{\hat{a}} = \eta_b|_{\hat{b}}$ and \bar{a}, \bar{b} are adjacent in \mathcal{M} .

(E2b): $\hat{a} = \hat{b}$, $\eta_a|_{\hat{a}} = \eta_b|_{\hat{b}}$ and $\hat{b} = t_\alpha^r \hat{a}$, where $\alpha \in \bar{a}$, t_α is the Dehn twist about α , and $|r| \leq 2^{\eta_a(\alpha)}$.

We refer to condition (E1) as “vertical adjacency” and to (E2) (that is (E2a) or (E2b)) as “horizontal adjacency”. Given that \mathcal{M} is connected, it is easily seen that \mathcal{R} is connected also. We write ρ for the combinatorial metric on \mathcal{R} (assigning each edge unit length).

We say that an element $a \in \mathcal{R}^0$ is *thick* if $\hat{a} = \emptyset$. The *thick part*, \mathcal{R}_T , of \mathcal{R} is the full subgraph of \mathcal{R} whose vertex set consists of thick decorated markings. Note that there is a natural embedding, $v : \mathcal{M} \rightarrow \mathcal{R}_T \subseteq \mathcal{R}$, extending this inclusion. It is easily seen that this is a quasi-isometry, with respect to the intrinsic path metric induced on \mathcal{R}_T . Note that this is again robust — if we take a different marking complex satisfying the conditions laid out in Section 2, we get a quasi-isometric space.

In Section 2 above we described subsurface projections, $\theta_X^\wedge : \mathcal{M}(\Sigma) \rightarrow \mathcal{G}(X)$. Here we need to modify that in the case where X is an annulus.

Recall, first, that we have associated to any annulus, graphs $\mathcal{G}(A)$ etc., as described in Section 2.3. From this, we can define the *decorated arc graph*, $\mathcal{H}(A)$. Its vertex set, $\mathcal{H}^0(A)$ is $\mathcal{G}^0(A) \times \mathbb{N}$, where (δ, i) and (ϵ, j) are deemed adjacent if either $\delta = \epsilon$ and $|i - j| = 1$, or if $i = j$ and $\sigma_A^\wedge(\delta, \epsilon) \leq 2^i$. We write σ_A for the induced combinatorial metric. One can see that $\mathcal{H}(A)$ is quasi-isometric to a horodisc in the hyperbolic plane. In fact, if $\mathcal{H}_0(A)$ is the full subgraph of $\mathcal{H}(A)$ with vertex set $\mathcal{G}_0^0(A) \times \mathbb{N}$, then the inclusion of $\mathcal{H}_0(A)$ in $\mathcal{H}(A)$ is an isometric embedding with 1-cobounded image. Moreover the map sending $(t^r \delta, i)$ to the point $(r, 1 + i \log 2)$ in the upper-half-plane model gives a quasi-isometry of $\mathcal{H}_0(A)$ to the horodisc $\mathbb{R} \times [1, \infty)$.

Now suppose that $X \in \mathcal{X}_A$. The open annular cover of Σ corresponding to X has a natural compactification to a compact annulus,

$A(X)$ (cf. [MaM2]). We write $(\mathcal{G}(X), \sigma_X^\wedge) = (\mathcal{G}(A(X)), \sigma_{A(X)}^\wedge)$, and $(\mathcal{H}(X), \sigma_X) = (\mathcal{H}(A(X)), \sigma_{A(X)})$. Write $\mathcal{H}_T(X) = \mathcal{G}(X) \subseteq \mathcal{H}(X)$.

Let $\theta_X^\wedge : \mathcal{M} \rightarrow \mathcal{G}_0(X)$ be the usual subsurface projection map (which we assume sends vertices to vertices). This commutes with the Dehn twist, t , about the core curve of X . In particular, it follows from the above discussion that $|\sigma_A^\wedge(\theta_X^\wedge m, \theta_X^\wedge t^r m) - |r|| \leq 1$ for all $r \in \mathbb{Z}$ and $m \in \mathcal{M}$. If $a \in \mathcal{R}^0$, set $\theta_X a = (\theta_X^\wedge \bar{a}, i)$, where $i = \eta_a(\alpha)$ if $\alpha \in \bar{a}$, and $i = 0$ if $\alpha \notin \bar{a}$.

If $X \in \mathcal{X}_N$, we simply set $(\mathcal{H}(X), \sigma_X) = (\mathcal{G}(X), \sigma_X^\wedge)$. We define $\theta_X : \mathcal{R}^0 \rightarrow \mathcal{H}^0(X)$ just by setting $\theta_X a = \theta_X^\wedge(\bar{a})$.

Lemma 3.1. *If $X \in \mathcal{X}$, then the map $\theta_X : \mathcal{R}^0 \rightarrow \mathcal{H}^0(X)$ extends to a coarsely lipschitz map $\theta_X : \mathcal{R} \rightarrow \mathcal{H}(X)$.*

Proof. In other words, we claim that if $a, b \in \mathcal{R}^0$ are adjacent in \mathcal{R} , then $\sigma_X(\theta_X a, \theta_X b)$ is bounded above (in terms of $\xi(\Sigma)$). We deal with the three types of edges in turn.

(E1): We have $\bar{a} = \bar{b}$. If $X \in \mathcal{X}_N$ then $\theta_X a = \theta_X b$. If $X \in \mathcal{X}_A$, then the first coordinates of $\theta_X a$ and $\theta_X b$ are equal, and their second coordinates are either differ by 1 or both equal 0 (depending on whether or not the core curve of X lies in \bar{a}). In all cases $\sigma_X(\theta_X a, \theta_X b) \leq 1$.

(E2a): We have $\hat{a} = \hat{b}$ and $\eta_a|\hat{a} = \eta_b|\hat{b}$, and that $\sigma_X^\wedge(\theta_X^\wedge a, \theta_X^\wedge b)$ is bounded. If $X \in \mathcal{X}_N$, we are done. If $X \in \mathcal{X}_A$, the first coordinates are a bounded distance apart in $\mathcal{G}(X)$, and the second coordinates are equal (to $\eta_a\alpha = \eta_b\alpha$ or to 0, depending on whether or not the core curve of X lies in \hat{a}).

(E2b): We have $b = t_\alpha^r a$ (so that $\hat{a} = \hat{b}$). Suppose first that X is not a regular neighbourhood of α . In this case, we have $\sigma_X^\wedge(\theta_X^\wedge \bar{a}, \theta_X^\wedge \bar{b})$ bounded. (To see this, let γ be any curve not homotopic into $\Sigma \setminus X$, with $\iota(\gamma, \alpha) = 0$, and with $\iota(\gamma, \delta)$ bounded for all $\delta \in \bar{a}$. Note that $t_\alpha \gamma = \gamma$, and we see that $\iota(\gamma, \epsilon) = \iota(\gamma, t_\alpha^{-r} \epsilon)$ is bounded for all $\epsilon \in \bar{b}$. It follows that $\theta_X^\wedge \bar{a}$ and $\theta_X^\wedge \bar{b}$ are both a bounded distance from the projection of the curve γ to X in $\mathcal{G}(X)$.) We see that in this case $\sigma_X(\theta_X a, \theta_X b)$ is bounded. We are therefore reduced to considering the case where X is an annulus with core curve α . From the earlier discussion, we know that $\sigma_X^\wedge(\theta_X^\wedge \bar{a}, \theta_X^\wedge \bar{b}) = \sigma_X^\wedge(\theta_X^\wedge \bar{a}, \theta_X^\wedge t_\alpha^r \bar{a})$ differs by at most 1 from $|r|$. Moreover, $|r| \leq 2^i$, where $i = \eta_a\alpha = \eta_b\alpha$. Note that i is also the second coordinate of $\theta_X a$ and $\theta_X b$. By construction of $\mathcal{H}(X)$, we see that $\sigma_X(\theta_X a, \theta_X b) \leq 2$ in this case. \square

If $X \in \mathcal{X}_N$, we set $\mathcal{H}(X) = \mathcal{G}(X)$, and $\theta_X^\wedge = \theta_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{H}(X)$. If $X = \Sigma$, we write $\chi_\Sigma = \theta_\Sigma : \mathcal{R}(\Sigma) \rightarrow \mathcal{G}(\Sigma)$. Up to bounded distance, this simple selects some curve from the marking.

Given $a \in \mathcal{R}$, $\alpha \in \hat{a}$ and $i \in \mathbb{N}$, write $a_i = a_i(\alpha) \in \mathcal{R}$ for the decorated marking obtained by changing the decoration on α to i . (That is, $\bar{a}_i = \bar{a}$, $\eta_{a_i}(\beta) = \eta_a(\beta)$ for all $\beta \in \bar{a} \setminus \{\alpha\}$, and $\eta_{a_i}(\alpha) = i$.) Write $t = t_\alpha$ for Dehn twist about α . Write $H_a(\alpha) = \{t^r a_i \mid r \in \mathbb{Z}, i \in \mathbb{N}\}$.

Lemma 3.2. *Suppose $a \in \mathcal{R}$, $\alpha \in \hat{a}$ and $X = X(\alpha)$. There is a quasi-isometric embedding $\kappa : \mathcal{H}(X) \rightarrow \mathcal{R}$ with image a bounded Hausdorff distance from $H_a(\alpha)$, and with $\theta_X \circ \kappa$ a bounded distance from the identity on $\mathcal{H}(X)$.*

Proof. Let $\delta = \theta_X^\wedge \bar{a}$. We can assume that $\delta \in \mathcal{H}_0^0(X) \subseteq \mathcal{H}^0(X)$. Thus, $\mathcal{H}_0^0(X) = \{(t^r \delta, i) \mid r \in \mathbb{Z}, i \in \mathbb{N}\}$. Define $\kappa|_{\mathcal{H}_0^0(X)}$ by $\kappa((t^r \delta, i)) = t^r a_i$. Thus, by construction, $\kappa(\mathcal{H}_0(X)) = H_a(\alpha)$ and $\theta_X \circ \kappa|_{(\mathcal{G}_0^0(X))}$ is the identity. If $b, c \in (\mathcal{H}_0^0(X))$, the $\kappa b, \kappa c$ are connected by an edge of \mathcal{R} (of type (E1) or (E2b)). Thus, we can extend this to an embedding of $\mathcal{H}_0(X)$ in \mathcal{R} , and hence to a coarsely lipschitz map, $\kappa : \mathcal{H}(X) \rightarrow \mathcal{R}$. Given that it has a left quasi-inverse, this must be a quasi-isometric embedding. \square

Note that, in fact, we can see that the multiplicative constant of the quasi-isometry is 1 in this case, i.e. distances agree up to an additive constant. In other words, we see that if $b, c \in H_a(\alpha)$, then $|\rho(b, c) - \theta_X(b, c)|$ is bounded.

We can extend this to a statement about twists on multicurves. Given $a, b \in \mathcal{R}$, suppose that there is some $\tau \subseteq \hat{a} \cap \hat{b}$ such that b is obtained from a by applying powers of Dehn twists about elements of τ , and changing the decorations on these curves. In this case, we get:

Lemma 3.3. *If $a, b \in \mathcal{R}$ are as above, then $|\rho(a, b) - \sum_{\alpha \in \tau} \sigma_{X(\alpha)}(a, b)|$ is bounded.*

(We will only really need that $\rho(a, b) \asymp \sum_{\alpha \in \tau} \sigma_{X(\alpha)}(a, b)$.)

Note that for all $X \in \mathcal{X}$, $\mathcal{H}(X)$ is uniformly hyperbolic (in the sense of Gromov). In particular, they each admit a median operation $\mu_X : \mathcal{H}(X)^3 \rightarrow \mathcal{H}(X)$, well defined up to bounded distance, and such that $(\mathcal{H}(X), \mu_X)$ is a coarse median space of rank 1.

We will need the following observation:

Lemma 3.4. *$\mathcal{R}_T(\Sigma)$ is uniformly embedded in $\mathcal{R}(\Sigma)$.*

Proof. In fact, $\mathcal{R}_T(\Sigma)$ is exponentially distorted in $\mathcal{R}(\Sigma)$. Given $a, b \in \mathcal{R}_T^0(\Sigma)$, let $n = \rho(a, b)$. Let $a = a_0, \dots, a_n = b$ be vertex path from a to b in $\mathcal{R}(\Sigma)$. Certainly, $\rho(a_i, \bar{a}_i) = \rho(a_i, \mathcal{R}_T^0(\Sigma)) \leq n$ for all i . So by construction of $\mathcal{R}(\Sigma)$, we have $\rho(a_i, a_{i+1}) \leq 2^n$, so $\rho(a, b) \leq 2^n n$. \square

There are many variations on the construction of \mathcal{R} which would give rise to quasi-isometrically equivalent graphs. Suppose that \mathcal{M}^0 is any $\text{Map}(\Sigma)$ -invariant collection of markings of Σ , where a “marking” here can be any collection of curves which fill Σ . We suppose that the action of $\text{Map}(\Sigma)$ is cofinite (which is equivalent to bounding the self-intersection of each element of \mathcal{M}^0), and that every complete multicurve is a subset of some element of $\text{Map}(\Sigma)$. Now suppose that \mathcal{M} is any connected locally finite graph with vertex set \mathcal{M} and that the action of $\text{Map}(\Sigma)$ extends to a cofinite action on \mathcal{M}_0 . We can define “decorated markings” similarly as before, and construct a graph \mathcal{R} of decorated markings. Again there are a number of variations. For example, in place of the exponent base 2 in (E2b) we could use any number bigger than 1. Note that \mathcal{M} embeds in $\mathcal{M}(p, q)$ for all sufficiently large p, q and that the inclusion is a quasi-isometry. From this one can construct a quasi-isometry from the new \mathcal{R} to the decorated marking graph as we have defined it. (This is where we use the fact that two markings a bounded distance apart in \mathcal{M} , both containing a given multicurve, τ , can be connected by a bounded-length path of markings in \mathcal{M} all containing τ .)

The construction of the “augmented marking graph” in [D1] fits (more or less) into this picture. There, the marking graph, \mathcal{M} , is taken to be the marking graph as defined in [MaM2]. There is a restriction that \hat{a} is required to be a subset of the “base” of a marking a (but any multicurve with bounded intersection with a will be the base of some nearby marking, so this requirement does not change things on a large scale). Our edges of type (E1), (E2a) and (E2b) correspond to “vertical moves”, “flip moves” and “twist moves” in the terminology there. Note that the exponent e is used instead of 2 for the twist moves. In any case, it is easily seen from the above, that the augmented marking complex of [D1] is equivariantly quasi-isometric to the decorated marking complex as we have defined it.

It was shown in [D1] that the augmented marking graph is equivariantly quasi-isometric to $\mathbb{T}(\Sigma)$. We deduce:

Proposition 3.5. *There is a $\text{Map}(\Sigma)$ -equivariant quasi-isometry of $\mathbb{T}(\Sigma)$ to $\mathcal{R}(\Sigma)$.*

Note that this quasi-isometry necessarily maps $\mathbb{T}_T(\Sigma)$ to within a bounded Hausdorff distance of $\mathcal{R}_T(\Sigma)$. (This can be seen explicitly from the various constructions, but also follows from the fact that the constructions are equivariant and that $\text{Map}(\Sigma)$ acts coboundedly on both $\mathbb{T}_T(\Sigma)$ and $\mathcal{R}_T(\Sigma)$.) In fact, one can see from Lemma 7.8 that any quasi-isometry from $\mathbb{T}_T(\Sigma)$ to $\mathcal{R}_T(\Sigma)$ must have this property.

As a special case of the above construction, we consider the situation where Σ is a $S_{1,1}$ or $S_{0,4}$. In this case, we can take $\mathcal{G}^0(\Sigma)$, as usual, to be the set of curves in Σ . We deem α, β to be adjacent in \mathcal{G}^0 if they have minimal intersection (that is, $\iota(\alpha, \beta) = 1$ for the $S_{1,1}$ and $\iota(\alpha, \beta) = 2$ for the $S_{0,4}$). Thus, $\mathcal{G}(\Sigma)$ is a Farey graph, which we can identify with the 1-skeleton of a regular ideal tessellation of the hyperbolic plane, \mathbb{H}^2 .

We can take $\mathcal{M}(\Sigma)$ to be the dual 3-regular tree. Its vertices are at the centres of the ideal triangles, and its edges are geodesic segments. In this way, an element of $\mathcal{R}(\Sigma)$ consists of a triple of curves $\{\alpha, \beta, \gamma\}$ corresponding to a triangle in $\mathcal{G}(\Sigma)$, with decorations assigned to these curves, at most one of which is non-zero. We define a map $f : \mathcal{R}^0 \rightarrow \mathbb{H}^2$ as follows. If all the decorations of $\{\alpha, \beta, \gamma\}$ are 0, then we map it to the centre, m , of the corresponding triangle. If the decoration on α , say, is $i > 0$, then we map it to $\lambda(i \log 2)$ where $\lambda : [0, \infty) \rightarrow [0, \infty)$ is the geodesic ray with $\gamma(0) = m$, and tending the ideal point of \mathbb{H}^2 corresponding to α . Mapping edges to geodesic segments, we get a map $f : \mathcal{R} \rightarrow \mathbb{H}^2$, which extends the inclusion of \mathcal{M} into \mathbb{H}^2 . It is easily checked that f is a quasi-isometry.

Note that each curve $\alpha \in \mathcal{G}^0$ corresponds to a component, $C(\alpha)$, of the complement of \mathcal{M} in \mathbb{H}^2 . Now $f|_{\mathcal{H}(\alpha)}$ is a quasi-isometry of $\mathcal{H}(\alpha)$ to $C(\alpha)$, which is in turn a bounded Hausdorff distance from a horodisc in \mathbb{H}^2 .

Finally note that there is a natural $\text{Map}(\Sigma)$ -equivariant identification of \mathbb{H}^2 with the Teichmüller space of Σ with the Teichmüller metric.

4. THE MEDIAN CONSTRUCTION

The main aim of this section will be to prove the following:

Theorem 4.1. *There is a coarsely lipschitz ternary operation $\mu : \mathcal{R}(\Sigma)^3 \rightarrow \mathcal{R}(\Sigma)$, unique up to bounded distance, such that for all $a, b, c \in \mathcal{R}(\Sigma)$ and all $X \in \mathcal{X}$, $\theta_X \mu(a, b, c)$ agrees up to bounded distance with $\mu_X(\theta_X a, \theta_X b, \theta_X c)$. Moreover, $(\mathcal{R}(\Sigma), \mu)$ is a finitely colourable coarse median metric space of rank $\xi(\Sigma)$. All constants involved in the conclusion depend only on $\xi(\Sigma)$ and the constants of the hypotheses.*

As we will see, the median is characterised up to bounded distance by the fact that all the subsurface projection maps, $\theta_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{H}(X)$ are uniform quasimorphisms for all $X \in \mathcal{X}$. Since this condition is $\text{Map}(\Sigma)$ -equivariant, the median is necessarily $\text{Map}(\Sigma)$ -equivariant up to bounded distance. In view of Proposition 3.5, we see that this will then imply Theorem 1.1.

It is not hard to deduce the existence of medians on $\mathcal{R}(\Sigma)$ from the existence of medians in $\mathcal{M}(\Sigma)$. However, the approach we take here is to verify the properties laid out in Section 7 of [Bo4]. The statement then follows directly from Theorem 7.2 of that paper. We begin with a general discussion of decorated markings.

To summarise so far, we have a graph, (\mathcal{R}, ρ) , and a collection of uniformly hyperbolic spaces, $(\mathcal{H}(X), \sigma_X)$, indexed by the set, \mathcal{X} , of subsurfaces of Σ , together with uniformly coarsely lipschitz maps $\theta_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{H}(X)$. The maps θ_X were constructed out of the uniformly lipschitz maps $\theta_X^\wedge : \mathcal{M}(\Sigma) \rightarrow \mathcal{G}(X)$, for the family of graphs, $(\mathcal{G}(X), \sigma_X^\wedge)$. Note that these constructions are all $\text{Map}(\Sigma)$ -equivariant up to bounded distance.

Given $a, b \in \mathcal{R}$, we will often abbreviate $\sigma_X(a, b) = \sigma_X(\theta_X a, \theta_X b)$ and $\sigma_X^\wedge(a, b) = \sigma_X^\wedge(\theta_X^\wedge \bar{a}, \theta_X^\wedge \bar{b}) = \sigma_X^\wedge(\theta_X^\wedge \bar{a}, \theta_X^\wedge \bar{b})$. We also write $\theta_X a = \theta_X \bar{a}$. If γ is a curve in Σ , we will abbreviate $\sigma_\gamma = \sigma_{X(\gamma)}$, $\sigma_\gamma^\wedge = \sigma_{X(\gamma)}^\wedge$, $\theta_\gamma = \theta_{X(\gamma)}$, $\theta_\gamma^\wedge = \theta_{X(\gamma)}^\wedge$, etc. We write $\mathcal{G}(\gamma) = \mathcal{G}(X(\gamma))$, $\mathcal{H}(\gamma) = \mathcal{H}(X(\gamma))$, etc.

The following two statements are immediate consequences of Theorem 4.8 below, though we offer a more direct proofs which only uses properties of the augmented marking complex.

Lemma 4.2. *There is some $l_0 \geq 0$ such that for all $a, b \in \mathcal{R}(\Sigma)$, $\{X \in \mathcal{X} \mid \sigma_X(a, b) \geq l_0\}$ is finite.*

Proof. This is an immediate consequence of the corresponding statement for $\mathcal{M}(\Sigma)$ [MaM2], given Lemma 2.1 here. We only need to worry about $X \in \mathcal{X}_A$. But then $\sigma_X(a, b)$ is bounded above in terms of $\sigma_X^\wedge(a, b)$, unless perhaps the core curve of X lies in $\hat{a} \cup \hat{b}$. But there are only finitely many such X . \square

Lemma 4.3. *For all $l \geq 0$, there is some $l' \geq 0$, depending only on l and $\xi(\Sigma)$ such that of $a, b \in \mathcal{R}$ satisfy $\sigma_X(a, b) \leq l$ for all $X \in \sigma_X$, then $\rho(a, b) \leq l'$.*

Proof. First note that we can assume that $\hat{a} = \hat{b}$. For suppose that $\alpha \in \hat{a} \setminus \hat{b}$. Let $X(\alpha)$ be the regular neighbourhood of α . Then $\eta_\alpha(\alpha) \leq \sigma_{X(\alpha)}(a, b) \leq l$. Let $a_0 = a_0(\alpha) \in \mathcal{R}^0$ (i.e. we reset the decoration on α to 0). Thus, $\rho(a, a_0) = \eta_\alpha(\alpha) \leq l$, and $\hat{a}_0 = \hat{a} \setminus \{\alpha\}$. Note that by construction, $\theta_X(a_0) = \theta_X(a)$ for all $X \in \mathcal{X} \setminus X(\alpha)$. We now replace a with a_0 . We continue with this process for all curves in $\hat{a} \setminus \hat{b}$ and in $\hat{b} \setminus \hat{a}$, until \hat{a} and \hat{b} are both equal to some (possibly empty) multicurve, τ , say. Note that this process does not change \bar{a} or \bar{b} , and moves a and b each a bounded amount.

If X is not a regular neighbourhood of any element of τ , then we see that $\sigma_X^\wedge(\bar{a}, \bar{b})$ is bounded in terms of l . Moreover, for each $\beta \in \tau$,

we can find some $r(\beta) \in \mathbb{Z}$ such that $\sigma_\beta^\wedge(\bar{b}, t_\beta^{r(\beta)}\bar{a})$ is bounded. Let g be the composition of the twists $t_\beta^{r(\beta)}$ as β ranges over τ , and set $e = \mathcal{G}(A) \in \mathcal{R}^0$. We now get that $\sigma_X^\wedge(\bar{b}, \bar{e})$ is bounded for all $X \in \mathcal{X}$. Therefore by Corollary 2.2, we get $\rho^\wedge(\bar{b}, \bar{e})$ is bounded in terms of l .

After moving b a bounded distance (across edges of type (E2a)) in \mathcal{M} , we assume that $\bar{b} = \bar{e}$. In other words, a, b now satisfy the hypotheses of Corollary 2.2, and so $\rho(a, b)$ is bounded above as a function of $\sum_{\alpha \in \tau} \sigma_\alpha(a, b)$, and hence bounded. \square

The following is an analogue of Behrstock's Lemma [Be], given as Lemma 2.3 here.

Lemma 4.4. *There is a constant, l_1 , depending only on $\xi(\Sigma)$ with the following property. Suppose that $X, Y \in \mathcal{X}$ with $X \pitchfork Y$, and that $a \in \mathcal{R}_T^0$. Then $\min\{\sigma_X(a, Y), \sigma_Y(a, X)\} \leq l_1$.*

Proof. Suppose, for contradiction, that $\sigma_X(a, Y)$ and $\sigma_Y(a, X)$ are both large. Now by Lemma 2.3, we can assume, without loss of generality that $\sigma_X^\wedge(a, X)$ is bounded. This can only happen if $X = X(\gamma)$ for some curve $\gamma \in \hat{a}$. By assumption, γ crosses Y , so we see that $\sigma_Y^\wedge(a, X)$ is also bounded. We must therefore have $Y = X(\beta)$ for some $\beta \in \hat{a}$. But $\beta \pitchfork \alpha$, giving a contradiction, since \hat{a} is assumed to be a multicurve. \square

We will write $\langle x, y:z \rangle_\sigma = \frac{1}{2}(\sigma(x, z) + \sigma(y, z) - \sigma(x, y))$ for the Gromov product of x, y with basepoint z . We will write $\langle x, y:z \rangle_X = \langle \theta_X x, \theta_X y: \theta_X z \rangle_{\sigma_X}$, and $\langle x, y:z \rangle_X^\wedge = \langle \theta_X^\wedge x, \theta_X^\wedge y: \theta_X^\wedge z \rangle_{\sigma_X^\wedge}$.

The following is an analogue of Lemma 11.4 of [Bo1]:

Lemma 4.5. *There are some l_1, l_2 , depending only on $\xi(\Sigma)$, with the following property. Suppose that $X, Y \in \mathcal{X}$ with $Y \prec X$, and suppose that $a, b \in \mathcal{R}$ with $\langle a, b:Y \rangle_X \geq l_1$. Then $\sigma_Y(a, b) \leq l_2$.*

Proof. Note that $X \in \mathcal{X}_N$, so we have that $\langle a, b:Y \rangle_X^\wedge = \langle a, b:Y \rangle_X$ is large. From the bounded geodesic image theorem of [MaM2] (see Lemma 11.4 of [Bo1]), it follows that $\sigma_Y^\wedge(a, b)$ is bounded. If $Y \in \mathcal{X}_N$, $\sigma_Y(a, b) = \sigma_Y^\wedge(a, b)$ and we are done. So suppose that $Y = X(\gamma)$ for some curve γ . If $\sigma_Y(a, b)$ is large, then we must have $\gamma \in \hat{a} \cup \hat{b}$, and so we suppose $\gamma \in \hat{a} \subseteq \bar{a}$. But then $\langle a, b:Y \rangle_X \leq \sigma_X(a, Y)$ is bounded, giving a contradiction. \square

Next, we need to define maps, $\psi_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{R}(X)$ for $X \in \mathcal{X}$. As noted in Section 2.2, we already have corresponding maps $\psi_X^\wedge : \text{Map}(\Sigma) \rightarrow \mathcal{M}(X)$. Suppose, first that $X \in \mathcal{X}_N$, and that $a \in \mathcal{R}^0(\Sigma)$. Write $\tau = \{\gamma \in \hat{a} \mid \gamma \prec X\}$. As noted in Section 2.2, we can suppose that $\tau \subseteq \psi_X^\wedge \bar{a}$. Now let $\psi_X a$ be the decorated marking with decorations

determined by $\eta_a|_\tau$. It is easily seen that ψ_X is uniformly coarsely lipschitz, so we also get a map $\psi_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{R}(X)$. Note also that $\chi_X \circ \psi_X \sim \theta_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{H}(X)$. (Recall that χ_X just selects a curve from the marking.) In the case where $X \in \mathcal{X}_A$, we set $\mathcal{R}(X) = \mathcal{H}(X)$ and $\psi_X = \theta_X$. In this case, we set χ_X to be the identity map, (so trivially, $\chi_X \circ \psi_X = \theta_X$). Note that we can perform the above constructions intrinsically to any $X \in \mathcal{X}$; so if $Y \preceq X$, we get maps $\psi_{YX} : \mathcal{R}(X) \rightarrow \mathcal{R}(Y)$, with $\chi_Y \circ \psi_{YX} \sim \theta_Y$. It is also immediate from the construction (and corresponding fact in $\mathcal{M}(\Sigma)$) that if $Z \preceq Y \preceq X$, the $\psi_{ZY} \circ \psi_{YX} \sim \psi_{ZX}$.

We are now in the set-up described Section 7 of [Bo4]. In the notation of that paper, we have $\Lambda(X) = \mathcal{R}(X)$ and $\Theta(X) = \mathcal{H}(X)$. The maps $\chi_X : \Lambda(X) \rightarrow \Theta(X)$ and $\psi_{YX} : \Lambda(X) \rightarrow \Lambda(Y)$ for $Y \preceq X$ are as defined above. In that paper, we defined $\theta_Y = \theta_{YX} : \Lambda(X) \rightarrow \Theta(Y)$ by $\theta_Y = \chi_Y \circ \psi_{YX}$, but that agrees with the above up to bounded distance (depending only on $\xi(\Sigma)$) which is all we need.

We need to verify properties (A1)–(A10) of [Bo4], most of which we have already done.

Property (A1) says that $\mathcal{H}(X)$ is Gromov hyperbolic. Properties (A2) and (A3) say respectively that χ_X and ψ_{YX} are uniformly coarsely lipschitz, and Property (A4) asserts that $\psi_{ZY} \circ \psi_{YX} \sim \psi_{ZX}$ whenever $Z \preceq Y \preceq X$. All of which, we have already observed. Property (A5) states that if $Y, Z \preceq X$ with either $Y \prec Z$ or $Y \pitchfork Z$, then $\sigma_X(\theta_X Y, \theta_X Z)$ is uniformly bounded, which is an elementary general property of subsurface projection (given that in this case, $X \in \mathcal{X}_N$). Properties (A6), (A7), (A8) and (A9) are respectively Lemmas 4.2, 4.3, 4.5 and 4.4 here. Finally property (A10) is an immediate consequence of the following:

Lemma 4.6. *There is some $r_0 \geq 0$, depending only on $\xi(\Sigma)$ with the following property. Suppose that $\mathcal{Y} \subseteq \mathcal{X}$ is a collection of pairwise disjoint subsurfaces, and to each $Y \in \mathcal{Y}$, we have associated some element, $a_Y \in \mathcal{R}(Y)$. Then there is some $a \in \mathcal{R}(\Sigma)$ with $\rho(a_Y, \psi_Y a) \leq r_0$ and with $\sigma_Z(a, Y) \leq r_0$ for all $Z \in \mathcal{X}$ satisfying $Z \pitchfork X$.*

Proof. Despite the somewhat technical statement, this is really an elementary observation about combining decorated markings. It was verified in [Bo4] that the corresponding statement holds in $\mathcal{M}(\Sigma)$. In particular, we can find a marking $\bar{a} \in \text{Map}(\Sigma)$ with $\iota(\bar{a}, \tau)$ bounded, and with $\psi_Y^\wedge \bar{a} \sim \bar{a}_Y$ for all $Y \in \mathcal{Y}$. Here, τ is the union of all relative boundary components of elements of \mathcal{Y} . In fact, we can assume that \bar{a} contains each \bar{a}_Y for $Y \in \mathcal{Y} \cap \mathcal{X}_N$ and the core curve of each element of

$\mathcal{Y} \cap \mathcal{X}_A$. Note that $\bigcup_Y \hat{a}_Y$ together with all these core curves form a multicurve in Σ , and we assign the prescribed markings to its components. We set all other markings equal to 0. It is now easily verified that the resulting decorated marking, a , has the required properties. \square

We have now verified the hypotheses of Theorem 7.2 of [Bo4]. We deduce:

Theorem 4.7. *There is a ternary operation, μ_X , defined on each space $\mathcal{R}(X)$ such that $(\mathcal{R}(X), \rho_X, \mu_X)$ is a coarse median space, and such that the maps $\theta_{YX} : \mathcal{R}(X) \rightarrow \mathcal{H}(Y)$ for $Y \preceq X$ are all median quasimorphisms. The median μ_X is unique with this property, up to bounded distance. The maps $\psi_{YX} : \mathcal{R} \rightarrow \mathcal{R}(Y)$ for $Y \preceq X$ are also median quasimorphisms. The coarse median space $(\mathcal{R}(X), \rho_X, \mu_X)$ is finitely colourable, and has rank at most $\xi(X)$.*

In particular, this implies Theorem 1.1.

For future reference (see Section 7), we note the following variation, due to Rafi, the distance formula of Masur and Minsky, mentioned in Section 2.

Given $a, b \in \mathcal{R}(\Sigma)$ and $r \geq 0$, let $\mathcal{A}(a, b; r) = \{X \in \mathcal{X} \mid \sigma_X(a, b) > r\}$. We have noted that this is finite. We have:

Proposition 4.8. *There is some $r_0 \geq 0$ depending only on $\xi(\Sigma)$ such that for all $r \geq r_0$, then for all $a, b \in \mathcal{R}(\Sigma)$, $\rho(a, b) \asymp \sum_{X \in \mathcal{A}(a, b; r)} \sigma_X(a, b)$.*

The corresponding statement for Teichmüller space is proven in [R]. Its reinterpretation for the augmented marking complex is used in [D1]. Given that this is (equivariantly) quasi-isometric to $\mathcal{R}(\Sigma)$, the statement can be easily deduced.

We remark that this is the only input that is not derived directly from the defining properties of $\mathcal{R}(\Sigma)$. (Though it can probably be deduced with some work.)

We can apply these results to the extended asymptotic cone.

We write $\mathcal{R}^\infty(\Sigma) \subseteq \mathcal{R}^*(\Sigma)$ for the asymptotic cone and extended asymptotic cone of $\mathcal{R}(\Sigma)$, and $\mathcal{R}^\infty(\Sigma)$ for some component thereof. We similarly have spaces $\mathcal{G}^\infty(\Sigma) \subseteq \mathcal{G}^*(\Sigma)$ etc.

For the remainder of this section, we will refer to an element of the ultraproduct, $\mathcal{UG}^0(\Sigma)$, as a *curve* and to an element of $\mathcal{G}(\Sigma)$ as a *standard curve*. We will similarly refer to elements of \mathcal{UX} and \mathcal{X} as *subsurfaces* and *standard subsurfaces* respectively. We apply similar terminology to multicurves etc.

Note that $\mathcal{U} \text{Map}(\Sigma)$ acts by isometry of $\mathcal{R}^*(\Sigma)$, and $\mathcal{U}^0 \text{Map}(\Sigma)$ acts on $\mathcal{R}^\infty(\Sigma)$. Unlike the case of the marking graph, however, these spaces are not homogeneous (as we will see in Section 6).

In fact, we have a $\mathcal{U}\text{Map}(\Sigma)$ -invariant *thick part*, \mathcal{R}_T^* , of \mathcal{R}^* . We say that a component, $\mathcal{R}^\infty(\Sigma)$, of $\mathcal{R}^*(\Sigma)$ is *thick* if $\mathcal{R}^\infty(\Sigma) \cap \mathcal{R}_T^*(\Sigma) \neq \emptyset$, in which case, we denote $\mathcal{R}^\infty(\Sigma) \cap \mathcal{R}_T^*(\Sigma)$ by $\mathcal{R}_T^\infty(\Sigma)$. In fact, $\mathcal{U}\text{Map}(\Sigma)$ acts transitively on $\mathcal{R}_T^*(\Sigma)$. Note that $\mathcal{R}_T^*(\Sigma)$ contains the standard basepoint of $\mathcal{R}^*(\Sigma)$. It follows that every thick component of $\mathcal{R}^*(\Sigma)$ is a $\mathcal{U}\text{Map}(\Sigma)$ -image of the standard component.

The various coarsely lipschitz quasimorphisms we described in Section 4 now give rise to lipschitz median homomorphisms. Specifically, we have maps: $\theta_X^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{G}^*(X)$, and $\psi_X^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{R}^*(X)$ for all $X \in \mathcal{UX}$.

5. SOME APPLICATIONS OF THE COARSE MEDIAN PROPERTY

In this section, we describe a few immediate consequences of what we have shown. In particular, we will give proofs of Theorems 1.2, 1.3, and 1.8, and the “if” part of Theorem 1.4. Theorems 1.2, 1.8 and the “only if” part of 1.3 are simple consequences of the preceding discussion, so we discuss these first.

The first result is a coarse isoperimetric inequality. This is a quasi-isometrically invariant property which one could equivalently formulate in a number of different ways. For definiteness we will say that a geodesic space, Λ , satisfies a *coarse quadratic isoperimetric inequality* if the following holds. There is some $r \geq 0$ and some $k \geq 0$ such that if γ is any closed path in Λ of length at most nr , where $n \geq 1$ is some natural number, then we can find a triangulation of the disc with at most kn^2 2-simplices and a map of its 1-skeleton into Λ such that the image of every 1-cell has length at most r and such that the map restricted to the boundary agrees with γ (thought of as a map of the boundary of the disc into Λ). One can easily check that this property is quasi-isometry invariant.

It is shown in [Bo1] (Proposition 8.2) that any coarse median space has this property. We immediately deduce from Theorem 1.1 that:

Proposition 5.1. *$\mathcal{R}(\Sigma)$ has a coarse quadratic isoperimetric inequality.*

This is, of course, equivalent to Theorem 1.2.

We next prove Theorem 1.8. Recall that it is shown in [Bo2] that any asymptotic cone of a finitely colourable coarse median space embeds as a closed subset of a finite product of \mathbb{R} -trees. The embedding can be taken to be a bilipschitz median homomorphism. The image of the embedding is a median metric space, in which all intervals are compact. One can go on to show that this is in turn bilipschitz equivalent to a CAT(0) metric [Bo3]. Note that, in view of Theorem 4.1, this applies in

particular to any asymptotic cone of $\mathcal{R}(\Sigma)$, hence also any asymptotic cone of $\mathbb{T}(\Sigma)$. This proves Theorem 1.8.

Some other consequences follow on from the fact that $\mathcal{R}_T^*(\Sigma)$ is a locally convex topological median algebra of finite rank. For example, the topological dimension of any locally compact subset of any asymptotic cone of $\mathcal{R}_T^\infty(\Sigma)$ is at most $\xi(\Sigma)$ (see Theorem 2.2 and Lemma 7.6 of [Bo1]). In particular, it does not admit any continuous injective map of $\mathbb{R}^{\xi+1}$. From this we get the following. (A similar statement can be found in [EMR1].) Write B_R^n for the ball of radius r in the euclidean space \mathbb{R}^n .

Proposition 5.2. *Given parameters of quasi-isometry, there is some constant $r \geq 0$, such that there is no quasi-isometric embedding of $B_r^{\xi+1}$ into $\mathcal{R}(\Sigma)$ with these parameters.*

Proof. This is a standard argument involving asymptotic cones. Suppose that, for each $i \in \mathbb{N}$, the ball, B_i , of radius i admits a uniformly quasi-isometric embedding, $\phi_i : B_i \rightarrow \mathcal{R}(\Sigma)$. Now pass to the asymptotic cone with scaling factors, i . We end up with a bilipschitz map, $\phi^\infty : B_1 \rightarrow \mathcal{R}^\infty(\Sigma)$, contradicting the dimension bound. \square

(Indeed, the above holds in any coarse median space of rank at most ξ .)

An immediate consequence is that $\mathcal{R}(\Sigma)$ does not admit any quasi-isometric embedding of a euclidean $(\xi(\Sigma) + 1)$ -dimensional half-space. This proves the “only if” part of Theorem 1.3.

For the “if” part, we need to construct such an embedding in dimension $\xi(\Sigma)$. We use the same construction as in [EMR1], though base the proof on the arguments here. This will show, in addition, that the image can be assumed quasiconvex in the coarse median structure.

Given any $a \in \mathcal{M}^0(\Sigma)$, and any $t \subseteq a$, let $O(t) = O_a(t) = \{b \in \mathcal{R}^0(\Sigma) \mid \bar{b} = a, \hat{a} \subseteq t\} \subseteq \mathcal{R}(\Sigma)$. In other words, we take all possible decorations on a subject to the constraint that all the decorated curves must lie in t .

Lemma 5.3. *$O(t)$ is quasiconvex in $\mathcal{R}(\Sigma)$.*

Proof. We define a map $\omega : \mathcal{R}(\Sigma) \rightarrow O(t)$ as follows. Given $x \in \mathcal{R}^0(\Sigma)$, let $\tau = \hat{x} \cap t \subseteq a$. Let $b \in \mathcal{R}^0$ be such that $\bar{b} = a$, $\eta_b|_\tau = \eta_x|_\tau$ and $\eta_b|_{(\bar{b} \setminus \tau)} \equiv 0$, and set $\omega(x) = b$. In other words, we take the base marking a , and decorate curves in a if and only if they also happen to be decorated curves of x . We claim that ω is a coarse gate map.

To this end, let $c \in O(t)$, so $\hat{c} \subseteq a \cap \hat{x}$. If $\gamma \in \tau$, then $\theta_\gamma \omega(x) \sim \theta_\gamma x$, so $\theta_\gamma \mu(x, \omega(x), c) \sim \theta_\gamma \omega(x)$. If $\gamma \in \hat{c} \setminus \tau$, then $\gamma \notin \hat{x}$, so $\theta_\gamma \omega(x) \sim \theta_\gamma a \sim \theta_\gamma c$

(since $\bar{c} = a$), and $\theta_\gamma \mu(x, \omega(x), c) \sim \theta_\gamma \omega(x)$. Finally, if $X \in \mathcal{X}$ does not have the form $X(\gamma)$ for such γ , then $\theta_X c \sim \theta_X a \sim \theta_X \omega(x)$, so $\theta_X \mu(x, \omega(x), c) \sim \theta_X \omega(x)$. By Lemma 4.3, $\mu(x, \omega(x), c) \sim \omega(x)$ as claimed. \square

Given a multicurve $\tau \in \mathcal{S}$, write $\mathcal{O}(\tau) = [0, \infty)^\tau$ and $\mathcal{O}^0(\tau) = \mathbb{N}^\tau \subseteq \mathcal{O}(\tau)$. Note that $\mathcal{O}(\tau)$ is a median algebra with the product structure, and that $\mathcal{O}^0(\tau)$ is a subalgebra.

Suppose $a \in \mathcal{M}^0 \equiv \mathcal{R}_T^0$, with $\tau \subseteq a$. Define a map $\lambda = \lambda_a : \mathcal{O}^0(\tau) \rightarrow \mathcal{R}(\Sigma)$ by setting $\lambda_a(v) = b$ where $\bar{b} = \bar{a}$, $\hat{b} = \tau$, $\eta_a|_\tau = v$, and $\eta_a|(a \setminus \tau) \equiv 0$. (In other words, we take the base marking a , with decorations determined by v .) The map λ is easily seen to be a quasimorphism. Note that $O(\tau) = O_a(\tau) = \lambda(\mathcal{O}^0(\tau))$. Also, if $x \in \mathcal{R}(\Sigma)$, then $\omega(x) = \lambda(u)$, where $u \in \mathcal{O}(\tau)$ is defined by $u|(\tau \cap \hat{x}) = \eta_x|(\tau \cap \hat{x})$ and $u|(\tau \setminus \hat{x}) \equiv 0$.

By Lemma 5.3, $O(\tau)$ is quasiconvex in $\mathcal{R}(\Sigma)$. Moreover, it follows that λ extends to a quasi-isometric embedding of $\mathcal{O}(\tau)$ into $\mathcal{R}(\Sigma)$.

We have shown that:

Lemma 5.4. *The map $\lambda_a : \mathcal{O}(\tau) \rightarrow \mathcal{R}(\Sigma)$ is a quasi-isometric embedding with quasiconvex image.*

If we take τ to be any complete multicurve and a to be any marking containing it, then we get a quasi-isometric embedding of a ξ -orthant (or ξ -dimensional half-space).

This proves the ‘‘if’’ part of Theorem 1.3.

Note that this shows that $\mathcal{R}(\Sigma)$ has coarse median rank exactly $\xi(\Sigma)$ (given the observation after Proposition 5.2).

For the ‘‘if’’ part of Theorem 1.4, we want to construct a quasi-isometric embedding of \mathbb{R}^ξ , in the cases described. (We will deal with the ‘‘only if’’ part at the end of Section 8.)

Suppose that Υ is a finite simplicial complex. We can construct a singular euclidean space, $\mathcal{O}(\Upsilon)$, by taking an orthant for every simplex of Υ and gluing them together in the pattern determined by Υ . This has vertex o and we can identify Υ as the spherical link of o in $\mathcal{O}(\Upsilon)$. (For example, the cross polytope gives a copy of euclidean space.) Note that if Υ is bilipschitz equivalent to the standard $(n - 1)$ -sphere, then $\mathcal{O}(\Upsilon)$ is bilipschitz equivalent to \mathbb{R}^n . We write $\mathcal{O}^0(\Upsilon)$ for the set of integer points in $\mathcal{O}(\Upsilon)$.

Now let $\mathcal{C}(\Sigma)$ be the curve complex associated to Σ . This is the flag complex with 1-skeleton $\mathcal{G}(\Sigma)$. In particular, $\mathcal{C}^0(\Sigma) = \mathcal{G}^0(\Sigma)$. We can identify the set of simplices of $\mathcal{C}(\Sigma)$ with the set, $\mathcal{S} \setminus \{\emptyset\}$, of non-empty multicurves in Σ .

Given a finite subcomplex, Υ , of $\mathcal{C}(\Sigma)$, write $\mathcal{S}(\Upsilon) \subseteq \mathcal{S}$ for the set of multicurves corresponding to the simplices of Υ . Write $\Upsilon^0 = \bigcup \mathcal{S}(\Upsilon)$ for the set of vertices. If $\tau \in \mathcal{S}$, we can identify $\mathcal{O}(\tau)$ as a subset of $\mathcal{O}(\Upsilon)$, and $\mathcal{O}^0(\tau)$ as a subset of $\mathcal{O}^0(\Upsilon)$.

Suppose $a \in \mathcal{M}^0$ is marking of Σ with $\Upsilon^0 \subseteq a$. We can define a map $\lambda = \lambda_a : \mathcal{O}^0(\Upsilon) \rightarrow \mathcal{R}(\Sigma)$ by combining the maps $\lambda_a : \mathcal{O}^0(\tau) \rightarrow \mathcal{R}(\Sigma)$ for $\tau \in \mathcal{S}$. Write $O(\Upsilon) = \lambda(\mathcal{O}(\Upsilon)) = \bigcup_{\tau \in \mathcal{S}(\Upsilon)} O(\tau)$.

Suppose now that Υ is a full subcomplex of $\mathcal{C}(\Sigma)$ (in other words, if the vertices of a simplex in $\mathcal{C}(\Sigma)$ are contained in Υ^0 , then the whole simplex is contained in Υ). In this case, we have $O(\Upsilon) = O(\Upsilon^0)$ as previously defined. In particular, Lemma 5.3 tells us that $O(\Upsilon)$ is coarsely convex.

As in the case of a single orthant, we now see:

Lemma 5.5. *If Υ is a full subcomplex of $\mathcal{C}(\Sigma)$, with $\Upsilon^0 \subseteq a \subseteq \mathcal{M}^0$, then $O(\Upsilon)$ is quasiconvex in $\mathcal{R}(\Sigma)$.*

We see that λ_a extends to a quasi-isometric embedding of $\mathcal{O}(\Upsilon)$ into $\mathcal{R}(\Sigma)$. (In fact, one can show that λ_a is a quasi-isometric embedding even if we do not assume that Υ is full, though of course, its image need not be quasiconvex in this case.)

Note that if Υ is PL homeomorphic to the standard $(n-1)$ -sphere, we get a bilipschitz embedding of \mathbb{R}^n into $\mathcal{R}(\Sigma)$.

We will show:

Proposition 5.6. *$\mathcal{C}(\Sigma)$ contains a full subcomplex PL homeomorphic to the standard $(\xi(\Sigma) - 1)$ -sphere if and only if Σ has genus at most 1, or is the closed surface of genus 2.*

This is based on results and constructions in [Har]. (For further related discussion, see [Br].) If $\Sigma \cong S_{g,p}$, write $\xi'(\Sigma) = 2g + p - 2$ if $g, p > 0$, $\xi'(\Sigma) = 2g - 1$ if $p = 0$ and $\xi'(\Sigma) = p - 3$ if $g = 0$. (Here, we are assuming that $\xi(\Sigma) \geq 2$.)

Note that $\xi'(\Sigma) \leq \xi(\Sigma)$ with equality precisely in the cases described by Proposition 5.6.

It is shown in [Har] (Theorem 3.5) that $\mathcal{C}(\Sigma)$ is homotopy equivalent to a wedge of spheres of dimension $\xi'(\Sigma) - 1$. In particular, the homology is trivial in dimension $\xi'(\Sigma)$. (It does not matter which homology theory we use here.) Now $\mathcal{C}(\Sigma)$ has dimension $\xi(\Sigma)$, so if $\xi'(\Sigma) < \xi(\Sigma)$, it follows that $\mathcal{C}(\Sigma)$ cannot contain any $(\xi(\Sigma) - 1)$ -dimensional homology cycle, and so in particular, no topologically embedded $(\xi(\Sigma) - 1)$ -manifold. This proves the “only if” part of Proposition 5.6.

For the “if” part, we need to construct such a sphere. In the planar (genus-0) case there is a simple explicit construction described in [AL],

which involves doubling the arc complex of a disc. The latter is known to be homeomorphic to a sphere, see [S]. (Another proof of this is given in [HuM]. Although not explicitly stated, it is easily checked that this gives a PL sphere.) However, it is unclear how to adapt this to the genus-1 case. Below, we give an argument which deals with all cases. It is based on an idea in Harer's proof of the result mentioned above. We first show:

Lemma 5.7. *Suppose $g, p, n \in \mathbb{N}$ and $p \geq 2$. Suppose that $\mathcal{C}(S_{g,p})$ contains a full subcomplex PL homeomorphic to the standard n -sphere. Then $\mathcal{C}(S_{g,p+1})$ contains a full subcomplex PL homeomorphic to the standard $(n+1)$ -sphere.*

Proof. For this discussion, it will be convenient to view $S_{g,p}$ as a closed surface, S , of genus g , with a set $A \subseteq S$ of p preferred points. Let $\Upsilon \subseteq \mathcal{C}(S_{g,p})$ be a full subcomplex PL homeomorphic to the standard n -sphere. Now realise the elements of Υ^0 as closed curves in $S \setminus A$, so that no three curves intersect in a point, and such that the total number of intersections is minimal, subject to this constraint. (It is well known that this necessarily minimises pairwise intersection numbers.)

Now let $I \subseteq S$ be an embedded arc meeting A precisely at its endpoints, a, b , say. We may assume that no point of I lies in two curves of Υ^0 and that (subject to this constraint) the total number, m , of intersections, $I \cap \bigcup \Upsilon^0 \subseteq \Sigma$, is minimal, in the homotopy class of I in $S \setminus A$ relative to its endpoints. Let I_0, I_1, \dots, I_m , be the components of $I \setminus (\{a, b\} \cup \bigcup \Upsilon^0)$, consecutively ordered, so that a and b lie respectively in the closures of I_0 and I_m .

Choose any point $c_i \in I_i$ and an arbitrary point $d \in I \setminus \{a, b\}$. Let $B = A \cup \{d\}$, and think of $S_{g,p+1}$ as S with the points of B removed.

The following can be thought of intuitively as sliding the point d from a to b along I . However, formally it is better expressed as keeping d fixed and applying an isotopy to the curves, as we now describe.

Given any $i \in \{0, \dots, m\}$, we obtain a map $f_i : \Upsilon \rightarrow \mathcal{C}(S_{g,p+1})$ as follows. Take an isotopy of S supported on a small neighbourhood of the interval $[d, c_i]$, fixing I setwise, and carrying c_i to d . At the end of the isotopy we get a map sending each curve in Υ^0 to a curve in $S \setminus B$, and so gives rise to a map $f_i : \Upsilon^0 \rightarrow \mathcal{C}(S_{g,p+1})$. Note that postcomposing with the map which forgets the point d , we get the inclusion of Υ^0 into $\mathcal{C}(S_{g,p})$. Now it is easily seen that f_i preserves disjointness of curves, and so extends to a map $f_i : \Upsilon \rightarrow \mathcal{C}(S_{g,p+1})$, which maps Υ isomorphically to a full subcomplex $\Upsilon_i = f_i(\Upsilon) \subseteq \mathcal{C}(S_{g,p+1})$. Now let α and β be, respectively, the boundary curves of small regular neighbourhoods of $[a, d]$ and $[d, b]$ in I . Let $\Omega^0 = \{\alpha, \beta\} \cup \bigcup_{i=0}^m \Upsilon_i^0$, and let

Ω be the full subcomplex of $\mathcal{C}(\Sigma)$ with vertex set Ω_0 . We claim that Ω is PL homeomorphic to the standard $(n + 1)$ -sphere.

Note first that Υ_0^0 and Υ_m^0 are respectively the sets of points adjacent to α and β .

Now, given $i \in \{0, \dots, m\}$, let $\Omega_i^0 = \{\alpha\} \cup \bigcup_{j=0}^i \Upsilon_j^0$, and let Ω_i be the full subcomplex with vertex set Ω_i^0 . Now Ω_0 is the cone on Υ_0 with vertex α , and so PL homeomorphic to a ball with boundary Υ_0 . We claim that, for all i , there is a PL homeomorphism of Ω_0 to Ω_i whose restriction to Υ_0 is the map $f_i \circ f_0^{-1} : \Upsilon_0 \rightarrow \Upsilon_m$.

Suppose, inductively, that this holds for i . Moving from Υ_i to Υ_{i+1} corresponds to pushing one of the curves of Υ_i^0 across the hole, d , of $S_{g,p+1}$. In other words, there is some $\gamma \in \Upsilon^0$ such that $f_i|_{(\Upsilon^0 \setminus \{\gamma\})} = f_{i+1}|_{(\Upsilon^0 \setminus \{\gamma\})}$, but with $\delta = f_i(\gamma) \neq \epsilon = f_{i+1}(\gamma)$. Thus, $\Omega_{i+1}^0 = \Omega_i^0 \cup \{\epsilon\}$. Now δ and ϵ are clearly adjacent. In fact, it is easily checked that the set of curves in Ω_i adjacent to ϵ are precisely those of the form $f_i(\zeta)$, where $\zeta \in \Upsilon^0$ is equal to or adjacent to γ . Thus, Ω_{i+1} is obtained from Ω_i by attaching a cone with vertex ϵ to the star of δ in Υ_i . Given that Ω_i is a PL $(n + 1)$ -ball with boundary Υ_i , we can find a PL homeomorphism of Ω_i to Ω_{i+1} which restricts to $f_{i+1} \circ f_i^{-1}$ on Υ_i . Precomposing this with the homeomorphism from Ω_0 to Ω_i proves the inductive step of the statement.

In particular, we have a PL homeomorphism from Ω_0 to Ω_m whose restriction to Υ_0 is $f_m \circ f_0^{-1}$. Now Ω is obtained from Ω_m by coning the boundary, Υ_m , with vertex β . Thus, Ω is PL homeomorphic to a suspension of Υ , hence a PL $(n + 1)$ -sphere. \square

Now it is easy to find a pentagon which is a full subcomplex of $\mathcal{C}(S_{0,5})$ or equivalently in $\mathcal{C}(S_{1,2})$. Note that these cases correspond to $\xi = \xi' = 2$. Therefore, applying Lemma 5.7 inductively, we find a $(\xi - 1)$ -sphere for all surfaces of genus at most 1, where $\xi \geq 2$. Note that $\mathcal{C}(S_{2,0}) \cong \mathcal{C}(S_{0,6})$ so this deals with that case also. (An explicit description of a sphere in the case of $S_{2,0}$ can be found in [Br].)

This completes the proof Proposition 5.6.

To construct our quasi-isometric embedding of \mathbb{R}^ξ into $\mathcal{R}(\Sigma)$ in these cases, we could assume that, in defining the marking complex, we have taken the intersection bounds large enough so that some marking a contains all the vertices of a $(\xi - 1)$ -sphere, Υ , in $\mathcal{C}(\Sigma)$. Now construct $\lambda_a : \mathcal{O}(\Upsilon) \rightarrow \mathcal{R}(\Sigma)$ as above and apply Lemma 5.5. (Alternatively, we could take different marking containing each multicurve of $\mathcal{S}(\Upsilon)$. Note that these markings are all a bounded distance apart. Thus, if $\tau' \subseteq \tau \in \mathcal{S}(\Upsilon)$, then the map of $\mathcal{O}(\tau')$ into $\mathcal{R}(\Sigma)$ agrees up to bounded distance with the restriction of the map of $\mathcal{O}(\Upsilon)$ into $\mathcal{R}(\Sigma)$. Therefore,

these maps again combine to give a quasi-isometric embedding of $\mathcal{O}(\Upsilon)$ into $\mathcal{R}(\Sigma)$.)

This proves the “if” part of Theorem 1.4.

6. PRODUCT STRUCTURE AND STRATIFICATION

In this section, we show how the extended asymptotic cone, $\mathcal{R}^*(\Sigma)$ can be partitioned into “strata” indexed by (ultralimits of) multicurves. The strata have a local product structure, where the factors correspond to extended asymptotic cones of subsurfaces. This structure arises from a kind of coarse stratification of $\mathcal{R}(\Sigma)$. We begin by describing a coarse product structure based on multicurves.

Recall that \mathcal{S} is the set of all multicurves in Σ . We allow $\emptyset \in \mathcal{S}$. If $\tau \in \mathcal{S}$, we will use the following notation (as in [Bo4]). We write $\mathcal{X}_A(\tau) = \{X(\gamma) \mid \gamma \in \tau\}$. We write $\mathcal{X}_N(\tau)$ for the set of components of $\Sigma \setminus \tau$ which are not $S_{0,3}$'s, and set $\mathcal{X}(\tau) = \mathcal{X}_N(\tau) \cup \mathcal{X}_A(\tau)$. We write $\mathcal{X}_T(\tau) = \{Y \in \mathcal{X} \mid Y \pitchfork \tau\}$; that is, there is some $\gamma \in \tau$ with $\gamma \pitchfork Y$ or $\gamma \prec Y$.

Let τ be a (possibly empty) multicurve. Let $L(\tau)$ be the set of $a \in \mathcal{R}(\Sigma)$ such that $\tau \subseteq a$ and τ does not cross \hat{a} (so that $\tau \cup \hat{a}$ is a multicurve). Thus $a \in L(\hat{a})$ for all $a \in \mathcal{R}(\tau)$. Given $r \geq 0$, let $L(\tau; r) = \{a \in \mathcal{R}(\Sigma) \mid \iota(\bar{a}, \tau) \leq r\}$. One can check that for all sufficiently large r (in relation to $\xi(\Sigma)$) we have $\text{hd}(L(\tau), L(\tau; r))$ is bounded. As in Lemma 7.8 of [Bo2], we see that $a \in \mathcal{R}(\Sigma)$ is a bounded distance from $L(\tau)$ if and only if $\sigma_Y(\theta_Y a, \theta_Y \tau)$ is bounded for all $Y \in \mathcal{X}_T(\tau)$.

Let $\mathcal{L}(\tau) = \prod_{X \in \mathcal{X}(\tau)} \mathcal{R}(X)$. We give this the l^1 metric (though any quasi-isometrically equivalent geodesic metric would serve for our purposes). Note that this has a product coarse median structure. Combining the maps $\psi_X : \mathcal{R}(\Sigma) \rightarrow \mathcal{R}(X)$, we get a coarsely lipschitz quasimorphism $\psi_\tau : \mathcal{R}(\Sigma) \rightarrow \mathcal{L}(\tau)$.

In the other direction, Lemma 4.6 gives us a map, $v_\tau : \mathcal{L}(\tau) \rightarrow \mathcal{R}(\Sigma)$, so that $\psi_Y \circ v_\tau$ gives us prescribed element of $\mathcal{R}(Y)$ for each coordinate Y . In other word, $\psi_\tau \circ v_\tau$ is the identity up to bounded distance. Note that v_τ is necessarily a quasimorphism.

We now set $\omega_\tau : v_\tau \circ \psi_\tau : \mathcal{R}(\Sigma) \rightarrow L(\tau)$. It is now an immediate consequence of Lemma 9.3 of [Bo4] that this is a coarse gate map the set $L(\tau)$; that is, $\rho(\omega_\tau x, \mu(x, \omega_\tau(x), c))$ is bounded for all $x \in \mathcal{R}(\Sigma)$ and all $c \in L(\tau)$.

We now apply this to the extended asymptotic cone, $\mathcal{R}^*(\Sigma)$. Recall that we have maps: $\theta_X^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{G}^*(X)$ and $\psi_X^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{R}^*(X)$ for all $X \in \mathcal{UX}$.

If τ is a multicurve, we have $\psi_\tau^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{L}^*(\tau)$ and $v_\tau^* : \mathcal{L}^*(\tau) \rightarrow \mathcal{L}^*(\Sigma)$ with $\psi_\tau^* \circ v_\tau^*$ the identity. We write $\omega_\tau^* = v_\tau^* \circ \psi_\tau^* : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{R}^*(\Sigma)$. Note that $\mathcal{L}^*(\tau)$ is a direct product of the spaces $\mathcal{R}^*(X)$ as X varies in $\mathcal{UX}(\tau)$.

Write $L^*(\tau) = v_\tau^*(\mathcal{L}^*(\tau)) = \omega_\tau^*(\mathcal{R}^*(\Sigma))$. This is also the limit of the sets $(L(\tau_\zeta))_\zeta$. Note that $L^*(\tau)$ is median convex in $\mathcal{R}^*(\Sigma)$, and that $\omega_\tau : \mathcal{R}^*(\Sigma) \rightarrow L^*(\tau)$ is the gate map (that is $\omega_\tau^*(x) \in [x, c]$ for all $x \in \mathcal{R}^*(\Sigma)$ and all $c \in L^*(\tau)$). In particular, $\omega_\tau^*|_{L^*(\tau)}$ is the identity.

Note that if τ is big, then $L^\infty(\tau)$ is the direct product of ξ almost furry \mathbb{R} -trees. (Since in this case each factor of $L(\tau_\zeta)$ is quasi-isometric to either a hyperbolic plane or a horodisc.)

We now move on to describe the stratification.

Let \mathcal{S} be the set of standard multicurves in Σ . We allow $\emptyset \in \mathcal{S}$. Given $\tau \in \mathcal{S}$, let $\Theta(\tau) = \{a \in \mathcal{R}(\Sigma) \mid \hat{a} = \tau\}$. Thus $\Theta(\tau) \subseteq L(\tau)$, and $\Theta(\emptyset) = \mathcal{R}_T(\Sigma)$.

Lemma 6.1. *Given any $\tau, \tau' \in \mathcal{S}$ and any $a \in \Theta(\tau)$ and $b \in \Theta(\tau')$, there is some $c \in \Theta(\tau \cap \tau')$ with $\mu(a, b, c) \sim c$.*

Proof. Let $d \in \mathcal{R}^0(\Sigma)$ be obtained from a by setting $\bar{d} = \bar{a}$, resetting the decorations on $\tau \setminus \tau' \subseteq \hat{a}$ equal to 0, and leaving all other decorations on a unchanged. Thus $d \in \Theta(\tau \cap \tau')$. Now apply Dehn twists to d about the curves of $\tau \setminus \tau'$ to give $c \in \mathcal{R}^0(\Sigma)$, so that $\theta_\gamma c \sim \theta_\gamma b$ for all $\gamma \in \tau \setminus \tau'$. We also have $c \in \Theta(\tau \cap \tau')$. Suppose $X \in \mathcal{X}$. If $X = X(\gamma)$ for some $\gamma \in \tau \setminus \tau'$, then $\theta_X c \sim \theta_X b$. If X is not of this form, we get $\theta_X c \sim \theta_X a$. In all cases, we have $\theta_X \mu(a, b, c) \sim \mu(\theta_X a, \theta_X b, \theta_X c) \sim \theta_X c$. It follows by Lemma 4.3 that $\mu(a, b, c) \sim c$. \square

We now pass to the asymptotic cone. Write \mathcal{US} for the ultraproduct of \mathcal{S} . Recall that we have an intersection operation defined on \mathcal{US} .

Given any $\tau \in \mathcal{US}$, let $\Theta(\tau)$ be the limit of the sets $\Theta^*(\tau)$ be the limit of $(\Theta(\tau_\zeta))_\zeta$. Note that $\Theta^*(\tau)$ is closed, and $\Theta^*(\tau) \subseteq L^*(\tau)$. Also, clearly $\mathcal{R}^*(\Sigma) = \bigcup_{\tau \in \mathcal{US}} \Theta^*(\tau)$.

Lemma 6.2. *Given any $\tau, \tau' \in \mathcal{US}$, and any $a \in \Theta^*(\tau)$, $b \in \Theta^*(\tau')$, then $[a, b] \cap \Theta^*(\tau \cap \tau') \neq \emptyset$.*

Proof. Choose $a_\zeta \in \Theta(\tau)$ and $b_\zeta \in \Theta(\tau')$ with $a_\zeta \rightarrow a$ and $b_\zeta \rightarrow b$. Let $c_\zeta \in \Theta(\tau \cap \tau')$ be as given by Lemma 6.1. Then $c_\zeta \rightarrow c \in \Theta^*(\tau \cap \tau')$ and $\mu(a, b, c) = c$, that is, $c \in [a, b]$. \square

In particular, it follows that $\Theta^*(\tau) \cap \Theta^*(\tau') \subseteq \Theta^*(\tau \cap \tau')$. Therefore, given any $a \in \mathcal{R}^*$, there is a unique minimal $\tau \in \mathcal{US}$ with $a \in \Theta^*(\tau)$. We write $\tau(a) = \tau$. Since the sets $\Theta^*(\tau)$ are all closed, we see that the

map $\tau : \mathcal{R}^*(\Sigma) \rightarrow \mathcal{US}$ is lower semicontinuous. Moreover, given any $a, b \in \mathcal{R}^*(\Sigma)$, there is some $c \in \mathcal{R}^*(\Sigma)$ with $\tau(c) \subseteq \tau(a) \cap \tau(b)$.

More generally, suppose that $C \subseteq \mathcal{R}^*$ is convex. Choose $a \in C$ with $\tau(a)$ minimal. If $b \in C$, then there is some $c \in [a, b] \subseteq C$ with $\tau(c) \subseteq \tau(a) \cap \tau(b)$, so $\tau(a) = \tau(c) \subseteq \tau(b)$. We write $\tau(C) = \tau(a)$. Thus, $\tau(C)$ is uniquely determined by the property that $\tau(C) \subseteq \tau(b)$ for all $b \in C$ and $\tau(C) = \tau(a)$ for some $a \in C$. Note that this applies in particular, if C is a component of \mathcal{R}^* . Note that a component, C , of \mathcal{R}^* is thick (i.e. $C \cap \mathcal{R}_T^*(\Sigma) \neq \emptyset$) if and only if $\tau(C) = \emptyset$.

Now, given $\tau \in \mathcal{US}$, let $\Xi(\tau) = \{a \in \mathcal{R}^*(\Sigma) \mid \tau(a) = \tau\}$. Clearly, $\Xi(\tau) \subseteq \Theta^*(\tau)$ and $\Xi(\emptyset) = \Theta^*(\emptyset) = \mathcal{R}_T^*$. Since $\tau : \mathcal{R}^* \rightarrow \mathcal{US}$ is lower semicontinuous, we have that $\Xi(\tau)$ is open in $\Theta^*(\tau)$. Also:

Lemma 6.3. *For all $\tau \in \mathcal{US}$, $\Xi(\tau)$ is dense in $\Theta^*(\tau)$.*

Proof. Let $a \in \Theta^*(\tau)$, and choose $a_\zeta \in \Theta^*(\tau_\zeta)$, with $a_\zeta \rightarrow a$. Thus $\tau(a_\zeta) \subseteq \tau_\zeta$. Given any $i \in \mathbb{N}$, let $a_{i,\zeta} \in \mathcal{R}$ be the decorated multicurve with $\bar{a}_{i,\zeta} = \bar{a}_\zeta$, and resetting the decoration, $\eta_{\alpha_{i,\zeta}}(\gamma)$, on each $\gamma \in \tau_\zeta$ equal to $\eta_{a_\zeta}(\gamma) + i$. Now $(a_{i,\zeta})_i$ is a (quasi)geodesic sequence in \mathcal{R} with $a_0 = a$ and with $N(a_{i,\zeta}, i) \subseteq \Theta(\tau_\zeta)$ for all i . In fact, $\rho(a_i, \Theta(\tau'_\zeta)) \geq i$ for all $\tau' \neq \tau$.

Passing to the asymptotic the cone, we see that from any $a \in \Theta^*(\tau)$, there is a bilipschitz embedded ray, λ , emanating for which $\lambda(t) \in \Xi(\tau)$ for all $t > 0$. \square

In fact, the argument shows that $\Xi(\tau) \cap C$ is dense in $\Theta^*(\tau) \cap C$ for any component, C of \mathcal{R}^* .

Write $\mathcal{S}_C \subseteq \mathcal{S}$ for the set of standard complete multicurves, and write $\mathcal{US}_C \subseteq \mathcal{US}$ for the set of complete multicurves.

Note that, if $a \in \mathcal{R}(\Sigma)$, then a is a bounded distance from a marking, b , which contains a complete multicurve, $\tau \supseteq \hat{a}$. Thus $\bigcup_{\tau \in \mathcal{S}_C} \Theta(\tau)$ is cobounded in $\mathcal{R}(\Sigma)$. We deduce:

Lemma 6.4. $\mathcal{R}^*(\Sigma) = \bigcup_{\tau \in \mathcal{US}_C} \Theta^*(\tau)$.

From this, we immediately get:

Lemma 6.5. *If $\tau \in \mathcal{US}_C$, then $\Xi(\tau)$ is open in \mathcal{R}^* . Also $\bigcup_{\tau \in \mathcal{US}_C} \Xi(\tau)$ is dense in any component of \mathcal{R}^* .*

In this way, we see that $(\Xi(\tau))_{\tau \in \mathcal{US}}$ defines a stratification of \mathcal{R}^* .

We also note:

Lemma 6.6. *For all $\tau \in \mathcal{US}$, $\Xi(\tau)$ lies in the interior of $L^*(\Sigma)$ in $\mathcal{R}^*(\Sigma)$.*

Proof. For all $a, b \in \Xi(\tau)$, then by lower semicontinuity, there is some open $U \subseteq \mathcal{R}^*$ with $\tau(b) \supseteq \tau$ for all $b \in U$. So $b \in L^*(\tau(b)) \subseteq L^*(\tau)$. This shows that $U \subseteq L^*(\tau)$. \square

Lemma 6.7. *Suppose $\tau \in \mathcal{S}$, $d \in \Theta(\tau)$ and $\tau' \subseteq \tau$. Then there exist $a, b, c \in \Theta(\tau') \cap L(\tau)$ with $d \sim \mu(a, b, c)$. Moreover, we can choose a, b, c so that $\rho(d, a)$, $\rho(d, b)$ and $\rho(d, c)$ are all bounded above by a fixed linear function of $\rho(d, \Theta(\tau'))$.*

Proof. Let $a \in \mathcal{R}$ be obtained from d by resetting all decorations on $\tau \setminus \tau'$ equal to 0. Thus $a \in \Theta(\tau') \cap L(\tau)$. By applying suitable powers of Dehn twists about each curve $\gamma \in \tau \setminus \tau'$, we obtain $b, c \in \Theta(\tau') \cap L(\tau)$, so that $\theta_\gamma d \sim \mu(\theta_\gamma a, \theta_\gamma b, \theta_\gamma c)$ in $\mathcal{H}(\gamma)$, and such that $\sigma_\gamma(d, a) \sim \sigma_\gamma(d, b) \sim \sigma_\gamma(d, c)$. Note that $\theta_X^* a \sim \theta_X^* b \sim \theta_X^* c \sim \theta_X^* d$ for all $X \in \mathcal{X}$ not of the form $X(\gamma)$ for $\gamma \in \tau \setminus \tau'$. It now follows that $\mu(a, b, c) \sim d$, so that $\rho(d, a) \asymp \rho(d, b) \asymp \rho(d, c) \asymp \sum_{\gamma \in \tau \setminus \tau'} \sigma(d, a) \asymp d(a, \Theta(\tau'))$. \square

Lemma 6.8. *Suppose $\tau \in \mathcal{US}$, $d \in \Theta^*(\tau)$ and $\tau' \subseteq \tau$. Then there exist $a, b, c \in \Theta^*(\tau') \cap L^*(\tau)$ with $d = \mu(a, b, c)$. Moreover, $\rho(d, a)$, $\rho(d, b)$ and $\rho(d, c)$ are all bounded above by a fixed multiple of $\rho(d, \Theta^*(\tau))$.*

Proof. Take $d_\zeta \in \Theta(\tau_\zeta)$ and apply Lemma 6.7 to give $a_\zeta, b_\zeta, c_\zeta \in \Theta(\tau'_\zeta) \cap L(\tau_\zeta)$. Let $a_\zeta \rightarrow a$, $b_\zeta \rightarrow b$ and $c_\zeta \rightarrow c$. The additive constants go away, and we get $d = \mu(a, b, c)$. \square

Note in particular, setting $\tau' = \emptyset$, we get that if $d \in \Theta^\infty(\tau) \subseteq \mathcal{R}^\infty$ lies in a thick component, \mathcal{R}^∞ , of \mathcal{R}^* , then we can find $a, b, c \in L^\infty(\tau) \cap \mathcal{R}_T^\infty$ with $d = \mu(a, b, c)$.

In what follows, let $\mathcal{R}^\infty(\Sigma)$ be any component of any extended asymptotic cone, $\mathcal{R}^*(\Sigma)$ of $\mathcal{R}(\Sigma)$.

If $\tau \in \mathcal{S}_B$, the $L^\infty(\tau)$ is a closed convex subset which is a direct product of ξ almost furry \mathbb{R} -trees. (Recall that ‘‘almost furry’’ means non-trivial and with no 2-valent element.) Now it was shown in [Bo2], that any continuous injective map of a direct product of ξ almost furry trees with closed image is necessarily a median isomorphism onto its range, which must be convex. In particular, we see that the median structure on $L^\infty(\tau)$ is determined by its intrinsic topology. We deduce:

Lemma 6.9. *Suppose that $f : \mathcal{R}^\infty(\Sigma) \rightarrow \mathcal{R}^\infty(\Sigma)$ is a homeomorphism and that $\tau \in \mathcal{US}_B$. Then $f|L^\infty(\tau)$ is a median isomorphism onto its range, $L^\infty(\tau)$, which is convex.*

We also get:

Lemma 6.10. *Suppose that \mathcal{R}^∞ is a thick component of \mathcal{R}^* and that $f : \mathcal{R}^\infty \rightarrow \mathcal{R}^\infty$ is a homeomorphism with $f|_{\mathcal{R}_T^\infty}$ the identity. Then f is the identity on \mathcal{R}^∞ .*

Proof. Let $x \in \mathcal{R}^\infty(\Sigma)$. By Lemma 6.4, $x \in \Theta^\infty(\tau)$ for some $\tau \in \mathcal{US}_C$. By Lemma 6.8, we can find $a, b, c \in \mathcal{R}_T^\infty \cap \Theta^\infty(\tau)$ with $x = \mu(a, b, c)$. By Lemma 6.9, $f|L^\infty(\tau)$ is a median homomorphism, and so $fx = x$. \square

7. QUASI-ISOMETRIC MAPS

In this section, we apply the results of Section 6 to quasi-isometries between Teichmüller spaces. In particular, we give proofs of Theorems 1.5, 1.6 and 1.7.

We remark that much of the initial discussion, in particular, Lemmas 7.1–7.4 can be placed in a more general setting, namely that of Section 7 of [Bo4]. Thus, they could be equally well applied to the mapping class group, or Weil-Petersson metrics.

Let M be a topological median algebra. We say that a subset $P \subseteq M$ is *square-free* if there is no square in M with a side contained in P .

Recall that a *gate map* to P is a (necessarily continuous) map $\omega : M \rightarrow P$ such that for all $x \in M$ and $y \in P$ we have $\omega(x) \in [x, y]$. In particular, it follows that P is convex $\omega|_P$ is the identity. (Such a map is unique if it exists.)

Note that if P is square-free and $x, y \in M$ with $[x, y] \cap P = \emptyset$, then $\omega(x) = \omega(y)$ (otherwise, we would have a square $\omega(x), \omega(y), \mu(x, y, \omega(y)), \mu(x, y, \omega(x))$ with side $\{\omega(x), \omega(y)\}$ in P). If M is weakly locally convex, it then follows that $\omega : M \rightarrow P$ is locally constant on $M \setminus P$.

Now suppose $P \subseteq M$ with $\omega : M \rightarrow P$ a locally constant retraction. If $p \in P$ separates $x, y \in P$ in P , then p also separates x, y in M . (Note that $\omega^{-1}(p) \setminus \{p\}$ is open M . Thus, if $P \setminus \{p\} = U \sqcup V$ is an open partition of $P \setminus \{p\}$, then $M \setminus \{p\} = (\omega^{-1}(U \cup \{p\}) \setminus \{p\}) \sqcup \omega^{-1}V$ is an open partition of $M \setminus \{p\}$. Taking U, V so that $x \in U$ and $y \in V$, the claim follows.)

Putting the above observations together, we have shown:

Lemma 7.1. *Suppose that M is a weakly locally convex topological median algebra, and $P \subseteq M$ is closed convex and square-free and admits a gate map. If $p, x, y \in P$, and p separates x from y in P , then it also separates x from y in M .*

In particular, any cut point of P will also be a cut point of M .

In our situation, \mathcal{R}^∞ is certainly (weakly) locally convex, see [Bo1].

We now consider some constructions of such P .

Definition. We say that a set $P \subseteq \mathcal{R}(\Sigma)$ is *coarsely square-free* if given any square, Q , and any quasimorphism $\phi : Q \rightarrow \mathcal{R}(\Sigma)$ which maps some side of Q into P , there is a (possibly different) side, c, d , of P with $\rho(\phi c, \phi d)$ bounded.

(Evidently, this entails implicit constants of quasimorphism and side-length.)

Definition. Given $k \geq 0$ and $P \subseteq \mathcal{R}(\Sigma)$, we say that P is k -straight if $\text{diam } \theta_X(P) \leq k$ for $X \in \mathcal{X} \setminus \{\Sigma\}$. We say P is *coarsely straight* if it is k -straight for some $k \geq 0$.

Note that by the distance formula of Rafi (Proposition 4.8 here), we see that if $a, b \in P$, then $\rho(a, b) \asymp \sigma_\Sigma(a, b)$. (Here \asymp denotes agreement to within linear bounds depending on k .) In other words, the map $\theta_\Sigma : P \rightarrow \mathcal{H}(\Sigma) = \mathcal{G}(\Sigma)$ is a quasi-isometric embedding.

Lemma 7.2. *A coarsely straight set is coarsely square-free.*

Proof. Let $Q = \{a, b, d, c\}$ be a square, with sides, $\{a, b\}$ and $\{a, c\}$, and let $\phi : Q \rightarrow \mathcal{R}$ be a quasimorphism, with $\phi a, \phi b \in P$. Now $\theta_\Sigma \circ \phi : Q \rightarrow \mathcal{H}(\Sigma)$ is also a quasimorphism. If $\rho_\Sigma(a, b) \sim \sigma_\Sigma(a, b)$ is sufficiently large, then an elementary property of hyperbolic spaces (essentially the fact the median is rank-1) tells us that $\sigma_\Sigma(\phi a, \phi c)$ and $\sigma_\Sigma(\phi b, \phi d)$ are both bounded. Suppose $X \in \mathcal{X}$. We claim that $\sigma_X(\phi a, \phi b)$ is bounded. This is because, by the above, we can assume (swapping a and b if necessary) that $\sigma_\Sigma(\phi a, X)$ is large, and so the Gromov product $\langle \theta_\Sigma \phi a, \theta_\Sigma \phi c; \theta_\Sigma Y \rangle_\Sigma$ is large. The claim then follows by Lemma 4.5. Since this holds for all $X \in \mathcal{X}$, it follows by Lemma 4.3 that $\rho(\phi a, \phi c)$ is bounded. In other words, we have shown that either $\rho(\phi a, \phi b)$ or $\rho(\phi a, \phi c)$ is bounded as required. \square

Lemma 7.3. *Suppose that P is coarsely straight. Then P is quasiconvex in $\mathcal{R}(\Sigma)$ if and only if $\theta_\Sigma P$ is coarsely median convex in $\mathcal{G}(\Sigma)$.*

Proof. The “only if” part is an immediate consequence of the fact that $\theta_\Sigma : \mathcal{R}(\Sigma) \rightarrow \mathcal{G}(\Sigma)$ is a quasimorphism. For the converse, suppose $a, b \in P$ and $c \in \mathcal{R}$. By hypothesis, there is some $d \in P$ with $\theta_\Sigma d \sim \mu(\theta_\Sigma a, \theta_\Sigma b, \theta_\Sigma c)$, and so $\theta_\Sigma d \sim \theta_\Sigma \mu(a, b, c)$. If $X \in \mathcal{X} \setminus \{\Sigma\}$, then $\theta_X a \sim \theta_X b \sim \theta_X d \sim \theta_X \mu(a, b, c)$, and so $\theta_X d \sim \mu(\theta_X a, \theta_X b, \theta_X c) \sim \theta_X \mu(a, b, c)$. It follows by Lemma 4.3, that $d \sim \mu(a, b, c)$. In other words, this shows that the coarse interval $[a, b]$ lies in a bounded neighbourhood of P as required. \square

Clearly, if P is quasiconvex and coarsely square free, then P^∞ is closed convex and square-free in \mathcal{R} .

We can construct examples of such P from coarsely straight sequences.

Definition. We say that a bi-infinite sequence, $(a_i)_{i \in \mathbb{Z}}$ in $\mathcal{R}(\Sigma)$ is *coarsely straight* if $\rho(a_i, a_{i+1})$ is bounded above for all i , and if $\sigma_\Sigma(a_i, a_j)$ is bounded below by an increasing linear function of $|i - j|$.

Note that, since the first condition implies also that $\sigma_X(a_i, a_{i+1})$ is bounded above, the second condition is equivalent to saying that the sequence $(\theta_\Sigma a_i)_i$ is quasigeodesic in $\mathcal{G}(\Sigma)$.

Lemma 7.4. *If $(a_i)_i$ is a coarsely straight sequence, then the set $\{a_i \mid i \in \mathbb{Z}\}$ is coarsely straight in $\mathcal{R}(\Sigma)$.*

Proof. Since $(\theta_\Sigma a_i)_i$ is quasigeodesic in $\mathcal{G}(\Sigma)$, we can find $m, n \in \mathbb{Z}$, with $0 \leq n - m$ bounded, and with $\sigma_\Sigma(\theta_\Sigma a_i, \theta_\Sigma(\partial X)) \geq t$ and $\sigma_\Sigma(\theta_\Sigma a_j, \theta_\Sigma(\partial X)) \geq t$ for all $i \leq m$ and all $j \geq n$, where t is an upper bound on $\sigma_X(a_p, a_{p+1})$. (Of course, this might hold for all $i, j \in \mathbb{Z}$.) By Lemma 4.5, it follows that $\sigma_X(a_i, a_m)$ and $\sigma_X(a_j, a_n)$ are bounded for all $i \leq m$ and all $j \geq n$. Also, if $m \leq i \leq j \leq n$, then $\rho(a_i, a_j)$ is bounded, so $\sigma_X(a_i, a_j)$ is bounded. It follows that $\sigma_X(a_i, a_j)$ is bounded for all $i, j \in \mathbb{Z}$ as required. \square

Note that it now follows that $(a_i)_i$ is quasiconvex, and it is also quasigeodesic in $\mathcal{R}(\Sigma)$.

We can also define a coarse gate map to $P = \{a_i \mid i \in \mathbb{Z}\}$. Given $c \in \mathcal{R}$, we can find $n \in \mathbb{Z}$ such that $\mu(\theta_\Sigma a_n, \theta_\Sigma c, \theta_\Sigma a_i) \sim \theta_\Sigma a_n$ for all $i \in \mathbb{Z}$. Then $\theta_\Sigma \mu(a_n, c, a_i) \sim \theta_\Sigma a_n$. Also, for all $X \in \mathcal{X} \setminus \{\Sigma\}$, we have $\theta_X a_i \sim \theta_X a_n$, so $\theta_X \mu(a_n, c, a_i) \sim \theta_X a_n$. It follows that by setting $\omega(c) = a_n$ we obtain a coarse gate map $\omega : \mathcal{R}(\Sigma) \rightarrow P$.

Now P^* is a convex subset of $\mathcal{R}^*(\Sigma)$ median isomorphic to \mathbb{R}^* . Restricting to the standard component, $\mathcal{R}^\infty(\Sigma)$, of $\mathcal{R}^*(\Sigma)$, we see that P^∞ is closed and convex in $\mathcal{R}^\infty(\Sigma)$ and median isomorphic to \mathbb{R} . In particular, it is a bi-lipschitz embedding of \mathbb{R} . Moreover, P^∞ is square-free and admits a gate map, $\omega^\infty : \mathcal{R}^\infty(\Sigma) \rightarrow P^\infty$. It follows that any point of P^∞ is a cut point of $\mathcal{R}^\infty(\Sigma)$.

Finally, note that if $g \in \text{Map}(\Sigma)$ is pseudoanovos, then the $\langle g \rangle$ -orbit of any point of $\mathcal{G}(X)$ is quasigeodesic (see [MaM1]). We see that if $a \in \mathcal{R}(\Sigma)$, then $(g^i a)_i$ is a coarsely straight sequence, hence quasiconvex by the above. This gives rise to a line in $\mathcal{R}^\infty(\Sigma)$ all of whose points are cut points of \mathcal{R}_T^∞ .

Since $\mathcal{U} \text{Map}(\Sigma)$ acts transitively on \mathcal{R}_T^* , we deduce:

Lemma 7.5. *Each point of \mathcal{R}_T^* is a cut point of the component of \mathcal{R}^* in which it lies.*

In fact, we have a converse:

Lemma 7.6. *If $x \in \mathcal{R}^* \setminus \mathcal{R}_T^*$, then x is not a cut point of the component in which it lies.*

Proof. Let $\tau = \tau(x) \in \mathcal{S}$. By assumption, $\tau \neq \emptyset$. By Lemma 6.6, x lies in the interior of $L^\infty(\tau)$ in \mathcal{R}_T^∞ . But $L^\infty(\tau)$ is a median, hence

topological, direct product of at least two non-trivial path-connected spaces, and so any two points lie in an embedded disc. \square

We see that \mathcal{R}_T^* is determined by the topology of \mathcal{R}^* as the set of cut points. We deduce:

Lemma 7.7. *Suppose that Σ, Σ' are compact surfaces, and $f : \mathcal{R}^*(\Sigma) \longrightarrow \mathcal{R}^*(\Sigma')$ is a homeomorphism, then $f(\mathcal{R}_T^*(\Sigma)) = \mathcal{R}_T^*(\Sigma')$.*

(Given that ξ is the locally compact dimension of $\mathcal{R}^\infty(\Sigma)$, if such a homeomorphism exists, then $\xi(\Sigma) = \xi(\Sigma')$.)

Note that the above holds for any extended asymptotic cones, for any choice of scaling factors.

Now suppose that Σ and Σ' are compact surfaces of complexity at least 2 and that $\phi : \mathcal{R}(\Sigma) \longrightarrow \mathcal{R}(\Sigma')$ is a quasi-isometry. This induces a (bilipschitz) homeomorphism, $f = \phi^* : \mathcal{R}^*(\Sigma) \longrightarrow \mathcal{R}^*(\Sigma')$.

Lemma 7.8. *There is some $k \geq 0$ such that $\text{hd}(f(\mathcal{R}_T(\Sigma)), \mathcal{R}_T(\Sigma')) \leq k$.*

Proof. By symmetry, it's enough to show that $f(\mathcal{R}_T(\Sigma))$ lies in a bounded neighbourhood of $\mathcal{R}_T(\Sigma')$. Suppose to the contrary that we can find an \mathbb{N} -sequence, $(x_i)_i$, in $\mathcal{R}(\Sigma)$ with $r_i = \rho(f(x_i), \mathcal{R}_T(\Sigma')) \rightarrow \infty$. Let \mathcal{R}^∞ and $\mathcal{R}^\infty(\Sigma')$ be asymptotic cones with $\mathcal{Z} = \mathbb{N}$, scaling factors $(r_i)_i$ and basepoints $(x_i)_i$ and $(\phi x_i)_i$. We get a homeomorphism, $f = \phi^\infty : \mathcal{R}^\infty(\Sigma) \longrightarrow \mathcal{R}^\infty(\Sigma')$ with $\rho(f(x), \mathcal{R}_T^\infty(\Sigma')) = 1$. But $f(x) \in f(\mathcal{R}_T^\infty(\Sigma)) = \mathcal{R}_T^\infty(\Sigma')$, giving a contradiction. \square

This proves Theorem 1.6.

It now follows that ϕ is a bounded distance from a map from $\mathcal{R}_T(\Sigma)$ to $\mathcal{R}_T(\Sigma')$. Recall (Lemma 3.4) that $\mathcal{R}_T(\Sigma)$ is a uniformly embedded copy of the marking complex $\mathcal{M}(\Sigma)$ of Σ . Thus ϕ gives rise to a quasi-isometry from $\mathcal{M}(\Sigma)$ to $\mathcal{M}(\Sigma')$.

Now, if $\mathcal{M}(\Sigma)$ and $\mathcal{M}(\Sigma')$ are quasi-isometric, then Σ and Σ' are homeomorphic, under the conditions described by Theorem 1.5. This follows using the result of [BeKMM, Ham], and is shown directly in [Bo4].

This proves Theorem 1.5.

Henceforth, we assume that $\Sigma = \Sigma'$, so $\phi : \mathcal{R}(\Sigma) \longrightarrow \mathcal{R}(\Sigma)$ is a quasi-isometry. By [BeKMM, Ham], there is some $g \in \text{Map}(\Sigma)$ such that $\rho(gx, \phi x)$ is bounded for all $x \in \mathcal{R}_T(\Sigma)$. Postcomposing with g^{-1} , we may as well assume that g is the identity, so $\rho(x, \phi x)$ is bounded. Thus, up to bounded distance, we can assume that $\phi|_{\mathcal{R}_T(\Sigma)}$ is the identity.

If we pass to any asymptotic cone, we will now have that $f = \phi^*|\mathcal{R}_T^*$ is the identity. By Lemma 7.7, it follows that f is the identity on any thick component of $\mathcal{R}_T^*(\Sigma)$. From this we get:

Lemma 7.9. *There is a function, $l : [0, \infty) \rightarrow [0, \infty)$ with $l(t)/t \rightarrow 0$ as $t \rightarrow \infty$, such that for all $x \in \mathcal{R}(\Sigma)$, we have $\rho(x, \phi x) \leq l(\rho(x, \mathcal{R}_T(\Sigma)))$.*

Proof. We claim that, for all $\epsilon > 0$, there is some $u = u(\epsilon) \geq 0$, such that if $\rho(x, \mathcal{R}_T(\Sigma)) \geq u$, then $\rho(x, \phi x) \leq \epsilon \rho(x, \mathcal{R}_T)$. For suppose not. We can find a \mathbb{N} -sequence $(r_i)_i$ with $r_i \rightarrow \infty$, $x_i \in \mathcal{R}$, $y_i \in \mathcal{R}_T$ with $\rho(x_i, y_i) \leq r_i$ and $\rho(x_i, \phi x_i) \geq \epsilon r_i$. Passing to an asymptotic cone, with scaling factors $(r_i)_i$ and basepoints x_i and ϕx_i , we get $f : \mathcal{R}^\infty \rightarrow \mathcal{R}^\infty$, and $x \in \mathcal{R}^\infty$, $y \in \mathcal{R}_T^\infty$ with $\rho^\infty(x, y) \leq 1$ and $\rho^\infty(x, f x) \geq \epsilon$. But, by the above, f is the identity on \mathcal{R}^∞ giving a contradiction.

We take $u_n = u(1/n)$ for $n \in \mathbb{N}$, $n \geq 1$. We can suppose that $u_{n+1} \geq u_n$. We take l equal to $2u_1$ on $[0, u_1]$, and equal to $[t \mapsto t/n]$ on $[u_n, u_{n+1}]$ for $n \geq 1$. \square

This now proves Theorem 1.7.

8. CONCLUSION OF THE PROOF OF THEOREM 1.4

In this section, we prove the “only if” part of Theorem 1.4. Before starting, we make a few general observations.

We first note that the results of Sections 10–12 of [Bo4] apply here (with $\Lambda(X) = \mathcal{R}(X)$ and with $\Theta(X) = \mathcal{H}(X)$). These only depend on Properties (A1)–(A10) and Property (B1) as laid on in Section 7 of that paper, and we have verified these for $\mathcal{R}(X)$ in Section 4 of the present paper. In particular, the general discussion of “quasicubes” in $\mathcal{R}(\Sigma)$, as well as cubes in $\mathcal{R}^*(\Sigma)$ applies here. For the purposes of this paper we just note the following.

Let $Q \subseteq \mathcal{R}^*(\Sigma)$ be an n -cube. If c, d and c', d' are both i th sides of Q , then the intervals $[c, d]$ and $[c', d']$ are parallel. That is, the maps $[x \mapsto \mu(c', d', x)]$ and $[x \mapsto \mu(c, d, x)]$ are inverse median isomorphisms between $[c, d]$ and $[c', d']$. Now if $x, y \in [c, d]$, let $x' = \mu(c', d', x)$ and $y' = \mu(c', d', y)$. Given $X \in \mathcal{UX}$, and $\theta_X^* x = \theta_X^* y$, then $\theta_X^* x' = \theta_X^* y'$. We see that if $\theta_X^*|[c', d']$ is injective, then so is $\theta_X^*|[c, d]$ and conversely by symmetry. The same applies to ψ_X^* . Thus $D([c, d]) = D([c', d'])$, so we can write this as $D_i(Q)$. We write $D_i^0(Q) = D_i(Q) \cap \mathcal{UG}^0(\Sigma)$, which we identify with the set of curves $\gamma \in \mathcal{UG}^0(\Sigma)$ such that $\theta_\gamma^*|[c, d]$ is injective.

We note that if $n = \xi$, and c, d is an i th side of Q , the $[c, d]$ is a rank-1 median algebra (that is, a totally ordered set). In fact, it is median isomorphic to an interval in \mathbb{R}^* . Moreover, the convex hull, $\text{hull}(Q)$, of

Q is a median direct product of such intervals. (In particular, if $Q \subseteq \mathcal{R}^\infty(\Sigma)$, then $\text{hull}(Q)$ is median isomorphic to the real cube $[-1, 1]^\xi$.)

The following is now just a restatement of Proposition 12.6 of [Bo4].

Proposition 8.1. *Let $Q \subseteq \mathcal{R}^*(\Sigma)$ be a $\xi(\Sigma)$ -cube. For any i , the set $D_i^0(Q)$ is either empty or consists of a single curve $\gamma_i \in \mathcal{UG}^0$. If it is empty, then there is a unique complexity-1 subsurface $Y_i \in D_i(Q)$. If the γ_i are all disjoint, and they form a big multicurve $\tau(Q)$. The Y_i are also disjoint, and are precisely the complexity-1 components of $\tau(Q)$.*

We note, in particular, that γ_i or Y_i is completely determined by any face of Q , without reference to Q itself.

For the proof of Theorem 1.4, we also note the following:

Lemma 8.2. *Suppose that $a, b, c \in \mathcal{R}^*(\Sigma)$, with $c \in [a, b] \setminus \{a, b\}$, and with $\theta_\gamma^* a$, $\theta_\gamma^* b$ and $\theta_\gamma^* c$ all distinct. Then $c \notin \mathcal{R}_T^*(\Sigma)$.*

Proof. Take $\gamma_\zeta \in \mathcal{G}^0(\Sigma)$ with $\gamma_\zeta \rightarrow \gamma$ and $a_\zeta, b_\zeta, c_\zeta \in \mathcal{R}(\Sigma)$ with $a_\zeta \rightarrow a$, $b_\zeta \rightarrow b$ and $c_\zeta \rightarrow c$. If $c \in \mathcal{R}_T^\infty$, then we could also take $c_\zeta \in \mathcal{R}_T$. Let $d_\zeta = \mu(a_\zeta, b_\zeta, c_\zeta)$. Thus $d_\zeta \rightarrow c$. Since $\mathcal{H}(\gamma_\zeta)$ is quasi-isometric to a horodisc, it is easily seen that $\min\{\sigma_{\gamma_\zeta}(a_\zeta, c_\zeta), \sigma_{\gamma_\zeta}(b_\zeta, c_\zeta)\}$, is bounded above by a linear function of $\sigma_{\gamma_\zeta}(c_\zeta, d_\zeta)$. Passing to the limit, we see that $\min\{\sigma_\gamma^\infty(a, c), \sigma_\gamma^\infty(b, c)\} = 0$, giving a contradiction. \square

We now proceed to the proof of the “only if” part of Theorem 1.4.

Suppose that $\phi : \mathbb{R}^\xi \rightarrow \mathcal{R}(\Sigma)$ is a quasi-isometric embedding. Passing to an asymptotic cone with fixed basepoint, we get a map $f = \phi^\infty : \mathbb{R}^\xi \rightarrow \mathcal{R}^\infty(\Sigma)$, which is bilipschitz onto its image, $\Phi = f(\mathbb{R}^\xi)$. Note that the basepoint, o , of $\mathcal{R}^\infty(\Sigma)$ lies in $\Phi \cap \mathcal{R}_T^\infty(\Sigma)$.

Now, by the Proposition 4.2 of [Bo4], Φ is “cubulated”, here meaning that it is a locally finite union of convex hulls of ξ -cubes. (As noted above, such a convex hull is median isomorphic to $[-1, 1]^\xi \subseteq \mathbb{R}^\xi$.) In fact, we can find a neighbourhood of the basepoint $o \in \Phi$, which has the structure of a finite cube complex, where each ξ -dimensional cell is the convex hull of a ξ -cube.

Now consider the link, Δ , of o in Φ . This is a simplicial complex which is a homology $(\xi - 1)$ -sphere. In particular, the $(\xi - 1)$ th dimensional homology of Δ is non-trivial.

Let Δ^0 be its vertex set. Each $x \in \Delta^0$ corresponds to a 1-cell of Φ , with one vertex o and the other denoted $a(x) \in \Phi \subseteq \mathcal{R}^\infty(\Sigma)$. Note that this 1-cell is precisely the median interval, $[o, a(x)]$. Now, as discussed after Proposition 8.2, we can canonically associate to x , either a curve $\gamma(x) \in \mathcal{UG}^0$, or a complexity-1 subsurface, $Y(x) \in \mathcal{UX}$. Any $(\xi - 1)$ -simplex in Δ corresponds to a ξ -cube, and so, by Proposition 8.1, we

see that the curves $\gamma(x)$ or subsurfaces $Y(x)$ are all disjoint, as x ranges over the vertices of the simplex.

Lemma 8.3. *Suppose that $x, y \in \Delta^0$ are distinct, and that both correspond to curves, $\gamma(x)$ and $\gamma(y)$. Then $\gamma(x) \neq \gamma(y)$.*

Proof. Suppose, for contradiction, that $\gamma(x) = \gamma(y) = \gamma$, say. Since the intervals $[o, a(x)]$ and $[o, a(y)]$ meet precisely in o , it follows that $o = \mu(a(x), a(y), o)$; in other words, $o \in [a(x), a(y)]$. Moreover, $\theta_\gamma^\infty|_{[o, a(x)]}$ and $\theta_\gamma^\infty|_{[o, a(y)]}$ are both injective (by construction of $\gamma(x)$ and $\gamma(y)$), and so, in particular, $\theta_\gamma^\infty o, \theta_\gamma^\infty a(x)$ and $\theta_\gamma^\infty a(y)$ are all distinct. We now apply Lemma 8.2 to give the contradiction that $o \notin \mathcal{R}_T^\infty$. \square

Lemma 8.4. *Each $x \in \Delta^0$ corresponds to some $\gamma(x) \in \mathcal{UG}^0(\Sigma)$.*

Proof. Suppose, to the contrary, that $x \in \Delta^0$ corresponds to a complexity-1 subsurface, $Y(x)$. Let Q be any ξ -cube of the cubulation containing $a(x)$, so that $\{o, a(x)\}$ is a face. (This corresponds to a $(\xi - 1)$ -simplex in Δ .) Let $\tau = \tau(Q)$ be the big multicurve described by Proposition 8.1. Let $\gamma \in \tau$ be a boundary curve of $Y(x)$ in Σ . Thus, $\gamma = \gamma(y)$ for some $y \in Q$ (adjacent to x in Σ). Let $Q_0 \subseteq Q$ be the $(\xi - 1)$ -face containing o but not containing $a(x)$. (This corresponds to a $(\xi - 2)$ -simplex of Δ .) Now, given that Φ is homeomorphic to \mathbb{R}^ξ , it is easily seen that there must be a (unique) ξ -cube, Q' , of the cubulation with $Q \cap Q' = Q_0$. Let $z \in \Delta^0$ be the unique vertex with $a(z) \in Q' \setminus Q_0$. Let $\tau' = \tau(Q')$. Now τ' is obtained from τ by replacing $\gamma(y)$ by $\gamma(z)$ and leaving all other curves alone. Note also that all complexity-1 components of the complement also remain unchanged (since these are the subsurfaces $Y(w)$ for those vertices w which do not correspond to curves). It therefore follows that, in fact, we must have $\gamma(z) = \gamma(y) = \gamma$, contradicting Lemma 8.3. In other words, this situation can never arise, and so each $x \in \Delta^0$ must correspond to a curve $\gamma(x)$. \square

By Lemmas 8.4 and 8.3, we therefore have an injective map $[x \mapsto \gamma(x)] : \Delta^0 \rightarrow \mathcal{UG}^0(\Sigma) = \mathcal{UC}^0(\Sigma)$. Now (again since Φ is homeomorphic to \mathbb{R}^ξ), every edge of Δ lies inside some $(\xi - 1)$ -simplex. So if $x, y \in \Delta^0$ are adjacent, $a(x), a(y)$ lie in some cube of the cubulation, and so $\gamma(x) \wedge \gamma(y)$. Now the ultraproduct, $\mathcal{UC}(\Sigma)$, of the curve complex $\mathcal{C}(\Sigma)$ is a flag complex (since $\mathcal{C}(\Sigma)$ is) so we get an injective simplicial map of Δ into $\mathcal{UC}(\Sigma)$. This gives us an injective map of Δ into $\mathcal{C}(\Sigma)$. Since $\mathcal{C}(\Sigma)$ has dimension $\xi - 1$, it follows that $\mathcal{C}(\Sigma)$ has non-trivial homology in dimension $\xi - 1$. But by the result of [Har] referred to above, the homology is trivial in all dimensions at least ξ' . It now follows that $\xi = \xi'$, and so Σ has genus at most 1, or is a closed surface of genus 2.

This proves the “only if” part of Theorem 1.4.

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MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL,
GREAT BRITAIN.