

# The Ending Lamination Theorem

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## Preface

This paper is based on a combination of two earlier manuscripts respectively entitled “Geometric models for hyperbolic 3-manifolds” and “End invariants of hyperbolic 3-manifolds”, both originally produced in 2005. Many of the ideas for the first preprint were worked out while visiting the Max Planck Institute in Bonn. The remainder and most of the write-up of this was carried out when I was visiting the Tokyo Institute of Technology at the invitation of Sadayoshi Kojima. Most of the second preprint was written while visiting the Centre Bernoulli, E.P.F. Lausanne, as part of the programme organised by Christophe Bavard, Peter Buser and Ruth Kellerhals. I thank all three institutions for their generous hospitality. I also thank Dave Gabai for his many helpful comments on the former preprint in particular, for the argument that appears at the end of Section 7, and his permission to include it. The final drafts of these preprints were prepared at the University of Southampton. The preprints were substantially reworked and combined, and some new material added at the University of Warwick. I thank Al Marden for his comments on the first few sections of the current manuscript.

## 0. Introduction.

In this paper, we give an account of Thurston’s Ending Lamination Conjecture regarding hyperbolic 3-manifolds. This was originally proven for indecomposable 3-manifolds by Minsky, Brock and Canary [Mi4,BrocCM1], who also announced a proof in the general case [BroCM2]. Since then, a number of other approaches have been proposed (see for example, [R,BrocBES,So2]). In the present work, we give a proof of the general case of the Ending Lamination Theorem. It is intended to be self-contained, given the background material discussed in Section 2.

The Ending Lamination Theorem is a major component of the classification of finitely generated Kleinian groups, or equivalently complete hyperbolic 3-manifolds with finitely generated fundamental group. This project is now essentially complete. The Ending Lamination Theorem can be viewed as the “uniqueness” part of the classification. It shows that such manifolds are determined by their “end invariants”. The other main components of the classification are the Tameness Theorem [Bon,Ag,CalG] and the existence of manifolds with prescribed end invariants (see [NS]).

We proceed with an informal statement of the Ending Lamination Theorem. For a more precise statement, see Theorem 2.4.

Let  $M$  be a complete hyperbolic 3-manifold with  $\pi_1(M)$  finitely generated. The “thin

part” of  $M$  is the (open) subset where the injectivity radius is less than some sufficiently small fixed “Margulis” constant. The unbounded components of the thin part form a (possibly empty) finite set of cusps of  $M$ . Removing these unbounded components, we obtain the “non-cuspidal” part,  $\Psi = \Psi(M)$  of  $M$ . The Tameness Theorem tells us that  $\Psi$  is topologically finite. This means that there is a compact manifold,  $\bar{\Psi}$ , with boundary  $\partial\bar{\Psi}$ , and a closed subsurface  $\partial_I\bar{\Psi} \subseteq \partial\bar{\Psi}$ , such that  $\Psi(M)$  is homeomorphic to  $\bar{\Psi} \setminus \partial_I\bar{\Psi}$ . Fixing some such (proper homotopy class of) homeomorphism, we can identify  $\partial\Psi(M)$  with  $\partial_V\Psi = \partial\bar{\Psi} \setminus \partial_I\bar{\Psi}$ , which we refer to as the *vertical boundary* of  $\Psi$ . Each torus component of  $\partial\Psi$  bounds a  $\mathbf{Z} \oplus \mathbf{Z}$ -cusp of  $M$ , and does not meet  $\partial_I\Psi$ . All other components of  $\partial\Psi$  have genus at least 2.

Note that the ends of  $\Psi$  are in bijective correspondence with the components of  $\partial_I\Psi$ . Each end  $e$  has a neighbourhood homeomorphic to  $\Sigma \times [0, \infty)$ , where  $\Sigma = \Sigma(e)$  is such a component. Note that this meets  $\partial_V\Psi$  in  $\partial\Sigma \times [0, \infty)$ . Associated to each such end we have a geometric “end invariant”, which is either a Riemann surface (for a “geometrically finite” end) or a geodesic lamination (for a “degenerate” end). The Ending Lamination Theorem asserts:

**Theorem 0 :**  *$M$  is determined up to isometry by the topology of its non-cuspidal part,  $\Psi(M)$ , together with its end invariants.*

Implicit in this is a preferred proper homotopy class of homeomorphism of  $\Psi$  with a given topological model. This gives rise to “markings” of the end invariants which we assume to be part of the data. The homotopy class of the isometry of the will respect these markings.

One can get a simpler picture by considering the case where  $M$  has no cusps, so that  $\Psi(M) = M$ . In this case,  $\bar{\Psi}$  is a compactification of  $M$ , obtained by adjoining a surface (of genus at least 2) to each end of  $M$ . These surfaces are just the components of  $\partial\Psi = \partial_I\bar{\Psi}$ , and each has an end invariant associated to it.

The above will be discussed in more detail in Section 2.

Particular cases of the Ending Lamination Theorem were known before the work of Minsky et al. If  $M$  is closed, so that  $\Psi(M) = M$  and there are no end invariants, then  $M$  is determined by its topology. This is Mostow rigidity [Most] in dimension 3. The same applies in the more general case, where  $M$  has finite volume, so that each boundary component of  $\Psi(M)$  is a torus corresponding to a  $\mathbf{Z} \oplus \mathbf{Z}$ -cusp of  $M$ . Again,  $M$  is determined by its topology. This was shown in [Mar] and [P]. More generally still, if  $M$  is geometrically finite (i.e. all its ends are geometrically finite) then the Ending Lamination Theorem is shown in [Mar]. (This used arguments similar to those of Section 16 here.) Indeed there was a complete classification in this case, following from the deformation theory of Alfors, Bers, Marden, Maskit etc. (see [Mar]).

Our account of the Ending Lamination Theorem broadly follows the strategy of the original, though the logic is somewhat different. Notably, we take the “a-priori bounds” theorem of Minsky [Mi4] as a starting point, rather than a result embedded in the proof. An independent argument for this is given in [Bow4]. (See also [So3] for some simplifications of this.) The model spaces we use are essentially the same as those in [Mi4], though we give

a combinatorial description that bypasses much of the theory of hierarchies as developed in [MaM2].

We shall first prove the theorem in the indecomposable case. Some additional ingredient will be needed for the general case, mainly to give a proper description of the end invariants, and to “isolate” an end of  $\Psi$  from the “core” of the manifold. Apart from that, it will only call for reinterpreting certain constructions.

We will present the main part of the argument in the specific context of a doubly degenerate manifold, namely, where  $\Psi(M)$  is a topological product,  $\Psi(M) \cong \Sigma \times \mathbf{R}$ , and both ends are degenerate. We do this for several reasons.

Firstly it greatly simplifies the exposition. Most of the main ideas can be seen in this context. What remains for the general indecomposable case is largely a matter of describing how the various bits fit together in a more complicated situation.

Secondly, these ideas have further applications to Teichmüller theory and the geometry of the curve complex, etc. As far as these are concerned, one only really needs to worry about such product manifolds. In the case of a doubly degenerate group, we get a somewhat cleaner, and stronger statement. In particular, one can show that the quasi-isometry constants are uniform, in that they depend only on the topology of the base surface. (A similar uniformity in this case is obtained in [BroCM1].) This is lost (at least without more work) in the general indecomposable case.

A third, though relatively minor, reason is that one is obliged to give some special consideration to the doubly degenerate case, since there we have to check that each end of the model gets sent to “right” end of  $M$  — a fact that is automatic from the topology in all other situations. (Indeed, precisely this issue caused Bonahon a certain amount of strife in [Bon].)

In the first five sections of this paper, we describe the relevant background, outline the main results, and describe the ingredients of the proof. In Sections 6 to 16, we proceed to a proof of the Ending Lamination Theorem in the indecomposable case. In Sections 17 to 24, we describe the modifications necessary to deal with the general case. In Section 25, we give an account of the Uniform Injectivity Theorem applicable to our situation.

## 1. Background.

We give a summary of the main ideas behind the Ending Lamination Conjecture and the classification of finitely generated Kleinian groups. We include some historical background, though our account is not strictly chronological. Much of the discussion of this section is not logically essential to understanding the statement or proof as presented in this paper. The bits that are will be reviewed again later. In particular, a more formal discussion of end invariants will be given in Section 2.

Before the late 1970s, much of the theory of 3-manifolds and of Kleinian groups had developed separately. Prior to this, most major results of 3-manifold theory were based on combinatorial or topological techniques. General accounts of the topological theory of 3-manifolds can be found in [He] and [J]. Meanwhile, the theory of Kleinian groups tended to use analytical machinery, focusing on the action of the group on the Riemann sphere.

An account of the state of the art with regard to Kleinian groups around this time can be found in [BersK]. In the background, though never fully exploited, was hyperbolic geometry arising from the fact that a Kleinian group acts properly discontinuously on hyperbolic 3-space. It was gradually recognised, through work of Riley, Jørgensen and others, that hyperbolic structures were natural and commonplace in 3-manifold theory. Marden was one of the first to bring 3-manifold theory properly into play in the theory of Kleinian groups, notably through the seminal work [Mar]. Then, in the late 1970s, the subject was revolutionised through work of Thurston [Th].

A key topological result is the Scott Core Theorem [Sc1,Sc2] and a generalisation thereof to the relative case due to McCullough [Mc]. The latter states:

**Theorem 1.1 :** *Suppose  $\Psi$  is a 3-manifold with (possibly empty) boundary,  $\partial\Psi$  and  $F \subseteq \partial\Psi$  be any compact subset. Then there is a compact connected submanifold  $\Psi_0 \subseteq \Psi$ , with  $\Psi_0 \cap \partial\Psi = F$ , such that the induced map from  $\pi_1(\Psi_0)$  to  $\pi_1(\Psi)$  is an isomorphism.*

We refer to  $\Psi_0$  as a “compact core”.

Suppose now that  $M$  is a complete hyperbolic 3-manifold with  $\Gamma = \pi_1(M)$  finitely generated. Then  $M = \mathbf{H}^3/\Gamma$ , where  $\Gamma \equiv \pi_1(M)$  acts properly discontinuously on hyperbolic 3-space. The action extends to the ideal sphere  $\partial\mathbf{H}^3$  which decomposes as the limit set,  $L(\Gamma)$ , and the discontinuity domain,  $D(\Gamma)$ . We write  $R(M)$  for the (possibly empty) quotient surface  $D(\Gamma)/\Gamma$ . Ahlfors’s Finiteness Theorem [Ah] says:

**Theorem 1.2 :**  *$R(M)$  is a finite disjoint union of Riemann surfaces of finite type.*

By “finite type” we mean compact with finitely many punctures (possibly none).

Let  $\Psi = \Psi(M)$  be the non-cuspidal part of  $M$ . One can show that  $\partial\Psi$  has finitely many components, each an annulus or torus. We can thus find a compact core  $\Psi_0 \subseteq \Psi$  which includes each torus component of  $\partial\Psi$  and which meets each annular component of  $\partial\Psi$  in a compact annular core. (This is a consequence of Theorem 1.1, together with a simple Euler characteristic argument.) The ends of  $\Psi$  are in bijective correspondence with the components of  $\partial\Psi_0 \setminus \partial\Psi$ . Such an end is “geometrically finite” if it has a neighbourhood which meets no closed geodesic in  $M$ . One can show that the geometry of such an end is relatively simple. In particular, it can be constructed from a finite-sided polyhedron by carrying out side identifications. The geometrically finite ends are in bijective correspondence with the components of  $R(M)$ . Indeed any component,  $S$ , of  $R(M)$  admits a natural homotopy equivalence to the corresponding component of  $\partial\Psi_0 \setminus \partial\Psi$ . This determines a marking of  $S$ , and hence identifies  $S$  as a point in the Teichmüller of this surface (at least in the indecomposable case).

**Definition :** We say that  $M$  is *geometrically finite* if each end of  $\Psi(M)$  is geometrically finite.

This can be reformulated in a number of equivalent ways. For example, it is equivalent to saying that  $M$  has a finite-sided fundamental polyhedron.

The deformation theory of geometrically finite manifolds has been well understood for some time, due the work of Ahlfors, Bers, Marden, Maskit (see [Mar]). In particular, we have:

**Theorem 1.3 :** *If  $M$  is geometrically finite, then it is determined up to isometry by the topological type of  $\Psi_0$ , and the collection of end invariants,  $R(M)$  (viewed as elements of Teichmüller space).*

The above two theorems, and their proofs, are essentially analytic in nature. Gradually the theory of 3-manifolds was brought into play, notably through work of Marden. In particular, in [Mar], Marden asked if every complete hyperbolic 3-manifold,  $M$ , with finitely generated fundamental group is topologically finite, that is, homeomorphic to the interior of a compact 3-manifold with boundary.

This question was given a geometric reinterpretation by Thurston. He defined the notion of a “simply degenerate” end of  $\Psi$ . Given this,  $M$  is said to be *tame* if each of its ends is either geometrically finite or simply degenerate. Using work of [BrinT] he showed that any tame manifold is topologically finite. He asked if any complete hyperbolic 3-manifold with finitely generated fundamental group is tame — the “Tameness Conjecture”. He also defined an end invariant associated to each simply degenerate end. Such an invariant is a geodesic lamination [Th,CasB]. (To first approximation a geodesic lamination can be thought of as an ideal point of Teichmüller space.) Thurston asked whether every tame hyperbolic 3-manifold was determined by its topology and its end invariants — the “Ending Lamination Conjecture”.

It turns out to be simpler to study the case of “indecomposable” manifolds. This can be defined algebraically in terms of  $\Gamma = \pi_1(M)$  and the collection of parabolic subgroups arising from its action on  $\mathbf{H}^3$ .

**Definition :** We say that  $M$  is indecomposable if, for every decomposition of  $\Gamma$  as a free product,  $\Gamma \cong A * B$ , there is a parabolic subgroup of  $\Gamma$  (acting on  $\mathbf{H}^3$ ) that cannot be conjugated into either  $A$  or  $B$ .

(Clearly if  $\Gamma$  has no non-trivial free product decomposition at all, then  $M$  is indecomposable.) Via the Dehn Lemma of Papakyriakopoulos [He], there is an equivalent topological formulation of indecomposability. This says that there is no disc in  $\bar{\Psi}$  whose boundary is an essential curve in  $\partial_I \bar{\Psi}$ .

In [Bon], Bonahon proved Marden’s conjecture for indecomposable manifolds. Using this work, Canary [Cana] proved a converse to Thurston’s result, thereby showing that tameness was equivalent to topological finiteness. After this, Marden’s conjecture became largely synonymous with the Tameness Conjecture.

The Tameness Conjecture was finally proven independently by Agol [Ag] and Calegari and Gabai [CaG]. (See also [So1,Bow7] for other accounts.)

**Theorem 1.4 :** *Let  $M$  be a complete hyperbolic 3-manifold with  $\pi_1(M)$  finitely generated. Then the non-cuspidal part,  $\Psi(M)$ , is topologically finite.*

It follows that  $M$  is also topologically finite.

In fact,  $\Psi$  is homeomorphic to the interior of the compact core  $\Psi_0$ . We can embed  $\Psi$  into a compact manifold,  $\bar{\Psi} \cong \Psi_0$  in such a way that  $\bar{\Psi} \cap \partial\Psi$  gets identified with  $\Psi_0 \cap \partial\Psi$ .

Thus, in retrospect, in describing the end invariants of  $\Psi$ , we can replace  $\Psi_0$  by  $\bar{\Psi}$ , and subsequently forget about  $\Psi_0$ . In fact, in our discussion, we will have no formal use of the Core Theorem, though, of course, it remains an essential ingredient in the proof of tameness. As before, we write  $\partial_V \bar{\Psi} = \Psi \cap \partial \bar{\Psi}$ . We note that, via the Dehn Lemma of Papakyriakopoulos [He], there is an equivalent topological formulation of indecomposability. This says that there is no disc in  $\bar{\Psi}$  whose boundary is an essential curve in  $\partial_I \bar{\Psi}$ .

Prior to the proof of tameness in the general case, Minsky had already made significant progress towards the Ending Lamination Conjecture. He proved it in the special case of manifolds of bounded geometry (where the injectivity radius is bounded below) [Mi1] and for punctured torus groups [Mi3]. The general indecomposable case was finally dealt with in [Mi4, BrocCM1]. They also announced the result for the decomposable case [BrocCM2].

The overall strategy of the above proofs are similar. Based on the topological data and end invariants, one constructs a “model space”, which is a (possibly singular) riemannian manifold,  $P$ . One then constructs a bilipschitz map from  $P$  to  $M$ . Given another hyperbolic manifold,  $M'$ , diffeomorphic to  $M$ , and with the same end invariants, one obtains, via  $P$ , a bilipschitz map from  $M$  to  $M'$ . This gives rise to a bilipschitz map between the universal covers, each isometric to  $\mathbf{H}^3$ , that is equivariant with respect to the respective actions of  $\pi_1(M)$  and  $\pi_1(M')$ . The earlier deformation theory (as with the geometrically finite case) can now be brought into play to show that these actions are in fact conjugate by isometry of  $\mathbf{H}^3$ . It follows that  $M$  and  $M'$  are isometric. In fact, we only really need an equivariant quasi-isometry of  $\mathbf{H}^3$  to make this work. (See Sections 16 and 24 for more details.)

In fact, most of the work is involved in understanding the geometry of the simply degenerate ends of  $\Psi$ . We can thus effectively reduce (at least in the indecomposable case) to the case where  $\Psi$  is diffeomorphic to  $\Sigma \times \mathbf{R}$ , where  $\Sigma$  is a compact surface. This is in turn closely related to understanding the large scale geometry of the Teichmüller space of  $\Sigma$ . Indeed, in the bounded geometry case [Mi1], Minsky originally constructed a model out of a Teichmüller geodesic (cf. the singular sol geometry manifold used in [CanT]). Another viewpoint on this is discussed in [Mosh] and [Bow8]. (We will not be explicitly using that material in this paper.)

The general case [Mi4, BrocCM1, BrocCM2] uses a different construction of the model, based on Harvey’s curve complex [Har]. This relies on work of Masur and Minsky [MasM1, MasM2] (some of which will be discussed in Section 6). This work has many other applications and potential applications, some of which we mention below.

The present work uses essentially the same model as Minsky, though described in a somewhat different way. A further variation of this, with some simplifications, is described in [So2]. Both take as starting point the a-priori bounds of Minsky [Mi4], reproven by a more direct argument in [Bow4]. For the decomposable case we will need a generalisation of this statement, which we describe in Section 21.

A different proof, in the general case, based on surgery arguments to reduce to the geometrically finite case, has been proposed in [BrocBES]. They also start with an a-priori

bounds theorem, but avoid the use of models.

The Ending Lamination Theorem does not complete the classification of finitely generated Kleinian groups. For this, one needs to describe the set of ending laminations that can arise for any given manifold. (There are no restrictions on the end invariants of geometrically finite ends.) For example, in the case where  $\Psi$  is a topological product,  $\Sigma \times \mathbf{R}$ , the only restriction is that we cannot have both end invariants equal to the same lamination. If  $\Psi$  is a handlebody (with  $\partial\Psi = \emptyset$ ) then the set permissible ending laminations is the “Masur domain” [Mas]. The general case can be described by a generalisation of Masur’s construction. To construct a manifold with specified admissible end invariants, the usual strategy is to take a sequence of geometrically finite manifolds whose end invariants tend to the prescribed ones, and prove that everything converges in an appropriate sense. This a culmination of work of various people (see for example [KleS,KiLO]). A general account of this can now be found in [NS].

As mentioned earlier, the Ending Lamination Conjecture, and the methods involved in its proof, have many other applications. One particularly notable consequence is the Density Conjecture [Brom,BrocB,NS]:

**Theorem 1.5 :** *Any finitely generated kleinian group is an algebraic limit of geometrically finite kleinian groups.*

The models can also be used to give a description of geometric limits of finitely generated kleinian groups. For product manifolds, this is described in [OhS].

Various other applications of this technology include [Bow6] and [Ta].

## 2. End invariants.

In this section, we describe more carefully how end invariants are defined, and give a more precise statement of the Ending Lamination Theorem as Theorem 2.4. We begin by recalling two structures associated to a surface, namely the Teichmüller space and the curve graph.

Let  $\Sigma$  be a compact orientable surface. We write  $\kappa(\Sigma) = 3g + p - 3$ , where  $g$  is the genus and  $p$  is the number of boundary components. The only significance of this quantity here is that it measures the “complexity” of the topological type of  $\Sigma$ . We will always assume here that  $\kappa(\Sigma) \geq 0$ . If  $\kappa(\Sigma) = 0$ , then  $\Sigma$  is a three-holed sphere, abbreviated to “3HS”. If  $\kappa(\Sigma) = 1$ , then  $\Sigma$  is either a one-holed torus or a four-holed sphere, abbreviated respectively to “1HT” and “4HS”.

We write  $\mathbf{T}(\Sigma)$  for the Teichmüller space of  $\Sigma$ . Here, we think of this as the space of marked finite-type conformal structures on  $\text{int}(\Sigma)$ .

We will define the end invariant of a simply degenerate ends in terms of the curve graph — the 1-skeleton of the curve complex introduced by Harvey [Harv]. Let  $X(\Sigma)$  be the set of simple non-trivial non-peripheral closed curves in  $\Sigma$ , defined up to homotopy. We shall frequently refer to elements of  $X(\Sigma)$  simply as “curves”. The curve graph,  $\mathcal{G} = \mathcal{G}(\Sigma)$  has vertex set  $X(\Sigma)$ , and two curves are deemed to be adjacent if they have minimal

possible intersection number in  $\Sigma$ . If  $\kappa(\Sigma) > 1$  (the “non-exceptional cases”) the minimal possible intersection number is 0. In other words, two curves are deemed adjacent if they can be realised disjointly in  $\Sigma$ . For the exceptional cases of the 1HT and 4HS, the minimal possible intersection number is 1 and 2 respectively. In each of these case, the curve graph,  $\mathcal{G}(\Sigma)$  is isomorphic to the Farey graph. In the case of the 3HS, the curve graph is empty.

It is easily seen that the Farey graph is hyperbolic in the sense of Gromov [GhH]. In fact, a remarkable theorem of Masur and Minsky [MasM1] tells us that all curve graphs have this property:

**Theorem 2.1 :** *If  $\Sigma$  is a compact surface with  $\kappa(\Sigma) \geq 1$ , then  $\mathcal{G}(\Sigma)$  is Gromov hyperbolic.*  $\diamond$

Other proofs can be found in [Bow2] and [Ham1].

One can associate to such a Gromov hyperbolic space its Gromov boundary,  $\partial\mathcal{G}(\Sigma)$ . It was shown in [Kl] that  $\partial\mathcal{G}(\Sigma)$  can be naturally identified with the space of arational laminations. The theory of laminations was introduced by Thurston [Th]. The general theory is discussed for example in the book [CasB]. The result of [Kl] provides the link with end invariants as they are more traditionally defined (as in [T,Bon] etc.). However, since we will define an end invariant directly as an element of  $\partial\mathcal{G}(\Sigma)$ , we won’t formally need it for our statement or proof of the Ending Lamination Theorem.

It is worth remarking that  $\partial\mathcal{G}(\Sigma)$  has a rich topological structure. This has recently come in for much study, see for example [Ga]. Again, that work is not directly relevant to the discussion of this paper.

We now move on to 3-manifolds. We need to clarify how we understand the “marking” of an end invariant. Suppose that  $\Psi$  is a 3-manifold with a topologically finite end,  $e$ . This means that there is a compact surface,  $\Sigma$ , and a proper injective map  $\theta : \Sigma \rightarrow \Psi$  so that  $\theta(\Sigma \times [0, \infty))$  is a neighbourhood of the end (and hence a homeomorphism to its range). If  $\theta' : \Sigma' \times [0, \infty) \rightarrow \Psi$  is another such map, then there is a canonically defined homotopy equivalence from  $\Sigma$  to  $\Sigma'$  — take any  $t \geq 0$  large enough so that  $\theta(\Sigma \times \{t\}) \subseteq \theta'(\Sigma' \times [0, \infty))$ , and postcompose  $(\theta')^{-1} \circ (\theta|_{\Sigma \times \{t\}})$  with projection to  $\Sigma'$ . This homotopy equivalence respects the peripheral structure of these surfaces. We note, in particular, that  $\Sigma$  and  $\Sigma'$  are homeomorphic. This gives us a basis for using  $e$  (thought of formally as a directed set of subsets of  $\Psi$ ) as a topological model for marking structures associated to  $\Sigma$ . In particular, we can define the Teichmüller space,  $\mathbf{T}(e)$  associated to  $e$  by canonically identifying it with  $\mathbf{T}(\Sigma)$  via  $\theta$ . Similarly, we define the curve graph  $\mathcal{G}(e)$  by identifying it with  $\mathcal{G}(\Sigma)$ . Note that any element of  $X(e)$  can be realised as a curve in any neighbourhood of  $e$ . We refer to  $\Sigma = \Sigma(e)$  as the *base surface* of the end (which implicitly implies a choice of map  $\theta$ ).

Note that if we embed  $\Psi$  in a compact manifold  $\bar{\Psi}$ , with  $\partial_V \bar{\Psi} = \bar{\Psi} \cap \partial\Psi$  and  $\partial_I \bar{\Psi} = \partial\bar{\Psi} \setminus \partial_V \bar{\Psi}$ , as described in Section 0, then we can identify  $\Sigma(e)$  (at least up to homotopy) with a component of  $\partial_I \bar{\Psi}$ . We can therefore also regard  $\partial_I \bar{\Psi}$  as a base surface for our marking. This ties in with the informal description given by Theorem 0.

Suppose we have a homeomorphism  $f : \Psi \rightarrow \Psi'$  between two such manifolds. This will associate to each end,  $e$ , of  $\Psi$ , an end of  $\Psi'$ , which we denote by  $f(e)$ . Moreover, there is a canonical homotopy equivalence between the base surfaces, respecting their

peripheral structure. Thus we get induced isomorphisms  $f_* : \mathbf{T}(e) \longrightarrow \mathbf{T}(f(e))$  and  $f_* : \mathcal{G} \longrightarrow \mathcal{G}(f(e))$ .

Now suppose that  $M$  is a complete hyperbolic 3-manifold, and that  $\Psi(M)$  is the non-cuspidal part with respect to some Margulis constant  $\eta > 0$ . For the purposes of this section, we can regard  $\eta$  as being fixed. The choice does not really matter since if we choose a smaller constant, then the difference between the two non-cuspidal parts will just be a topological product, with a very simple geometrical structure. (In fact, it will be convenient later to allow  $\eta$  to depend on the topological type of  $\Psi$ .)

For the purposes of this paper, we shall make the following definition:

**Definition :** Let  $e \in \mathcal{E}(M)$ . We say that  $e$  is *geometrically finite* if there is a neighbourhood of the end in  $\Psi(M)$  which meets no closed geodesic on  $M$ . Otherwise, we say that  $e$  is *degenerate*.

We partition  $\mathcal{E}(M) = \mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$  into geometrically finite and degenerate ends accordingly.

In view of the Tameness Theorem, there are many other equivalent ways of describing these two types of ends. We begin by discussing the geometrically finite case. This is well understood, and the observations we make below are now standard.

Let  $e \in \mathcal{E}_F(M)$ . We can find a neighbourhood,  $E$ , of the end  $e$  in  $\Psi(M)$  and a homeomorphism of  $E$  with  $\Sigma(e) \times [0, \infty)$  such that each surface  $\Sigma(e) \times \{t\}$  is convex outwards. As a result,  $E$  has a fairly simple geometry. This is described in more detail in Section 16. (Also as observed in Section 0, we can construct a neighbourhood of the end by taking a finite sided polyhedron and identifying faces.) Recall that we can write  $M = \mathbf{H}^3/\Gamma$  where  $\Gamma \equiv \pi_1(M)$ . Let  $R(M) = D(\Gamma)/\Gamma$  be the quotient of the discontinuity domain, as described in Section 0. Note that  $M \cup R(M)$  carries a quotient topology as  $(\mathbf{H}^3 \cup D(\Gamma))/\Gamma$ . In this topology, the closure of  $E$  is equal to  $E \cup R(e)$ , where  $R(e)$  is a component of  $R(M)$ . In fact, we can extend the above homeomorphism to a homeomorphism of  $E \cup R(e)$  with  $(\Sigma(e) \times [0, \infty]) \setminus (\partial\Sigma(e) \times \{\infty\})$ , where  $R(e)$  corresponds to  $(\Sigma(e) \setminus \partial\Sigma(e)) \times \{0\}$ . Throwing in the (possibly empty) union of curves  $\partial\Sigma \times \{\infty\}$  we get a compactification of  $E$  as  $\Sigma(e) \times [0, \infty]$ . In this way, it is natural to view  $R(e)$  as describing the structure of  $e$  at infinity. Note that this comes equipped with a marking, as described earlier, so we get a well defined point,  $a(e) \in \mathbf{T}(e)$ .

The geometric structure of a degenerate end is much more subtle. Understanding it is the main task in proving the Ending Lamination Theorem. We shall start from the following:

**Proposition 2.2 :** *Suppose that  $e \in \mathcal{E}_D(M)$ . There is a sequence  $(\gamma_i)_i$  of elements of  $X(e)$  which have representatives of length at most  $l_0$  in  $\Psi(M)$  and which tend out the end of  $e$ , where  $l_0$  depends only on  $\kappa(\Sigma(e))$ .*

To define “representative” in the above, choose a neighbourhood  $E$  of  $e$  in  $\Psi(M)$  with a homeomorphism of  $E$  with  $\Sigma(e) \times [0, \infty)$ . Then any element of  $\mathcal{G}(e)$  determines a free homotopy class in  $\Sigma(e)$  hence in  $E$ . The statement that “ $\gamma_i$  goes out the end  $e$ ” means in

particular  $\gamma_i$  lies in  $E$  for all sufficiently large  $i$ . In fact, we shall see that we could replace “representative” by “closed geodesic representative”, in the above statement.

It turns out that the limit point of the sequence  $(\gamma_i)_i$  in  $\partial\mathcal{G}(e)$  is well defined. In fact:

**Proposition 2.3 :** *There is some  $a \in \partial\mathcal{G}(e)$  such that if  $(\gamma_i)_i$  is any sequence with bounded length representatives going out the end  $e$ , then  $(\gamma_i)_i$  tends to  $a$  in  $\mathcal{G}(e) \cup \partial\mathcal{G}(e)$ .*

Here we can allow any bound on the length — it may depend on our sequence.

We postpone further comment on Propositions 2.2 and 2.3 for the moment, and just note that they determine  $a$  uniquely. We can therefore denote it by  $a(e)$ .

**Definition :** Given  $e \in \mathcal{E}(M)$ , we refer to  $a(e)$  as the *end invariant* of  $e$ .

Note that if  $e \in \mathcal{E}_F(M)$ , then  $a(e) \in \mathbf{T}(e)$  and if  $e \in \mathcal{E}_D(M)$ , then  $a(e) \in \partial\mathcal{G}(e)$ .

In the case where  $\Sigma(e)$  is a 3HS, then necessarily,  $e \in \mathcal{E}_F(M)$ . In fact, we can find a neighbourhood  $E$  of the end so that  $\partial E$  is a totally geodesic. Moreover,  $\mathbf{T}(e)$  is just a singleton. In other words, the end invariant carries no information in this case. We can effectively discard the end invariants of such ends.

We are now in a position to give a formal statement of the Ending Lamination Theorem as follows:

**Theorem 2.4 :** *Suppose that  $M$  and  $M'$  are complete hyperbolic 3-manifolds with finitely generated fundamental groups. Suppose that  $f : \Psi(M) \rightarrow \Psi(M')$  is a proper homotopy equivalence between the respective non-cuspidal parts such that for each  $e \in \mathcal{E}(M)$ , we have  $f_*(a(M, e)) = a(M', f(e))$ . Then there is an isometry  $g : M \rightarrow M'$  such that  $g|_{\Psi(M)} : \Psi(M) \rightarrow \Psi(M')$  is properly homotopic to  $f$ .*

(Here, a “proper homotopy” is a continuous map,  $F : \Psi(M) \rightarrow \Psi(M')$  with  $F^{-1}(\partial\Psi(M')) = \partial\Psi(M)$  which is proper in the usual topological sense, i.e. the preimage of every compact set is compact.)

Theorem 2.4, will be proven in the indecomposable case in Section 17, and in general in Section 24.

There is special case we should mention. Suppose that  $\Psi(M) \cong \Sigma \times \mathbf{R}$ . Then,  $\Psi(M)$  has two ends, say  $e^-$  and  $e^+$ . If these are both degenerate, we say that  $M$  is *doubly degenerate*. In that case, we have the following (see [Bon]):

**Theorem 2.5 :** *If  $M$  is doubly degenerate with ends  $e^+$  and  $e^-$ , then  $a(M, e^-) \neq a(M, e^+)$ .*

As observed in Section 0, this case calls for some special attention, in that we need to keep track of which end is which.

We should make some more comments on the end invariant of a degenerate end, and relate this to earlier work.

First, we note that, under the correspondence described in [Kl], an element  $a \in \partial\mathcal{G}(e)$  corresponds to an arational lamination  $\lambda$ . That is, a geodesic lamination on  $\Sigma = \Sigma(e)$  with the property that any essential non-peripheral curve in  $\Sigma$  must cross  $\lambda$ . (For a discussion of laminations, see for example [CasB].) A sequence of curves,  $(\gamma_i)_i$ , in  $X(e)$  converges to  $a$  in  $\mathcal{G}(e) \cup \partial\mathcal{G}(e)$  if and only if it converges to  $\lambda$  in the Hausdorff topology on  $\Sigma$ .

The discussion is simplified in the following situation. Let  $E$  be a neighbourhood of  $e$  homeomorphic to  $\Sigma \times [0, \infty)$ .

**Definition :** We say that  $e$  is *incompressible* if the inclusion of  $E$  into  $\Psi(M)$  is  $\pi_1$ -injective.

It is easily seen that this is independent of the choice of  $E$ . It is also equivalent to saying that  $\partial E$  is an incompressible surface in  $\Psi(M)$ . If  $M$  is indecomposable, then every end is incompressible.

Suppose that  $e \in \mathcal{E}_D(M)$  is incompressible. It was shown in [Bon] that  $e$  is “simply degenerate”. (Of course, Bonahon did not assume a-priori that  $e$  is topologically finite. That is a consequence of being simply degenerate. We shall however take that as given here.) One way of saying this is that there is a sequence,  $(\gamma_i)_i$ , in  $X(e)$  whose geodesic representatives in  $M$  all lie in  $\Psi(M)$  and tend out the end  $e$ . (For this, we need to assume that the constant  $\eta$  defining  $\Psi(M)$  is sufficiently small in relation to the complexity of  $\Sigma$ .) Moreover, Bonahon showed that any such sequence must converge on a well defined arational lamination  $\lambda$  in  $\Sigma$ . This therefore gives an end invariant  $\lambda = \lambda(e)$ .

To relate this to the above, we need some other ideas, also found in [Th] and [Bon]. First, we can extend each of the closed geodesic representatives of the  $\gamma_i$  to a “pleated surface”. This notion was originally due to Thurston — see [CanEG] for a detailed discussion. A pleated surface is (in particular) a 1-lipschitz map of  $\Sigma$  into  $\Psi(M)$ , with respect to some hyperbolic structure,  $\sigma_i$ , in the domain. These pleated surfaces also go out the end  $e$ . (For further discussion of surfaces of this type, see Section 18.) We can now find a simple closed curve in  $(\Sigma, \sigma_i)$  whose length is bounded by some constant  $l(e)$ , depending only on the complexity of  $\Sigma$ . This gives us an element  $\beta_i$  in  $X(\Sigma) = X(e)$  represented by a curve of bounded length at most  $l(e)$  in  $\Psi(M)$ . These curves also go out  $e$ , and in fact, so do their geodesic representatives in  $M$ . Thus, in retrospect, we could have chosen our curves  $\gamma_i$  all to have bounded length. Note that all these sequences tend to the same lamination  $\lambda$ . Reinterpreting in terms of  $\mathcal{G}(e)$ , we recover the formulation given by Propositions 2.2 and 2.3 for an incompressible end.

If  $e \in \mathcal{E}_D(M)$  is compressible, then one needs to modify the above. The essential ingredients are contained in the general proof of tameness, as in [Ag,CalG]. We give a direct proof of this (without explicit reference to laminations) in Section 17 here. This is also based on ideas in [So1], as formulated in [Bow7].

In fact the uniqueness of the point  $a \in \partial\mathcal{G}(\Sigma)$  is quite subtle in the compressible case. The proof we give here will involve some machinery from the proof of the Ending Lamination Theorem, and is postponed until Section 23. It is possible to give a statement of Theorem 2.4 without assuming the uniqueness of  $a$  as follows. Instead of a point  $a \in \partial\mathcal{G}(\Sigma)$  we could take the end invariant to be a nonempty subset  $\mathbf{a}(e) \subseteq \partial\mathcal{G}(\Sigma)$  (namely, the set of all possible limits of curves of bounded length that go out the end). The hypothesis of

Theorem 2.4 then becomes  $f_*(a) \in \mathbf{a}(e)$  for some  $a \in \partial\mathcal{G}(\Sigma)$ . Given this, the fact that  $\mathbf{a}(e)$  must be a singleton becomes more apparent. This is formally proven here as Proposition 23.2 (see also Proposition 23.3). This discussion is also applicable to the incompressible case.

We remark that there are yet other ways of interpreting degenerate end invariants, for example via “non-realizability” of laminations in  $M$  (see for example [NS]).

We also note that one can define the Thurston boundary of Teichmüller space  $\mathbf{T}(\Sigma)$ , as the space of projective laminations on  $\Sigma$ . This is clearly related to (though not the same as) the space of arational laminations on  $\Sigma$ , as identified with  $\partial\mathcal{G}(\Sigma)$ , via [Kl]. This suggests that one can informally view a degenerate end as some kind of limit of geometrically finite ends. A formal statement of this requires the Density Theorem, alluded to at the end of Section 2.

### 3. Ingredients of the proof.

We list below some general topological and geometrical ingredients of our argument. First, we describe a few well known topological facts.

We have the following procedure for replacing maps of surfaces by embeddings in 3-manifolds. Let  $\Psi$  be an aspherical 3-manifold with possibly empty boundary,  $\partial\Psi$ . By a “proper map”  $f : \Phi \rightarrow \Psi$  of a surface  $\Phi$  into  $\Psi$  we mean that  $f$  is continuous and that  $f^{-1}(\partial\Psi) = \partial\Phi$ . (We can always assume that  $f$  is in general position.) By a “proper homotopy” we mean a map  $F : \Phi \times [0, 1] \rightarrow \Psi$  such that  $F^{-1}(\partial)\Psi = \partial\Phi \times [0, 1]$ .

**Theorem 3.1 :** *Suppose that  $f : \Phi \rightarrow \Psi$  is a  $\pi_1$ -injective proper map, and that  $f$  is properly homotopic to an embedding of  $\Phi$ . Suppose that  $U \subseteq \Psi$  is any open subset containing  $f(\Phi)$ . Then  $f$  is properly homotopic in  $\Psi$  to a proper embedding  $f' : \Phi \rightarrow \Psi$  with  $f'(\Phi) \subseteq U$ .  $\diamond$*

As observed in [Bon], this follows from the construction of [FHS]. The general result given in [FHS] makes use of minimal surfaces. If  $f$  is a homotopy equivalence, the relevant part of their argument in this case is a purely combinatorial tower construction. This works equally well in the relative case (whereas [FHS] deals with closed surfaces). Also [FHS] assumes that  $f$  is an immersion. However (as observed in [Bow6] for example) the combinatorial argument is readily adapted to (general position) maps. One, (somewhat artificial) way to deal with this would be to lift  $f$  to the cover corresponding to  $f_*(\pi_1(\Sigma))$ , so that  $f$  becomes a homotopy equivalence. The tower argument then gives us an embedding in this cover, which projects back to an immersion in an arbitrarily small neighborhood of the original map. We can then apply [FHS] as in the original form.

In applying Theorem 1.1, some caution is needed in that it gives us no geometric control on the homotopy between  $f$  and  $f'$ . In principle, it could go all over the place. With more work, we could place some restrictions on the homotopy, but we won't be needing them here.

We also note Waldhausen's cobordism theorem (see [He]):

**Theorem 3.2 :** *Let  $\Psi$  be an aspherical 3-manifold with (possibly empty) boundary,  $\partial\Psi$ . Suppose that  $\Sigma_1$  and  $\Sigma_2$  are disjoint homotopic properly embedded surfaces  $\Sigma_1 \cap \Sigma_2 = \emptyset$ . Then  $\Sigma_1 \cup \Sigma_2$  bounds a submanifold,  $Q$ , of  $\Psi$ , such that  $(Q, Q \cap \partial\Psi)$  is homeomorphic to  $(\Sigma_i \times [0, 1], \partial\Sigma_i \times [0, 1])$ .  $\diamond$*

For product manifolds, we have the following specific results. Let  $\Sigma$  be a compact surface. By a *homotopy fibre* we mean a map  $f : \Sigma \longrightarrow \Sigma \times \mathbf{R}$  with  $f^{-1}(\partial\Sigma \times \mathbf{R}) = \partial\Sigma$  such that relative homotopy class of  $(\Sigma, \partial\Sigma) \longrightarrow (\Sigma \times \mathbf{R}, \partial\Sigma \times \mathbf{R})$  is the same as the inclusion  $[x \mapsto (x, 0)]$ .

We have the following result of Brown [Brow]:

**Theorem 3.3 :** *An injective homotopy fibre in  $\Sigma \times \mathbf{R}$  is ambient isotopic to  $\Sigma \times \{0\}$ .  $\diamond$*

(Note that, if  $\Psi \cong \Sigma \times \mathbf{R}$ , and  $\Sigma_1$  and  $\Sigma_2$  are the images of embedded homotopy fibres, then Theorem 3.2 in this case is a consequence of Theorem 3.3.)

There is a refinement of Theorem 3.1 available in this case:

**Theorem 3.4 :** *Suppose that  $f : \Sigma \longrightarrow \Sigma \times \mathbf{R}$  is a homotopy fibre, and that  $\alpha \subseteq \Sigma$  is a curve with  $f^{-1}(f(\alpha)) = \alpha$ . Let  $U \subseteq \Sigma \times \mathbf{R}$  be any open set containing  $f(\Sigma)$ . Then there is an injective homotopy fibre  $f' : \Sigma \longrightarrow \Sigma \times \mathbf{R}$  with  $f'(\Sigma) \subseteq U$  and  $f'|_{\alpha} = f|_{\alpha}$ .  $\diamond$*

The above is observed in [O2]. It is a consequence of the tower argument used in [FHS] — at the top of the tower, and then at each stage in descending the tower, we can assume that we can perform the surgeries so as to retain the curve  $\alpha$  as a subset of our surface. (It is likely that Theorem 3.4 is true more generally, though we won't explore that here.)

Recall that a 3-manifold is “haken” if it contains an embedded incompressible surface. We have the following theorem of Waldhausen [Wal]:

**Theorem 3.5 :** *Suppose that  $\Lambda$  and  $\Lambda'$  are compact haken 3-manifolds with boundary. Then any relative homotopy equivalence between  $(\Lambda, \partial\Lambda)$  and  $(\Lambda', \partial\Lambda')$  is relatively homotopic to a homeomorphism.  $\diamond$*

In Section 14 we will be applying Theorem 3.5 to the complement of a set of unknotted tori in  $\Sigma \times \mathbf{R}$ . Some further properties of product manifolds are discussed there.

We will need various facts concerning the geometry of hyperbolic 3-manifolds. First we note the well known thick-thin decomposition.

Let  $M$  be a complete hyperbolic 3-manifold. The open  $\eta$ -thin part of  $M$  set of points of injectivity radius at less than  $\eta$ . The (closed)  $\eta$ -thin part is its closure. The following is a consequence of the Margulis Lemma:

**Theorem 3.6 :** *There is a universal constant,  $\eta_0 > 0$ , such that for all  $\eta \leq \eta_0$ , each component of the thin part of  $M$  is homeomorphic to one of  $D^2 \times S^1$ ,  $S^1 \times \mathbf{R} \times [0, \infty)$  or  $S^1 \times S^1 \times [0, \infty)$ .  $\diamond$*

Here  $D^2$  and  $S^1$  denote the unit disc and circle respectively. We refer to the three homeomorphism types as *Margulis tubes*,  $\mathbf{Z}$ -*cusps* and  $\mathbf{Z} \oplus \mathbf{Z}$ -*cusps* respectively.

We denote by  $\Theta(M)$  the complement of the open thin part, and by  $\Psi(M)$ , the closure of the complement of the union of open cusps. In other words,  $\Psi(M)$  is  $\Theta(M)$  union all the Margulis tubes. We refer to  $\Theta(M)$  and  $\Psi(M)$  respectively as the *thick part* and the *non-cuspidal* part of  $M$ . In practice, it will be convenient to generalise the construction, by allowing different components of the thin part to be defined by different “constants”, provided each of these “constants” lie between two fixed positive constants less than  $\eta_0$ . This makes no essential difference to our arguments, and will be discussed further in Section 10.

For completeness, we state again the Tameness Theorem of Bonahon, Agol, Calegari and Agol [Bon,Ag,CalG]:

**Theorem 3.7 :** *If  $\pi_1(M)$  is finitely generated, then  $\Psi(M)$  is topologically finite.*  $\diamond$

In other words, we have a compactification  $\bar{\Psi}(M)$  of  $\Psi(M)$  as described in Section 0.

We can write  $M = \mathbf{H}^3/\Gamma$ , where  $\Gamma \equiv \pi_1(M)$  acts properly discontinuously on  $\mathbf{H}^3$ . We decompose  $\partial\mathbf{H}^3$  as the limit set,  $L(\Gamma)$ , and discontinuity domain,  $D(\Gamma)$ . Suppose that  $M'$  is another complete hyperbolic 3-manifold, with  $M' = \mathbf{H}^3/\Gamma'$ , where  $\Gamma \cong \Gamma'$ .

For the final part of the proof of the Ending Lamination Theorem we need the rigidity result of Sullivan [Sul] which says that there is no quasi-conformal deformation supported on the limit set. More formally:

**Theorem 3.8 :** *Suppose that  $f : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$  is a quasiconformal map, equivariant with respect to the respective actions of  $\Gamma$  and  $\Gamma'$  (so that  $f(D(\Gamma)) = D(\Gamma')$ ), and that  $f|_{D(\Gamma)}$  is conformal. Then  $f$  is conformal (hence a Möbius transformation).*  $\diamond$

(We remark that a consequence of the Tameness Theorem is the Ahlfors Measure Conjecture. This tells us that either  $D(\Gamma) = D(\Gamma')$  are both empty, or both have full measure. In the latter case, Theorem 3.8 is an immediate consequence of the fact that a quasiconformal map that is conformal almost everywhere is conformal everywhere [LV]. Of course, Sullivan’s rigidity theorem predates the Tameness Theorem.)

We recall the notion of a quasi-isometry of between two path-metric spaces (see, for example, [GhH] and Section 9). The following is well known:

**Theorem 3.9 :** *Let  $f : \mathbf{H}^3 \rightarrow \mathbf{H}^3$  be a quasi-isometry, then there is a unique extension  $f : \mathbf{H}^3 \cup \partial\mathbf{H}^3 \rightarrow \mathbf{H}^3 \cup \partial\mathbf{H}^3$  which is continuous at every point of  $\mathbf{H}^3$ . Moreover,  $f|_{\partial\mathbf{H}^3}$  is a quasiconformal homeomorphism of  $\partial\mathbf{H}^3$ .*

We will need a version of Thurston’s Uniform Injectivity Theorem for decomposable manifolds. This is discussed in Section 25.

Central to both the statement and the proof of the Ending Lamination Theorem is the curve graph,  $\mathcal{G}(\Sigma)$ , associated to a compact surface  $\Sigma$ . We have already noted in Section 2 that this is Gromov hyperbolic and that its boundary can be identified with the space of arational laminations. Further analysis of the structure of  $\mathcal{G}(\Sigma)$  is given in [MasM2].

The fact that  $\mathcal{G}(\Sigma)$  is Gromov hyperbolic is a central ingredient to the proof of the Ending Lamination Theorem. For the most part, this is already exploited in the proving other results quoted here — notably the a-priori bounds theorem. We will revisit this in Section 21 to prove a variant of the a-priori bound theorem applicable in the decomposable case. For this, we will need the following standard fact about Gromov hyperbolic spaces.

**Theorem 3.10 :** *Suppose that  $(\mathcal{G}, d)$  is a  $k$ -hyperbolic space in the sense of Gromov. Given any  $a > 0$ , there exist  $r, c > 0$  depending only on  $k$  and  $a$  with the following property. Suppose that  $\alpha \subseteq \mathcal{G}$  is a geodesic segment, and denote its  $r$ -neighbourhood by  $N(\alpha, r)$ . Let  $\beta$  be any path in  $\mathcal{G}$  with  $d(\alpha, \beta) \geq r$ , connecting any two points,  $x, y \in \partial N(\alpha, r)$ . Then the length of  $\beta$  is at least  $ad(x, y) - c$ .*

This is a standard fact, used for example in proving the “stability of quasigeodesics” in a Gromov hyperbolic, see for example [GhH].

A key ingredient to the proof of the Ending Lamination Theorem is the “a-priori bounds” result of Minsky [Mi4]. This uses the notion of a “tight geodesic” in  $\mathcal{G}(\Sigma)$ , [MasM2] which will be defined in Section 6. (We use a slightly more general notion from the original.) We quote the following results directly from [Bow4], though they essentially follow from the arguments of [Mi4].

Let  $M$  be a complete hyperbolic 3-manifold with non-cuspidal part  $\Psi(M)$  homeomorphic to  $\Sigma \times \mathbf{R}$ , where  $\Sigma$  is a compact surface. Given a free homotopy class  $\alpha$  of closed curve in  $\Sigma$ , we write  $\alpha^*$  for its closed geodesic realisation in  $M$ . If  $\alpha$  is simple in  $\Sigma$ , then we can assume this to lie in  $\Psi(M)$ , provided the Margulis constant is chosen small enough in relation to  $\kappa(\Sigma)$  (see for example, the discussion in Section 18). We write  $l_M(\alpha) = \text{length}(\alpha^*)$ .

**Theorem 3.11 :** *Suppose that  $\alpha, \beta, \gamma \in X(\Sigma)$ , and that  $\gamma$  lies on some tight geodesic from  $\alpha$  to  $\mathcal{G}(\Sigma)$ . Then  $l_M(\gamma)$  is bounded above in terms of  $\kappa(\Sigma)$  and  $\max\{l_M(\alpha), l_M(\beta)\}$ .*

The following weakens the conditions on the endpoints of the geodesic at the cost of requiring us to stay at least a fixed distance from the endpoints.

**Theorem 3.12 :** *Given  $r, \kappa$  there exists  $R$  with the following property. Suppose that  $\alpha, \beta, \gamma, \delta, \epsilon \in X(\Sigma)$ , and that  $\gamma$  lies on some tight geodesic from  $\delta$  to  $\epsilon$  in  $\mathcal{G}(\Sigma)$ , where  $d_{\mathcal{G}}(\alpha, \delta) \leq r$ ,  $d_{\mathcal{G}}(\beta, \epsilon) \leq r$ ,  $d_{\mathcal{G}}(\gamma, \delta) \geq R$ ,  $d_{\mathcal{G}}(\gamma, \epsilon) \geq R$ ,  $l_M(\alpha) \leq l$  and  $l_M(\beta) \leq l$ . Then  $l_M(\gamma)$  is bounded above in terms of  $l, r$  and  $\kappa(\Sigma)$ .*

(In the above two theorems we can allow for “accidental parabolics”. These correspond to simple closed curves in  $\Sigma$  which give rise to cusps of  $M$  and are deemed to have 0 length. This was included in the description given in [Bow4], though we won’t need them here.)

There are also versions applicable to subsurfaces, see Theorem 15.2 here.

We remark that we make no use in this paper of the important tool of “subsurface projections” introduced in [MasM], and central to the original proof of [Mi4, BrocCM]. Of course, this notion finds many other applications elsewhere. A discussion of subsurface projections in connection with the models we construct here can be found in [Bow5].

Finally, for the decomposable case, we use ideas from the theory of  $\text{CAT}(-1)$  spaces. These are variations of notions used in [Som1]. These are discussed further in [Bow6] and in Section 19 of the present paper. A standard reference for  $\text{CAT}(-1)$  spaces is [BrH].

Various other constructions will be outlined where needed.

#### 4. Outline of proof.

As mentioned in Section 1, the proof involves constructing a “model” manifold,  $P$ , based on a topological data, together with a prescribed set of “end invariants”. Such a manifold will approximate the geometry of a complete hyperbolic 3-manifold,  $M$ . We assume that  $\pi_1(M)$  is finitely generated, and write  $\Psi = \Psi(M)$  for the non-cuspidal part.

We construct the model, starting with  $\Psi$ , viewed as a topological 3-manifold. We assume that we have a formal partition of its ends into two sets deemed “geometrically finite” or “degenerate”, and an assignment of an element  $a(e) \in \partial\mathcal{G}(e)$  to each simply degenerate end,  $e$ . From this data, we construct a “geometric model”,  $P$ . This a riemannian manifold, with a preferred submanifold,  $\Psi(P)$ , called the “non-cuspidal part” of  $P$ . The non-cuspidal part is homeomorphic to  $\Psi$  and there is a preferred homotopy class of homeomorphism from  $\Psi$  to  $\Psi(P)$ . This model has the following property:

**Proposition 4.1 :** *Suppose that  $M$  is a complete hyperbolic 3-manifold, and that  $g' : \Psi \rightarrow \Psi(M)$  is a homeomorphism. Suppose that  $g$  respects the partition of ends into geometrically finite and simply degenerate, and suppose that  $g_*(a(e)) = a(M, g(e))$  for each simply degenerate end  $e$ . Then there is a proper lipschitz homotopy equivalence  $f : P \rightarrow M$  such that  $f^{-1}(\Psi(M)) = \Psi(P)$ , and  $f|_{\Psi(P)} : \Psi(P) \rightarrow \Psi(M)$  is properly homotopic to  $g$ , and such that a lift of  $f$  to the universal covers,  $\tilde{f} : \tilde{P} \rightarrow \tilde{M} \equiv \mathbf{H}^3$  is a quasi-isometry.*

We note that we have not used the end invariants of the geometrically finite ends in the construction of  $P$ . These are not needed to construct a quasi-isometry of the type described. In our approach, the geometrically finite end invariants are only brought into play at the final stage of the proof (see Sections 16 and 23). (Alternatively, one could use this information to construct a model with stonger properties at this stage. This is the approach taken by Minsky [Mi4].)

We first outline how the construction of  $P$ , and the remainder of the argument works in the case where  $\Psi(M)$  is homeomorphic to  $\Sigma \times \mathbf{R}$ . For simplicity we focus on the doubly degenerate case, i.e. where both ends of  $\Psi(M)$  are degenerate.

To begin with, we describe the construction of the model manifold. Thus,  $\Psi(M) \cong \Sigma \times \mathbf{R}$  where  $\Sigma$  is compact orientable surface. Recall that  $\Theta(M)$  denotes the thick part of  $M$ .

Let  $\mathcal{G}(\Sigma)$  be the curve graph associated to  $\Sigma$  as described in Section 2. The end invariants of  $M$  give us two distinct points,  $a, b \in \partial\mathcal{G}(\Sigma)$  (Section 11). We now have a tight bi-infinite geodesic in  $\mathcal{G}(\Sigma)$  from  $a$  to  $b$ . This consists of a sequence of curves,  $(\alpha_i)_{i \in \mathbf{Z}}$ , in  $\Sigma$  with  $\alpha_i \cap \alpha_{i+1} = \emptyset$  for all  $i$ . One way to imagine this would be to construct what we

call a “ladder” in  $\Sigma \times \mathbf{R}$ . This is a sequence of annuli,  $\Omega_i = \alpha_i \times I_i$ , where  $I_i \subseteq \mathbf{R}$  is a closed interval, and  $I_i \cap I_j$  is a non-trivial interval if  $|i - j| \leq 1$  and  $I_i \cap I_j = \emptyset$  otherwise. A consequence of the a-priori bounds theorem (see below) is that the closed geodesics in  $M$  in the corresponding homotopy classes all have bounded length. Thus, the ladder in some way reflects the geometry of  $M$ . However, we need more than this to determine a riemannian metric on our model. We can extend  $\{\Omega_i \mid i \in \mathbf{Z}\}$  to a locally finite set,  $\mathcal{W}$ , of disjoint annuli of the form  $\alpha \times I$ , where  $\alpha \subseteq \Sigma$  is a curve and  $I \subseteq \mathbf{R}$  is a closed interval, and which is “complete”. Completeness means that for each  $t \in \mathbf{R}$ ,  $(\Sigma \times \{t\}) \cap \bigcup \mathcal{W}$  is either a pants decomposition of  $\Sigma$ , or a pants decomposition missing one curve. In the former case, each complementary component is a three-holed sphere. In the latter case there will be one component that is either a four-holed sphere or a one-holed torus. (Combinatorially this is essentially the same as a path in the pants graph. In [MasM2], the analogous procedure is expressed in terms of a resolution of a hierarchy.)

There are many ways one might construct such a complete system of annuli. The only properties we need are laid out in Theorem 4.1, labelled (P1)–(P4). (Some additional conditions are added in Lemmas 4.2 and 4.3 for the purposes of giving an inductive construction, but these are not needed for applications.) We have (P1) that the annuli arise from a bounded iteration of the tight geodesic construction. This is needed in order to obtain the a-priori bound on the length of the corresponding curves in  $M$ . More precisely:

**Theorem 4.2 :** *There is some  $L$  depending only on the topological type of  $\Sigma$  such that if  $\Omega \in \mathcal{W}$ , then the corresponding closed geodesic in  $M$  has length at most  $L$ .*

We require (P2) that no two annuli are homotopic, i.e. have the same base curve. This ensures that no two tubes in the model will correspond to the same Margulis tube in  $M$ . This is in turn needed to ensure that the map constructed between thick parts is a homotopy equivalence. We require a “tautness” condition (P3) and (P4), expressed in terms of ladders, which says that the annuli follow a geodesic in the curve graph up to bounded distance. This will ensure that the map from our model to  $M$  does not crumple up or fold back over large distances. For the inductive structure of the proof we also require this to hold on a class of subsurfaces of  $\Sigma$ , in the appropriate sense. This is incorporated into the statement of properties (P3) and (P4).

Given our annulus system  $\mathcal{W}$ , the construction of our model space,  $P$ , is relatively straightforward. First, we cut open each annulus of  $\mathcal{W}$  so as to give us a manifold  $\Lambda = \Lambda(\mathcal{W})$  with a toroidal boundary component for each annulus. The fact that the local combinatorics of  $\mathcal{W}$  are bounded means that we can give  $\Lambda$  a riemannian metric that is natural up to bilipschitz equivalence, and such that each toroidal boundary is euclidean. The exact construction doesn’t much matter, but a precise prescription is given in Section 8. We can now glue in a “model” Margulis tube to each toroidal boundary component. This gives the non-cuspidal part  $\Psi(P) \cong \Sigma \times \mathbf{R}$  of the model space. To obtain  $P$  from  $\Psi(P)$  we simply attach cusp to each boundary component. (This construction essentially just a variant of [Mi4], except that our combinatorial requirements are weaker.)

The construction of a map  $f : P \rightarrow M$  now uses the “a-priori bound” result. Each model Margulis tube gets sent either to a Margulis tube in  $M$ , or to a closed geodesic of

bounded length (possibly still quite long though). We write  $\mathcal{T}(P)$  for the former set of model tubes, and write  $\Theta(P) = P \setminus \text{int} \bigcup \mathcal{T}(P)$  for the “thick” part of the model space. The construction tells us that  $\Theta(P) = f^{-1}\Theta(M)$  and that  $f|_{\Theta(P)}$  is lipschitz. These constructions are described in Sections 7 and 8. (It is possible that  $\Theta(P)$  may depend on  $M$ , but it doesn’t matter what strategy we adopt to construct a map from the model space, once the model space has been defined.)

Topological considerations described in Section 3 now imply that the map  $f : \Theta(P) \longrightarrow \Theta(M)$  is a homotopy equivalence, and so we get an equivariant map  $\tilde{f} : \tilde{\Theta}(P) \longrightarrow \tilde{\Theta}(M)$  of universal covers. Some effort is now invested in showing that  $\tilde{f}$  is a quasi-isometry. Given this, the argument can be completed as follows. We first use the fact (Lemma 4.8) that, when lifted to an appropriate cover, the boundary of a model tube is quasi-isometrically embedded. This then implies that the map between this boundary and the corresponding boundary from  $M$  is a quasi-isometry in the induced euclidean path metrics. This in turn gives us a means (Section 9) of arranging that  $f$  is lipschitz on each tube, and a quasi-isometry between the universal covers of tubes. We can also extend over cusps by sending geodesic rays locally isometrically to geodesics rays. We then have a lipschitz map  $P \longrightarrow M$ , and the fact that the lift to the universal covers is a quasi-isometry is relatively straightforward given what we have shown. The details are described in Section 15.

It still remains to explain why the lift,  $\tilde{f} : \tilde{\Theta}(P) \longrightarrow \tilde{\Theta}(M)$  is a quasi-isometry. This is where tautness comes into play. As noted in Section 5 one can show (see Section 13) that two curves of bounded length and a bounded distance apart in  $M$  are also a bounded distance apart in the curve graph,  $\mathcal{G}(\Sigma)$ . Tautness gives us some control on how far apart such curves can be in the model space. This is a start, but is not sufficient. We need to construct topological barriers in  $P$  so that points separated by a barrier in  $P$  get mapped to points separated by a similar barrier in  $M$  — taking appropriate account of homotopy classes of paths, since we are really interested in the lift to universal covers. Our barriers are called “bands”. A band in  $\Psi(M)$  is a product,  $\Phi \times I$ , where  $\Phi$  is a subsurface of  $\Sigma$ , and  $I$  a compact interval, and such that  $\partial\Phi \times I$  lies in the boundary of a model tube. Its intersection with  $\Theta(M)$  is a “band” in  $\Theta(M)$ . We shall usually insist that  $\partial\Phi \times \{0, 1\} \subseteq \Theta(M)$ . Much of the second half of Section 8 and Section 14 are devoted to analysing such bands.

To give an idea of how this works, we consider a very simple case. There is a vague sense in which a point,  $x$ , of  $M$  approximately determines a “fibre” hyperbolic structure on  $\Sigma$  — the domain of a lipschitz “pleated surface” with image close to  $x$ . In the bounded geometry case, where there are no Margulis tubes, this structure progresses at a roughly uniform rate in the  $\mathbf{R}$  direction. (Indeed it stays close to a Teichmüller geodesic [Mi1, Mi2].) A slightly more complicated situation is where we have just one (unknotted) Margulis tube,  $T$ , corresponding to a curve  $\alpha \subseteq \Sigma$ . Let us suppose that  $\alpha$  separates  $\Sigma$  into two subsurfaces  $\Phi_1$  and  $\Phi_2$ . One possibility it that the area of  $\partial T$  is bounded. As we cross the tube, we will be twisting our fibre structure along  $\alpha$ , but doing little to the structure on  $\Phi_1$  and  $\Phi_2$ . Alternatively, while we are crossing  $\alpha$ , it may be that the structure on  $\Phi_1$  changes a lot. In this case, the length of  $\partial T$  in the direction transverse to the longitude of  $T$ , becomes very large. We get a band,  $B_1 \cong \Phi_1 \times I \subseteq \Theta(M)$ , with  $B_1 \cap \partial T = \partial\Phi_1 \times I$ . This will be very long in the  $I$  direction, where the change in the structure takes place. We might get another similar band,  $B_2 \cong \Phi_2 \times I$ . In this case,  $B_1$  and  $B_2$  together act as barriers

between the two ends of  $\Theta(M)$ . If there is no band on the  $\Phi_2$  side of  $\alpha$ , then we could sneak around  $B_1$  on the other side of  $T$ . However, this path will be in the “wrong” homotopy class, and the lift of  $B_1$  to the cover,  $\tilde{\Theta}(M)$ , will still serve as a barrier there. In general we may get a very complicated system of nested bands. A general decomposition of  $M$  into bands is discussed in [Bow3], though we won’t be needing any result from that paper here.

The logic of our argument is somewhat different from the motivation of the last paragraph. We construct our bands first in the model space (Section 8). We then show that they correspond to bands in  $M$  (Lemma 13.6). The manner in which they form “barriers” is rephrased in terms of pushing around paths and discs (Section 14). We need some general principles of bounded geometry to complete the argument. These are discussed in Section 12.

Finally in Sections 16 we discuss how all this works in the general indecomposable case. The only really new ingredient needed is an analysis of the geometry of geometrically finite ends, but this is something pretty well understood.

The remainder of the paper (Sections 17 to 25) will be aimed at dealing with the compressible case. Here some of the results we cited from elsewhere, such as the a-priori bound theorem, and the fact that Margulis tubes are unlinked, are not immediately available, so we have to revisit these. The technique we use will be to isolate the ends of the three manifolds by cutting out a compact polyhedral complex. For this we use the theory of CAT(-1) spaces, described in Section 19.

We remark that most of the arguments presented here should be adaptable to the case of pinched negative curvature (cf. [Can]), though of course, the final rigidity conclusion is no longer valid. One will need to reinterpret various things, for example, the boundaries of Margulis tubes and cusps will no longer be euclidean.

In fact, as is shown in [Bow8] one can also generalise to the case where one assumes the universal cover is Gromov hyperbolic. For this we need a few additional assumptions, notably that the thin part is standard.

## 5. Surface groups.

We describe some additional information we can obtain from the Ending Lamination Theorem, or its proof, in the case where  $\pi_1(M) \cong \pi_1(\Sigma)$ , where  $\Sigma$  is a compact surface. Indeed this has applications beyond hyperbolic geometry. We will assume that the cusps of  $M$  are in bijective correspondence with the components of  $\partial\Sigma$ . We are assuming that everything is orientable, so Bonahon’s Tameness Theorem tells us that  $\Psi(M) \cong \Sigma \times \mathbf{R}$ , where  $\Psi(M)$  is the  $\eta$ -non-cuspidal part of  $M$ . We write  $d$  for the path metric on  $\Psi(M)$  induced from  $M$ .

We write  $\mathcal{T}$  for the set of  $\eta$ -Margulis tubes in  $M$ . Thus,  $\bigcup \mathcal{T} \subseteq \text{int } \Psi(M)$ , and  $\Theta(M) = \Psi(M) \setminus \text{int}(\bigcup \mathcal{T})$  is the  $\eta$ -thick part of  $M$ .

The following result of Otal [O2] will be reproven in Section 11 (though our argument will not give a computable estimate on the constant  $\eta$  which is implicit in the original).

**Theorem 5.1 :** *There is a constant,  $\eta(\Sigma)$ , depending only on the topological type of  $\Sigma$ , such that if  $\eta \leq \eta(\Sigma)$ , then the set of  $\eta$ -Margulis tubes in  $\Psi(M)$  is topologically unlinked.*

◇

In other words, we can choose the homeomorphism of  $\Psi(M)$  with  $\Sigma \times \mathbf{R}$  in such a way that the core of each Margulis tube lies in  $\Sigma \times \mathbf{Z}$ . (See Proposition 7.1 of this paper.)

For the purposes of this section, we will assume that  $\Psi(M)$  and  $\Theta(M)$  are defined by some fixed  $\eta \leq \eta(\Sigma)$ . (One can allow for some flexibility, without any essential change, as discussed in Section 10.)

Given  $\gamma \in X(\Sigma)$ , let  $\gamma_M$  be the geodesic representative of  $\gamma$  in  $M$ , and write  $l_M(\gamma) = \text{length}(\gamma_M)$ . Let  $X(M, l) = \{\gamma \in X(\Sigma) \mid l_M(\gamma) \leq l\}$ . Note that it is a consequence of Theorem 5.1, that  $X(M, \eta)$  is precisely the set of core curves of Margulis tubes in  $M$ .

Recall that a subset,  $Y$ , of  $\mathcal{G}(\Sigma)$  is called  $r$ -quasiconvex if any geodesic in  $\mathcal{G}(\Sigma)$  with endpoints in  $Y$  lies in the  $r$ -neighbourhood  $N(Y, r)$  of  $Y$ . The following is a consequence of the arguments used in proving the a-priori bounds theorem:

**Theorem 5.2 :** *There are constants,  $l_0$  and  $r_0$  depending only on  $\kappa(\Sigma)$  such that  $X(M, l_0)$  is  $r_0$ -quasiconvex in  $\mathcal{G}(\Sigma)$ , and such that for all  $l \geq l_0$  we have  $X(M, l) \subseteq N(X(M, l_0), t)$ , where  $t$  depends only on  $\kappa(\Sigma)$  and  $l$ .*

This is given explicitly in [Bow4], though it is also a consequence of the arguments given in [Mi4]. Note that it follows that  $X(M, l)$  is  $r$ -quasiconvex, where  $r$  depends only on  $\kappa(\Sigma)$  and  $l$ .

One can say more:

**Theorem 5.3 :** *There is some  $t_0 \geq 0$  depending only on  $\kappa(\Sigma)$  such that  $X(M, l_0) \subseteq N(\pi, t_0)$ , where  $\pi$  is a geodesic segment in  $\mathcal{G}(\Sigma)$ .*

Since  $\mathcal{G}(\Sigma)$  is hyperbolic,  $\pi$  is determined up to bounded Hausdorff distance, depending on  $\kappa(\Sigma)$ .

Recall that  $\text{core}(M)$  is the convex core of  $M$ . The manifold  $\Psi(M) \cap \text{core}(M)$  is homeomorphic to  $\Sigma \times I$ , where  $I \subseteq \mathbf{R}$  is a compact interval, a ray or all of  $\mathbf{R}$ , depending on whether  $\Psi(M)$  has 0, 1 or 2 degenerate ends. These cases are termed respectively “geometrically finite” (or “quasifuchsian”), “singly degenerate” or “doubly degenerate”. In the three cases, the geodesic  $\pi$  will be a compact interval, a ray, or a bi-infinite geodesic. From the discussion of Section 2, we see that the unbounded ends of  $\pi$  converge to the end invariants of  $M$  in  $\partial\mathcal{G}(\Sigma)$ .

We can relate these more explicitly as follows. Given a path,  $\xi$ , in  $\Psi(M)$ , let  $l_\rho(\xi) = \text{length}(\xi \cap \Theta(M))$ . Given  $x, y \in \Psi(M)$ , let  $\rho_M(x, y) = \inf\{l_\rho(\xi)\}$ , as  $\xi$  ranges over all paths from  $x$  to  $y$  in  $\Psi(M)$ . (In fact, the minimum is attained.) Thus,  $\rho_M$  is a pseudometric on  $\Psi(M)$ , with each Margulis tube having zero diameter.

**Definition :** We refer to  $\rho_M$  as the *electric pseudometric* on  $\Psi(M)$ .

We will refer to a curve,  $\gamma$ , in  $\Psi(M)$  as “simple” if it is homotopic to a simple non-peripheral curve in  $\Sigma$ . We shall write  $[\gamma] \in X(\Sigma)$  for its homotopy class. Now each point of  $\Psi(M)$ , lies in some simple curve in  $\Psi(M)$  of  $d$ -length bounded by some constant depending only on  $\kappa(\Sigma)$ . We may as well denote the length bound by  $l_0$  (as in Theorem 5.2). We shall write  $\gamma_x$  for some choice of such curve. Moreover,  $\gamma_x$ , can be taken to lie in some homotopy fibre of  $\Psi(M)$ , of bounded  $\rho_M$ -diameter. These facts follow from interpolation of pleated surfaces described by Thurston, and one can give explicit computable estimates for the bounds. They are also consequences of the results of Section 13 (though these do not give computable bounds). One immediate consequence is the fact that  $(\Psi(M) \cap \text{core}(M), \rho_M)$  is quasi-isometric to an interval in the real line (again, see Section 13).

One can elaborate on this. The following discussion briefly explains how the various pieces fit together. While it is not need directly for the Ending Lamination Theorem, it finds application elsewhere.

In fact, one can show:

**Theorem 5.4 :** *Suppose that  $\alpha_M, \beta_M$  are simple curves in  $M$  of  $d$ -length at most  $l \geq 0$ . Write  $\alpha = [\alpha_M] \in X(\Sigma)$  and  $\beta = [\beta_M] \in X(\Sigma)$ . Then:*

$$(1) \rho_M(\alpha_M, \beta_M) \leq k_1 d_{\mathcal{G}}(\alpha, \beta) + k_2$$

$$(2) d_{\mathcal{G}}(\alpha, \beta) \leq k_3 \rho_M(\alpha_M, \beta_M) + k_4$$

where  $k_1, k_2, k_3, k_4$  depend only on  $\kappa(\Sigma)$  and  $l$ . ◇

Part (1) of Theorem 5.4 can also be proven using pleated surfaces. This arises from the fact, alluded to in Section 2, that any pair of disjoint simple geodesics in  $M$  can be realised in a pleated surface in  $M$ , and the intersection of such a pleated surface  $\Psi(M)$  has bounded  $\rho_M$ -diameter (see for example, the discussion in [Bow4] and Section 18 here). This argument gives computable bounds on  $k_1$  and  $k_2$ . Part (2) involves quite bit more work. It follows from the existence of model spaces. In fact, in our account of the Ending Lamination Theorem, we use a closely related result as the first step in proving lower bounds in Section 13. As we will mention there, by a slight variation of the argument we can deduce Theorem 5.4 as a corollary (thereby bypassing the remainder of the proof). Unfortunately, this argument does not give us computable bounds on  $k_3$  or  $k_4$ .

Putting Theorem 5.4 together with the previous paragraph, we can relate these various facts. First, we can define a map from  $(\Psi(M) \cap \text{core}(M), \rho_M)$  to  $\mathcal{G}(\Sigma)$  by sending  $x$  to  $[\gamma_x]$ . By Theorem 5.4, this is well defined up to bounded distance. Its image lies in  $X(M, l_0)$ . Moreover, every point in  $X(M, l_0)$  lies a bounded distance from some point in this image. (If  $\alpha \in X(M, l_0)$ , choose any point  $x$  in the closed geodesic,  $\alpha_M^* \subseteq M$ , then  $d_{\mathcal{G}}(\alpha, [\gamma_x])$  is bounded.) The map  $x \mapsto [\gamma_x]$  is therefore a quasi-isometric embedding of  $(\Psi(M) \cap \text{core}(M), \rho_M)$  into  $\mathcal{G}(\Sigma)$ , whose image is a bounded Hausdorff distance from  $X(M, l_0)$ , and hence also from the geodesic  $\pi$  described by Theorem 5.3. We therefore have a quasi-isometry from  $(\Psi(M) \cap \text{core}(M), \rho_M)$  to  $\pi$ , which is natural up to bounded Hausdorff distance. This fits in with the earlier observation that  $(\Psi(M) \cap \text{core}(M), \rho_M)$

is quasi-isometric to a real interval (though the constants we get by this argument are no longer computable).

We can also tie this in with the description of the model space, used in the proof of the Ending Lamination Theorem. This is a riemannian manifold,  $P$ , which a preferred “non-cuspidal” part,  $\Psi(P)$ , diffeomorphic to  $\Sigma$  times a real interval. The construction of  $P$  starts with a (tight) geodesic in  $\mathcal{G}(\Sigma)$ . One can similarly define an electric pseudometric,  $\rho_P$ , on  $P$ . It is easily seen from the construction that every point  $x \in P$ , lies in a curve  $\gamma_x$  of bounded length, which in turn lies in a fibre,  $S(x)$ , of bounded  $\rho_P$ -diameter. By the construction of  $P$ , the distance between two curves,  $[\gamma_x]$  and  $[\gamma_y]$  in  $\mathcal{G}(\Sigma)$  agrees with  $\rho_P(x, y)$  up to linear bounds. These facts can then be translated across to  $M$  via Theorem 15.9.

These observations have applications or potential applications beyond hyperbolic geometry. Some of these are based on the following observation. Although it is not directly relevant to the Ending Lamination Theorem, we note:

**Theorem 5.5 :** *Given any  $\alpha, \beta$  in  $X(\Sigma)$  and any  $\epsilon > 0$ , there is a complete hyperbolic 3-manifold,  $M$ , with  $\Psi(M) \cong \Sigma \times \mathbf{R}$ , and with  $\alpha, \beta \in X(M, \epsilon)$ .  $\diamond$*

In other words, we can realise any pair of curves  $\alpha, \beta$  as arbitrarily short geodesics,  $\alpha_M^*, \beta_M^*$ , in some such manifold. Theorem 5.5 is a simple consequence of the deformation theory of quasifuchsian groups. It is given explicitly in [Bow5].

By Theorem 5.4,  $d_{\mathcal{G}}(\alpha, \beta)$  agrees with  $\rho_M(\alpha_M, \beta_M)$ . up to linear bounds depending only on  $\kappa(\Sigma)$ . This finds application, for example, in [Bow4], [Bow5] and [Ta].

## 6. The curve graph.

Let  $\Sigma$  be a compact surface, and  $\kappa(\Sigma)$  be the complexity, as defined in Section 2. We assume that  $\kappa(\Sigma) \geq 1$ . We recall the definition of the curve graph,  $\mathcal{G}(\Sigma)$ , with vertex set  $X(\Sigma) = V(\mathcal{G}(\Sigma))$  as defined in Section 2. We write  $d = d_{\mathcal{G}}$  for the combinatorial metric on  $\mathcal{G}(\Sigma)$ . Given  $\alpha, \beta \in \mathcal{G}(\Sigma)$ , we write  $\iota(\alpha, \beta)$  for the geometric intersection number (i.e., the minimal number of intersections among realisations of  $\alpha$  and  $\beta$  in  $\Sigma$ ). It follows from work of Lickorish that  $d(\alpha, \beta)$  is bounded above in terms of  $\iota(\alpha, \beta)$ . In fact, one can show that  $d(\alpha, \beta) \leq \iota(\alpha, \beta) + 1$ .

It is often convenient to fix some hyperbolic structure on  $\Sigma$  with geodesic boundary,  $\partial\Sigma$ . In this way, each curve is realised uniquely as a closed geodesic in  $\Sigma$ , and we can use the same notation for a curve and its realisation. This serves purely to simplify the description of certain combinatorial constructions, and bears no relation to the various geometric structures in which we have a genuine interest.

**Definition :** A *multicurve*,  $\gamma$ , in  $\Sigma$  is a non-empty disjoint union of curves.

We write  $X(\gamma) \subseteq X(\Sigma)$  for the set of components of  $\gamma$ . A multicurve is *complete* if it has maximal cardinality. In this case, each component of  $\Sigma \setminus \gamma$  is a 3HS. We note that the number of curves in a complete multicurve is equal to  $\kappa(\Sigma)$ .

We noted in Section 2 that  $\mathcal{G}(\Sigma)$  is Gromov hyperbolic [MasM1], and write  $\partial\mathcal{G}(\Sigma)$  for the Gromov boundary.

Suppose, for the moment, that  $\kappa(\Sigma) \geq 2$ . A *multigeodesic* is a sequence  $(\gamma_i)_i$  of multicurves such that for all  $i, j$  and all  $\alpha \in X(\gamma_i)$  and  $\beta \in X(\gamma_j)$ ,  $d(\alpha, \beta) = |i - j|$ . It is *tight* if for all non-terminal indices,  $i$ , any curve that crosses  $\gamma_i$  must also cross either  $\gamma_{i-1}$  or  $\gamma_{i+1}$ . A *tight geodesic*,  $(\alpha_i)_i$  is a sequence of curves such that there exists a tight multigeodesic  $(\gamma_i)_i$  such that  $\alpha_i \in X(\gamma_i)$  for all  $i$ . (Note that this “tight” terminology is now standard, if a little confusing: A tight geodesic is a geodesic in  $\mathcal{G}$  in the usual sense. However it need not be a tight as a multigeodesic, in the sense defined. In most cases we will be talking about tight geodesics. We will only need to specify tight multigeodesic for a construction in Section 8: see Lemma 8.2.) The notion of tightness was introduced in [MasM2]. They show that the set of tight geodesics between any two curves is non-empty and finite. Other arguments for finiteness are given in [Bow4] and [Sh], the latter giving explicit bounds. For a closed surface, they also follow from [L].

One can make a stronger statement, for example:

**Lemma 6.1 :** *Suppose that  $(\alpha_i)_{i \in \mathbf{N}}$  and  $(\beta_i)_{i \in \mathbf{N}}$  are sequences of curves, each converging to a point of either  $\partial\mathcal{G}(\Sigma)$  or  $X(\Sigma)$ . Then for any bounded set  $A \subseteq X$ , there is a finite subset,  $B \subseteq A$ , such that for all sufficiently large  $i, j$  any curve in  $A$  also lying on any tight geodesic from  $\alpha_i$  to  $\beta_j$  must lie in  $B$ .*

**Proof :** Let  $a, b \in V(\mathcal{G}) \cup \partial\mathcal{G}$  be the limits of  $(\alpha_i)_i$  and  $(\beta_i)_i$ . If  $a, b \in V(\mathcal{G})$ , then the result is immediate from the finiteness of tight geodesics between two points [MasM2]. Suppose  $a, b \in \partial\mathcal{G}$ . If  $a = b$ , we can take  $B = \emptyset$ . If  $a \neq b$ , the result follows from other finiteness results for tight geodesics. For example, in [Bow4], it is shown that if  $\alpha, \beta \in X(\Sigma)$  and  $r \in \mathbf{N}$ , then there is some finite  $C \subseteq X(\Sigma)$  such that if  $(\gamma_i)_{i=0}^p$  is a tight geodesic with  $d(\alpha, \gamma_0) \leq r$  and  $d(\beta, \gamma_p) \leq r$ , then  $\gamma_i \in C$  for all  $i$  with  $12r \leq i \leq p - 12r$ . The rest is just an exercise in hyperbolic spaces. We can take  $\alpha$  and  $\beta$  arbitrarily close to  $a$  and  $b$ , and  $r \geq 0$  so that for all  $i$  and  $j$  sufficiently large, any geodesic from  $\alpha_i$  to  $\beta_j$  meets both  $N(\alpha, r)$  and  $N(\beta, r)$ . Choosing  $\alpha$  and  $\beta$  far away from our bounded set  $A$ , the result now follows. Finally the case where  $a \in V(\mathcal{G})$  and  $b \in \partial\mathcal{G}$  follows by a variation on the above result, namely if  $\gamma_0 = \alpha$  and  $d(\beta, \gamma_p) \leq r$  and then  $\gamma_i$  lies in a finite subset for all  $i \leq p - 12r$ .  $\diamond$

This result allows us to use diagonal sequence arguments. For example, we obtain the fact [MasM2] that any two boundary points are connected by a bi-infinite tight geodesic.

In the case where  $\kappa(\Sigma) = 1$  we have noted that  $\mathcal{G}(\Sigma)$  is a Farey graph. In this case, every geodesic is deemed to be tight. The above statement, in particular Lemma 6.1, remain valid, and can be verified directly in that case.

By a *subsurface* of  $\Sigma$  we mean the closure, in  $\Sigma$ , of a non-empty connected open subsurface  $\text{int}(\Phi)$  of  $\Sigma$  with geodesic boundary,  $\partial\Phi$ . We write  $\partial^\Sigma\Phi = \partial\Phi \setminus \partial\Sigma$ . We express it in this way since we want to allow the possibility of two boundary curves in  $\partial^\Sigma\Phi$  being identified to a single curve in  $\Sigma$ . (We could homotope  $\Phi$  to an embedded surface with a complementary annulus so that these boundary curves become genuinely distinct, though for most purposes, it will be convenient to realise things with respect to some

fixed hyperbolic structure.) We are allowing  $\Sigma$  as a subsurface of itself. A subsurface is not allowed to be a disc or an annulus. Note that if  $\Phi'$  is a proper subsurface of  $\Phi$  then  $\kappa(\Phi') < \kappa(\Phi)$ .

The following definitions arise out of the discussion of “hierarchies” in [MasM2]. Let  $\kappa = \kappa(\Sigma)$ . Given  $Q \subseteq X(\Sigma)$  and  $k \in \mathbf{N}$ , let  $Y_k(Q)$  be  $Q$  together with all those curves  $\gamma \in X(\Sigma)$  such that there is some subsurface  $\Phi \subseteq \Sigma$  with  $X(\partial^\Sigma \Phi) \subseteq Q$  and  $2 \leq \kappa(\Phi) \leq \kappa - k + 1$ , and two curves  $\alpha, \beta \in Q \cap X(\Phi)$ , such that  $\gamma$  lies on some tight geodesic in  $\mathcal{G}(\Phi)$  from  $\alpha$  to  $\beta$ . Note that, for  $k \geq \kappa$  there is no such subsurface, so  $Y_k(Q) = Q$ . For any  $k$ , we set  $Y^k(Q) = Y_k Y_{k-1} Y_{k-2} \cdots Y_1(Q)$ , and set  $Y^\infty(Q) = Y^\kappa(Q)$ .

Note that  $Y^\infty(Q)$  contains the union,  $Y_0(Q)$ , of all tight geodesics between any pair of points of  $Q \subseteq X(\Sigma)$ . (For the first step, we are allowing  $\Phi = \Sigma$ ). However, all constructions involving proper subsurfaces occur in a 1-neighbourhood in  $X(\Sigma)$  of a curve already constructed. In particular, we see that  $Y^k(Q) \subseteq N(Y_0(Q), k)$  and so  $Y^\infty(Q) \subseteq N(Y_0(Q), \kappa)$ . If  $Q$  is locally finite, then it follows that  $Y(Q)$  is locally finite (since only finitely many subsurfaces enter into the construction in any bounded set). Thus, inductively,  $Y_k(Q)$  is locally finite, and it follows that  $Y^\infty(Q)$  is locally finite. Also, Lemma 6.1, tells us that:

**Lemma 6.2 :** *Suppose that  $(\alpha_i)_{i \in \mathbf{N}}$  and  $(\beta_i)_{i \in \mathbf{N}}$  are sequences of curves, each converging to a point of either  $\partial \mathcal{G}(\Sigma)$  or  $X(\Sigma)$ . Then for any bounded set  $A \subseteq X$ , there is a finite subset,  $B \subseteq X$ , such that for all sufficiently large  $i, j$ ,  $A \cap Y^\infty(\{\alpha_i, \beta_j\}) \subseteq B$ .  $\diamond$*

Thus, after passing to a subsequence, we can assume that  $Y^\infty(\{\alpha_i, \beta_i\})$  stabilises on each bounded set to some finite set, thereby giving us a locally finite limit.

We have the following variation for subsurfaces of complexity 1. Given  $Q \subseteq X(\Sigma)$ , define  $\bar{Y}(Q)$  as for  $Y_k(Q)$ , replacing the statement “ $2 \leq \kappa(\Phi) \leq \kappa - k + 1$ ” by the statement “ $\kappa(\Phi) = 1$ ”. (All geodesics in  $\mathcal{G}(\Phi)$  are deemed to be “tight” in this case.) We write  $\bar{Y}^\infty(Q) = \bar{Y}(Y^\infty(Q))$ . A similar discussion applies. In particular, Lemma 2.2 holds with  $\bar{Y}^\infty$  replacing  $Y^\infty$ .

A number of variations on the above definitions are possible, and would probably serve just as well. We have chosen a formulation that works well with our inductive constructions below, and in Section 4.

A *path of multicurves* is a sequence  $(\gamma_i)_i$  of multicurves such that for each  $i$ ,  $\gamma_{i+1}$  is obtained from  $\gamma_i$  by either adding or deleting one component. A path  $(\alpha_i)_i$  in  $\mathcal{G}$  determines a path of multicurves by inserting  $\alpha_i \cup \alpha_{i+1}$  between  $\alpha_i$  and  $\alpha_{i+1}$ .

**Lemma 6.3 :** *Suppose that  $\alpha, \beta \in X(\Sigma)$  with  $d(\alpha, \beta) \geq 3$ . Then there is a complete multicurve,  $\gamma$ , with  $\alpha \subseteq \gamma$  and  $X(\gamma) \subseteq Y^\infty(\{\alpha, \beta\})$ .*

**Proof :** Let  $\underline{\gamma} = (\gamma_i)_{i \in D}$  be a path of multicurves with indexing set  $D = \{0, \dots, m\}$ , so that  $\gamma_0 = \alpha$  and  $\gamma_m = \beta$ . Given any  $i \in D$ , write  $c_i = |X(\gamma_i)|$  for the number of components of  $\gamma_i$ . Thus  $c_0 = c_m = 1$ . We say that  $i$  is a “local minimum” if  $c_{i-1} = c_{i+1} = c_i + 1$ . We assume:

(\*) If  $i$  is not a local minimum, then  $X(\gamma_i) \subseteq Y^{c_i-1}(\{\alpha, \beta\})$ .

Note that such a path of multicurves exists: just take any tight geodesic from  $\alpha$  to  $\beta$  and construct a path of multicurves from it as described earlier.

Let  $\kappa = \kappa(\Sigma)$ . Given any  $n \in \{1, 2, \dots, \kappa\}$  let  $v_n(\underline{\gamma}) = |\{i \in D \mid c_i = n\}|$  and  $v(\underline{\gamma}) = (v_1(\underline{\gamma}), \dots, v_\kappa(\underline{\gamma})) \in \mathbf{N}^\kappa$ . We order  $\mathbf{N}^\kappa$  lexicographically. It is thus well-ordered. We now choose  $\underline{\gamma}$  so as to minimise  $v(\underline{\gamma})$  among all paths of multicurves (of any length) satisfying (\*) above.

Given any  $i \in D$ , let  $F_i$  be the union of  $\gamma_i$  and all those components of  $\Sigma \setminus \gamma_i$  which are three-holed spheres.

First, we claim that if  $i$  is a local minimum, then  $F_{i-1} = F_{i+1}$ . To see this let  $\gamma_{i-1} = \gamma_i \cup \delta$  and  $\gamma_{i+1} = \gamma_i \cup \epsilon$ . Note that  $i-1$  and  $i+1$  are not local minima, and so  $\delta, \epsilon$  and all the components of  $\gamma_i$  lie in  $Y^{c_i}(\{\alpha, \beta\})$ . Now  $\delta$  and  $\epsilon$  must cross, otherwise we could replace  $\gamma_i$  by  $\gamma_i \cup \delta \cup \epsilon$ , thereby decreasing  $v(\underline{\gamma})$ . Thus,  $\delta$  and  $\epsilon$  lie in the same component,  $\Phi$ , of  $\Sigma \setminus \gamma_i$ . If  $\kappa(\Phi) > 1$ , then we can connect  $\delta$  to  $\epsilon$  by some tight geodesic  $\delta = \delta_0, \delta_1, \dots, \delta_p = \epsilon$  in  $\mathcal{G}(\Phi)$ . Now each  $\delta_j \in Y_{c_{i+1}}(Y^{c_i}(\{\alpha, \beta\})) = Y^{c_i+1}(\{\alpha, \beta\})$ . We can now replace  $\gamma_i$  by the sequence

$$\gamma_i \cup \delta_0 \cup \delta_1, \quad \gamma_i \cup \delta_1, \quad \gamma_i \cup \delta_1 \cup \delta_2, \quad \dots, \quad \gamma_i \cup \delta_{p-1}, \quad \gamma_i \cup \delta_{p-1} \cup \delta_p.$$

To verify (\*), note that the  $\gamma_i \cup \delta_j$  are all at local minima, and that  $\gamma_i \cup \delta_j \cup \delta_{j+1}$  has  $c_i + 2$  components. We have therefore again reduced  $v(\underline{\gamma})$ . We conclude that  $\kappa(\Phi) = 1$ . From this it follows that  $F_{i-1} = F_i \cup \Phi = F_{i+1}$ , as claimed.

Next, we claim that there does not exist any  $i$  with  $c_{i+2} = c_{i+1} + 1 = c_i + 2 = c_{i-1} + 1$ . For in such a case, we have  $F_{i-1} = F_{i+1}$  by the above, and so we could replace  $\gamma_i$  by  $\gamma_{i-1} \cup \delta$  and  $\gamma_{i+1}$  by  $\gamma_i \cup \delta$ , where  $\delta = \gamma_{i+2} \setminus \gamma_{i+1}$ . Note that since  $\gamma_{i+1}$  and  $\gamma_{i+2}$  were not local minima in the original path, all curves lie in  $Y^{c_i+1}(\{\alpha, \beta\})$ . We have reduced  $v(\underline{\gamma})$  giving a contradiction. By the same argument, we cannot have any  $i$  with  $c_{i-2} = c_{i-1} + 1 = c_i + 2 = c_{i+1} + 1$ .

We have thus shown that  $c_i$  increases monotonically from  $c_0 = 1$  up to a maximum value  $c_p = k$  and then alternates between  $k$  and  $k-1$  before decreasing monotonically from  $c_q = k$  down to  $c_m = 1$ . By the first observation, we see that  $F_p = F_q$ . But  $\alpha \subseteq \gamma_p \subseteq F_p$  and  $\beta \subseteq \gamma_q \subseteq F_q$ . Since  $\alpha \cup \beta$  fills  $\Sigma$ , it follows that  $F_p = F_q = \Sigma$ . Thus  $\gamma_i$  is a complete multicurve (i.e.  $c_p = k = \kappa$ ). Moreover,  $\alpha \subseteq \gamma_p$  and  $X(\gamma_p) \subseteq Y^{\kappa-1}(\{\alpha, \beta\})$ . We can thus set  $\gamma = \gamma_p$ .  $\diamond$

## 7. Topology of products.

In this section, we give some discussion to the topology of a product  $\Psi = \Sigma \times \mathbf{R}$ , where  $\Sigma$  is a compact surface. We study collections of unlinked curves in  $\Psi$ . The main result we are aiming for will be Proposition 7.2. We also give a criterion for unlinking (Proposition 7.7), which can be used to reprove Otal's theorem [Ot2], and generalise to the decomposable case.

We will write  $\partial\Psi = \partial\Sigma \times \mathbf{R}$ . We write  $\pi_\Sigma : \Psi \longrightarrow \Sigma$  and  $\pi_V : \Psi \longrightarrow \mathbf{R}$  for the vertical and horizontal projections respectively.

**Definition :** A *horizontal fibre* in  $\Psi$  is a subset of the form  $\Sigma \times \{t\}$  for some  $t \in \mathbf{R}$ .

More generally we have:

**Definition :** A *fibre*,  $S \subseteq \Psi$ , in  $\Sigma$  is the image  $f(\Sigma)$  in  $\Psi$  of an embedding  $f : \Sigma \longrightarrow \Psi$  with  $f^{-1}(\partial\Psi) = \partial\Sigma$  such that  $\pi_\Sigma \circ f$  is homotopic to the identity on  $\Sigma$  (rel  $\partial\Sigma$ ).

If  $S, S'$  are two fibres, we write  $S < S'$  if they are disjoint and  $S'$  can be homotoped to the positive end of  $\Psi$  in  $\Psi \setminus S$ .

The theorem of Brown [Brow] (given as Theorem 3.3 here) tells us that if  $S$  is any fibre in  $\Psi$ , then there is an ambient isotopy sending  $S$  to a horizontal fibre. Inductively, we see that if  $S_1, S_2, \dots, S_n$  are disjoint fibres, then there is an ambient isotopy of  $\Psi$  sending each  $S_i$  to a horizontal fibre,  $S \times \{t_i\}$ . After permuting the indices, we can assume that  $t_1 < t_2 < \dots < t_n$ , and so  $S_1 < S_2 < \dots < S_n$ . From this, we see easily that  $<$  defines a total order on any locally finite set of disjoint fibres. If  $S < S'$  we write  $[S, S']$  for the compact region bounded by  $S$  and  $S'$ .

By a *homotopy fibre*, we mean a map  $f : \Sigma \longrightarrow \Psi$  with  $f^{-1}(\partial\Psi) = \partial\Sigma$  and with  $\pi_\Sigma \circ f$  homotopic to the identity. (We can always take such a map to be in general position.) We will also sometimes refer to its image as a homotopy fibre. We can also define an order on homotopy fibres by writing  $f(\Sigma) < f(\Sigma')$  if  $f(\Sigma) \cap f(\Sigma') = \emptyset$  and  $f(\Sigma')$  can be homotoped out the positive end in  $\Psi \setminus f(\Sigma)$ . This is again a total order on any disjoint, locally finite collection. Moreover, if  $S, S'$  are fibres close to  $f(\Sigma)$  and  $f(\Sigma')$ , then  $S < S'$  if and only if  $f(\Sigma) < f(\Sigma')$ .

**Definition :** By a *curve* in  $\Psi$  we mean a simple closed curve that is not homotopic to a point or into  $\partial\Psi$ .

We want to define the notion of unlinking of curves in  $\Psi$ . This is based on the following:

**Proposition 7.1 :** Let  $\mathcal{L}$  be a locally finite disjoint collection of curves in  $\Psi = \Sigma \times \mathbf{R}$ . The following are equivalent:

- (1) There is a homeomorphism  $f : \Psi \longrightarrow \Psi$  such that  $f(\bigcup \mathcal{L}) \subseteq \Sigma \times \mathbf{Z}$ .
- (2) There is a collection of pairwise disjoint embedded fibres,  $(S(\alpha))_{\alpha \in \mathcal{L}}$ , such that  $\alpha \subseteq S(\alpha)$  for all  $\alpha \in \mathcal{L}$ .
- (3) There is a collection,  $(f_\alpha)_{\alpha \in \mathcal{L}}$ , of homotopy fibres  $f_\alpha : \Sigma \longrightarrow \Psi$ , such that  $f_\alpha|_{f_\alpha^{-1}(\alpha)}$  is a homeomorphism from  $f_\alpha^{-1}(\alpha)$  to  $\alpha$ , and such that  $f_\alpha(\Sigma) \cap \beta = \emptyset$  for all  $\beta \in \mathcal{L} \setminus \{\alpha\}$ .

Note that, in (1), we can take  $f$  to be properly homotopic to the identity. Also, there is no loss in assuming that for each  $i \in \mathbf{Z}$ ,  $f^{-1}(\Sigma \times \{i\})$  contains at most one element of  $\mathcal{L}$ . It is a consequence of the theorem that in (2) and (3), we can take the fibres to be locally finite in  $\Psi$ , though this is not assumed a-priori.

**Proof :**

From the above observation, it is clear that (1) implies (2). Trivially, (2) implies (3).

(2) implies (1):

Note that we do not assume a-priori that the fibres are locally finite in  $\Psi$ . We claim that we can, if necessary, modify them so that they become locally finite. From this, (1) can be deduced using Waldhausen's cobordism theorem (stated as Theorem 3.2 here).

Let  $<$  be the total order on  $\mathcal{L}$  induced by the order of the fibres  $(S(\gamma))_\gamma$ . Since  $\mathcal{L}$  is locally finite, we see that this order is discrete. Thus, we can index it by a subset  $I \subseteq \mathbf{Z}$  such that  $\gamma_i < \gamma_j$  if and only if  $i < j$ . We can assume that  $0 \in I$ . By Brown's result (Theorem 3.3 here), we can assume that  $S(\gamma_0) = \Sigma \times \{0\}$ . We will obtain  $(S(\gamma_i))_i$  locally finite for  $i \geq 0$  and for  $i \leq 0$  separately. To this end, we may as well assume that  $I = \mathbf{N}$ .

Given  $i \in \mathbf{N}$ , let  $R(i)$  be the closed unbounded subset of  $\Psi$  with relative boundary  $S(\gamma_i)$ . Thus  $(R(i))_i$  is a decreasing sequence of subsets of  $\Psi$ . Since  $\mathcal{L}$  is locally finite, for each  $n \in \mathbf{N}$ , there is some  $i(n)$  such that  $R(n) \cap \bigcup \mathcal{L} \subseteq \Sigma \times [n+1, \infty)$ . Inductively over  $n$ , we now find ambient isotopies of the  $S(\gamma_i)$  for  $i \geq i(n)$ , supported on  $\Sigma \times [n-1, n+1]$ , which push the  $S(\gamma_i)$  into  $\Sigma \times [n, \infty)$ , while fixing the curves  $\gamma_i$ . Note that the  $S(\gamma_i)$  remain disjoint. Moreover, the process stabilises on a disjoint locally finite set of embedded fibres as required.

(3) implies (2):

Note that by Theorem 3.4, we can assume each of the  $f_\gamma$  to be injective. We write  $Z(\gamma) = f_\gamma(\Sigma)$ . Let  $\mathcal{L} = (\gamma_i)_{i \in I}$  be an arbitrary indexing of  $\mathcal{L}$ , where  $I$  is an initial segment of  $\mathbf{N}$ . We inductively replace  $Z(\gamma_i)$  by another fibre,  $S(\gamma_i) \supseteq \gamma_i$ , such that the  $S(\gamma_i)$  are all pairwise disjoint. Suppose that we have found disjoint  $S(\gamma_0), \dots, S(\gamma_n)$ , and that  $S(\gamma_i) \cap \gamma_j = \emptyset$  for all  $j > n$ . Let  $R$  be the component of  $\Psi \setminus \bigcup_{i=0}^n S(\gamma_i)$  containing  $\gamma_{n+1}$ . Using Brown's result (Theorem 3.3) we can isotope  $\gamma_{n+1}$  into  $R$  to obtain a fibre  $S(\gamma_{n+1}) \supseteq \gamma_{n+1}$  contained in an arbitrarily small neighbourhood of  $Z(\gamma_{n+1}) \cup \partial R$ . In particular,  $S(\gamma_{n+1})$  is disjoint from  $S(\gamma_i)$  for  $i < n$  and from all  $\gamma_j$  for  $j > n$ . We now proceed inductively to give  $S(\gamma_i)$  for all  $i$ .  $\diamond$

**Definition :** We say that a locally finite collection,  $\mathcal{L}$ , of curves in  $\Psi$  is *unlinked* if it satisfies any of the conditions of Proposition 7.1.

We say that a curve  $\alpha$  is *unknotted* if  $\{\alpha\}$  is unlinked.

Note that an unknotted curve can be isotoped to be *horizontal*, that is, of the form  $\gamma \times \{t\}$ , where  $t \in \mathbf{R}$  and  $\gamma \subseteq \Sigma$  is a simple closed curve.

Suppose  $\gamma \subseteq \Psi$  is a curve. We can locally compactify  $\Psi \setminus \gamma$  by adjoining a toroidal boundary component  $\Delta(\gamma)$ . We can think of  $\Delta(\gamma)$  as the unit normal bundle to  $\gamma$ . We write  $\Lambda(\gamma)$  for the resulting manifold. Note that it comes equipped with a natural homotopy class of meridian curve,  $m(\gamma)$ . We can recover  $\Psi$  up to homeomorphism by gluing in a solid torus,  $T(\gamma)$ , along  $\Delta(\gamma)$  so that the meridian bounds a disc in  $T(\gamma)$ . If  $\gamma$  is unknotted, it also has a natural class of longitude. It can be defined as a simple curve that can be homotoped to infinity in  $\Lambda(\gamma)$ . It can also be determined by sitting the curve in some (indeed any) fibre. More generally, if  $\Lambda$  is any locally finite set of curves, we can form the

manifold  $\Lambda(\mathcal{L})$  by adding toroidal boundaries to  $\Psi \setminus \bigcup \mathcal{L}$ . Thus  $\partial\Lambda = \partial\Psi \cup \bigcup_{\gamma \in \mathcal{L}} \Delta(\gamma)$ . (We can also think  $\Lambda(\mathcal{L})$  as the complement of an open regular neighbourhood of  $\bigcup \mathcal{L}$ .)

We will need the following:

**Proposition 7.2 :** *Suppose that  $\mathcal{L}$  and  $\mathcal{L}'$  are unlinked collections of curves. Suppose that no two elements of  $\mathcal{L}$  are homotopic in  $\Psi$ , and similarly for  $\mathcal{L}'$ . Suppose that  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L}')$  is a proper degree-one map with  $f^{-1}\partial\Lambda(\mathcal{L}') = \partial\Lambda(\mathcal{L})$ . Suppose that  $f|_{\partial\Lambda(\mathcal{L})}$  is a homotopy equivalence and sends the meridian and longitude of each toroidal boundary component to the meridian and longitude of its image. Then  $f$  is homotopic to a homeomorphism from  $\Lambda(\mathcal{L})$  to  $\Lambda(\mathcal{L}')$ .*

Note that  $f$  necessarily extends to a degree-one map from  $\Psi$  to itself. Since the induced homomorphism on the surface group  $\pi_1(\Psi) \cong \pi_1(\Sigma)$  is surjective, it follows from the residual finiteness for such groups [Sc3] (via the hopfian property) that this extension is a homotopy equivalence of  $\Psi$ . (In fact, this will be immediate from the construction in our applications.) Thus, there is no loss in taking the extension to be homotopic to the identity on  $\Psi$ .

The issue of degree-one maps between 3-manifolds has been investigated by a number of authors (see, for example, [Wan] for a survey). There are certainly many examples which are not homotopy equivalences. Some positive results are also known, but I know of no result that directly implies the statement given above.

If we can show that  $\Lambda(\mathcal{L})$  and  $\Lambda(\mathcal{L}')$  are homeomorphic, then we are in reasonably good shape. Consider the case where  $\mathcal{L}$  is finite. Then  $\Lambda$  is a topologically finite and admits a hyperbolic structure hyperbolic by the work of Thurston [Ot1,Ka]. Thus  $\pi_1(\Lambda)$  is residually finite. Since it is finitely generated it is hopfian by a result of Malcev. It follows that  $f$  induces an isomorphism of fundamental groups (see [He2]). Since  $\pi_2(\Lambda)$  is trivial (by the sphere theorem),  $f$  is a homotopy equivalence. Now using the work of Waldhausen [Wal], it follows that  $f$  is homotopic to a homeomorphism. In the case where  $\mathcal{L}$  is infinite, we will need a bit more explanation as to why the map on fundamental groups is injective, but this is relatively simple. Of course, there is a highly non-trivial input into this. An argument, suggested by Gabai, that bypasses hyperbolisation will be outlined at the end of this section.

Thus, most of the additional work is involved in showing that  $\Lambda(\mathcal{L})$  and  $\Lambda(\mathcal{L}')$  are indeed homeomorphic. For this we need to define a partial order on the link components, and show that this is preserved. It is fairly intuitive, but the details are a bit subtle.

Suppose  $\alpha, \beta \subseteq \Psi$  are unlinked curves. Write  $\alpha \approx \beta$  if they do not cross homotopically  $\Sigma$ , i.e. if  $d(\pi_\Sigma\alpha, \pi_\Sigma\beta) \leq 1$ . This is equivalent to asserting that there is some fibre of  $\Psi$  containing both  $\alpha$  and  $\beta$ . We will write  $\alpha \preceq \beta$  (respectively  $\alpha \succeq \beta$ ) if  $\beta$  can be homotoped out the positive (respectively negative) end of  $\Psi$  in  $\Psi \setminus \alpha$ .

Suppose that  $\alpha \preceq \beta$  and  $\alpha \succeq \beta$ . Then  $\beta$  can be homotoped from the negative to the positive end of  $\Psi$  without ever meeting  $\alpha$ . Such a homotopy must intersect any fibre of  $\Psi$  in at least some (not necessarily embedded) curve homotopic to  $\beta$ . From this one can see that  $\alpha \approx \beta$  by the above definition.

**Lemma 7.3 :** *Suppose  $\alpha$  and  $\beta$  are unlinked. Then the following are equivalent:*

- (1)  $\alpha \preceq \beta$
- (2)  $\beta \succeq \alpha$
- (3) *We can find disjoint fibres  $S \supseteq \alpha$  and  $S' \supseteq \beta$  with  $S < S'$ .*

**Proof :** It's clearly enough to show that (1) implies (3). By hypothesis we have disjoint fibres,  $Z \supseteq \alpha$  and  $Z' \supseteq \beta$ . If  $Z' < Z$ , then  $\alpha \succeq \beta$ , and so by the above observation,  $\alpha \approx \beta$ . It follows that  $\alpha$  and  $\beta$  are contained in some common fibre. We can now push these fibres slightly so that they become disjoint in the order required.  $\diamond$

We write  $\alpha \prec \beta$  to mean that  $\alpha \preceq \beta$  and  $\alpha \not\approx \beta$ . Thus, for any two unlinked curves,  $\alpha$  and  $\beta$ , exactly one of the relations  $\alpha \prec \beta$ ,  $\beta \prec \alpha$  or  $\alpha \approx \beta$  holds.

Suppose that  $\gamma_1, \gamma_2, \dots, \gamma_n$  are unlinked curves, with  $\gamma_i \prec \gamma_{i+1}$  for all  $i$ . By definition, we have disjoint fibres,  $S_i \supseteq \gamma_i$ , and by the above observation, we must have  $S_i < S_{i+1}$  for all  $i$ . From this we can deduce that there does not exist any finite cycle of unlinked curves,  $\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0$  with  $\gamma_i \prec \gamma_{i+1}$  for all  $i$ .

If  $\mathcal{L}$  is any unlinked set of curves, we can let  $<$  be the transitive closure of the relation  $\prec$  on  $\mathcal{L}$ . We see that this is a strict partial order on  $\mathcal{L}$ . Moreover,  $\alpha < \beta$  implies  $\alpha \preceq \beta$ , and so if  $\alpha \not\approx \beta$  we also get  $\alpha \prec \beta$ .

We aim to show that this order on  $\mathcal{L}$ , together with the natural map of  $\mathcal{L}$  to  $X(\Sigma)$  determines  $\mathcal{L}$  up to ambient isotopy, and so, in particular,  $\Lambda(\mathcal{L})$  up to homeomorphism. We use the following observation.

We say that a strict total order  $\ll$  is *compatible* with the partial order  $<$  if  $\alpha < \beta$  implies  $\alpha \ll \beta$ . We say that it is *discrete* if all intervals are finite.

**Lemma 7.4 :** *Suppose that  $\mathcal{L}$  is an unlinked set of curves. Suppose that  $\ll$  is a discrete total order on  $\mathcal{L}$  compatible with  $<$ . Then we can find a set of disjoint fibres,  $(S(\gamma))_{\gamma \in \mathcal{L}}$  with  $\gamma \subseteq S(\gamma)$  for all  $\gamma$ , and  $S(\alpha) < S(\beta)$  if and only if  $\alpha \ll \beta$ .*

**Proof :** Since  $\mathcal{L}$  is unlinked, we can find a locally finite disjoint collection of fibres  $Z(\gamma) \supseteq \gamma$ . Write  $\alpha \ll' \beta$  to mean that  $Z(\alpha) < Z(\beta)$ . This defines another discrete total order compatible with  $<$ . We can now find a sequence,  $(\ll_n)_n$ , of discrete total orders, all compatible with  $<$ , with  $\ll_0 = \ll'$ , with  $\ll_n$  stabilising on  $\ll$  on any finite subset of  $\mathcal{L}$ , and with  $\ll_{n+1}$  obtained from  $\ll_n$  by interchanging the order on a pair of consecutive elements of  $\mathcal{L}$ . We suppose inductively that  $Z_n(\gamma) \supseteq \gamma$  is a collection of disjoint fibres inducing the order  $\ll_n$ . Suppose that  $\ll_{n+1}$  is obtained by interchanging the order on  $\alpha$  and  $\beta$ . These are consecutive, which means that the region  $[Z_n(\alpha), Z_n(\beta)]$  contains no other curve of  $\mathcal{L}$ . Since both orders are compatible with  $\leq$ , we must have  $\alpha \approx \beta$ . We can now construct two new disjoint fibres,  $Z_{n+1}(\alpha) \supseteq \alpha$  and  $Z_{n+1}(\beta) \supseteq \beta$ , both in  $[Z_n(\alpha), Z_n(\beta)]$ , but with the opposite order. We set  $Z_{n+1}(\gamma) = Z_n(\gamma)$  for all  $\gamma \neq \alpha, \beta$ . Now the above process stabilises on any compact subset of  $\Psi$ , and so we eventually end up with a collection of fibres,  $(S(\gamma))_\gamma$  inducing  $\ll$  as required.  $\diamond$

**Lemma 7.5 :** *Suppose  $\mathcal{L}$  and  $\mathcal{L}'$  are unlinked sets of curves with a bijection  $[\gamma \mapsto \gamma']$  from  $\mathcal{L}$  to  $\mathcal{L}'$ . Suppose that  $\gamma$  and  $\gamma'$  are homotopic in  $\Psi$  for all  $\gamma$ . Suppose also that  $\alpha \preceq \beta$  implies  $\alpha' \preceq \beta'$ . Then there is an end-preserving self-homeomorphism of  $\Psi$  sending  $\gamma$  to  $\gamma'$  for all  $\gamma$ .*

**Proof :** First note that by the condition on homotopies, we have  $\alpha \approx \beta$  if and only if  $\alpha' \approx \beta'$ . By the trichotomy, we see that  $\alpha \prec \beta$  if and only if  $\alpha' \prec \beta'$ , and so  $\alpha < \beta$  if and only if  $\alpha' < \beta'$ .

Now, let  $S'(\gamma') \supseteq \gamma'$  be a locally finite disjoint set of fibres of  $\Psi$ . Applying Lemma 3.4, we can find a locally finite disjoint set of fibres  $S(\gamma) \supseteq \gamma$  such that  $S(\alpha) < S(\beta)$  if and only if  $S'(\alpha') < S'(\beta')$ . We can now find an isotopy of  $\Psi$  sending each  $S(\gamma)$  to  $S'(\gamma')$ . Since  $\gamma'$  is homotopic to  $\gamma$  in  $\Psi$  and hence in  $\Sigma$ , we can isotop  $\gamma$  to  $\gamma'$  in  $S(\gamma)$  and extend to an ambient isotopy in a small neighbourhood of  $S(\gamma)$ . The resulting homeomorphism sends  $\mathcal{L}$  to  $\mathcal{L}'$  as required.  $\diamond$

**Lemma 7.6 :** *Suppose that  $\mathcal{L}$  and  $\mathcal{L}'$  are unlinked sets of curves, and that  $f : \Psi \rightarrow \Psi$  is an end-preserving proper homotopy equivalence with  $f^{-1}(\bigcup \mathcal{L}') = \bigcup \mathcal{L}$ , and with  $f|_{\bigcup \mathcal{L}}$  a homeomorphism to  $\bigcup \mathcal{L}'$ . Then there is an end-preserving homeomorphism  $g : \Psi \rightarrow \Psi$  homotopic to  $f$  in  $\Psi$  with  $g|_{\bigcup \mathcal{L}} = f|_{\bigcup \mathcal{L}}$ .*

**Proof :** We may as well suppose that  $f$  is homotopic to the identity on  $\Psi$ . Suppose  $\alpha, \beta \in \mathcal{L}$ . If  $\alpha \preceq \beta$ , then we can homotope  $\beta$  out the positive end of  $\Psi$  in  $\Psi \setminus \alpha$ . The image of this homotopy under  $f$  sends  $f(\beta)$  out the positive end in  $\Psi \setminus f(\alpha)$ . Thus  $f(\alpha) \preceq f(\beta)$ . Lemma 3.5 now gives us a homeomorphism of  $\Psi$  sending each  $\gamma \in \mathcal{L}$  to  $f(\gamma)$ . By isotopy in a neighbourhood of  $\gamma$  we can assume that  $f|_{\gamma} = g|_{\gamma}$ .  $\diamond$

We remark that we do not in fact need to assume that  $\mathcal{L}'$  is unlinked in  $\Psi$ . This is a consequence of the other hypotheses:

**Proposition 7.7 :** *Suppose that  $f : \Psi \rightarrow \Psi$  is an end-preserving homotopy equivalence and that  $\mathcal{L}$  is an unlinked collection of curves in  $\Psi$ . Suppose that  $f^{-1}(f(\bigcup \mathcal{L})) = \bigcup \mathcal{L}$ , and that  $f|_{\bigcup \mathcal{L}}$  is injective. Then  $\{f(\gamma) \mid \gamma \in \mathcal{L}\}$  is an unlinked collection of curves in  $\Psi$ .*

**Proof :** We take a collection of disjoint fibres for  $\mathcal{L}$  as given by (2) of Proposition 7.1, and map them by  $f$  to give us a collection of homotopy fibres in  $\Psi$  satisfying (3) for the collection  $\{f(\gamma) \mid \gamma \in \mathcal{L}\}$ .  $\diamond$

**Lemma 7.8 :** *Let  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L}')$  be as in the hypothesis of Proposition 7.2. Then  $f$  is a homotopy equivalence.*

**Proof :** By extending over the tori  $T(\gamma)$  we get a map satisfying the hypotheses of Lemma 3.5, and so it follows that  $\Lambda(\mathcal{L})$  and  $\Lambda(\mathcal{L}')$  are homeomorphic. Let  $\Gamma = \pi_1(\Lambda(\mathcal{L}))$ . The map  $f$  induces an epimorphism of  $\Gamma$ , and we need to show that this is also injective.

If  $\mathcal{L}$  were finite, then this follows from the fact that  $\pi_1(\Lambda(\mathcal{L}))$  satisfies the hopfian property as described earlier.

In general suppose  $g \in \Gamma$  were in the kernel. We can represent it by a closed curve  $\delta \subseteq \Lambda(\mathcal{L})$ , which lies between two fibres, say  $S < Z$  in  $\Psi$ . We can take these disjoint from  $\bigcup \mathcal{L}$ . Let  $\mathcal{L}_0 = \{\gamma \in \mathcal{L} \mid \gamma \subseteq [S, Z]\}$ , and write  $\mathcal{L}'_0$  for the corresponding subset of  $\mathcal{L}'$ . Now  $f$  extends to degree one map between  $\Lambda(\mathcal{L}_0)$  and  $\Lambda(\mathcal{L}'_0)$ . By the finite case, it now follows that  $\delta$  bounds a disc in  $\Lambda(\mathcal{L}_0)$ . But we can now push that disc into the region  $[S, Z]$ , and so we see that  $\delta$  bounds a disc in  $\Lambda(\mathcal{L})$ . In other words,  $g$  is trivial in  $\Gamma$ .

The result now follows by Whitehead's theorem, given that the higher homotopy groups are trivial.  $\diamond$

**Proof of Proposition 7.2 :** We have shown (Lemma 7.6) that  $f$  is a homotopy equivalence. Suppose first that  $\mathcal{L}$  is finite. We can compactify  $\Psi$  to  $\Sigma \times [-\infty, \infty]$ , and we can modify  $f$  in a neighbourhood of the ends of  $\Psi$  so that it extends to a map from  $\Sigma \times [-\infty, \infty] \rightarrow \Sigma \times [-\infty, \infty]$ , without changing it on  $\bigcup \mathcal{L}$ . Now the compactified manifold  $\Lambda(\mathcal{L}) \cup (\Sigma \times \{-\infty, \infty\})$  is Haken. If  $\mathcal{L}$  is finite, since  $\Lambda(\mathcal{L}) \cong \Lambda(\mathcal{L}')$  is Haken, the result then follows by the result of Waldhausen [Wal] (stated here as Theorem 3.5).

To deal with the general case, we first show that if  $F \subseteq \Lambda(\mathcal{L})$  is a surface with  $\partial F = F \cap \partial \Lambda(\mathcal{L})$ , then  $f|_F$  is homotopic to an embedding in  $\Lambda(\mathcal{L}')$  relative to  $\partial \Lambda(\mathcal{L}')$ . (In fact, this is all we need for our applications.) To see this, choose fibres  $S, Z \subseteq \Lambda(\mathcal{L}')$ , disjoint from  $\bigcup \mathcal{L}'$ , so that  $f(F) \subseteq [S, Z]$ . Let  $\mathcal{L}'_0 = \{\gamma \in \mathcal{L}' \mid \gamma \subseteq [S, Z]\}$ , and let  $\mathcal{L}_0 \subseteq \mathcal{L}$  be the corresponding curves in  $\mathcal{L}$ . Now  $f$  is a homotopy equivalence from  $\Lambda(\mathcal{L}_0)$  to  $\Lambda(\mathcal{L}'_0)$ . By the finite case,  $f|_F$  is homotopic to an embedding in  $\Lambda(\mathcal{L}'_0)$ . By Theorem 3.4, we can take this embedding in a small neighbourhood of  $f(F)$ , and so, in particular, in  $[S, Z]$ . But  $\Psi$  retracts onto  $[S, Z]$ , so we can now push the homotopy into  $[S, Z]$ , and so these surfaces are also homotopic in  $\Lambda(\mathcal{L}')$ .

To complete the proof, we can apply this result to a sequence of fibres,  $(S_i)_{i \in \mathbf{Z}}$  in  $\Lambda(\mathcal{L})$ , whose images are disjoint, so as to find disjoint fibres  $Z_i$  in  $\Lambda(\mathcal{L}')$ , homotopic to  $f(S_i)$ . We can now apply the finite case to the regions between these fibres. We omit the details, since we have already shown what we need for the following corollary.  $\diamond$

As we have noted, the fact we really need is the following. It was proven in the course of Proposition 7.2, and can also be viewed as a combination of Proposition 7.2 and Theorem 3.1.

**Corollary 7.9 :** *Let  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L}')$  be as in the hypotheses of Proposition 3.1. Suppose that  $F \subseteq \Lambda(\mathcal{L})$  is a properly embedded  $\pi_1$ -injective compact surface (so that  $F \cap \partial \Lambda(\mathcal{L}) = \partial F$ ). Let  $U$  be any neighbourhood of  $f(F)$  in  $\Lambda(\mathcal{L}')$ . Then there is a proper embedding  $g : F \rightarrow U$  such that  $f|_F$  is homotopic in  $\Lambda(\mathcal{L}')$  to  $g$  relative to  $\partial F$ .  $\diamond$*

We conclude this section with an outline of how one can bypass the use of hyperbolisation in the proof of Proposition 7.2. It elaborates on a suggestion of Dave Gabai, and I thank him for his permission to include it here.

### **Proof of Proposition 7.2 bypassing hyperbolisation.**

We can suppose that  $\mathcal{L} = \mathcal{L}'$ , and that  $f : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L})$  extends to a map homotopic to the identity on  $\Psi$ . We claim that  $f$  is homotopic to a homeomorphism (in fact, the

identity) on  $\Lambda(\mathcal{L})$ .

We write  $\Sigma_t = \Sigma \times \{t\}$ . We can assume that  $\mathcal{L} = \{\alpha_i \mid i \in I\}$ ,  $I \subseteq \mathbf{Z}$  and each  $\alpha_i$  is a curve in  $\Sigma_i$ . We can also assume that  $\Sigma \times (\mathbf{Z} + \frac{1}{2}) \subseteq \Lambda(\mathcal{L})$ .

We will first show that if  $t \in \mathbf{Z} + \frac{1}{2}$ , then  $f|_{\Sigma_t}$  is homotopic in  $\Lambda(\mathcal{L})$  to the inclusion  $\Sigma_t \hookrightarrow \Lambda(\mathcal{L})$ . Let  $J$  be the vertical range of  $f(\Sigma_t)$ , i.e. the compact interval  $\{u \in \mathbf{R} \mid \Sigma_u \cap f(\Sigma_t) \neq \emptyset\}$ . Suppose first that  $I \cap J \cap [t, \infty) \neq \emptyset$ . Let  $i$  be its maximal element, and let  $A = \alpha_i \times [i, \infty)$ . We can assume that  $(f|_{\Sigma_t})^{-1}A$  consists of simple closed curves. These will be either trivial or homotopic to  $\alpha_i$ . Since  $\Lambda(\mathcal{L})$  is aspherical, after homotopy in  $\Lambda(\mathcal{L})$ , we can get rid of trivial curves, and since it is atoroidal, we can get rid of pairs of non-trivial curves with opposite orientations. We are left with either  $p$  positively oriented curves or  $-p$  negatively oriented curves homotopic to  $\alpha$ , where  $p \in \mathbf{Z}$  is the number of times  $f(\Sigma_t)$  wraps around  $\alpha_i$ . More precisely,  $p = \langle \omega, f(\Sigma_t) \rangle$ , where  $\omega \in H^2(\Lambda(\{\alpha_i\}))$  measures the intersection with a ray in  $\Lambda(\{\alpha_i\})$  from  $\alpha_i$  to  $+\infty$ . For sufficiently negative  $u$ ,  $f(\Sigma_u) \subseteq \Sigma \times (-\infty, n)$  and so  $\langle \omega, f(\Sigma_u) \rangle = 0$ . Since  $f(\Sigma_u)$  is homotopic to  $f(\Sigma_t)$  in  $\Lambda(\{\alpha_i\})$  it follows that  $p = \langle \omega, f(\Sigma_t) \rangle = 0$ . In other words, we have pushed  $f(\Sigma_t)$  off  $A$ . We can now homotope it (in  $\Lambda(\mathcal{L})$ ) below  $\Sigma_i$ . We can continue inductively until  $I \cap J \cap [t, \infty) = \emptyset$ . Proceeding similarly below, we push  $f(\Sigma_t)$  so that its vertical range lies in the component of  $\mathbf{R} \setminus I$  containing  $t$ . After a further homotopy, we will get  $f(\Sigma_t) \subseteq \Sigma_t$ . Now,  $f|_{\Sigma_t} : \Sigma_t \rightarrow \Sigma_t$  is homotopic to the identity in  $\Sigma \times \mathbf{R}$  and hence in  $\Sigma_t$ . This proves the claim.

Performing such homotopies for all  $t \in \mathbf{Z} + \frac{1}{2}$ , we can assume that  $f|_{\Sigma \times (\mathbf{Z} + \frac{1}{2})}$  is just the inclusion  $\Sigma \times (\mathbf{Z} + \frac{1}{2}) \hookrightarrow \Sigma \times \mathbf{R}$ .

Now let  $P_n = \Sigma \times [n - \frac{1}{2}, n + \frac{1}{2}]$ , and let  $R_n = P_n \cap \Lambda(\mathcal{L})$ . (In other words,  $R_n$  is  $P_n$  with at most one tube drilled out.) We next homotope  $f$  so that  $f(R_n) = R_n$ . The idea is to proceed as we did for the surfaces  $\Sigma_t$ . Given  $i \in I \cap J \cap [n + 1, \infty)$  maximal, let  $A_i = \alpha_i \times [i, \infty)$  as before. This time, we can assume that each component of  $f^{-1}(A_i)$  is a surface in the interior of  $R_n$ . Moreover, by standard 3-manifold topology, we can assume it to be incompressible. It's not hard to see that any incompressible surface in  $R_n$  must be boundary parallel. Thus it must either be a fibre or be homotopic in  $P_n$  to  $\alpha_n$ . But it must be homotopic in  $\Sigma \times \mathbf{R}$  to  $\alpha_i$ , giving a contradiction. We have arranged that  $f(R_n) \cap A_i = \emptyset$ . Continuing as with  $\Sigma_t$ , we eventually homotope  $f(R_n)$  into  $R_n$  as claimed.

Next, if  $n \notin I$ , then  $R_n = P_n$ , and by Waldhausen [Wal], we can homotope  $f|_{R_n}$  fixing  $R_{n-\frac{1}{2}} \cup R_{n+\frac{1}{2}}$  to a homeomorphism (in fact the identity). If  $n \in I$ , let  $B = (\alpha_n \times [n, n + \frac{1}{2}]) \cap R_n$ . Since  $R_n$  is atoroidal, we can assume  $f(B) = B$ . Again, we can assume that  $f^{-1}(B) \setminus B$  consists of incompressible surfaces in  $R_n \setminus B$ , and therefore empty. In other words  $f^{-1}(B) = B$ . After a homotopy, holding the curve  $\Sigma_{n+\frac{1}{2}} \cap B$  fixed (though perhaps rotating the other boundary component of  $B$ ) we get  $f|_B$  to be inclusion. Cutting  $R_n$  along  $B$ , we get a homeomorphic copy of  $\Sigma \times [0, 1]$  and so, again, we are done by Waldhausen. Doing this for all  $n$ , we homotope  $f$  to the identity in  $\Lambda(\mathcal{L})$ .

## 8. Annulus systems.

In this section, we will talk about “annulus systems” in  $\Psi \cong \Sigma \times \mathbf{R}$ . This is the combinatorial structure we use to construct the model manifold. It can be thought of as a simplified version of the “heirarchies” introduced in [MaM2] and applied in [Mi1]. As noted earlier, we make no use of “subsurface projections” of [MaM2] which were key to Minsky’s construction. (A discussion of subsurface projections and their relation to our annulus systems can be found in [Bow6].)

Let  $\Sigma$  be a compact surface. Let  $\Psi = \Sigma \times \mathbf{R}$ . We write  $\pi_\Sigma : \Psi \longrightarrow \Sigma$  and  $\pi_V : \Psi \longrightarrow \mathbf{R}$  be vertical and horizontal projections respectively. We refer to the two ends of  $\Psi$  as the *positive* and *negative* ends. We are only really interested in the total order on the vertical coordinate in  $\mathbf{R}$ . We are thus free to adjust by any orientation preserving homeomorphism of  $\mathbf{R}$ .

**Definition :** A *horizontal curve* in  $\Psi$  is a subset of the form  $\gamma \times \{t\}$  for some curve  $\gamma \subseteq \Sigma$ , and some  $t \in \mathbf{R}$ .

A *horizontal surface* in  $M$  is a subset of the form  $\Phi \times \{t\}$  for some subsurface  $\Phi \subseteq \Sigma$  and some  $t \in \mathbf{R}$ . If  $\Phi = \Sigma$ , it is called a *horizontal fibre*.

A *vertical annulus* is a subset of the form  $\gamma \times I$  where  $I \subseteq \mathbf{R}$  is a non-trivial compact interval.

A *strip* is a subset of the form  $\Phi \times I$  where  $\Phi \subseteq \Sigma$  and  $I \subseteq \mathbf{R}$  is a non-trivial compact interval.

More generally we speak about a subset or path etc. being *horizontal* if it is entirely contained in a horizontal fibre.

As before, in what follows we shall fix some hyperbolic structure on  $\Sigma$  with geodesic boundary, and realise curves as geodesics. In this way, curves will automatically intersect minimally, so it will simplify the combinatorial arguments.

Let  $\Omega = \gamma \times I$  be a vertical annulus, with  $I = [\partial_- I, \partial_+ I] \subseteq \mathbf{R}$ . We write  $\partial_\pm \Omega = \gamma \times \partial_\pm I$ . These are horizontal curves. (We shall be a bit sloppy about distinguishing points of  $\mathbf{R}$  from 1-element subsets.) We write  $\partial_H \Omega = \gamma \times \partial I = \partial_- \Omega \sqcup \partial_+ \Omega$ .

Let  $B = \Phi \times I$  be a strip. We write  $\partial_\pm B = \Phi \times \partial_\pm I$ ,  $\partial_H B = \Phi \times \partial I = \partial_- B \sqcup \partial_+ B$ ,  $\partial_V B = \partial \Phi \times I$  and  $\partial_V^\Sigma B = \partial^\Sigma \Phi \times I$ . (Recall that  $\partial^\Sigma \Phi = \partial \Phi \setminus \partial \Sigma$  is the relative boundary of  $\Phi$  in  $\Sigma$ .) We refer to  $\partial_H B$  as the *horizontal boundary* of  $B$ . It consists of two horizontal surfaces. We refer to  $\partial_V B$  as the *vertical boundary*. Note that the relative boundary of  $B$  in  $\Psi$  is  $\partial_V^\Sigma B \cup \partial_H B$ . We define the complexity  $\kappa(B)$  as  $B$  as  $\kappa(\Phi)$ . We refer to  $\pi_\Sigma B$  as the *base surface* of  $B$ .

**Definition :** An *annulus system*,  $\mathcal{W}$ , in  $\Psi$  is a locally finite collection of disjoint vertical annuli.

Let  $W = \bigcup \mathcal{W}$ . Given  $t \in \mathbf{R}$ , let  $\gamma_t = \pi_\Sigma(W \cap (\Sigma \times t))$ . This is either empty or a multicurve in  $\Sigma$ . Clearly,  $W$  is completely determined by the piecewise constant map  $[t \mapsto \gamma_t]$ . For many purposes, it will be convenient to assume that  $\mathcal{W}$  is *vertically generic*, that is, if  $\Omega, \Omega' \in \mathcal{W}$  with  $\pi_V \partial_H \Omega \cap \pi_V \partial_H \Omega' \neq \emptyset$ , then  $\Omega = \Omega'$ . This is easily achieved by pushing the horizontal boundaries of annuli up or down a little. In this case,  $\Omega$  is

combinatorially equivalent to sequence  $(\gamma_i)_i$  where each  $\gamma_i$  is either empty or a multicurve, and  $\gamma_{i+1}$  is obtained from  $\gamma_i$  by either adding or deleting a multicurve. We say that  $\Omega$  is *vertically full* if  $\mathbf{R} = \pi_V W$ . In this case, the  $\gamma_i$  are all multicurves. In Section 6, we referred to such a sequence  $(\gamma_i)$  as a *path of multicurves*. In other words, there is a bijective correspondence between paths of multicurves and vertically full (and generic) annulus systems up to vertical reparametrisation.

A *ladder* is a minimal vertically full annulus system. This corresponds to a path of multicurves where the number of components alternates between 1 and 2. Those with one component constitute a path in the curve complex. Thus a ladder is combinatorially equivalent to a (bi-infinite) path in  $\mathcal{G}(\Sigma)$ .

**Definition :** A strip,  $B \subseteq \Psi$  is a *band* (with respect to  $\mathcal{W}$ ), if  $\partial_V^\Sigma B \subseteq W$ .

We can view  $W_B = (W \cap B) \setminus \partial_V B$  as a finite annulus system on  $B$  (at least if  $\partial_H B \cap \partial_H \Omega = \emptyset$  for all  $\Omega \in \mathcal{W}$ ), and a similar discussion applies. In this case if  $W_B$  is full then it corresponds to a finite path of multicurves in  $\Phi$ . We write  $\mathcal{W}_B$  for the set of components of  $W_B$ .

**Definition :** An annulus system is *complete* if for any horizontal fibre  $S \subseteq \Psi$ , each component of  $S \setminus W$  has complexity at most 1.

In this case, a *brick* is a maximal band whose interior does not meet  $W$ .

Any such brick has complexity at most 1. We refer a brick as *type 0* or *type 1* depending on whether its complexity is 0 or 1, i.e. its base surface is a 3HS or a 1HT or 4HS. Let  $\mathcal{D} = \mathcal{D}(\mathcal{W})$  be the set of all bricks. One sees easily that this is a locally finite collection of bands with disjoint interiors, and that  $\Psi = \bigcup \mathcal{D}$ . If two bands meet in a horizontal surface, then one is of type 0 and the other of type 1. We can recover  $\mathcal{W}$  from  $\mathcal{D}$  as the set of components of  $W = \bigcup_{B \in \mathcal{D}} \partial_V^\Sigma B$ . We write  $\mathcal{W} = \mathcal{W}(\mathcal{D})$ .

**Remark :** The notion of a complete annulus system is closely related to that of a path in the pants graph. Suppose that  $\mathcal{W}$  is a complete annulus system. Suppose moreover that the horizontal projections of the type 1 bricks are pairwise disjoint. (This can always be achieved by vertical isotopy of the annuli.) Let  $(\gamma_i)_i$ , as described as described above. For even indices  $\gamma_i$  is a complete multicurve, (or ‘‘pants decomposition’’) and for odd indices it is a complete multicurve with one curve removed. Thus, at odd indices we have exactly one complementary component of complexity 1 (a 1HT or 4HS). By interpolating additional curves in this component, we arrive at the situation where consecutive complete multicurves differ by replacing one curve (say  $\alpha$ ) by another (say  $\beta$ ) such that a regular neighbourhood of  $\alpha \cup \beta$  is either a 1HT or a 4HS. (A particular case of this interpolation process is used again below.) The sequence of complete multicurves is then a path in the pants graph. Conversely, a path in the pants graph gives rise to a complete annulus system of this sort.

Given an annulus system,  $\mathcal{W}$ , we can define  $\Lambda = \Lambda(\mathcal{W})$  as the metric completion of  $\Psi \setminus W$  in its induced path metric. Thus there is a toroidal boundary component,  $\Delta(\Omega)$ , of  $\Lambda$ ,

associated to each  $\Omega \in \mathcal{W}$ . Indeed,  $\partial\Lambda = \partial\Psi \cup \bigcup_{\Omega \in \mathcal{W}} \Delta(\Omega)$ . There is natural projection,  $\pi_\Psi : \Lambda \rightarrow \Psi$  that is injective on  $\text{int } \Lambda$ . On each  $\Delta(\Omega)$  it is injective on  $\pi_\Psi^{-1}\partial_H\Omega$  and two-to-one elsewhere. As in Section 7,  $\Delta(\Omega)$  comes equipped with a free homotopy class of longitude, denoted  $l(\Omega)$  and meridian denoted  $m(\Omega)$ . We refer to this procedure as “opening out” the annuli of  $\mathcal{W}$ . We can also fill them back in again.

Given any subset,  $\mathcal{W}_0 \subseteq \mathcal{W}$  let  $\Lambda(\mathcal{W}, \mathcal{W}_0)$  be the manifold obtained from  $\Lambda(\mathcal{W})$  by gluing in a solid torus,  $T(\Omega)$  to  $\Delta(\Omega)$  for each  $\Omega \in \mathcal{W}_0$ . so that the meridian bounds a disc. Thus, we recover  $\Psi$  up to homeomorphism as  $\Lambda(\mathcal{W}, \mathcal{W})$ . From a purely topological point of view this is a rather fruitless exercise. However, we will want to view these spaces as having different structures. We will be regarding  $\Psi$  together with  $\mathcal{W}$  as an essentially combinatorial object, whereas  $\Lambda(\mathcal{W}, \mathcal{W})$  will be given a geometric structure.

Suppose that  $\mathcal{W}$  is a complete annulus system. We obtain a *brick decomposition* of  $\Lambda = \Lambda(\mathcal{W})$  by lifting each brick in  $\Psi$  to a *brick* in  $\Lambda$ . Note that two vertical boundary components of such a (lifted) brick may become identified under the projection map,  $\pi_\Psi$ , but the projection is otherwise injective on bricks. By abuse of notation we will also denote this lifted brick decomposition by  $\mathcal{D}$ . In this case, if two bricks meet one is of type 0 and the other of type 1. They meet along a horizontal 3HS.

We want to describe a particular construction of complete annulus systems. Note that Lemma 5.3 gave us a means of connecting two curves by a path of multicurves giving us a complete annulus system in some compact region of  $\Psi$ . However this will not be sufficient for our purposes. We will require some additional properties. To describe these, we need some further definitions.

Let  $\mathcal{W}$  be an annulus system and  $B \subseteq \Psi$  a band. We write  $X_\pm(B) = \pi_\Sigma(W \cap \partial_\pm B \setminus \partial_V B) \subseteq X(\Phi) \subseteq X(\Sigma)$ . Recall that a ladder in  $B$  is minimal set of annuli in  $\mathcal{W}_B$  which is vertically full in  $B$  (i.e.  $\pi_V W_B = \pi_V B$ ).

**Definition :** The *height*,  $H(B)$  of  $B$  is the minimal length of a ladder in  $B$ .

More intuitively, the height can be thought of as the minimal number of annuli we need to cross to get from one horizontal boundary component of  $B$  to the other, where we are allowed to jump between annuli along horizontal paths.

We also write  $H_0(B) = d_{\mathcal{G}}(X_-(B), X_+(B))$ . Note that  $H_0(B) \leq H(B)$ . In the case where  $\mathcal{W}$  is complete and  $\kappa(B) \geq 2$ , these quantities are finite. We define the *slackness* of  $B$  as  $H(B) - H_0(B)$ .

**Definition :** We say that  $B$  is *k-taut* if its slackness is at most  $k$ .

In the case where  $\kappa(B) = 1$ ,  $\mathcal{W}_B$  consists of a sequence  $\Omega_0, \dots, \Omega_n$  of annuli whose vertical projections are disjoint and occur in this order. In this case, we say that  $B$  is *taut* if  $(\pi_\Sigma \Omega_i)_i$  is a geodesic segment in  $\mathcal{G}(\Phi)$ .

The main result about existence of complete annulus systems in the bi-infinite case can be stated as follows.

Let  $\Sigma$  be a compact surface with complexity,  $\kappa(\Sigma) \geq 2$ . Suppose  $a, b \in \partial\mathcal{G}(\Sigma)$  are distinct and that  $(\alpha_i)_i$  and  $(\beta_i)_i$  are sequences of curves converging on  $a$  and  $b$  respectively.

As observed in section 2, the sets  $Y^\infty(\{\alpha_i, \beta_i\})$  converge locally on some locally finite subset  $Y \subseteq B$ . This lies a bounded distance (depending only on  $\kappa(\Sigma)$ ) from any bi-infinite geodesics from  $a$  to  $b$  in  $\mathcal{G}(\Sigma)$ .

We will prove:

**Theorem 8.1 :** *There is a constant  $c$  depending only on  $\kappa(\Sigma)$  such that for any  $a, b \in \partial\mathcal{G}(\Sigma)$  and  $Y$  constructed as above, we can find a complete annulus system  $\mathcal{W}$  such that*

- (P1)  $X(\mathcal{W}) \subseteq \bar{Y}^\infty(Y)$ .
- (P2) If  $\Omega, \Omega' \in \mathcal{W}$  with  $\pi_\Sigma \Omega = \pi_\Sigma \Omega'$ , then  $\Omega = \Omega'$ .
- (P3) Every band in  $\Psi$  of complexity at least 2 in  $\Psi$  is  $c$ -taut.
- (P4) Every band of complexity 1 is taut.

Note that in (P3) we are allowing bands with base surface  $\Sigma$ . Tautness then tells us that the path of multicurves associated to  $\mathcal{W}$  is quasi-geodesic.

Property (P1) will eventually serve to show that the map from the model space is lipschitz (using the “a-priori bounds”, Theorem 14.2). Property (P2) is needed to show that the map has degree one on the “thick parts” of these space. Properties (P3) and (P4) are needed for the reverse coarse inequalities, to show that our map is a quasi-isometry.

Using the local finiteness of our sets,  $Y$  (Lemma 5.4), we can see that we can reduce to a finite case of Theorem 8.1. Here we consider only a finite band,  $O = \Sigma \times I$ . We interpret “completeness” to include the statement that  $X_-(\mathcal{W})$  and  $X_+(\mathcal{W})$  are both complete multicurves. In this case, we started with two curves  $\alpha, \beta$  which fill  $\Sigma$ . We can replace  $Y$  by  $Y^\infty(\{\alpha, \beta\})$ , and insist that  $\alpha \in X_-(\mathcal{W})$  and  $\beta \in X_+(\mathcal{W})$ . To get us started, we can apply Lemma 5.3 to give us our a multicurve that will serve as  $X_-(\mathcal{W})$ . (This is the only reason we need the two  $Y$ ’s in property (1). One can clearly formulate other versions that would not involve us in constructing quite so many tight geodesics, but there seems little point for our purposes.)

We now have the basis for proving Theorem 8.1. The argument will be by induction on complexity. We need to state the induction hypothesis in a different way. This formulation is mostly stronger than that already given. However, it is weaker in the sense that we are assuming we are given an initial complete multicurve, and we will also forget, for the moment about bands of complexity 1.

We say that a horizontal curve  $\gamma \subseteq O$  is *compatible* with an annulus system  $W = \bigcup \mathcal{W}$  if either  $\gamma \subseteq W$  or  $\gamma \cap W = \emptyset$ . We say that a vertical annulus  $\Omega \subseteq O$  is *compatible* with  $W$  if  $\Omega \cap W$  is empty, a single vertical annulus or a boundary curve of  $\Omega$ . (This implies that  $W \cup \Omega$  is also an annulus system.)

**Lemma 8.2 :** *Suppose that  $\kappa(\Sigma) \geq 1$ , and that  $\alpha, \beta$  are multicurves in  $\Sigma$  with  $\alpha$  complete. Then there is a complete (vertically generic) annulus system,  $W = \bigcup \mathcal{W} \subseteq O = \Sigma \times I$  satisfying:*

- (1)  $\alpha = X_-(\mathcal{W})$  and  $\beta \subseteq X_+(\mathcal{W})$ .
- (2) If  $\Omega \subseteq O$  is a vertical annulus with  $\partial_-\Omega$  and  $\partial_+\Omega$  both compatible with  $W$ , then  $\Omega$  is compatible with  $W$ .

- (3) If  $\Omega$  is a vertical annulus with  $\partial_- \Omega$  compatible with  $W$  and with  $\partial_+ \Omega \subseteq \partial_+ O$  and not crossing  $\beta$ , then  $\Omega$  is compatible with  $W$  (so that by completeness,  $W \cap \partial_+ O \subseteq \Omega$ ).
- (4) If  $B = \Phi \times J \subseteq O$  is a band of complexity at least 2, then  $B$  is  $2(\kappa(\Sigma) - \kappa(\Phi) + 1)$ -taut.
- (5)  $X(W) \subseteq Y^\infty(X(\alpha) \cup X(\beta))$ .

**Proof :** The proof will be induction on  $\kappa(\Sigma)$ .

In the case where  $\kappa(\Sigma) = 1$ , there is not much to be said. Here,  $\alpha, \beta$  are just single curves, and we enlarge them slightly to be vertical annuli. We put them in a single annulus if they happen to be equal.

The case where  $\kappa(\Sigma) = 2$  is not much harder. We choose  $\gamma_0 \in X(\alpha)$  and  $\gamma_n \in X(\beta)$  with  $d(\gamma_0, \gamma_n) = d(X(\alpha), X(\beta)) = n$ , say, and connect  $\gamma_0$  to  $\gamma_n$  by a tight geodesic  $\gamma_0, \gamma_1, \dots, \gamma_n$  in  $\mathcal{G}(\Sigma)$ . The path of multicurves,

$$\alpha, \quad \gamma_0, \quad \gamma_0 \cup \gamma_1, \quad \gamma_1, \quad \dots, \quad \gamma_{n-1} \cup \gamma_n \quad [, \quad \gamma_n, \quad \beta]$$

gives us a ladder in  $O$ , which in this case, is a complete annulus system. (The last two bracketed terms are omitted if  $\beta$  is a single curve.) The properties stated are all easily verified in this case.

Now suppose that  $\kappa(\Sigma) \geq 3$  and that the lemma holds for all surfaces of smaller complexity. To apply the induction hypothesis, first consider the special case where  $X(\alpha) \cap X(\beta) \neq \emptyset$ . Let  $\gamma \subseteq \alpha$  be the union of all components of  $\alpha$  that do not cross any curve of  $\beta$ . Thus,  $\gamma$  and  $\gamma \cup \beta$  are multicurves. Now,  $\gamma \times I$  cuts  $O$  into subsets of the form  $\Phi \times I$  where  $\Phi$  is (the closure of) a component of  $\Sigma \setminus \gamma$ . If  $\beta \cap \text{int } \Phi = \emptyset$ , let  $W_\Phi = (\alpha \cap \text{int } \Phi) \times I$ . Otherwise, we apply the inductive hypothesis to  $\Phi \times I$  and the curves  $\alpha \cap \text{int } \Phi$  and  $\beta \cap \text{int } \Phi$  to give us a complete annulus system  $W_\Phi$  in  $\Phi \times I$ . We now set  $W = (\gamma \times I) \cup \bigcup_\Phi W_\Phi$  as  $\Phi$  ranges over all such subsurfaces. (We can modify  $W$  so that it becomes vertically generic.) It is complete, and all the above properties are easily verified. For (4), note that if a band  $B$  does not lie in any of the components,  $\Phi \times I$ , then it is crossed by a single annulus and so  $H(B) = H_0(B) = 0$ , so it is 0-taut.

The general construction is as follows. Let  $n = d(X(\alpha), X(\beta))$ . Similarly as in the complexity 2 case, we choose  $\gamma_0 \in X(\alpha)$  and  $\gamma_n \in X(\beta)$  with  $d(\gamma_0, \gamma_n) = n$ . Now connect  $\gamma_0$  to  $\gamma_n$  by a tight multigeodesic  $\gamma_0, \gamma_1, \dots, \gamma_n$ . (This needs to be a bona fide tight multigeodesic, as discussed in Section 5.) Now write  $I = [0, n+1]$  and set  $O_i = \Sigma \times [i, i+1]$  for  $i \in \{0, 1, \dots, n\}$ . Thus  $O = \bigcup_i O_i$  and  $\partial_+ O_i = \partial_- O_{i+1}$ . We apply the above construction to give us a complete annulus system,  $W_0 \subseteq O_0$  with  $\alpha = X_-(W_0)$  and  $\gamma_0 \cup \gamma_1 \subseteq X_+(W_0)$ . Let  $\alpha_1$  be the complete multicurve  $X_+(W_0)$ . We now do the same thing in  $O_1$  to get a complete annulus system,  $W_1 \subseteq O_1$  with  $\alpha_1 = X_-(W_1)$  and  $\gamma_1 \cup \gamma_2 \subseteq X_+(W_1)$ . Set  $\alpha_2 = X_+(W_1)$ , and continue inductively. For the final step,  $O_n$ , we get  $W_n \subseteq O_n$  with  $\gamma_{n-1} \cup \gamma_n \subseteq \alpha_n = X_-(W_n)$  and  $\beta \subseteq X_+(O_n)$ . Note that for all  $i$ ,  $X_+(O_i) = X_-(O_{i+1})$  and so the annulus systems match up. We set  $W = \bigcup_{i=0}^n W_n \subseteq O$ , and let  $\mathcal{W}$  be the set of components of  $W$ .

By construction,  $W$  is complete annulus system satisfying (1). We verify the remaining properties in turn.

We need the following observation. Suppose  $\Omega$  is any vertical annulus with  $\partial_H \Omega$  compatible with  $W$ . If  $\partial_- \Omega \subseteq O_i$  and  $\partial_+ \Omega \subseteq O_j$ , then  $d(X(\gamma_i), \pi_\Sigma \Omega) \leq 1$  and  $d(X(\gamma_j), \pi_\Sigma \Omega) \leq 1$  and so  $j - i = d(X(\gamma_i), X(\gamma_j)) \leq 2$ . In other words,  $\Omega$  can enter at most three of the  $O_i$ .

(2) Suppose that  $\partial_-\Omega \subseteq O_i$  and  $\partial_+\Omega \subseteq O_j$  are both compatible with  $W$ . By the above observation, there are three cases:

Case (2a):  $j = i$  so  $\Omega \subseteq O_i$ , and we are done by the inductive procedure, i.e. (1) applied to  $W_i$ .

Case (2b):  $j = i + 1$ . Note that  $\pi_\Sigma\Omega$  does not cross either  $\gamma_i$  or  $\gamma_{i+1}$  (by compatibility of  $\partial_-\Omega$  with  $W_i$  and  $\partial_-\Omega$  with  $W_{i+1}$  respectively). By (3) applied to  $W_i$ , we see that  $\Omega \cap O_i$  is compatible with  $W_i$ , and that  $\Omega \cap \partial_+O \subseteq W$ . We now apply (2) to  $W_{i+1}$ , showing that  $\Omega \cap O_{i+1}$  is compatible with  $W_{i+1}$ . Thus  $\Omega$  is compatible with  $W$  as required.

Case (2c):  $j = i + 2$ . In this case,  $\pi_\Sigma(\Omega)$  does not cross either  $\gamma_i$  or  $\gamma_{i+2}$ . By tightness of the multigeodesic  $(\gamma_i)_i$  we see that it cannot cross  $\gamma_{i+1}$  either. As in (2b), we see that  $\Omega \cap O_i$  is compatible with  $W_i$ , and that  $\Omega \cap \partial_+O_i \subseteq W$ . Applying the inductive hypothesis (3) to  $W_{i+1}$ , we see that  $\Omega \cap O_{i+1} \subseteq W$ . Finally applying (2) to  $W_{i+2}$ ,  $\Omega \cap O_{i+2}$  is compatible with  $W_{i+2}$ . Thus  $\Omega$  is compatible with  $W$  as required.

(3) Suppose  $\Omega$  is a vertical annulus with  $\partial_-\Omega$  compatible with  $W$  and  $\partial_+\Omega \subseteq \partial_+O$  and compatible with  $\beta$ . Let  $\partial_-\Omega \subseteq O_i$ . As with (2), there are three possibilities.

Case (3a)  $i = n$ . We just apply (3) to  $W_n$ .

Case (3b)  $i = n - 1$ . As in (2b), we see that  $\Omega \cap W_i$  is a vertical annulus meeting  $\partial_+O_i = \partial_-O_n$ , and applying (3) to  $W_n$ , we see that  $\Omega \cap O_n \subseteq W$ .

Case (3c)  $i = n - 2$ . We argue as in (2c). This time, we get  $\Omega \cap (O_{n-1} \cup O_n) \subseteq W$ .

(4) Let  $B$  be a band. First, consider the case where  $\pi_\Sigma(B) = \Sigma$ . Let  $\partial_-B \subseteq O_i$  and  $\partial_+B \subseteq O_j$ . Thus,  $X(\gamma_i) \subseteq X_-(W \cap B)$  and  $X(\gamma_j) \subseteq X_+(W \cap B)$ , and so  $H_0(B) = d(X_-(W \cap B), X_+(W \cap B)) \geq d(X(\gamma_i), X(\gamma_j)) - 2 = j - i + 2$ . By construction we have a ladder crossing in  $W \cap B$  of length at most  $j - i$ . Thus  $H(B) - H_0(B) \leq 2$ . In other words this is 2-taut, so we are done.

For the case where  $\pi_\Sigma B \neq \Sigma$ , we make the following observation. Suppose that a band  $B$  is a union of two subbands,  $B = B_1 \cup B_2$ , meeting at a common horizontal boundary. If  $H(B_2) = 0$  (i.e. some annulus in  $\mathcal{W}$  crosses  $B \setminus \partial_V B$ ) then  $H(B) = H(B_1)$  and  $H_0(B) \geq H_0(B_1) - 1$ . Thus, if  $B_1$  is  $k$ -taut, then  $B$  is  $(k + 1)$ -taut.

Suppose now that  $B$  is a band with  $\Phi = \pi_\Sigma B \neq \Sigma$ . Since  $\partial_V^\Sigma B$  lies in  $W$ ,  $B$  can meet at most three  $O_i$ . Suppose  $\partial_-B \subseteq O_i$  and  $\partial_+B \subseteq O_j$ . We have three cases.

Case (4a)  $j = i$ .  $B \subseteq O_i$ , so we apply the inductive hypothesis (4) to  $W_i$ .

Case (4b)  $j = i + 1$ . If  $\gamma_{i+1} \cap \text{int } \Phi = \emptyset$ , then (by (3) applied to  $W_i$ )  $B \cap W_i$  is just a product:  $(B \cap W \cap \partial_+O_i) \times \pi_V(B \cap O_i)$  and so  $B \cap W$  is combinatorially identical to  $B \cap W_{i+1}$ . By (4) applied to  $W_{i+1}$ ,  $B \cap W_{i+1}$  is  $2(\kappa(\Sigma) - \kappa(\Phi))$ -taut, and so the same applies to  $B \cap W$ , and we are happy. In the case where  $\gamma_{i+1} \cap \text{int } \Phi \neq \emptyset$ , then  $H(B \cap O_i) = 0$ . By (4) applied to  $W_i$ ,  $B \cap W_i$  is  $2(\kappa(\Sigma) - \kappa(\Phi))$ -taut, and so by above observation,  $B = (B \cap O_i) \cup (B \cap O_{i+1})$  is  $(2(\kappa(\Sigma) - \kappa(\Phi)) + 1)$ -taut.

Case (4c).  $j = i + 2$ . Now,  $\gamma_i \cup \gamma_{i+2}$  is connected. Since neither multicurve can cross  $\partial^\Sigma \Phi$ ,  $(\gamma_i \cup \gamma_{i+2}) \cap \partial \Phi = \emptyset$ . We are again reduced to two subcases. First, if  $(\gamma_i \cup \gamma_{i+2}) \cap \text{int } \Phi$ , then by tightness of  $\gamma_i$ , we also have  $\gamma_i \cap \text{int } \Phi = \emptyset$ . Applying (3) to  $B \cap O_i$ , and then again to  $B \cap O_{i+1}$ , we see that  $B \cap W \cap (O_i \cup O_{i+1})$  is just a product. Thus,  $B \cap W$  is combinatorially identical to  $B \cap W_{i+2}$ . By (4) applied to  $W_{i+2}$ , the latter is  $2(\kappa(\Sigma) - \kappa(\Phi))$ -taut. The

second subcase is when  $\gamma_i \cup \gamma_{i+2} \subseteq \text{int } \Phi$ . In this case,  $H(B \cap O_i) = H(B \cap O_{i+2}) = 0$ . Also, by (4) applied to  $W_{i+1}$ ,  $B \cap W_{i+1}$  is  $2(\kappa(\Sigma) - \kappa(\Phi))$ -taut. Thus, the above observation applied twice tells us that  $B$  is  $2(\kappa(\Sigma) - \kappa(\Phi)) + 2$ -taut, as required.

(5) By construction,  $X(\gamma_i) \subseteq Y_1(X(\alpha) \cup X(\beta))$  for all  $i$ . By (5) applied to  $W_i$ , we have  $X(W_i) = Y_\kappa Y_{\kappa-1} \cdots Y_3 Y_2 (X(\gamma_{i-1}) \cup X(\gamma_i) \cup X(\gamma_{i+1}))$ . Thus  $X(W) = \bigcup_i X(W_i) \subseteq Y_\kappa \cdots Y_2 Y_1 (X(\alpha) \cup X(\beta)) = Y^\infty(X(\alpha) \cup X(\beta))$  as required.  $\diamond$

We can now include the case of complexity 1 bands as an afterthought:

**Lemma 8.3 :** *Suppose that  $\alpha, \beta \subseteq \Sigma$  are multicurves with  $\alpha$  complete. Then there is a complete annulus system,  $W = \bigcup \mathcal{W} \subseteq O$ , satisfying:*

- (1)  $\alpha = \pi_\Sigma(W \cap \partial_- O)$  and  $\beta = \pi_\Sigma(W \cap \partial_- O)$ ,
- (2) If  $\Omega, \Omega' \in \mathcal{W}$  with  $\pi_\Sigma \Omega = \pi_\Sigma \Omega'$ , then  $\Omega = \Omega'$ .
- (3) If  $B \subseteq O$  is a band with  $\kappa(B) \geq 2$ , then  $B$  is  $2(\kappa(\Sigma) - \kappa(B) + 2)$ -taut.
- (4) If  $B \subseteq O$  is a band with  $\kappa(B) = 1$ , then  $B$  is taut.
- (5)  $X(\mathcal{W}) \subseteq \bar{Y}^\infty(X(\alpha) \cup X(\beta))$ .

**Proof :** Let  $\hat{W}$  be the complete annulus system given by Lemma 4.2. Suppose that  $A$  is a type 1 brick with respect to  $\hat{W}$ , in other words, a band with  $\kappa(A) = 1$  and with  $W \cap A \setminus \partial_V A$  consisting of two curves,  $\delta \subseteq \partial_- A$  and  $\epsilon \subseteq \partial_+ A$ . By property 4.2(1),  $\pi_\Sigma \delta \neq \pi_\Sigma \epsilon$ . Let  $\delta = \gamma_0, \gamma_1, \dots, \gamma_n = \epsilon$  be a geodesic in  $\mathcal{G}'(\pi_\Sigma A)$ . Let  $\Omega_i = \gamma_i \times I_i \subseteq A$  be disjoint annuli occurring in this order vertically. This give an annulus system,  $W_A = \bigcup_i \Omega_i$  in  $A$ . We perform this construction for all type 1 bricks. Since these bricks are disjoint, we get an annulus system  $W = \hat{W} \cup \bigcup_A W_A$ , and  $A$  varies over all type 1 bricks. We need to verify the above properties.

- (1) Since  $W \cap \partial_H O = \hat{W} \cap \partial_H O$ , this follows by construction.
- (2) Suppose that  $\Omega, \Omega' \in \mathcal{W}$ , with  $\pi_\Sigma \Omega = \pi_\Sigma \Omega'$ . Let  $\Omega''$  be a vertical annulus connecting  $\Omega$  to  $\Omega'$  (so that  $\Omega \cup \Omega'' \cup \Omega'$  is a vertical annulus). Now  $\partial_H \Omega' \subseteq W$ , so  $\partial_\pm \Omega'$  are compatible with  $\hat{W}$ . It follows by 4.2(2) that  $\Omega'$  is compatible with  $\hat{W}$  and so by the construction, the only way that can happen is if  $\Omega = \Omega'$ .
- (3) Let  $B$  be a band with  $\kappa(B) \geq 2$ . We can assume that  $\partial_V^\Sigma B \subseteq \hat{W}$ , for if a boundary component were in  $W \setminus \hat{W}$ , then  $B$  would be crossed by an annulus of  $W$  and so it would have height 0, and there is nothing to prove. In other words,  $B$  is a band with respect to  $\hat{W}$ . With respect to  $\hat{W}$ ,  $B$  is  $2(\kappa(\Sigma) - \kappa(B) + 1)$ -taut. In passing to  $W$ ,  $H(B)$  can only increase. It is possible there may be some new curves in  $\partial_\pm B \cap W$ , but this could decrease  $H_0(B)$  by at most 2. It follows that, with respect to  $W$ ,  $B$  is  $2(\kappa(\Sigma) - \kappa(B) + 2)$ -taut.
- (4) Let  $B$  be a band with  $\kappa(B) = 1$ . As in (3), we need only consider the case where  $\partial B \subseteq \hat{W}$ . We note that there is no component of  $\hat{W}$  contained in the interior of  $B$ . For if  $\Omega$  were such a component, we could construct another annulus,  $\Omega'$  in  $B$  with  $\pi_\Sigma \Omega' \neq \pi_\Sigma \Omega$ , so that  $\partial_H \Omega' \cap \mathcal{W} = \emptyset$ , and with  $\pi_V \Omega$  contained in the interior of  $\pi_V \Omega'$ . Thus  $\Omega'$  crosses  $\Omega$ . But  $\partial_\pm \Omega$  is compatible with  $\hat{W}$ , so this contradicts 8.2(2). We see that the only element of  $\mathcal{W}$  in the interior of  $B$  were those added in some brick,  $A$ , of our construction. Since these were made out a geodesic in  $\mathcal{G}'(\pi_\Sigma B)$ , it follows that  $B$  is, by definition, taut.

(5) Clearly  $X(\mathcal{W}) \subseteq \bar{Y}(X(\hat{\mathcal{W}}))$ . Since  $X(\hat{\mathcal{W}}) \subseteq Y^\infty(X(\alpha) \cup X(\beta))$ , the result follows.  $\diamond$

Putting Lemma 5.3 together with Proposition 7.3, we see that we have proven the finite analogue of Theorem 7.1. The bi-infinite case now follows using Lemma 5.2, as discussed earlier.

There are various combinatorial properties of bands that we will need. In what follows, we can define the *height*  $H(B)$  of a complexity 1 (4HS or 1HT) band to be the number of elements of  $\mathcal{W}$  it contains. The height of a band of higher complexity is defined as above.

We shall say that two bands,  $A, B \subseteq \Phi$  are *parallel* if they have the same base surface,  $\pi_\Sigma A = \pi_\Sigma B$ . We say that  $B$  is a parallel subband if also  $B \subseteq A$ . Note that in this case, the closures of  $B \setminus A$  are parallel bands which we refer to as the *collars* of  $B$  (in  $A$ ). We denote them by  $B_-$  and  $B_+$ . We write  $D(B, A) = \max\{H(B_-), H(B_+)\}$  for the (*combinatorial*) *depth* of  $B$  in  $A$ . We say that a band  $A$  is *maximal* if it is not contained in any larger parallel band. Every band  $B$  is contained in a unique maximal parallel band,  $M(B)$ . We write  $D(B) = D(B, M(B))$ . We say that  $B$  is *r-collared* if  $D(B) \geq r$ .

**Lemma 8.4 :** *Suppose that  $B, B'$  are 1-collared with base surfaces  $\Phi$  and  $\Phi'$  respectively. If  $\text{int } B \cap \text{int } B' \neq \emptyset$ , then  $\Phi$  and  $\Phi'$  are nested, i.e. either  $\Phi \subseteq \Phi'$  or  $\Phi' \subseteq \Phi$ .*

**Proof :** Let  $S \subseteq \Psi$  be a horizontal fibre through a point of  $\text{int } B \cap \text{int } B'$ . Now  $S \cap B$  and  $S \cap B'$  are fibres of  $B$  and  $B'$  respectively, and so the boundaries of  $\Phi$  and  $\Phi'$  cannot cross. If  $\Phi$  and  $\Phi'$  are not nested, then we can find curves  $\alpha \subseteq \partial\Phi \setminus \partial\Phi'$  and  $\alpha' \subseteq \partial\Phi' \setminus \partial\Phi$ . Let  $\Omega, \Omega' \in \mathcal{W}$  be the vertical annuli with  $\pi_\Sigma \Omega$  and  $\pi_\Sigma \Omega'$ . Thus  $\Omega$  and  $\Omega'$  contain boundary components of the maximal bands,  $M(B)$  and  $M(B')$  respectively. By considering the horizontal projections of these bands to  $\mathbf{R}$ , we see easily that either  $\Omega$  crosses one of the collars  $M(B) \setminus B$  or  $\Omega'$  crosses one of the collars  $M(B') \setminus B'$ . This contradicts the assumption that  $B$  and  $B'$  are 1-collared.  $\diamond$

Let  $\mathcal{B}$  be the set of maximal bands in  $\Psi$  with  $H(B) > 0$ . Given  $n \in \mathbf{N}$ , if  $A \in \mathcal{B}$  and  $r \in \mathbf{N}$  with  $H(A) \geq 2r + 1$ , we can find a parallel subband,  $B \subseteq A$ , so that each of the collars,  $B_\pm$ , has height  $H(B_\pm)$  exactly  $r$ . For each  $A \in \mathcal{B}$ , and each such  $r$ , we choose such a band  $B$ , and write  $\mathcal{B}(r)$  for the set of all bands that arise in this way for a fixed  $r \in \mathbf{N}$ .

Given a subsurface,  $\Phi$ , of  $\Sigma$ , write  $\mathcal{B}_\Phi(r) \subseteq \mathcal{B}(r)$  for those bands in  $\mathcal{B}(r)$  whose base surface is a proper subsurface of  $\Phi$ .

Let  $\mathcal{D}$  be the brick decomposition of  $\Psi$  described earlier. Given a subset  $Q \subseteq \Psi$  we define the *size* of  $Q$ , denoted  $\text{size}(Q)$ , to be the number of bricks of  $\mathcal{D}$  whose interiors meet the interior of  $Q$ . We view  $\text{size}(Q)$  as a combinatorial measure of the volume of  $Q$ .

**Lemma 8.5 :** *Given  $h, r, \kappa$ , there is some  $\nu = \nu(h, r, \kappa)$  such that if  $B$  is a band with  $H(B) \leq h$  and base surface  $\Phi$ , then*

$$\text{size}(B \setminus \bigcup \mathcal{B}_\Phi(r)) \leq \nu(h, r, \kappa(\Phi)).$$

$\diamond$

Here we are allowing the case where  $\Phi = \Sigma$ , in which case,  $\mathcal{B}_\Phi(r) = \mathcal{B}(r)$ .

**Proof :** We proceed by induction on  $\kappa(\Phi)$ . If  $\kappa(\Phi) = 1$ , then we see explicitly that  $\text{size}(B) \leq 3H(B) + 2$ .

Now suppose that  $\kappa(\Phi) \geq 2$ . We can cut  $B$  into a set of  $H(B) + 1$  parallel bands each of height 0. It is thus sufficient to deal with the case where  $H(B) = 0$ , in other words, some vertical annulus of  $\mathcal{W}$  cuts through  $B$ . Now the set of all such annuli that cut through  $B$  cut  $B$  into set of bands of lower complexity. The number (possibly just 1) of such bands is bounded by  $\kappa(\Phi)$ . Let  $A \subseteq B$  be such a band, and let  $\Phi' = \pi_\Sigma A \subseteq \Phi$ . Note that  $\mathcal{B}_{\Phi'}(r) \subseteq \mathcal{B}_\Phi(r)$ . If  $H(A) \leq 2r$ , then  $\text{size}(A \setminus \bigcup \mathcal{B}_\Phi(r))$  is bounded (by  $\nu(2r, r, \kappa(\Phi) - 1)$ ). If  $H(A) \geq 2r + 1$ , then  $H(M(A)) \geq 2r + 1$ , and so there is some  $C \in \mathcal{B}(r)$  with base surface  $\Phi'$ , so that each of the collars  $M(A) \setminus C$  has height  $r$ . Now  $C \in \mathcal{B}_\Phi(r)$  and  $A \setminus C$  consists of at most two bands each of height at most  $r$ . Applying the inductive hypothesis again, we see that  $\text{size}(A \setminus \bigcup \mathcal{B}_\Phi(r))$  is bounded (by  $2\nu(r, r, \kappa(\Phi) - 1)$ ). Since there are at most  $\kappa(\Phi)$  such bands  $A$ , this bounds  $\text{size}(B \setminus \bigcup \mathcal{B}_\Phi(r))$ , and the result follows by induction.  $\diamond$

To each  $\Omega \in \mathcal{W}$ , we have associated a toroidal boundary component,  $\Delta(\Omega)$ , of  $\Lambda(\mathcal{W})$  as described above. We can lift the brick decomposition,  $\mathcal{D}$ , to a brick decomposition of  $\Lambda(\mathcal{W})$  which we also denote by  $\mathcal{D}$ . Let  $\mathcal{D}(\Omega)$  be the set of components of  $D \cap \Delta(\Omega)$  which are annuli, as  $D$  runs over the set of bricks,  $\mathcal{D}$ . Thus  $\Delta(\Omega) = \bigcup \mathcal{D}(\Omega)$  is a decomposition of this torus. We can view  $|\mathcal{D}(\Omega)|$  as a combinatorial measure of its length.

Similarly suppose  $P$  is a non-compact boundary component of  $\Lambda(\mathcal{W})$ . This must be a bi-infinite cylinder, identified with a boundary component of  $\Psi$ . We get a decomposition of  $P$  into a collection  $\mathcal{D}(P)$  of compact annuli, by taking the intersection with bricks.

We observed that we can recover  $\Psi$  up to homeomorphism by gluing a solid torus  $T(\Omega)$  to each  $\Delta(\Omega)$ , so as to obtain the space  $\Lambda(\mathcal{W}, \mathcal{W})$ . We can describe this more explicitly as follows. Given  $\Omega \in \mathcal{W}$ , we choose an explicit homeomorphism of  $T(\Omega) \setminus \partial_H \Omega$  with  $S^1 \times [0, 1] \times (0, 1)$ , and foliate  $T(\Omega) \setminus \partial_H \Omega$  with annuli of the form  $S^1 \times [0, 1] \times \{t\}$  for  $t \in (0, 1)$ . We set up the homeomorphism so that the two circles  $S^1 \times \{0\} \times \{t\}$  and  $S^1 \times \{1\} \times \{t\}$  are horizontal in  $\Delta(\Omega)$  and get identified with the same horizontal circle in  $\Omega$ , under the projection of  $\Lambda(\Omega)$  to  $\Psi$ . We add in two degenerate leaves,  $\partial_- \Omega$  and  $\partial_+ \Omega$ , to complete the foliation of  $T(\Omega)$ .

Now if  $S$  is a horizontal fibre in  $\Psi$ , then pulling back to  $\Lambda(\mathcal{W})$  we get a union of ‘‘horizontal’’ surfaces. We can now use the foliations on the tori  $T(\Omega)$  to complete this to a fibre of  $\Lambda(\mathcal{W}, \mathcal{W})$ . These fibres foliate  $\Lambda(\mathcal{W}, \mathcal{W})$ . We refer to them as *horizontal fibres*. We denote by  $S(x) \subseteq \Lambda(\mathcal{W}, \mathcal{W})$  the fibre containing  $x \in \Lambda(\mathcal{W}, \mathcal{W})$ . There is a natural projection of  $\Lambda(\mathcal{W}, \mathcal{W})$  to  $\Psi$  collapsing each torus  $T(\Omega)$  to  $\Lambda$ , so that the fibres of  $T(\Omega)$  are preimages of horizontal curves. The horizontal fibres of  $\Lambda(\mathcal{W}, \mathcal{W})$  are preimages of horizontal fibres of  $\Psi$ . By a *band* in  $\Lambda(\mathcal{W}, \mathcal{W})$  we mean the preimage of a band in  $\Lambda(\mathcal{W}, \mathcal{W})$ .

Suppose now that there is some  $L \geq 0$  and a partition  $\mathcal{W} = \mathcal{W}_0 \sqcup \mathcal{W}_1$  of  $\mathcal{W}$  such that  $|\mathcal{D}(\Omega)| \leq L$  for all  $\Omega \in \mathcal{W}_1$ . (Such a situation will arise in Section 8 — see Theorem 8.1.) We write

$$\mathcal{T} = \{T(\Omega) \mid \Omega \in \mathcal{W}_0\},$$

$$\mathcal{T}_1 = \{T(\Omega) \mid \Omega \in \mathcal{W}_1\}.$$

Let

$$\Theta = \Lambda(\mathcal{W}, \mathcal{W}_1) = \Lambda(\mathcal{W}) \cup \bigcup \mathcal{T}.$$

(Thus  $\Theta$  is homeomorphic to  $\Lambda(\mathcal{W}_0)$ .)

Note that  $\Theta$  is made out of a collection  $\mathcal{D}$  of bricks and “tubes”  $\mathcal{T}_1$ . We refer to the elements of  $\mathcal{D} \cup \mathcal{T}_1$  collectively as the *building blocks* of  $\Theta$ . Similarly, if  $R$  is a boundary component of  $\Theta$  (either a cylinder or a torus) we refer to the elements of  $\mathcal{D}(R)$  as the *building blocks* of  $R$ . If  $\beta$  is a path in  $\Theta$ , or in  $R$ , we define the *combinatorial length* to be equal to the number of building blocks that it meets (counting multiplicities).

Note that (since we are assuming that  $|\mathcal{D}(\Omega)| \leq L$  for all  $\Omega \in \mathcal{W}_1$ ), each building block of  $\Theta$  meets boundedly many others (in fact at most  $\max\{8, L\}$ ). It follows that given any  $x \in \Theta$  and any  $r \in \mathbf{N}$ , there is bound, depending on  $r$ , on the number of building blocks that can be connected to  $x$  by a path of combinatorial length at most  $r$ .

We want to make a couple of observations concerning the embedding of the boundary components of  $\Theta$  into  $\Theta$ . We begin with the non-compact components, since the description is somewhat simpler. The geometrical interpretation of these statements is made more apparent by Lemma 8.8, which will eventually be used in Section 14 (see Lemmas 14.6 and 14.7).

**Lemma 8.6 :** *Suppose that  $\Pi$  is a non-compact boundary component of  $\Theta$ . Suppose that  $\beta$  is a path of combinatorial length  $n$  in  $\Theta$  connecting two points,  $x, y \in \Pi$ , and homotopic into  $\Pi$ , relative to  $\{x, y\}$ . Then  $x$  and  $y$  are connected by a path in  $\Pi$  whose combinatorial length is bounded above by some uniform linear function of  $n$ .*

**Proof :** Let  $C_x, C_y \in \mathcal{D}(\Pi)$  be annular blocks containing  $x$  and  $y$  respectively, and let  $\mathcal{D}_{x,y} \subseteq \mathcal{D}(\Pi)$  be the set of annular blocks between  $C_x$  and  $C_y$ . We want to bound  $|\mathcal{D}_{x,y}|$  linearly in terms of  $n$ .

Given  $z \in \Pi$ , recall that  $S(z)$  is the horizontal fibre of  $\Lambda(\mathcal{W}, \mathcal{W})$  containing  $z$ . Let  $F(z)$  be the component of  $S(z) \cap \Theta$  containing  $z$ . There is a bound, say  $l_0$ , on the number of blocks that  $F(z)$  can meet, depending only on  $\kappa(\Sigma)$ . Now if  $z \in \bigcup \mathcal{D}_{x,y}$  we see that  $F(z)$  must meet  $\beta$  (from the assumption that  $\beta$  is homotopic into  $\Pi$ ). It follows that  $z$  is connected to  $\beta$  by a path in  $\Theta$  of combinatorial length at most  $l_0$ . By the earlier observation (on the uniform local finiteness of our system of building blocks) we see that this gives some bound on  $|\mathcal{D}_{x,y}|$  in terms of  $n$ .

To make this a linear bound, let us fix our favourite positive integer, say 10, and let  $l_1$  be the bound when  $n$  is at most  $10 + 2l_0$ . This means that if  $z, w \in \Pi$  are separated by at least  $l_1$  blocks, then if  $\gamma$  is any path from  $F(z)$  to  $F(w)$  in  $\Theta$ , which can be homotoped into  $\Pi$  by sliding its endpoints along  $F(z)$  and  $F(w)$  respectively, then  $l(\gamma) \geq 10$ .

For the general case, we now choose a sequence of points  $x = z_0, z_1, \dots, z_p = y$  in  $\bigcup \mathcal{D}_{x,y}$  so that  $z_i$  and  $z_{i+1}$  are separated by at least  $l_1$  annular blocks in  $\mathcal{D}(\Omega)$ , and with  $|\mathcal{D}_{x,y}|$  bounded above by a fixed linear function of  $p$ . Now the path  $\beta$  must cross each of the surfaces  $F(z_i)$  and so  $p$  is in turn bounded above by a linear function of  $l(\beta)$ .  $\diamond$

We need a version of this where  $\Pi$  is replaced by a toroidal boundary component,  $\Delta = \Delta(\Omega)$  for some  $\Omega \in \mathcal{W}_0$ .

**Lemma 8.7 :** *Suppose that  $\Delta$  is a compact boundary component of  $\Theta$ . Suppose that  $\beta$  is a path of combinatorial length  $n$  in  $\Theta$  connecting two points,  $x, y \in \Delta$ . If  $\beta$  is homotopic to a path in  $\Delta$ , relative to  $\{x, y\}$ , then we can find such a path in  $\Delta$  in this relative homotopy class whose combinatorial length is bounded above by some uniform linear function of  $n$ .*

**Proof :** The argument is a slight refinement of that used for Lemma 4.6. If  $z \in \Delta$ , we can define the surface  $F(z)$  exactly as in Lemma 4.6.

Note that  $\partial_H \Omega$  cuts  $\Delta = \Delta(\Omega)$  into to annuli,  $A$  and  $A'$ , say. Suppose first that  $x, y \in A$  and that  $\beta$  is homotopic into  $A$  relative to  $x, y$ . Essentially the same argument as before gives as a bound on the number of building blocks separating  $x$  and  $y$  in  $A$ . We can therefore construct surfaces  $F(z_i)$  as before so as to obtain a linear bound in terms of  $n$ .

The general case is complicated by the fact that  $\beta$  might wrap around  $\Delta$  many times in the vertical direction (that of a meridian curve). However, we can construct surfaces,  $F(z_i)$ , on both sides of  $\Delta$  (the annuli  $A$  and  $A'$ ), and we note that  $\beta$  must cross all of these surfaces in sequence (counting multiplicities). We should note that nothing we have said excludes the possibility that the total vertical length,  $\Delta(\Omega)$ , is small, (maybe smaller than  $l_0$ , for example) so we need to take at least one such surface.  $\diamond$

We next want to translate some of these combinatorial observations into more geometrical terms. To this end, we shall put a riemannian metric,  $d$ , on  $\Lambda(\mathcal{W}, \mathcal{W})$ . The construction of  $d$  will be explained more carefully in Section 10, where we construct the model space. For the purposes of this section, we only care about the metric restricted to  $\Theta$ . The key points (which can be taken as hypotheses for the moment) are as follows.

The local geometry of the decomposition of  $(\Theta, d)$  is bounded. In particular, there is a uniform lower bound on the injectivity radius of  $(\Theta, d)$ . Each building block of  $\Theta$  (in  $\mathcal{D} \cup \mathcal{T}_1$ ) has bounded diameter. Moreover, there is a positive lower bound on the  $d$ -distance between any two disjoint building blocks. We can also assume that each of the building blocks of any boundary component of  $\Theta$  is a fixed isometry class of annulus, say  $S^1 \times [0, 1]$ . If  $x \in \Theta$ , the fibre  $S(x)$  meets each block of  $\Theta$  is a surface of bounded diameter. In particular, the diameter of each component of  $S(x) \cap \Theta$  is bounded.

Note that an immediate consequence is that, at least up to homotopy, the combinatorial length of a path in  $\Theta$  is bounded above by a linear function of its  $d$ -length. Since the building blocks are not simply connected, we do not have a converse statement keeping control of homotopy. However, distances between points, and diameters of sets, in the metric  $d$  are bounded above by a linear function of their combinatorial counterparts.

We can immediately translate Lemmas 8.6 and 8.7 into geometrical terms and express them in a unified fashion as follows.

Suppose that  $R$  is a boundary component of  $\Theta$ , and let  $H \subseteq \pi_1(\Theta)$  be the subgroup generated by a horizontal longitude. (Thus  $H \equiv \pi_1(R)$  if  $R$  is a bi-infinite cylinder). Let  $\hat{\Theta}$  be the cover corresponding to  $H$ . Thus  $R$  lifts to a bi-infinite cylinder,  $\hat{R} \subseteq \hat{\Theta}$  (so that  $\hat{R} \equiv R$  in the non-compact case).

**Lemma 8.8 :** *If  $R$  is a boundary component of  $\Theta$ , and  $\hat{R} \subseteq \hat{\Theta}$  constructed as above, then  $\hat{R}$  is quasi-isometrically embedded in  $\hat{\Theta}$ .*  $\diamond$

For applications, we will need another riemannian metric,  $\rho$ , on  $\Lambda(\mathcal{W}, \mathcal{W})$ , and its restriction to  $\Theta$ . This can be taken to be equal to  $d$  on  $\Theta$  and equal to 0 on each tube  $T \in \mathcal{T}$ . In other words we force each element of  $T$  to have diameter 0. As stated, we just get a pseudometric, and it will be discontinuous at the toroidal boundary components. If we want, we can smooth it out in a small neighbourhood of these boundaries. The only important requirement is that each element of  $\mathcal{T}$  should have bounded diameter with respect to  $\rho$ .

In summary, at this point, we have a manifold,  $\Lambda(\mathcal{W}, \mathcal{W})$  diffeomorphic to  $\Sigma \times \mathbf{R}$ , a collection,  $\mathcal{T}$ , of unlinked solid tori in  $\Lambda(\mathcal{W}, \mathcal{W})$ , the “thick part”,  $\theta = \Lambda(\mathcal{W}, \mathcal{W}) \setminus \text{int } \mathcal{T}$ . We have an electric pseudometric,  $\rho$ , in  $\Lambda(\mathcal{W}, \mathcal{W})$ , which is identically zero restricted to each  $T \in \mathcal{T}$ .

We can list some geometric properties of  $\Theta$  as follows:

(W1) Every point  $x \in \Theta$  lies in a fibre  $S(x)$  of uniformly bounded  $\rho$ -diameter.

We can simply take  $S(x)$  to be the horizontal fibre as described above. In this way,  $S(x)$ , will vary continuously in  $x$ . (This will be used in Section 9.) Alternatively, we can push the fibre off each torus of  $\mathcal{T}_0$  so as to give us a surface,  $S(x)$ , in  $\Theta$ , while retaining a bound on its  $\rho$ -diameter in  $\Theta$ . (Here we are referring to the extrinsic diameter in  $\Theta$ , and not the induced path-metric in  $\Theta$ , which may be arbitrarily large.) In this case, however, we can no longer assume that  $S(x)$  varies continuously in  $x$ . (This alternative construction will be useful in Section 13.)

(W2) Each  $x \in \Theta$  is contained in a loop  $\gamma_x \subseteq \Theta$  of bounded  $d$ -length, and homotopic to a curve,  $[\gamma_x] \in X(\Sigma)$ . If  $x$  lies in a component,  $R$ , of  $\partial\Theta \setminus \partial\Psi$ , then we can take  $\gamma_x$  to be the horizontal curve in  $R$  containing  $x$ .

In fact, if  $x \in D \in \mathcal{D}$ , we take  $\gamma_x$  to be freely homotopic into one of the vertical boundary components of  $D$ . Thus,  $\gamma_x$  is freely homotopic into an annulus  $\Omega_x \in \mathcal{W}$ . (In all cases, we can assume that  $\gamma_x \subseteq S(x)$ .)

(W3) If  $x, y \in \Theta$ , with  $d(x, y) \leq \eta$ , then  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$  is bounded above.

Here we can take  $\eta > 0$  to be the lower bound on injectivity radius, but any positive constant would do.

(W4) If  $x, y \in \Theta$ , then  $\rho(x, y)$  is bounded above by a uniform linear function of  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$ . ■

This is our geometric interpretation of the tautness condition (Theorem 8.1(P3)) in the case where the base surface is  $\Sigma$ . Note that  $\gamma_x$  and  $\gamma_y$  are homotopic to  $\Omega_x$  and  $\Omega_y$  and a bounded  $d$ -distance from the the corresponding tubes  $T(\Omega_x)$  and  $T(\Omega_y)$  (these tubes might lie in either  $\mathcal{T}$  or  $\mathcal{T}_1$ ). By tautness, there is a ladder,  $\Omega_x = \Omega_0, \Omega_1, \dots, \Omega_n = \Omega_y$  in  $\mathcal{W}$ , with  $n$  bounded above by a linear function of  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$  (in fact,  $n \leq d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y]) + c$ , where  $c$  is the tautness constant). Two consecutive  $\Omega_i$  and  $\Omega_{i+1}$  meet a horizontal fibre which has bounded  $\rho$ -diameter. We see that  $\rho(T(\Omega_i), T(\Omega_{i+1}))$  is bounded above, and so  $\rho(x, y)$  is linearly bounded in terms of  $n$  and hence in terms of  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$  as claimed. (One needs to rephrase this slightly if  $\Sigma$  is 4HS or 1HT, but the argument is essentially the same — consecutive tubes are a bounded distance apart, since they meet a common building block. Here we use the metric on the modified curve graph  $\mathcal{G}(\Sigma)$ .)

All these statements have analogues for the case of a band  $B \subseteq \Lambda(\mathcal{W}, \mathcal{W})$ . We are only really interested in the case where all vertical boundary components of  $B$  lie in  $\mathcal{W}_0$ . In this case,  $\partial_H B$  is the relative boundary of  $B$  in  $\Theta$ . Let  $\Phi$  be the base surface of  $B$ .

Let  $\mathcal{T}_B^0$  be the set of tubes in  $\mathcal{T}$  whose interiors meet  $B$ . We let  $d_B$  be the riemannian metric on  $B$  induced from  $d$ , and let  $\rho_B$  be the metric obtained from  $d_B$  by forcing each set  $T \cap B$  for  $T \in \mathcal{T}_B^0$  to have diameter 0 (the intrinsic electric pseudometric in  $B$ .) We can now perform the above constructions inside  $B$ . We get:

(W5) If  $x \in B \cap \Theta$ , then  $x$  lies in a fibre,  $F(x) \subseteq B$  of  $B$ , of uniformly bounded  $\rho_B$ -diameter.

(W6) If  $x \in B \cap \Theta$ , then  $x$  lies in a loop  $\gamma_x^B \subseteq B \cap \Theta$  of bounded  $d_B$ -length, with  $[\gamma_x^B] \subseteq X(\Phi) \subseteq X(\Sigma)$ . If  $x \in B \cap \partial\Theta$  we can take  $\gamma_x^B$  to be a horizontal curve of that boundary component.

(W7) If  $x, y \in B \cap \Theta$ , and  $d_B(x, y) \leq \eta$ , then  $d_{\mathcal{G}(\Phi)}([\gamma_x^B], [\gamma_y^B])$  is bounded above.

(W8) If  $x, y \in B \cap \Theta$ , then  $\rho_B(x, y)$  is bounded above by a uniform linear function of  $d_{\mathcal{G}(\Phi)}([\gamma_x^B], [\gamma_y^B])$ .

Finally we need to be able to recognise when a point of  $\Theta$  does not lie in a given maximal band. We can define a *maximal band* in  $\Lambda(\mathcal{W}, \mathcal{W})$  to be the preimage of a maximal band in  $\Psi$ .

(W9) If  $x \in \Theta \setminus B$ , then  $x$  lies in a loop  $\delta_x^B \subseteq B \setminus \Theta$  of bounded  $d_B$ -length such that either  $[\delta_x^B]$  is homotopic to a torus  $T \in \mathcal{T}$  not lying entirely inside  $B$ , or else  $[\delta_x^B]$  is not homotopic into  $\Phi$ .

To see this, let  $S(x)$  be the horizontal fibre through  $x$ , and let  $F$  be the component of  $S(x) \cap \Theta$  containing  $x$ . This has bounded  $d$ -diameter. If  $\pi_\Sigma F$  is not a subsurface of  $\Phi$ , then we can choose  $\delta_x^B \subseteq F$ , not homotopic into  $\Phi$ . If  $\pi_\Sigma F \subseteq \Phi$ , then by the maximality of  $B$ , at least one of the boundary components of  $F$  must be a torus  $T \in \mathcal{T}$ , and this is not contained in  $B$ . We can take  $\delta_x^B \subseteq F$ , freely homotopic to this boundary component.

## 9. Margulis tubes.

The constructions described in Section 8 are the basis of the model of the “thick part”. To complete the picture we will need some description of the “thin part”. There may be parabolic cusps, but the main thing we have to worry about is the existence of Margulis tubes.

Recall that *quasi-isometry* between two geodesic spaces,  $(X, d)$  and  $(X', d')$  is a map  $f : X \rightarrow X'$ , for which  $c_1 > 0, c_2, c_3, c_4, c_5$  with  $c_1 d(x, y) - c_2 \leq d'(f(x), f(y)) \leq c_3 d(x, y) + c_4$  for all  $x, y \in X$ , and with  $X' \subseteq N(f(X), c_5)$ . Here we shall make the following definition:

**Definition :** A *sesquilateral* map is a surjective lipschitz quasi-isometry.

In other words, in the definition of quasi-isometry we put  $c_4 = c_5 = 0$ .

**Definition :** A *universally sesquilipschitz* map between two spaces is a homotopy equivalence whose lifts to the universal covers are sesquilipschitz.

One can check that a universally sesquilipschitz map is indeed sesquilipschitz.

Throughout this section our results refer to implicitly assumed constants. We will take it as implied that the constants outputted are explicit functions of the constants inputted, though we will not bother to calculate these function explicitly. (Here they are all computable.) We shall use the adjective “uniform” if we want to stress this point.

We shall begin our discussion in dimension 1. It is well known that any quasi-isometry of the real line  $\mathbf{R}$  is a bounded distance from a bilipschitz homeomorphism. We note the following variation:

**Lemma 9.1 :** *Let  $f : \mathbf{R} \longrightarrow \mathbf{R}$  is sesquilipschitz then there is a sesquilipschitz homotopy to a bilipschitz map.*

In other words there is a sesquilipschitz map  $F : \mathbf{R} \times [0, 1] \longrightarrow \mathbf{R}$  with  $f = [x \mapsto F(x, 0)]$  and with  $g = [x \mapsto F(x, 1)]$  bilipschitz. Note that it follows that  $f$  is a bounded distance from  $g$ .

**Proof :** We can assume that  $f$  is end-preserving. We fix some sufficiently large, but bounded, constant,  $k \geq 0$ , so that  $f(x + k) > f(x) + 1$  for all  $x \in \mathbf{R}$ . We set  $g|_{k\mathbf{Z}} = f|_{k\mathbf{Z}}$ , and interpolate linearly. We then take a linear homotopy between  $f$  and  $g$ .  $\diamond$

A similar argument can be carried out equivariantly. We write  $S(r) = \mathbf{R}/r\mathbf{Z}$  for the circle of length  $r$ . We obtain:

**Lemma 9.2 :** *Suppose that  $r, s > 0$ , and that  $f : S(r) \longrightarrow S(s)$  is a universally sesquilipschitz map. Then there is a universally sesquilipschitz homotopy from  $f$  to a bilipschitz map from  $S(r)$  to  $S(s)$ .*  $\diamond$

In particular, the ratios  $s/r$  and  $r/s$  are bounded. Here, all constants depend on those of those of  $f$ .

More will be said about 1-dimensional quasi-isometries in Section 10, but this will do for the moment. We move on to 2 dimensions.

Let  $\Delta$  be a euclidean torus equipped with a preferred basis,  $(l, m_0)$  for the integral first homology. We refer to  $l$  as the *longitude* of  $\Delta$  and to  $m_0$  as the *standard meridian*. More generally a *meridian* will be a curve of the form  $m_0 + nl$  for some  $n \in \mathbf{Z}$ . In situations of interest to us the length of the longitude will be bounded both above and below, and so it is often convenient to normalise so that its length is 1. In this case the structure on  $\Delta$  is determined by a complex *modulus*  $\lambda \in \mathbf{C}$  with  $\Im(\lambda) > 0$ , so that  $\Delta = \Delta(\lambda) = \mathbf{C}/\langle [z \mapsto z + 1], [z \mapsto z + \lambda] \rangle$ , with  $[z \mapsto z + \lambda]$  giving us the standard meridian. Note that the shortest meridian in  $\Delta$  has length between  $\Im(\lambda)$  and  $\Im(\lambda) + \frac{1}{2}$ .

We refer to a geodesic longitude as being *horizontal*: it is the projection of a line parallel to the real axis. These foliate  $\Delta$  and we write  $S(\Delta)$  for the leaf space obtained by collapsing each leaf to a point. It is a circle of length  $\Im(\lambda)$ .

In most cases of interest, the injectivity radius of  $\Delta(\lambda)$  will be bounded below by some positive constant. One can see that this is equivalent to putting a lower bound on  $\mathfrak{S}(\lambda)$ . Moreover if there is an equivariant quasi-isometry between the universal covers of two such tori, a lower bound on the injectivity radius of one gives a lower bound for the other.

Let  $\Delta = \Delta(\lambda)$  and  $\Delta' = \Delta(\lambda')$ .

**Definition :** A map  $f : \Delta \longrightarrow \Delta'$  is *horizontally straight* if it sends each horizontal longitude of  $\Delta$  isometrically to a horizontal longitude of  $\Delta'$ .

In formulae, this means that, writing  $\tilde{\Delta} = \mathbf{R}^2 = \tilde{\Delta}'$ , we have  $f(x, y) = (x + f_H(y), f_V(y))$ , where  $f_H, f_V : \mathbf{R} \longrightarrow \mathbf{R}$  satisfy  $f_H(y + \mathfrak{S}(\lambda)) = f_H(y) + \mathfrak{R}(\lambda') - \mathfrak{R}(\lambda)$  and  $f_V(y + \mathfrak{S}(\lambda)) = f_V(y) + \mathfrak{S}(\lambda')$ .

Note that such a map induces a map,  $S(f) : S(\Delta) \longrightarrow S(\Delta')$  (lifting to  $f_V : \mathbf{R} \longrightarrow \mathbf{R}$ ). One can easily check that if  $f$  is lipschitz (respectively, sesquilipschitz, universally sesquilipschitz) then so is  $S(f)$ .

**Lemma 9.3 :** *Suppose  $\mathfrak{S}(\lambda) \geq \epsilon > 0$ . Suppose  $f : \Delta \longrightarrow \Delta'$  is a lipschitz map sending the longitude (homotopically) to the longitude. Then there is a lipschitz homotopy of  $f$  to a (lipschitz) horizontally straight map. Moreover if  $f$  is (universally) sesquilipschitz, we can take the homotopy to be (universally) sesquilipschitz. Here the constants only depend on  $\epsilon$  and the initial (sesqui)lipschitz constants.*

**Proof :** Let  $m$  be a shortest meridian on  $\Delta$ . The lower bound on  $\mathfrak{S}(\lambda)$  means that there is a lower bound on its slope with respect to any horizontal longitude. We now define  $g : \Delta \longrightarrow \Delta'$  by taking  $g|m = f|m$ , and extending in the unique way to a horizontally straight map. We now take a linear homotopy between  $f$  and  $g$ . The above properties are easily verified.  $\diamond$

**Lemma 9.4 :** *Suppose that  $f : \Delta \longrightarrow \Delta'$  is a universally sesquilipschitz horizontally straight map. Then there is a universally sesquilipschitz homotopy from  $f$  to a horizontally straight bilipschitz homeomorphism.*

**Proof :** By lemma 9.2 there is a universally sesquilipschitz homotopy  $F$  of  $S(f)$  to a bilipschitz map. Lifting to  $\mathbf{R}$  gives us a homotopy  $\tilde{F} : \mathbf{R} \times [0, 1] \longrightarrow \mathbf{R}$  from  $f_V : \mathbf{R} \longrightarrow \mathbf{R}$  to a bilipschitz map  $h$ . Now define  $\tilde{G} : \mathbf{R}^2 \times [0, 1] \longrightarrow \mathbf{R}^2$  by  $\tilde{G}(x, y, t) = (x + f_H(y), \tilde{F}(y, t))$ . Projecting back down gives us a bilipschitz map,  $g : \Delta \longrightarrow \Delta$ , with  $g_H = f_H$  and  $g_V = h$ .  $\diamond$

We have assumed that  $f$  lifts to a quasi-isometry of universal covers  $\tilde{\Delta} \longrightarrow \tilde{\Delta}'$ . However, we only really require that it is a quasi-isometry for the covers corresponding to the longitudes of  $\Delta$  and  $\Delta'$ , which are bi-infinite cylinders. This is enough to show that  $f_V$  is a quasi-isometry. If we assume that  $f$  is lipschitz, then the same conclusion holds.

We remark that the existence of a  $k$ -bilipschitz map from  $\Delta(\lambda)$  to  $\Delta(\lambda')$  implies both that  $k^{-1}\mathfrak{S}(\lambda) \leq \mathfrak{S}(\lambda') \leq k\mathfrak{S}(\lambda)$  and  $k^{-1}|\lambda| \leq |\lambda'| \leq k|\lambda|$ .

We need also to consider lipschitz maps to the circle. Suppose that  $f : \Delta \rightarrow S(1)$  is  $k$ -lipschitz. Let  $m$  be a shortest meridian on  $\Delta$ . It has length at most  $\mathfrak{S}(\lambda) + \frac{1}{2}$ , and so its image has length at most  $k(\mathfrak{S}(\lambda) + \frac{1}{2})$ . Thus the degree of  $f|m$  is at most  $k(\mathfrak{S}(\lambda) + \frac{1}{2})$  in absolute value.

**Lemma 9.5 :** *Suppose the torus  $\Delta(\lambda)$  admits a  $k$ -lipschitz map to  $S(1)$  which has degree 1 on the longitude and degree 0 on the standard meridian. Then  $|\lambda| \leq (k + 1)(\mathfrak{S}(\lambda) + \frac{1}{2})$ .*

**Proof :** Let  $l$  be the longitude, and  $m_0$  and  $m$  be the standard and shortest meridians respectively. Thus,  $m = m_0 + pl$  for some  $p \in \mathbf{Z}$ , and so  $\deg(f|m) = \deg(f|m_0) + p \deg(f|l) = p$ , where  $f$  is the  $k$ -lipschitz map. By the above observation,  $|p| \leq k(\mathfrak{S}(\lambda) + \frac{1}{2})$ . Now

$$\begin{aligned} |\lambda| &= \text{length}(m_0) \\ &\leq \text{length}(m) + |p| \\ &\leq (\mathfrak{S}(\lambda) + \frac{1}{2}) + k(\mathfrak{S}(\lambda) + \frac{1}{2}) \\ &= (k + 1)(\mathfrak{S}(\lambda) + \frac{1}{2}). \end{aligned}$$

◇

**Lemma 9.6 :** *Given  $c, k > 0$ , there is some  $h > 0$  such that if a map  $f : \Delta(\lambda) \rightarrow S(1)$  is  $k$ -lipschitz and degree 1 on the longitude and degree 0 on the meridian, and if  $\mathfrak{S}(\lambda) \leq c$ , then there is a  $h$ -lipschitz homotopy of  $f$  to a  $k$ -lipschitz map,  $g$ , sending every geodesic standard meridian to a point.*

**Proof :** Let  $l$  be some horizontal longitude. There is a unique map  $g : \Delta(\lambda) \rightarrow S(1)$  so that  $g|l = f|l$  and sending every standard meridian to a point. This is also  $k$ -lipschitz. Clearly  $f$  and  $g$  are homotopic, and we take a linear homotopy between them. To bound its lipschitz constant, it is enough to note that every geodesic standard meridian of  $\Delta(\lambda)$  gets mapped under  $f$  to a curve of length at most  $k|\lambda| \leq k(k + 1)(c + \frac{1}{2})$  by Lemma 9.6. ◇

We now move into 3-dimensions to consider Margulis tubes. For the purposes of this section, a ‘‘Margulis tube’’ is just a particular kind of hyperbolic structure on the solid torus.

Let  $r \geq 0$  and  $R = 2\pi \sinh r$ . Given  $t \geq 0$ , set  $a_r(t) = \cosh(rt)/\cosh r$  and set  $b_r(t) = \sinh(rt)$ . Define a riemannian metric on  $\mathbf{R} \times S^1 \times (0, 1]$  by  $ds^2 = a_r(t)^2 dx^2 + b_r(t)^2 dy^2 + r^2 dt^2$ , where  $(x, y, t)$  are the local coordinates. Let  $N$  be the metric completion of this space.

What we have defined is just the  $r$ -neighbourhood,  $N = N(\tilde{\alpha})$  of a bi-infinite geodesic,  $\tilde{\alpha}$  in  $\mathbf{H}^3$ . Its boundary,  $\partial N = \mathbf{R} \times S^1 \times \{1\}$  is isometric to  $\mathbf{R} \times S(R)$ , by an isometry that is the identity on the first co-ordinate.

A loxodromic isometry,  $g$ , with axis  $\tilde{\alpha}$  acts by translating the  $x$ -coordinate and rotating the  $y$ -coordinate. We refer to the quotient,  $T = N/\langle g \rangle$  as a *Margulis tube*. Thus,  $\partial T$  is a euclidean torus. The quotient,  $\alpha = \tilde{\alpha}/\langle g \rangle$  is the *core* of  $T$ . We refer to  $r = d(\alpha, \partial T)$  as the

depth of  $T$ . We define the *standard meridian*,  $m_0$ , of  $\partial T$ , to be homotopically trivial in  $T$ . It has length  $R$ . We deem another curve,  $l$ , in  $\partial T$ , homotopic to the core curve,  $\alpha$ , to be the *longitude* of  $\partial T$ . (Its homotopy class gives us some additional structure to  $T$ .) Note that the length of the core curve is equal to  $\text{area}(\partial T)/R \cosh r = 2\pi \text{area}(\partial T)/R\sqrt{R^2 + 4\pi^2}$ .

The following seems well known:

**Lemma 9.7 :** *Given any euclidean torus  $\Delta$  with preferred longitude and standard meridian, there is a unique (up to isometry) Margulis tube,  $T$ , with  $\partial T = \Delta$ .*

**Proof :** The cover of  $\Delta$  corresponding to the standard meridian is isometric to  $\mathbf{R} \times S(R)$  for some  $R > 0$ . Set  $r = \sinh^{-1}(R/2\pi)$  and construct  $N$  as above. The action of the longitude on  $\partial N = \mathbf{R} \times S(R)$  extends to a loxodromic on  $N$ , and we take the quotient.

Uniqueness is easily established (see the remark after Lemma 9.8 below).  $\diamond$

**Lemma 9.8 :** *Suppose  $T, T'$  are Margulis tubes and  $f : \partial T \rightarrow \partial T'$  is a  $k$ -bilipschitz map sending the standard meridian of  $\partial T$  (homotopically) to the standard meridian of  $\partial T'$ . Then  $f$  extends to a  $k'$ -bilipschitz map,  $f : T \rightarrow T'$ , where  $k'$  depends only on  $k$ .*

**Proof :** Let  $r, s$  be the depths of  $T, T'$  respectively. The lengths of the standard meridians are  $2\pi \sinh r$  and  $2\pi \sinh s$ . Their ratios are bounded by  $k$ . This also gives bounds on the ratios of  $r$  and  $s$  and of  $\cosh r$  and  $\cosh s$ . For  $t \in (0, 1]$ ,  $a_r(t)/a_s(t)$  varies between  $\cosh s/\cosh r$  and 1 and  $b_r(t)/b_s(t)$  varies between  $r/s$  and  $\sinh r/\sinh s$ . We can thus define a bilipschitz map  $[(x, t) \mapsto (f(x), t)] : \partial T \times (0, 1] \rightarrow \partial T' \times (0, 1]$  and extend over the completions.  $\diamond$

We remark that in the case where  $k = 1$ , we can take  $k' = 1$ , giving the uniqueness part of Lemma 9.7.

In the cases of interest to us the length of the longitude will be bounded above and below, and so, up to bilipschitz equivalence we can normalise so that it has length 1. (This is really just for notational convenience.) In this case, we can identify  $\partial T$  with  $\mathfrak{S}(\lambda)$  for some modulus  $\lambda \in \mathbf{C}$ . Note that  $|\lambda| = R$  and that  $\text{area} \partial T = \mathfrak{S}(\lambda)$ . We see that the length of the core curve is  $L(\lambda) = 2\pi \mathfrak{S}(\lambda)/|\lambda| \sqrt{|\lambda|^2 + 4\pi^2}$ .

We earlier defined the leaf space,  $S(\partial T)$  of  $\partial T$  by collapsing each geodesic longitude to a point. Its length is  $\mathfrak{S}(\lambda)$ . We can define another leaf space,  $S_0(\partial T)$  by collapsing each geodesic standard meridian to a point. It has length  $\text{area}(\partial T)/R = \mathfrak{S}(\lambda)/|\lambda|$ . There is a natural linear homeomorphism from  $S_0(\partial T)$  to the core curve,  $\alpha$ , given by orthogonal projection. It contracts distances by a factor  $L(\lambda)/(\mathfrak{S}(\lambda)/|\lambda|) = |\lambda|L(\lambda)/\mathfrak{S}(\lambda) = 2\pi/\sqrt{|\lambda|^2 + 4\pi^2}$ . By precomposing the inverse of this projection with the projection of  $T$  to the core curve, we see that the projection  $\partial T \rightarrow S_0(\Delta)$  extends to a  $(\sqrt{|\lambda|^2 + 4\pi^2}/2\pi)$ -lipschitz map  $T \rightarrow S_0(\Delta)$ .

**Lemma 9.9 :** *Suppose that  $T$  is a Margulis tube where the longitude of  $\partial T$  has length 1. Suppose that  $\partial T$  has area at most  $c$ . Let  $f : \partial T \rightarrow S(1)$  be a  $k$ -lipschitz map which has degree 1 on the longitude and degree 0 on the standard meridian. Then  $f$  extends to*

a  $k'$ -lipschitz map  $f : T \rightarrow S(1)$ , where  $k'$  depends only on  $k$  and  $c$ .

**Proof :** Using Lemma 9.6, we can reduce to the case where  $f$  sends each geodesic meridian to a point — since the lipschitz homotopy given by Lemma 9.6 can be carried out in a uniformly small neighbourhood of  $\partial T$  in  $T$ .

We can thus assume we have a  $k$ -lipschitz map  $f : \partial T \rightarrow S(1)$  which factors through the projection  $\partial T \rightarrow S_0(\partial T)$ . But the latter projection extends to a  $(\sqrt{|\lambda|^2 + 4\pi^2}/2\pi)$ -lipschitz map  $T \rightarrow S_0(\partial T)$ . Composing this gives us a  $k(\sqrt{|\lambda|^2 + 4\pi^2}/2\pi)$ -lipschitz map  $T \rightarrow S(1)$ . Finally, we note that, by Lemma 9.6, we have  $|\lambda| \leq (k + 1)(\Im(\lambda) + \frac{1}{2}) \leq (k + 1)(c + \frac{1}{2})$ .  $\diamond$

## 10. Systems of convex sets.

The purpose of this section is to describe some constructions of lipschitz maps which we will apply in Section 11 to get a lipschitz map from the thick part of our model space into the thick part of our 3-manifold. Since the actual set-up in which we are interested is somewhat complicated to describe, we will present most of it in the fairly general setting of systems of convex sets, only adding assumptions as we need them. The main application we have in mind here is described by Lemma 10.6, whose explicit hypotheses are laid out before its statement. In practice all the convex sets we deal with will be either horoballs or uniform neighbourhoods of geodesics, which are lifts of closed geodesics or Margulis regions in our 3-manifold. The domain for our map will be a locally finite polyhedral complex, which in applications arises out of the combinatorial construction of the model space. We note that any two sensible path metrics on such a space will be bilipschitz equivalent, and so the actual choice doesn't much matter to us. We will describe a specific metric for definiteness. Much of the argument will apply in any dimension, though we restrict our attention to 3.

Let  $\Pi$  be a 3-dimensional simplicial complex with vertex set  $\Pi^0$ . We write  $\Pi^i$  for the set of  $i$ -simplices. We assume  $\Pi^i$  to be locally finite away from  $\Pi^0$ . We write  $|\Pi|$  for its realisation. We are really interested in a *truncated realisation* of  $\Pi$ , denoted  $R(\Pi)$ , built out of truncated simplices. We can construct a truncated simplex by taking a regular euclidean simplex of side length 3, and removing a regular simplex of side length 1 about each vertex. The resulting polyhedron has all side lengths 1. In dimension 2, for example, we get a regular hexagon. Gluing these together we get a locally finite polyhedral complex,  $R(\Pi)$ , which we can view as a closed subset of  $|\Pi|$ . Associated to each  $x \in \Pi^0$ , we have a polyhedral subset  $D(x) \subseteq R(\Pi)$  — the boundary of a neighbourhood of  $x$  in  $|\Pi|$ . This is a 2-dimensional simplicial complex.

Given two convex subsets  $P, Q \subseteq \mathbf{H}^3$  we write  $\text{par}(P, Q) = \text{diam}(N(P, 1) \cap N(Q, 1))$ . (This is 0 if the intersection is empty.) We view this as a convenient measure of the extent to which  $P$  and  $Q$  remain close (or “parallel”). An upper bound in  $\text{par}(P, Q)$  means that they must diverge uniformly. (More precisely, for all  $t \geq 0$ , there is some set  $R$ , whose diameter is bounded in terms of  $t$  and  $\text{par}(P, Q)$  such that  $d(P \setminus R, Q \setminus R) \geq t$ .)

We start with a fairly simple construction that will be refined later. Suppose to each  $x \in \Pi^0$  we associate a closed convex set  $Q(x) \subseteq \mathbf{H}^3$ . We assume:

(A1) If  $x, y \in \Pi^0$  are distinct, then  $Q(x) \cap Q(y)$  consists of at most one common boundary point, and  $\text{par}(Q(x), Q(y))$  is bounded above.

(B1) If  $xy \in \Pi^1$ , then  $d(Q(x), Q(y))$  is bounded above.

We will later look into other hypotheses that imply (B1). Implicit in these statements are constants which give the respective bounds. The constant of (B1) will eventually depend on the hypotheses (A2)–(A7) described later (see Lemma 10.7).

Given  $xy \in \Pi^1$ , let  $\beta(xy)$  be the shortest geodesic from  $Q(x)$  to  $Q(y)$ . A key observation that is a simple exercise in hyperbolic geometry is the following:

**Lemma 10.1 :** *If  $xyz \in \Pi^2$ , then  $\text{diam}(\beta(xy) \cup \beta(yz) \cup \beta(zx))$  is bounded in terms of the constant of (A1).*

**Proof :** Consider the hexagonal path  $\beta(xy) \cup \alpha(y) \cup \beta(yz) \cup \alpha(z) \cup \beta(zx) \cup \alpha(x)$  with  $\alpha(x) \subseteq Q(x)$  etc. all geodesics. The lengths of the  $\beta$ -paths are bounded. So, if the hexagon were very long, two of the  $\alpha$  paths would have to run close together over a long distance, contradicting (A1).  $\diamond$

**Lemma 10.2 :** *With the above hypotheses ((A1) and (B1)) there is a canonical uniformly lipschitz map  $\phi : R(\Pi) \rightarrow \mathbf{H}^3$  such that  $\phi(D(x)) \subseteq Q(x)$  for all  $x \in \Pi^0$ , and such that if  $a \in Q(x) \cap \phi(R(\Pi))$  then  $d(a, \partial Q)$  is uniformly bounded above.*

Here the lipschitz constant and bound depend on the bounds assumed in (A1) and (B1).

Note that every point of  $\phi(R(\Pi))$  lies a bounded distance from two distinct sets  $Q(x)$ , from which it follows that this image can only boundedly enter any such convex set. Explicitly we note:

(\*) There is some constant  $K \geq 0$  depending only on the constants of (A1) and (B1) such that for all  $x \in \Pi^0$ , if  $a \in Q(x) \cap \phi(R(\Pi))$  then  $d(a, \partial Q) \leq K$ .

**Proof of Lemma 10.2 :**

Our construction of  $\phi$  is as follows. For any  $xy \in \Pi^1$ , length  $\beta(xy) = d(Q(x), Q(y))$  is bounded. Let  $p(x, y) = \beta(xy) \cap Q(x) \in \partial Q(x)$  be the nearest point in  $Q(x)$  to  $Q(y)$ . We map the corresponding edge of  $R(\Pi)$  linearly to  $\beta(xy)$ . By this process, we will map the vertex set of each  $D(x)$  into  $\partial Q(x)$ . We now extend linearly over  $\phi(D(x))$ . By convexity,  $\phi(D(x)) \subseteq Q(x)$ . Applying Lemma 10.1, we see that the images of simplices in  $D(x)$  are bounded.

The *centre* of a finite diameter subset,  $B \subseteq \mathbf{H}^3$ , can be defined as the unique point  $c \in \mathbf{H}^3$  such that  $B \subseteq N(c, r)$  with  $r$  minimal. Given  $xyz \in \Pi^2$ , write  $c(xyz)$  for the centre of  $\beta(xy) \cup \beta(yz) \cup \beta(zx)$ . Associated to  $xyz$ , we have a hexagonal 2-cell in  $R(\Pi)$ , and we

have already defined  $\phi$  on its boundary. We now extend over the interior by sending its centre to  $c(xyz)$  and coning linearly over the boundary.

Similarly, given  $xyzw \in \Pi^3$ , we let  $c(xyzw)$  be the centre of  $\beta(xy) \cup \beta(yz) \cup \beta(zx) \cup \beta(xw) \cap \beta(yw) \cup \beta(zw)$ . We have already defined  $\phi$  on the boundary of the associated 3-cell of  $R(\Pi)$  and now cone linearly over the centre  $c(xyzw)$ .

This gives us our lipschitz map  $\phi$ , proving Lemma 10.2.  $\diamond$

We want to refine the above construction to push  $\phi$  off the interiors of convex sets. For this we need some additional assumptions.

Suppose that  $A \subseteq \mathbf{H}^3$  is convex, and that  $Q = N(A, t)$  for some  $t \geq 0$ . Given any  $r \in (0, t)$  write  $Q_r = N(A, t - r)$ . Thus  $Q_0 = Q$ . We can define an outward projection  $\pi : Q_r \setminus A \rightarrow \partial Q$  so that each  $a \in Q_r$  lies on the shortest geodesic from  $\pi(a)$  to  $A$ . This projection is  $(\sinh t / \sinh(t - r))$ -lipschitz.

We can refine Lemma 10.2 as follows. Suppose that to each  $x \in \Pi^0$  we have associated some convex set,  $A(x)$  and some  $t(x) \geq 0$ . Let  $Q(x) = N(A(x), t(x))$ . We suppose that the collection  $(Q(x))_{x \in \Pi^0}$  satisfies the assumptions (A1) and (B1). Suppose that  $t_0 \geq K + 1$ , where  $K$  is the constant of (\*) above, and suppose that  $t_1 \geq t_0$ .

Now suppose that we decompose  $\Pi^0$  into two subsets,  $\Pi_0^0 \sqcup \Pi_1^0$  satisfying:

For all  $x \in \Pi_0^0$ ,  $t(x) \leq t_0$ , and

For all  $x \in \Pi_1^0$ ,  $t(x) \geq t_1$ .

**Lemma 10.3 :** *With the above hypotheses, we can find a canonical uniformly lipschitz map,  $\phi : R(\Pi) \rightarrow \mathbf{H}^3$  such that*

- (1) if  $x \in \Pi_1^0$ , then  $\phi(D(x)) \subseteq A(x)$ ,
- (2) if  $x \in \Pi_0^0$ , then  $\phi(D(x)) \subseteq \partial Q(x)$ , and
- (3) if  $x \in \Pi_0^0$ , then  $Q(x) \cap \phi(R(\Pi)) \subseteq \partial Q(x)$ .

**Proof :** We start with a map  $\psi : R(\Pi) \rightarrow \mathbf{H}^3$  as given by Lemma 10.3. If  $x \in \Pi_0^0$ , then  $Q(x) \cap \phi(R(\Pi)) \subseteq Q_{t_0}(x)$ . By composing with the outward projection  $\pi : Q_{t_0}(x) \rightarrow \partial Q(x)$  described above, we can push the image off the interior of  $Q(x)$ , while maintaining a control on the lipschitz constant.

If  $x \in \Pi_1^0$ , we have  $\phi(D(x)) \subseteq Q(x) \subseteq N(A(x), t_1)$ . We can now project  $\phi(D(x))$  to  $A(x)$  by nearest point projection, and extending by linear homotopy carried out in a uniformly small neighbourhood of  $D(x)$  in  $R(\Pi)$ . In this way we can arrange that  $\phi(D(x)) \subseteq A(x)$ , again maintaining control over the lipschitz constant.  $\diamond$

In applying these results, we will start from slightly different hypotheses. We suppose we have convex sets,  $(Q(x))_x$ , satisfying (A1), but we do not a-priori assume (B1). This we will need to deduce.

We begin by assuming:

(A2)  $(\forall x \in \Pi^0)(\forall g \in \Gamma)(Q(gx) = gQ(x))$ .

(A3) The setwise stabiliser of each element of  $\Pi^1$  and of  $\Pi^2$  is trivial.

(Note that (A3) implies that  $\Gamma$  acts freely and isometrically of  $R(\Pi)$ .)

We write  $\Gamma(x)$  for the stabiliser of  $x$  in  $\Gamma$ .

(A4) If  $g \in \Gamma(x) \setminus \{1\}$ , then for all  $a \in \partial Q(x)$ ,  $d(a, ga) \geq \epsilon$  for some fixed constant  $\epsilon > 0$ .

Note that an immediate consequence of this is that if  $y \in \Pi^0$  with  $xy \in \Pi^1$ , and  $g \in \Gamma(x) \setminus \{1\}$ , then the geodesic segments  $\beta(xy)$  and  $g\beta(xy)$  diverge uniformly. More precisely, given any  $t > 0$ , they can remain  $t$ -close only over a distance bounded above in terms of  $t$  and  $\epsilon$ .

We also suppose we have a  $\Gamma$ -equivariant subset  $\Pi_0^2 \subseteq \Pi^2$  satisfying:

(A5) If  $xyz \in \Pi_0^2$ , then there are non-trivial elements  $g(x, y, z) \in \Gamma(x)$ ,  $g(y, z, x) \in \Gamma(y)$  and  $g(z, x, y) \in \Gamma(z)$  with  $g(x, y, z)g(y, z, x)g(z, x, y) = 1$  in  $\Gamma$ .

**Lemma 10.4 :** *We assume (A1)–(A5). If  $xyz \in \Pi_0^2$ , then the lengths of  $\beta(xy)$ ,  $\beta(yz)$  and  $\beta(zx)$  are all bounded above in terms of the constants of (A1) and (A4).*

**Proof :** It is enough to bound  $\beta(xy)$ . Let  $g = g(x, y, z) \in \Gamma(x)$  and  $h = g(y, z, x) \in \Gamma(y)$ , so that  $gh = g(z, x, y)^{-1} \in \Gamma(z)$ . Let  $w = g^{-1}z = hz$ . Since  $\Gamma(x) \cap \Gamma(z)$  is trivial,  $w \neq z$ .

There are geodesics segments  $\alpha(x), \alpha(y), \alpha(z), \alpha(w)$  in  $\mathbf{H}^3$ , respectively contained in  $Q(x), Q(y), Q(z), Q(w)$ , so that

$$\alpha(x) \cup \beta(xz) \cup \alpha(z) \cup \beta(zx) \cup \alpha(y) \cup \beta(yw) \cup \alpha(w) \cup \beta(wx)$$

forms a closed path — an “octagon”. Consecutive edges of the octagon meet at an angle at least  $\pi/2$ , and can remain close only over a bounded distance. Moreover,  $\beta(xz)$  and  $\beta(xw)$  have the same length, and as mentioned above, (A4) implies that they can and remain close only over a bounded distance. The same also applies to  $\beta(yz)$  and  $\beta(yw)$ .

Now a simple exercise in hyperbolic geometry shows that if  $\alpha(x)$  and  $\alpha(y)$  are far apart, then  $\alpha(z)$  and  $\alpha(w)$  remain close over a large distance. In particular, if  $\beta(xy)$  is very long, then  $\text{par}(Q(z), Q(w))$  is large, contradicting (A1).

This shows that  $\beta(xy)$  has bounded length, as claimed.  $\diamond$

Next we assume:

(A6) If  $xyz \in \Pi^2$ , at least two of the edges  $xy, yz, zx$  lie in simplices of  $\Pi_0^2$ .

(A7) Each edge of  $\Pi^1$  lies in at least two simplices of  $\Pi^2$ .

**Lemma 10.5 :** *We assume (A1)–(A7). If  $xy \in \Pi^1$ , then the length of  $\beta(xy)$  is bounded above in terms of the constants of (A1) and (A4).*

**Proof :** By (A7) there are distinct  $z, w \in \Pi^0$  with  $xyz, xyw \in \Pi^2$ . Now if  $\beta(xy)$  is very long, then by Lemma 10.4,  $xy$  cannot lie in any simplex of  $\Pi_0^2$ . Thus, by (A7),  $xy, yz, xw, yw$  must all lie in simplices in  $\Pi_0^2$ . By Lemma 10.4 again, the lengths of each of  $\beta(xy)$ ,  $\beta(yz)$ ,  $\beta(xw)$  and  $\beta(yw)$  are bounded.

As in the proof of Lemma 10.4 we consider the octagon

$$\alpha(x) \cup \beta(xz) \cup \alpha(z) \cup \beta(zx) \cup \alpha(y) \cup \beta(yw) \cup \alpha(w) \cup \beta(wx).$$

This time, we note that all the  $\beta$ -edges have bounded length, and that  $\alpha(x)$  and  $\alpha(y)$  are far apart. Again by a simple exercise in hyperbolic geometry shows that  $\alpha(z)$  and  $\alpha(w)$  remain close over a large distance, giving a contradiction as before.  $\diamond$

In other words, we have shown (A1)–(A7) imply (B1). In particular, in Lemma 10.3, we can substitute hypothesis (B1) with (A2)–(A7).

Since the construction was canonical, the map we get is  $\Gamma$ -equivariant.

We now finally get to the specific application we have in mind.

To be clear about our hypotheses, we go back to the beginning. Let us suppose that  $\Pi$  is a simplicial complex, and let  $R(\Pi)$  and  $(D(x))_{x \in \Pi^0}$  be as constructed earlier. We assume that  $R(\Pi)$  is locally finite. Let  $\Gamma$  be a group acting simplicially on  $\Pi$ , and so we get an induced isometric action on  $R(\Pi)$ . Given  $x \in \Pi^0$ , we write  $\Gamma(x)$  for the stabiliser of  $x$  in  $\Gamma$ . Let  $\Pi_0^2$  be a  $\Gamma$ -invariant subset of  $\Pi^2$ . We now suppose:

- (C1) Every edge of  $\Pi$  is contained in at least two simplices in  $\Pi^2$ .
- (C2) If  $x \in \Pi^0$ , then  $\Gamma(x)$  is infinite cyclic.
- (C3) If  $x, y \in \Pi^0$  are distinct, then  $\Gamma(x) \cap \Gamma(y)$  is trivial.
- (C4) If  $xyz \in \Pi_0^2$ , we can choose the generators,  $g(x, y, z), g(y, z, x), g(z, x, y)$  respectively for  $\Gamma(x), \Gamma(y), \Gamma(z)$  so that  $g(x, y, z)g(y, z, x)g(z, x, y) = 1$ .
- (C5) If  $xyz \in \Pi^2$ , then at least two of its edges,  $xy, yz, zx$ , are also edges of elements of  $\Pi_0^2$ .

We now suppose that  $\Gamma$  also acts freely and properly discontinuously on hyperbolic 3-space,  $\mathbf{H}^3$ . Given  $x \in \Pi^0$ , we let  $l(x)$  for the infimum translation distance of  $g(x)$  on  $\mathbf{H}^3$ . We assume:

- (C6) There is some  $L \geq 0$  such that for all  $x \in \Pi^0$  we have  $l(x) \leq L$ .

We note that this “translation bound” constant,  $L$ , is the only constant we are inputting into the proceedings. All other constants arising can be chosen dependent on that (though we will be free to exercise some choice).

Let  $\Pi_P^0 = \{x \in \Pi^0 \mid l(x) = 0\}$ . Thus if  $x \in \Pi_P^0$ , then  $g(x)$  is parabolic with fixed point in  $\partial\mathbf{H}^3$ . If  $x \in \Pi^0 \setminus \Pi_P^0$ , then  $g(x)$  is loxodromic and translates some axes,  $\alpha$ , a distance  $l(x)$ .

Now fix some  $\epsilon_0$  less than the 3-dimensional Margulis constant, and sufficiently small in relation to  $L$  as we describe shortly. Suppose  $x \in \Pi^0$  with  $l(x) < \epsilon_0$ . Let  $P_0(x) \subseteq \mathbf{H}^3$  be the set of points,  $y$ , such that there exists a non-trivial  $h \in \Gamma(x)$  with  $d(y, hy) \leq \epsilon_0$ . (This of course, need not be a generator.) Thus,  $P_0(x)$  is the *Margulis region* corresponding to  $\Gamma(x)$ . (We refer to the  $\epsilon_0$ -Margulis region if we need to specify the constant.) If  $x \notin \Pi_P^0$  this is has the form  $N(\alpha(x), r(x))$  for some  $r(x) > 0$ . We refer to  $r(x)$  as the *depth* of  $P_0(x)$ . If  $x \in \Pi_P^0$  this is a horoball centred at the fixed point. We set  $r(x) = \infty$ . The Margulis lemma tells us that distinct Margulis regions are disjoint. Indeed we can assume the distance between them to be bounded below. (For the moment we are not excluding the possibility that there may be other points of  $\mathbf{H}^3$  translated a very small distance by

some element of  $\Gamma \setminus \{1\}$  outside the regions we have described.) In the above situation ( $l(x) < \epsilon_0$ ), we set  $Q(x) = P_0(x)$ . If  $l(x) \geq \epsilon_0$ , we set  $Q(x) = \alpha(x)$ .

Our choice of  $\epsilon_0$  depends on the following standard fact of hyperbolic geometry. Given any  $L > 0$ , there is some  $\epsilon(L) > 0$  such that if  $g, h$  are hyperbolic isometries generating a discrete group with  $d(x, gx) \leq L$  and  $d(x, hx) \leq \epsilon(L)$ , then  $g$  and  $h$  generate an elementary group. In our case this will be cyclic. Thus, if we choose  $\epsilon_0 \leq \epsilon(L)$ , then no axis  $\alpha(x)$  can enter any Margulis region  $P_0(y)$ . It follows that for all distinct  $x, y \in \Pi^0$ ,  $Q(x) \cap Q(y)$  is at most one point.

The following construction is most conveniently described now, even though its logical place in the argument comes a bit later. We fix some constant  $r_0$  sufficiently large (to be specified later). Let  $\Pi_0^0$  be the set of  $x \in \Pi^0$  such that  $l(x) < \epsilon_0$  and  $r(x) \geq r_0$ , and let  $\Pi_1^0 = \Pi \setminus \Pi_0^0$ . Note that  $\Pi_P^0 \subseteq \Pi_0^0$ . If  $x \in \Pi_P^0$ , let  $A(x)$  be the horoball about the fixed point, so that  $Q(x) = N(A(x), r_0)$ . If  $x \in \Pi_0^0 \setminus \Pi_P^0$ , we let  $A(x) = N(\alpha(x), r(x) - r_0)$ , so that again  $Q(x) = N(A(x), r_0)$ . If  $x \in \Pi_1^0$ , then we set  $A(x) = \alpha(x)$ . In this case, we have  $Q(x) = N(A(x), r)$  for some  $r \leq r_0$ .

Suppose now that  $x, y \in \Pi^0$  are distinct. It follows that  $\text{par}(Q(x), Q(y))$  is bounded above. If these are both loxodromic axes, this is a standard fact following from the upper bound on translation lengths (C6) and the discreteness of  $\Gamma$ . Otherwise it is a standard fact about the geometry of Margulis regions: they cannot remain parallel over large distance, nor can they remain parallel to any loxodromic axis of bounded translation length. We have thus verified property (A1).

Now suppose that  $xyz \in \Pi^2$ . We obtain (A5) is the same as (C4).

We are now in a set-up applicable to Lemma 10.5. Note that this makes reference only to the sets  $Q(x)$ , and so our construction of the sets  $A(x)$  is irrelevant for the moment. In particular, Lemma 10.5 us an upper bound on  $d(Q(x), Q(y))$  for all  $xy \in \Pi^1$ . This ultimately depends only on  $L$ . We are now free to choose  $r_0$  so that  $r_0 - 1$  is greater than the constant given by Lemma 10.2. We are now in a position to apply Lemma 10.3 with  $t_0 = t_1 = r_0$ .

We finally note that for all  $x \in \Pi_1^0$ , we have  $l(x) > \epsilon_1$  for some  $\epsilon_1$  depending only on  $\epsilon_0$  and  $r_0$ , and thus ultimately only on  $r_0$ .

Let us summarise what we have shown:

**Lemma 10.6 :** *Let  $\Gamma$  be a group acting on  $\Pi$  and on  $\mathbf{H}^3$  in the manner described above, in particular satisfying (C1)–(C6). Then there are positive constants,  $k, \epsilon_0, \epsilon_1$ , depending only on the translation bound of property (C6), such that we can write  $\Pi^0$  as an  $\Gamma$ -invariant disjoint union  $\Pi^0 = \Pi_0^0 \sqcup \Pi_1^0$  such that there exists an equivariant  $k$ -lipschitz map,  $\phi : R(\Pi) \rightarrow \mathbf{H}^3$  satisfying:*

- (1) *If  $x \in \Pi_1^0$ , then the generator of  $\Gamma(x)$  translates an axis  $\alpha(x)$  a distance at least  $\epsilon_1$ , and  $\phi(D(x)) \subseteq \alpha(x)$ .*
- (2) *If  $x \in \Pi_0^0$ , the  $\epsilon_0$ -Margulis region,  $P_0(x)$ , corresponding to  $\Gamma(x)$  is non-empty and  $\phi(D(x)) \subseteq \partial P_0(x)$ .*
- (3) *For all  $x \in \Pi_0^0$ ,  $P_0(x) \cap \phi(R(\Pi)) \subseteq \partial P_0(x)$ .* ◇

This map projects to a map,  $f : R(\Pi)/\Gamma \longrightarrow \mathbf{H}^3/\Gamma$ . We write  $\tilde{\Theta} = \mathbf{H}^3 \setminus \bigcup_{x \in \Pi^0} \text{int } P_0(x)$ , and let  $\Theta = \tilde{\Theta}/\Gamma$ . Thus  $f(R(\Pi)/\Gamma) \subseteq \Theta$ .

We note:

**Lemma 10.7 :** *There is some  $\epsilon_2 > 0$  depending only on  $L$ , such that if  $f(R(\Pi)/\Gamma) = \Theta$ , then the injectivity radius of  $\Theta$  is at least  $\epsilon_2$ .*

**Proof :** In this case, every point of  $\Theta$  is a bounded distance from some set of the form  $\phi(D(x))/\Gamma(x)$  for some  $x \in \Pi^0$ . This is either a closed geodesic whose length is bounded below by  $\epsilon_1$  and above by  $L$ , or else the boundary of an  $\epsilon_0$ -Margulis region. This places an upper bound on the depth of any Margulis region contained in  $\Theta$  and hence a lower bound on injectivity radius.  $\diamond$

In fact, this argument shows in general that the injectivity radius is bounded below in the image of  $f$ .

## 11. The model space.

In this section, we give a description of the model space for a doubly degenerate manifold, and show how the results of Section 8 can be used to construct a lipschitz map into such a hyperbolic 3-manifold. We will mostly deal with the doubly degenerate case for the purposes of exposition. We give some discussion to other cases at the end of the section.

Let  $W = \bigcup \mathcal{W}$  be a complete annulus system in  $\Psi = \Sigma \times \mathbf{R}$ . Let  $\Lambda = \Lambda(W)$  be the completion of  $\Psi \setminus W$ . In Section 8, we described the associated “brick decomposition”,  $\mathcal{D} = \mathcal{D}(W)$  of  $\Lambda$ . Each element  $B \in \mathcal{D}$  has the form  $\Phi \times [0, 1]$ , where  $\Phi$  is a 3HS (“type 0”) or a 4HS or 1HT (“type 1”). Suppose  $B$  is of type 1. There is a curve  $\gamma_+ \subseteq \partial_+ B$  that cuts  $\partial_+ B$  into one or two 3HS components, each the lower boundary of an adjacent type 0 brick. We have a similar curve,  $\gamma_- \subseteq \partial_- B$ . By construction, the intersection number  $\iota(\gamma_-, \gamma_+)$  is minimal (1 for a 1HT and 2 for a 4HS). Thus, if we forget about the marking (the map to  $\Sigma$ ), then the local combinatorics of  $\mathcal{D}$  is bounded.

We want to put a path-metric on  $\Lambda$  using our combinatorial structure. Since we are only interested in the metric up to bilipschitz equivalence, it doesn’t much matter how we do this, but a fairly specific procedure is as follows. We fix the unique hyperbolic metric on the 3HS so that every boundary component has length 1. This will be our *standard* 3HS. For a type 0 brick, we just take a product with the unit interval. Suppose  $B$  is a type 1 brick. We put hyperbolic structures on  $\partial_{\pm} B$  so that each component of  $\partial_{\pm} B \setminus \gamma_{\pm}$  is a standard 3HS and so that there is no “twisting” (the topological symmetries give geometric symmetries). We now choose (once and for all) our favourite path between these structures in the space of pointwise smooth riemannian metrics for which the boundary components are always geodesic of length 1. This gives us a riemannian metric on  $B = \Phi \times [0, 1]$ . (It would be natural to do this in such a way that the topological symmetries of  $B$  give geometric symmetries, though this doesn’t really matter to us.)

We can now glue all the bricks back together to give us a riemannian metric on  $\Lambda$ . Each boundary component,  $\Delta(\Omega)$  is a locally geodesic euclidean torus of the form  $\Delta(\lambda)$  with respect to the longitude and standard meridian (in the notation of Section 5). Note that  $\text{area}(\Delta(\lambda)) = \mathfrak{S}(\lambda)$  is the same as the “combinatorial length” of  $\Delta(W)$  as defined in Section 8.

Recall that  $\Lambda(\mathcal{W}, \mathcal{W}) = \Lambda(\mathcal{W}) \cup \bigcup_{\Omega \in \mathcal{W}} T(\Omega)$  is obtained by gluing in a solid torus,  $T(\mathcal{W})$  to each  $\Delta(\Omega)$  so that the standard meridian is trivial in  $T(\Omega)$ . The standard meridian was defined in such a way that  $\Lambda(\mathcal{W}, \mathcal{W})$  gives us back  $\Psi$  up to homeomorphism. In particular, there is a projection map  $\pi_\Sigma : \Lambda(\mathcal{W}, \mathcal{W}) \rightarrow \Sigma$ , well defined up to homotopy.

Now Lemma 9.7 gives us a riemannian metric on  $T(\Omega)$  isometric to a standard Margulis tube. In this way, we get a riemannian metric on all of  $\Lambda(\mathcal{W}, \mathcal{W})$ . (It is comforting to observe that by Lemma 9.8, if we chose a different metric on  $\Lambda(\mathcal{W})$  with euclidean boundary and in the same bilipschitz class we would get a bilipschitz equivalent metric on  $\Omega$ , so the construction is quite “robust”. However, once we have chosen our model space, we don’t formally need to know this.)

Finally to arrive at our model space,  $P = \text{int } \Sigma \times \mathbf{R}$ , we glue in a Margulis cusp to each boundary component of  $\Lambda(\mathcal{W}, \mathcal{W})$ . Any such boundary component is a bi-infinite cylinder,  $S(1) \times \mathbf{R}$ , and the Margulis cusp is the quotient of a horoball by a  $\mathbf{Z}$ -action. We can regard  $\Lambda(\mathcal{W}, \mathcal{W})$  as a subset of  $P(\mathcal{W})$ , which we shall denote by  $\Psi(P) = \Psi(\mathcal{W})$  and refer to it as the *non-cuspidal* part of  $P(\mathcal{W})$ .

This is all we need to know about the model space to understand the statements of the main results of this section, notably Theorem 11.1. For proofs, however, we need more combinatorial constructions, in order to apply the results of Section 10. We go on to describe these next.

We will first need to cut up  $\Lambda(\mathcal{W})$  into truncated simplices. This is done in a number of steps.

First, we replace our brick decomposition with a “block decomposition” — which is the same combinatorial structure as the “block decomposition” of Minsky [Mi4]. This is a fairly trivial adjustment which gets rid of the 3HS bricks. For each type 0 brick,  $B \equiv F \times [0, 1]$ , we take the horizontal 3HS,  $F_B = F \times \{\frac{1}{2}\} \subseteq B$ . The union of all these surfaces,  $\bigcup_B F_B$  as  $B$  ranges over all type 0 bricks, cuts  $\Lambda(\mathcal{W})$  into a collection of compact *blocks*, each of which is a type 1 brick with either two or four type 0 half bricks attached to it. Such a block,  $A$ , is homeomorphic to  $\Phi \times [0, 1]$ , where  $\Phi$  is a 4HS. We write  $\partial_V A = \partial\Phi \times [0, 1]$ ,  $\partial_- A = \Phi \times \{0\}$  and  $\partial_+ A = \Phi \times \{1\}$ . We can choose the homeomorphism in such a way that  $A \cap \partial\Lambda = \bigcup A \sqcup \Upsilon_- \sqcup \Upsilon_+$ , where  $\Upsilon_\pm$  is an annulus in  $\partial_\pm A$ , with core curves  $\gamma_\pm$ , say. These core curves correspond to components of  $W$  (meeting the original type 1 brick). By the tautness assumption on the type 1 band, these must have minimal intersection in the surface,  $\Phi$ .

Before proceeding to the second step, we make the following observation regarding a 3HS,  $F$ , which we can take to have the standard hyperbolic structure. If  $\alpha \subseteq \partial F$  is a boundary component, write  $\sigma(\alpha)$  for the shortest geodesic from  $\alpha$  to itself that separates the other two boundary components of  $F$ . If  $\beta$  is another boundary component, write  $\sigma(\alpha, \beta)$  for the shortest geodesic from  $\alpha$  to  $\beta$ . We can cut  $F$  into two right-angled hexagons in four different ways. We can cut it along  $\sigma(\alpha, \beta) \cup \sigma(\beta, \gamma) \cup \sigma(\gamma, \alpha)$ , or we can cut it along

$\sigma(\alpha) \cup \sigma(\alpha, \beta) \cup \sigma(\alpha, \gamma)$  for any boundary curve  $\alpha$ . We say these decompositions are of type  $D_0$  or type  $D_\alpha$  respectively.

The second step is to cut each block into truncated octahedra. The process is more conveniently described in reverse. Let  $O$  be a truncated octahedron — it has six square and eight hexagonal faces. We label the edges 1, 2, 3 so that the edges of each square face are alternately labelled 2 and 3, and the edges of each hexagonal face are either labelled alternately 1 and 2 or alternately 1 and 3. Thus all three labels appear at each vertex. Any two hexagons meet, if at all, in a 1-edge. There are four 12-hexagons and four 13-hexagons arranged alternately.

To describe a 4HS block, we take two copies of  $O$ , and identify the corresponding pairs of 13-hexagons. This gives us a genus-3 handlebody,  $H$ . The square faces turn into a set of six disjoint annuli embedded in  $\partial H$  (each bounded by two curves labelled 2). Each component of the complement of these annuli in  $\partial H$  is a 3HS and is cut into two hexagons by three 1-arcs. This decomposition is of type  $D_0$ . To identify  $H$  as a 4HS block, we select four annuli which cut  $\partial H$  into two 4HS's, and deem them to be vertical. The other two annuli give us our non-vertical annuli. (They correspond to a pair of opposite squares in each copy of  $O$ .)

To describe a 1HT block, take a copy of  $O$  and partition the 13-hexagons into two pairs. Thus, the two hexagons in a pair meet a common square face. Now identify the two hexagons of a pair in such a way that their common adjacent square turns into an annulus. This gives us a genus-2 handlebody,  $H$ , and two annuli in  $\partial H$ . The other four square faces of  $O$  get strung together to form a third annulus, which we deem to be vertical. It separates  $\partial H$  into two 1HT's each containing a non-vertical annulus. This gives us a 1HT block. Each of the four 3HS components of the complement of these annuli is cut into two hexagons by three 1-arcs, as before. This time, these decompositions are of type  $D_\alpha$ , where  $\alpha$  is the boundary component in the vertical annulus.

We note that, in fact, the decomposition of a block as two octahedra in the manner described above is combinatorially canonical. We can thus reverse the process to cut each block of our decomposition up in this way.

The third step arises from the complication that the two decompositions of a horizontal 3HS into two hexagons (arising from the blocks on either side) might not match up. For example if a 4HS block meets a 1HT block along a horizontal 3HS  $F$ , one decomposition will be of type  $D_0$  and the other of type  $D_\alpha$ . We can fix this by replacing  $F$  by a truncated simplex. Writing  $\partial F = \alpha \cup \beta \cup \gamma$ , we can think of one pair of opposite edges of this simplex as corresponding to  $\sigma(\alpha)$  and  $\sigma(\beta, \gamma)$ . Another pair of opposite edges corresponding to  $\sigma(\alpha, \beta)$  get identified, and a third pair, corresponding to  $\sigma(\alpha, \gamma)$  also get identified. It is also possible to get two decompositions of type  $D_\alpha$  and  $D_\beta$  arising from two 1HT blocks. In this case we replace  $F$  by two truncated simplices via an intermediate  $D_0$  decomposition by applying the above construction.

This gives a polyhedral decomposition of  $\Lambda$  into truncated simplices and truncated octahedra. Since our discussion in Section 7 only considered truncated simplices, we should apply a fourth step. Each octahedron can be cut into four simplices by connecting two opposite vertices by an edge. This cuts a truncated octahedron into four truncated simplices. There are choices involved, but the manner in which we do it is not important.

To relate this to the discussion of Section 10, we need to pass to covers. Let  $\Gamma = \pi_1(\Sigma)$ , and let  $\Lambda(\mathcal{W}, \mathcal{W}) = \Psi(\mathcal{W})$  for the non-cuspidal part of our model space. We have  $\Lambda(\mathcal{W}) \subseteq \Psi(\mathcal{W})$ , and we let  $R$  be the lift of this to the universal cover of  $\Psi(\mathcal{W})$ . Thus  $\Gamma$  acts on  $R$  with quotient  $\Lambda(\mathcal{W})$ . We can lift the polyhedral decomposition of  $\Lambda(\mathcal{W})$  we just constructed to a polyhedral decomposition of  $R$ . This has the form  $R = R(\Pi)$ , where  $\Pi$  is the simplicial complex obtained by shrinking each boundary component of  $R$  to a point. These points become the vertices,  $\Pi^0$ , of  $\Pi$ . The higher dimensional truncated simplices turn into simplices of  $\Pi$ . Thus, for each  $x \in \Pi^0$ , the complex  $D(x)$  described in Section 10 is a boundary component of  $R(\Pi)$ . We let  $\Pi_P^0$  be the set of  $x \in \Pi^0$  such that  $D(x)$  is homeomorphic to  $\mathbf{R}^2$ . In this case,  $D(x)/\Gamma(x)$  is a bi-infinite cylinder, in fact a boundary component of  $\Psi(\mathcal{W})$ . If  $x \in \Pi^0 \setminus \Pi_P^0$ , then  $D(x)$  is a bi-infinite cylinder, and  $D(x)/\Gamma(x)$  is a torus of the form  $\Delta(\Omega) = \partial T(\Omega)$  for some  $\Omega \in \mathcal{W}$ . In all cases,  $\Gamma(x)$  is infinite cyclic (Property (C2)).

In Section 7, we defined a polyhedral metric on  $R(\Pi)$ , which also gives us a polyhedral metric on  $R(\Pi)/\Gamma = \Lambda(\mathcal{W})$ . Provided we carry out the subdivision of  $\Lambda(\mathcal{W})$  in a geometrically sensible way, this will be bilipschitz equivalent to the model metric on  $\Lambda(\mathcal{W})$  we described above. (Note that we are carrying out very explicit, locally bounded, combinatorial operations.)

We set  $\Pi_0^2$  to be the set of 2-simplices that arose from hexagons in horizontal 3HS. In other words, these are 12-hexagons in the truncated octahedra constructed by the end of the second step, together with the hexagons introduced in the truncated simplices of the third step. (There will be other simplices arising from 13-hexagons in octahedra as well as those arising in the fourth step of the construction.) Note that every 2-simplex of  $\Pi$  has at least two edges in 2-simplices in  $\Pi_0^2$ . This is property (C5). Clearly every edge lies in a 2-simplex, and so (C1) holds.

If  $xyz \in \Pi_0^2$ , then we can choose generators,  $g(x), g(y), g(z)$ , of  $\Gamma(x), \Gamma(y), \Gamma(z)$  with  $g(x)g(y)g(z) = 1$ . This is just an observation about the boundary curves in the fundamental group of a 3HS. This is property (C4).

For property (C3), we need another assumption on  $\mathcal{W}$ , namely that no two annuli are parallel, i.e. if  $\pi_\Sigma \Omega = \pi_\Sigma \Omega'$  then  $\Omega = \Omega'$ . In this case, distinct boundary components of  $\Lambda(\mathcal{W})$  project to distinct curves in  $\Sigma$  (allowing peripheral curves for the cusp boundaries).

We have thus verified all the combinatorial hypotheses, (C1)–(C5), of Lemma 10.7. For the final hypothesis we need finally to introduce group actions on  $\mathbf{H}^3$ .

Suppose that  $M = \mathbf{H}^3/\Gamma$  is a complete hyperbolic 3-manifold, with a strictly type-preserving homotopy equivalence  $\pi_\Sigma^M : M \rightarrow \Sigma$ . Now every curve  $\alpha \in X(\Sigma)$  can be realised as a closed geodesic  $\alpha^* = \alpha_M^*$  in  $M$ . We will abuse notation and write  $\alpha^* \subseteq M$ , even if it is not simple. We write  $l_M(\alpha)$  for the length of  $\alpha^*$ . Given  $r \geq 0$ , we write  $X(M, r) = \{\alpha \in X(\Sigma) \mid l_M(\alpha) \leq r\}$ .

Suppose that  $\mathcal{W}$  is a complete annulus system such that no two annuli are parallel. Recall the notation  $X(\mathcal{W}) = \{\pi_\Sigma \Omega \mid \Omega \in \mathcal{W}\}$ . We shall make the following ‘‘a-priori bounds’’ assumption:

(APB) There is some constant  $L \geq 0$ , such that  $X(\mathcal{W}) \subseteq X(M, L)$ .

In other words, the curves corresponding to the annuli of  $W$  have bounded length when realised in  $M$ .

The constant,  $L$ , now gives us another constant,  $\epsilon_0$ , arising from Lemma 10.8. This is less than the Margulis constant. We write  $\Psi(M)$  for the non-cuspidal part of  $M$  with respect to this constant, in other words  $M$  minus the  $\epsilon_0$ -cusps. By tameness [Bon] this is homeomorphic to  $P = \Sigma \times \mathbf{R}$ . Given  $\alpha \in X(M, \epsilon_0)$  we write  $T_0(\alpha^*)$  for the  $\epsilon_0$ -Margulis tube about  $\alpha$ .

Let  $P(\mathcal{W})$  be the model space constructed above, and let  $\Psi(P) = \Psi(P(\mathcal{W}))$  be its non-cuspidal part. The closures of components of  $P(\mathcal{W}) \setminus \Psi(P)$  we shall refer to as *cusps*. Given  $\Omega \in \mathcal{W}$ , write  $\Omega^* = \pi_\Sigma(\Omega)^*$  for the corresponding closed geodesic in  $M$ .

**Theorem 11.1 :** *Let  $\mathcal{W}$  and  $M$  be as above, in particular, satisfying (APB). Then there is a proper map  $f : P(\mathcal{W}) \rightarrow M$  such that  $\pi_\Sigma^M \circ f$  is homotopic to  $\pi_\Sigma$  with the following properties. Each cusp of  $P(\mathcal{W})$  gets sent to a  $\epsilon_0$ -cusp of  $M$ , and  $f(\Psi(\mathcal{W})) \subseteq \Psi(M)$ . Moreover, we can write  $\mathcal{W} = \mathcal{W}_0 \sqcup \mathcal{W}_1$  so that if  $\Omega \in \mathcal{W}_1$ , then  $f(T(\Omega)) = \Omega^*$  and  $\Omega^*$  has length at least  $\epsilon_1 > 0$ . If  $\Omega \in \mathcal{W}_0$ , then  $\Omega^*$  has length less than  $\epsilon_0$  and  $T(\Omega) = f^{-1}T_0(\Omega^*)$ . The map  $f$  is  $k$ -lipschitz on the complement of  $\bigcup_{\Omega \in \mathcal{W}_0} \text{int } T(\Omega)$ . Here, the constants,  $\epsilon_0, \epsilon_1, k$  depend only on the constant  $L$  of the (APB) hypothesis.*

**Proof :** We first pass to the covers corresponding to  $\Gamma = \pi_1(\Sigma)$ , and construct the polyhedral complex,  $R(\Pi)$  as above. This satisfies (C1)–(C5) and (APB) gives us (C6). Thus, Lemma 10.6 gives us an equivariant map  $\tilde{f} : R(\Pi) \rightarrow \mathbf{H}^3$ , which projects to a map  $f : \Lambda(\mathcal{W}) \rightarrow M$ . This is lipschitz with respect to the polyhedral metric on  $R(\Pi)/\Gamma = \Lambda(\mathcal{W})$ , and hence also with respect to the model metric on  $\Lambda(\mathcal{W})$ .

Extending over a cusp,  $C$ , of  $\Psi(\mathcal{W})$  is a fairly trivial operation. Note that we can write  $C = \partial C \times [0, \infty)$ , where  $\{x\} \times [0, \infty)$  is a (riemannian) geodesic ray in  $C$ . Now,  $f$  sends  $\partial C \subseteq \partial \Psi$  to a horosphere in  $M$  bounding a cusp. We extend  $f$  by mapping  $\{x\} \times [0, \infty)$  linearly to the ray in the cusp of  $M$ . This process does not change the lipschitz constant.

The  $\Gamma$ -invariant partition  $\Pi^0 \setminus \Pi_P^0 = (\Pi_0^0 \setminus \Pi_P^0) \sqcup \Pi_1^0$  gives us a partition of  $\mathcal{W}$  as  $\mathcal{W}_0 \sqcup \mathcal{W}_1$ . If  $\Omega \in \mathcal{W}_0$ , then, by construction  $f(\partial T(\Omega)) \subseteq \partial T_0(\Omega^*)$ , and no other part of  $\Lambda(\mathcal{W})$ , nor indeed the cusps, can enter  $\text{int}(T_0(\Omega^*))$ . We can now extend  $f$  topologically over  $T(\Omega)$ . By slight adjustment in a neighbourhood of the boundary torus, we can assume that  $f^{-1}(T_0(\Omega^*)) = T(\Omega)$ . (This will not be affected by our remaining construction.)

Now suppose that  $\Omega \in \mathcal{W}$ . Now  $\partial T(\Omega) = \Delta(\Omega)$  is a euclidean torus of the form  $\Delta(\lambda)$  for some modulus  $\lambda$  (in the notation of Section 9). Moreover,  $f|_{\partial T}$  is a lipschitz map to a circle,  $\Omega^* \subseteq M$ , whose length is bounded above (by  $L$ ), and below by the constant  $\epsilon_1 > 0$  of Lemma 10.8. Thus, if we can place an upper bound on  $\mathfrak{S}(\lambda)$  we can apply Lemma 9.9 to extend it to a lipschitz map of  $T(\Omega)$  to  $\Omega^*$ .

To bound  $\mathfrak{S}(\lambda)$  we need again the assumption that no two annuli in  $\Omega$  are parallel. From this it is a simple matter to construct a set of  $n$  curves in  $\Lambda(\mathcal{W})$ , all a bounded distance from  $\Delta(\Omega)$  and of bounded length, which correspond to distinct elements of  $X(\Sigma)$  and such that  $\mathfrak{S}(\lambda)$  is bounded above by fixed (linear) function of  $n$ . For example we can take these curves to be boundary curves of type 0 bricks meeting  $\Delta(\Omega)$  and deleting repetitions.

Now the images of these curves in  $M$  also have bounded length, and are a bounded distance from  $\Omega^*$ . Since  $\Omega^*$  has length at most  $L$ , these curves all lie in a subset of  $M$  of bounded diameter. Moreover, they are all homotopically distinct in  $\Sigma$  and hence in  $M$ . But a set of bounded diameter in any hyperbolic 3-manifold contains boundedly many distinct curves of bounded length (unless they are all multiples of a very short geodesics, which cannot arise here). Thus,  $n$  is bounded, and so is  $\mathfrak{S}(\lambda)$  as required.

We still need to show that  $f$  is proper. First, to see that  $f|_{\Lambda(\mathcal{W})}$  is proper, we can use a variation on the above argument. Any bounded set of  $M$  can meet only finitely many toroidal boundaries, and hence only finitely many sets of the form  $f(\Delta(\Omega))$  for  $\Omega \in \mathcal{W}$ . Since every point of  $\Lambda(\mathcal{W})$  is a bounded distance from some  $\Delta(\Omega)$  the properness of  $f|_{\Lambda(\mathcal{W})}$  follows. The fact that  $f$  is proper on all of  $P(\mathcal{W})$  now follows easily from the manner in which we have extended over tubes and cusps.  $\diamond$

Now let  $\Theta(M) = \Psi(M) \setminus \bigcup_{\Omega \in \mathcal{W}_0} \text{int } T_0(\Omega^*)$  and  $\Theta(P) = \Theta_M(P) = \Lambda(\mathcal{W}, \mathcal{W}_1) = \Psi(\mathcal{W}) \setminus \bigcup_{\Omega \in \mathcal{W}_0} \text{int } T(\Omega)$ . Thus,  $\Theta(P) \subseteq f^{-1}(\Theta(M))$ . Note that the definition of  $\Theta(P)$  uses the partition of  $\mathcal{W}$  as  $\mathcal{W}_0 \sqcup \mathcal{W}_1$  coming from Theorem 11.1, and so (at least a-priori) may depend on  $M$ .

Now the map  $f : \Psi(P) \rightarrow \Psi(M)$  is proper, and both spaces are homeomorphic to  $\Psi = \Sigma \times \mathbf{R}$ . It thus sends each end of  $\Psi(P)$  to an end of  $\Psi(M)$ . We make the following “end consistency” assumption:

(EC) Distinct ends of  $\Psi(P)$  get sent to distinct ends of  $\Psi(M)$ .

Note that, in this case  $f|_{\Psi(P)}$  has degree 1 to  $\Psi(M)$ , and in particular is surjective. It also follows that  $f : P \rightarrow M$  has degree 1 and is surjective. In this case, the manifold  $M$  must be doubly degenerate.

**Lemma 11.2 :** *If (EC) is satisfied, then the set of Margulis tubes  $T_0(\Omega^*)$  for  $\Omega \in \mathcal{W}_0$  is unlinked in  $\Psi(M)$ .*

**Proof :** The preimage of this set under  $f$  is a set of Margulis tubes in  $\Psi(P)$ , which is unlinked by construction. The statement now follows from Proposition 6.1.  $\diamond$

**Proposition 11.3 :** *If (EC) is satisfied, then the injectivity radius of  $\Theta(M)$  is bounded below by some constant  $\epsilon_2$  depending only in  $L$ .*

**Proof :** This is an immediate consequence of Lemma 10.9.  $\diamond$

**Proposition 11.4 :** *The map  $f|_{\Theta(P)} : \Theta(P) \rightarrow \Theta(M)$  is homotopic to a homeomorphism.*

**Proof :** By Lemma 11.2, or by the result of Otal [O2], the set of Margulis tubes  $T_0(\Omega^*)$  for  $\Omega \in \mathcal{W}_0$  is unlinked in  $\Psi(M)$ . We can thus apply the results of Section 7, in particular, Proposition 7.1.  $\diamond$

Thus, applying Corollary 7.8, we get:

**Lemma 11.5 :** *Suppose  $F$  is a properly embedded  $\pi_1$ -injective surface in  $\Theta(P)$ . Let  $U$  be any neighbourhood of  $f(F)$  in  $\Theta(M)$ . Then there is a proper embedding  $g : F \rightarrow U$  such that  $f|_F$  is homotopic to  $g$  in  $\Theta(M)$  relative to  $\partial M$ .*  $\diamond$

We shall write  $\mathcal{T}(P) = \{T(\Omega) \mid \Omega \in \mathcal{W}_0\}$  and write  $\mathcal{T}(M) = \{f(T) \mid T \in \mathcal{T}(P)\}$  for the corresponding set of Margulis tubes in  $M$ . Thus  $\Theta(P) = \Psi(P) \setminus \text{int} \bigcup \mathcal{T}(P)$  and  $\Theta(M) = \Psi(M) \setminus \text{int} \bigcup \mathcal{T}(M)$ .

Before leaving this section, we describe how the constructions can be adapted to finite or semi-infinite model spaces. This will be used in Section 13 to prove some of the statements of Section 5 (though they are incidental to the proof of the Ending Lamination Theorem). It is also relevant to the construction of the model in the general case (Sections 16 and 23).

We start with  $\Sigma \times I$ , where  $I \subseteq \mathbf{R}$  is any interval. Let  $W = \bigcup \mathcal{W} \subseteq \Sigma \times I$  be an annulus system. We assume that  $\partial_V \Omega \neq I$  for all  $\Omega \in \mathcal{W}$  (i.e. no annulus crosses  $\Sigma \times I$ .) Let  $\mathcal{W}_\partial = \{\Omega \in \mathcal{W} \mid \partial_V(\Omega) \cap \partial I \neq \emptyset\}$  be the set of ‘‘boundary annuli’’ in  $W$ . Let  $\Lambda(\mathcal{W})$  be the space obtained from  $\Sigma \times I$  by opening up each annulus as before, and let  $p : \Lambda(\mathcal{W}) \rightarrow \Sigma \times I$  be the natural quotient map. Given  $\Omega \in \mathcal{W}$  let  $\Delta(\Omega) = p^{-1}(\Omega) \subseteq \partial \Lambda(\mathcal{W})$ . If  $\Omega \in \mathcal{W} \setminus \mathcal{W}_\partial$ , then  $\Delta(\Omega)$  is a torus, and we glue in a solid torus as before. If  $\Omega \in \mathcal{W}_\partial$ , then  $\Delta(\Omega)$  is an annulus, and we leave it alone. This gives us a model space,  $\Psi(P) = \Psi(P(\mathcal{W}))$ . (We have no need to construct  $P$  here, but use this notation in order to maintain consistency.) Note that  $\Psi(P)$  is homeomorphic to  $\Sigma \times I$ , and we can write  $\partial \Psi(P) = \partial_V \Psi(P) \cup \partial_H \Psi(P)$ , where the vertical and horizontal boundaries correspond to  $\partial \Sigma \times I$  and  $\Sigma \times \partial I$  respectively. The annuli  $\Delta(\Omega)$  for  $\Omega \in \mathcal{W}_\partial$  give us a disjoint collection of annuli in  $\partial_H \Psi(P)$ . As before,  $\Psi(P)$  has a riemannian metric,  $d = d_P$ .

Suppose that  $M$  is a complete hyperbolic 3-manifold with  $\Psi(M) \cong \Sigma \times \mathbf{R}$ . Suppose that  $M$  satisfies the a-priori bound condition (APB) with constant  $L$ . In other words,  $\text{length}(\Omega^*) \leq L$  for all  $\Omega \in \mathcal{W}$ . Thus, as before, we construct a lipschitz homotopy equivalence,  $f : \Psi(P) \rightarrow \Psi(M)$ , with  $f^{-1}(\Psi(M)) = \partial_V \Psi(P)$ . Each margulis tube of  $\Psi(P)$ , as well as each boundary annulus  $\Delta(\Omega)$  for  $\Omega \in \mathcal{W}_\partial$ , gets sent either to a Margulis tube in  $\Psi(M)$ , or to a closed geodesic (of length at most  $L$ ) in  $\Psi(M)$ .

As an example, suppose that  $\alpha, \beta \in X(M, l)$  and that  $d_G(\alpha, \beta) \geq 3$  (i.e.,  $\alpha \cup \beta$  fills  $\Sigma$ , and the corresponding geodesics,  $\alpha_M$  and  $\beta_M$  in  $M$  have length at most  $L$ ). Applying Lemmas 6.1 and 8.2, we can construct an annulus system  $W = \bigcup \mathcal{W}$  in  $\Sigma \times [0, 1]$ , with  $\alpha$  and  $\beta$  homotopic to boundary annuli in the horizontal boundary components. All the curves in this construction lie in a hierarchy built out of tight geodesics. As a result, it satisfies the (APB) condition, where the bound,  $L$ , depends only in  $\kappa(\Sigma)$  and  $l$ . (See Theorem 3.11.) We thus get a lipschitz homotopy equivalence,  $f : \Psi(P) \rightarrow \Psi(M)$ , where the lipschitz constant depends only on  $\kappa(\Sigma)$  and on  $l$ . This will be used at the end of

Section 13, to give a proof of Theorem 5.4.

## 12. Bounded geometry.

In this section, we describe some properties of “bounded geometry” manifolds. In Sections 13 and 14, these will be applied to the “thick parts”,  $\Theta(P)$  and  $\Theta(M)$  of our model space and hyperbolic 3-manifolds respectively. Much of the discussion is quite general. The discussion introduces constants which are not computable (at least by the arguments we give).

Let  $\Theta$  be a riemannian  $n$ -manifold (for us  $n = 3$ ) with boundary  $\partial\Theta$ . Here, for simplicity, we shall assume that everything is smooth.

Write  $B(\mathbf{R}^n) = \{\underline{x} \mid \|x\| \leq 1\}$  for the unit ball in  $\mathbf{R}^n$ , and write  $B^+(\mathbf{R}^n) = B(\mathbf{R}^n) \cap \{\underline{x} \mid x_n \geq 0\}$  for the unit half ball.

**Definition :** We say that  $\Theta$  has *bounded geometry* if there is some  $\mu > 0$  such that every  $x \in \Theta$  has a neighbourhood  $N \ni x$ , with a smooth  $\mu$ -bilipschitz homeomorphism to either  $B(\mathbf{R}^n)$  or  $B^+(\mathbf{R}^n)$  taking  $x$  to a point a distance at most  $\frac{1}{2}$  from the origin.

(If we allow ourselves to modify  $\mu$ , we can equivalently replace  $\frac{1}{2}$  by any constant strictly between 0 and 1.)

One can draw a few immediate conclusions. The neighbourhood  $N$  contains and is contained in a ball of uniform positive radius about  $x$ . In particular, the injectivity radius,  $\text{inj}(\Theta)$ , is bounded below by some positive constant. We fix some positive constant,  $\eta_0 = \eta_0(\theta) < \frac{1}{2} \text{inj}(M)$  depending only on  $\mu$ . If  $x, y \in \Theta$  with  $d(x, y) \leq \eta_0$ , we write  $[x, y]$  for the unique geodesic between them.

There are increasing functions,  $V_{\pm} : [0, \eta_0) \rightarrow [0, \infty)$  with  $V_{\pm}(0) = 0$  such that for all  $x \in \Theta$  and  $r \in [0, \infty)$ , we have  $V_-(r) \leq \text{vol}(N(x, r)) \leq V_+(r)$ .

**Definition :** Given  $\epsilon > 0$ , a subset  $P \subseteq \Theta$  is  $\epsilon$ -separated if  $d(x, y) > \epsilon$  for all distinct  $x, y \in P$ .

We see that  $|P| \leq \text{vol}(N(P, \epsilon/2))/V_-(\epsilon/2)$ . If  $\text{diam}(P) \leq r$ , then  $|P| \leq V_+(r + \epsilon/2)/V_-(\epsilon/2)$ .

**Definition :** If  $P \subseteq Q \subseteq M$ , we say that  $P$  is  $\epsilon$ -dense in  $Q$  if  $Q \subseteq N(P, \epsilon)$ .

**Definition :**  $P$  is an  $\epsilon$ -net in  $Q$  if it is  $(\epsilon/2)$ -separated and  $\epsilon$ -dense.

Note that any maximal  $(\epsilon/2)$ -separated subset of  $Q$  is an  $\epsilon$ -net in  $Q$ . We shall be taking  $\epsilon < \eta_0$ . The cardinality of any  $\epsilon$ -net is thus bounded in terms of  $\text{vol}(N(Q, \eta_0))$ . We will use the following technical lemma in Section 14.

**Lemma 12.1 :** *Suppose  $\Theta, \Theta'$  are bounded geometry manifolds, and  $f : \Theta \rightarrow \Theta'$  is  $\lambda$ -lipschitz. If  $Q \subseteq \Theta$ , then  $\text{vol}(N(f(Q), \eta_0))$  is bounded above in terms of  $\text{vol}(N(Q, \eta_0))$ ,  $\lambda$  and the bounded-geometry constants.*

(Here we are taking the same constant  $\eta_0$  for  $\Theta$  and  $\Theta'$ .)

**Proof :** Let  $P \subseteq Q$  be an  $\eta_0$ -net. Thus  $|Q|$  is bounded in terms of  $\text{vol}(N(Q, \eta_0))$ . Now  $f(Q) \subseteq N(f(P), \lambda\eta_0)$ , so  $N(Q, \eta_0)$  lies inside  $N(f(P), \lambda\eta_0 + \eta_0)$  whose volume is bounded above in terms of  $|f(P)| \leq Q$  and  $\lambda, \eta_0$ .  $\diamond$

The following “nerve” construction will serve as a substitute for certain “geometric limit” arguments. (Indeed it is the basis of many precompactness results in bounded geometry, cf. [Gr2].)

Given  $P \subseteq \Theta$  and  $\epsilon > 0$ , let  $\Upsilon = \Upsilon_\epsilon(P)$  be the simplicial 2-complex with vertex set  $V(\Upsilon) = P$  and with  $A \subseteq P$  deemed to be a simplex in  $\Upsilon$  if  $\text{diam}(A) \leq 3\epsilon$ . For us,  $P$  will be discrete, and so  $\Upsilon$  will be locally finite. We write  $\Upsilon^1$  for its 1-skeleton. If  $\epsilon < \eta_0$ , then the inclusion of  $P$  into  $M$  extends to a map  $\theta : \Upsilon \rightarrow \Theta$ . This can be taken to send each edge of  $\Upsilon$  linearly to a geodesic segment. We then extend over each 2-simplex by coning over a vertex. (The latter construction may entail putting some order on the vertices, and so may not be canonical.) We easily verify that  $\theta(\Upsilon) \subseteq N(P, 3\epsilon)$ .

**Definition :** We say that two paths  $\alpha$  and  $\beta$  in  $\Theta$  are  $\eta$ -close if we can parametrise them so that  $d(\alpha(t), \beta(t)) \leq \eta$  for all parameter values,  $t$ .

Note that if  $\alpha$  and  $\beta$  have the same endpoints, and  $\eta \leq \eta_0$ , then this implies that  $\alpha$  and  $\beta$  are homotopic relative to their endpoints.

Suppose now that  $Q \subseteq \Theta$  and that  $P$  is  $\epsilon$ -dense in  $Q$ , with  $3\epsilon \leq \eta_0$ . If  $\alpha$  is a path in  $Q$  with endpoints in  $P$ , then we can find a path  $\bar{\alpha}$  in  $\Upsilon^1$  with the same endpoints of combinatorial length at most  $3((\text{length}(\alpha)/\epsilon) + 1)$  so that  $\theta \circ \bar{\alpha}$  is  $3\epsilon$ -close to  $\alpha$ . In particular, if  $Q$  is connected, then so is  $\Upsilon$ , and the image of  $\pi_1(Q)$  in  $\pi_1(M)$  is contained in  $\theta_*(\pi_1(\Upsilon))$ .

If  $\pi_1(Q)$  injects into  $\pi_1(M)$ , it would be nice to say that  $\pi_1(Q)$  were isomorphic to  $\pi_1(\Upsilon)$ , but this is complicated by the fact that  $Q$  may have wiggly boundary. To help us cope with this problem, we make the following definition:

**Definition :** We say that  $Q$  is  $r$ -convex if given any  $x, y \in Q$  with  $d(x, y) \leq \eta_0$ , there is some arc  $\alpha$  in  $Q$  from  $x$  to  $y$  so that  $\alpha \cup [x, y]$  bounds a disc of diameter at most  $r$  in  $\Theta$ .

Note that an immediate consequence is that if  $P \subseteq Q$  is  $\epsilon$ -dense for some  $\epsilon \leq \eta_0$ , then if  $\beta$  is any path in  $\Upsilon^1$  then  $\theta \circ \beta$  is homotopic relative to its endpoints into  $Q$ . In particular, given our previous observation, we see that the image of  $\pi_1(Q)$  in  $\pi_1(M)$  must equal  $\theta_*(\pi_1(\Upsilon))$ . The problem remains that  $\pi_1(\Upsilon)$  may have lots of non-trivial loops “near the boundary”.

Suppose then that  $Q$  is  $r$ -convex, and let  $Q' = N(Q, r)$ . Let  $P \subseteq Q$  be an  $\epsilon$ -net in  $P$  and extend to an  $\epsilon$ -net  $P' \subseteq Q'$ . We thus have an inclusion of  $\Upsilon$  in  $\Upsilon'$ . Write  $\Gamma(\Upsilon, \Upsilon')$  for

the image of  $\pi_1(\Upsilon)$  in  $\pi_1(\Upsilon')$ . Note that  $\theta$  induces a natural map of  $\Gamma(\Upsilon, \Upsilon')$  into  $\pi_1(M)$ .

**Lemma 12.2 :** *Suppose that  $Q$  is  $r$ -convex and suppose that  $\pi_1(Q)$  injects into  $\pi_1(\Theta)$ . Let  $\Upsilon$  and  $\Upsilon'$  be as constructed above. Then the natural map of  $\Gamma(\Upsilon, \Upsilon')$  into  $\pi_1(\Theta)$  is injective, and its image equals the image of  $\pi_1(Q)$ . (In particular,  $\Gamma(\Upsilon, \Upsilon')$  is isomorphic to  $\pi_1(Q)$ .)*

**Proof :** We have already observed that the image of  $\pi_1(\Upsilon)$  and hence of  $\Gamma(\Upsilon, \Upsilon')$  in  $\pi_1(\Theta)$  equals the image of  $\pi_1(Q)$ . We thus need to show that the map of  $\Gamma(\Upsilon, \Upsilon')$  into  $\pi_1(\Theta)$  is injective. If  $\beta$  is any closed curve in  $\Upsilon^1$ . then  $\theta \circ \beta$  consists of a sequence of geodesics arcs of length at most  $3\epsilon \leq \eta_0$  connecting points of  $P \subseteq Q$ . By  $r$ -convexity,  $\theta \circ \beta$  can be homotoped into  $Q$  inside  $Q' = N(Q, r)$ . If  $\theta \circ \beta$  is trivial in  $\pi_1(\Theta)$ , then the homotoped curve is also trivial in  $\pi_1(Q)$ . It thus follows that  $\theta \circ \beta$  bounds a disc in  $Q'$ . We can now pull back this disc to  $\Upsilon'$  showing that  $\beta$  is trivial in  $\pi_1(\Upsilon')$ , and hence in  $\Gamma(\Upsilon, \Upsilon')$  as required.  $\diamond$

As an application, we have the following lemma to be used in Section 10. For the purposes of this lemma, we can define a “band” in a 3-manifold,  $\Theta$ , to be a closed subset,  $B \subseteq \Theta$ , homeomorphic to  $\Sigma \times [0, 1]$ , where  $\Sigma$  is compact surface such that  $B \cap \partial\Theta = \partial_V B$ , where  $\partial_V B = \partial\Sigma \times [0, 1]$  is the “vertical boundary”. Note that the relative boundary of  $B$  in  $\Theta$  is the “horizontal boundary”  $\partial_H B = \Sigma \times \{0, 1\}$ .

**Lemma 12.3 :** *Let  $\Theta$  be a bounded geometry 3-manifold, with  $\eta_0 \leq \frac{1}{2} \text{inj}(\Theta)$  as before. Suppose that  $B \subseteq \Theta$  is a band with  $\pi_1(B)$  injecting into  $\pi_1(\Theta)$  and that  $B$  is  $r$ -convex. Suppose there is a constant  $s \geq 0$  such that each component of  $\partial_V B$  is homotopic to a curve of length at most  $s$ . Suppose that  $\alpha_B, \beta_B$  are curves in  $B$  of length at most  $t$  for some other constant  $t \geq 0$ , and such that  $\alpha$  and  $\beta$  are homotopic to curves  $\alpha, \beta$  in  $X(\Sigma)$ . Then the intersection number,  $\iota(\alpha, \beta)$ , of  $\alpha$  and  $\beta$  in  $\Sigma$  is bounded above in terms of  $r, s, t, \text{diam}(B)$  the constant of bounded geometry (including  $\eta_0$ ), and the complexity,  $\kappa(\Sigma)$ , of the surface  $\Sigma$ .*

Note that the intersection number is independent of the choice of homeomorphism of  $B$  with  $\Sigma \times [0, 1]$ . The bound also places a bound on the distance,  $d_X([\alpha], [\beta])$ , between  $\alpha$  and  $\beta$  in the curve graph (which is what we are really interested in).

The proof relies on the observation that the intersection number of two curves is a function of the pair of conjugacy classes in  $\pi_1(\Sigma)$  representing their free homotopy class. (We need not explicitly describe what this function is, though of course this is in principle computable.) In the case where  $\pi_1(\Sigma)$  has boundary, we need also to take into account the peripheral structure — the set of conjugacy classes of boundary curves.

In practice the “short” peripheral curves in  $B$  will just be core curves of the corresponding annuli.

**Proof :** Fix some  $\epsilon \leq \eta_0/6$ . Let  $P \subseteq B$  be an  $\epsilon$ -net and extend to an  $\epsilon$ -net  $P'$  of  $Q' = N(Q, r)$ , and construct  $\Upsilon = \Upsilon_\epsilon(P)$  and  $\Upsilon' = \Upsilon_\epsilon(P')$  as above. Note that the diameter of  $Q'$  is bounded, and so  $|V(\Upsilon')| = |P'|$  is bounded. By Lemma 12.2, there is a

natural isomorphism of  $\pi_1(Q) \equiv \pi_1(\Sigma)$  with  $\Gamma(\Upsilon, \Upsilon')$ . Note that  $\alpha_B$  and  $\beta_B$  correspond to curves,  $\bar{\alpha}$  and  $\bar{\beta}$  of bounded length in the 1-skeleton of  $\Upsilon$ . If  $\Omega_1, \dots, \Omega_n$  is the (possibly empty) set of vertical boundary components, then each  $\Omega_i$  is homotopic to a curve,  $\gamma_i$  in  $B$  of bounded length, and thus corresponds to some bounded length curve,  $\bar{\gamma}_i$ , in the 1-skeleton of  $\Upsilon$ . We see that there are boundedly many combinatorial possibilities for  $\Upsilon, \Upsilon', \bar{\alpha}, \bar{\beta}, (\bar{\gamma}_i)_i$ . Among all such possibilities for which  $\Gamma(\Upsilon, \Upsilon')$  is isomorphic to  $\pi_1(\Sigma)$  with the  $\gamma_i$  peripheral, there is a maximal intersection number of  $\alpha$  and  $\beta$  which will serve as our bound.  $\diamond$

We remark that this argument does not give us a computable bound, since it involves sifting out those pairs,  $\Upsilon, \Upsilon'$  for which  $\Gamma(\Upsilon, \Upsilon')$  is a surface group, and this is not algorithmically testable. In principle, the above argument could be translated into a “geometric limit” argument.

Here is another application of this construction, to be used in Section 14 (see Proposition 14.11).

**Lemma 12.4 :** *Suppose that  $f : \Theta \rightarrow \Theta'$  is a surjective lipschitz homotopy equivalence between two bounded geometry manifolds  $\Theta$  and  $\Theta'$ . Suppose there is some positive  $\epsilon < \eta_0$  such that if  $x, y \in \Theta$  with  $d'(f(x), f(y)) \leq \epsilon$ , then there is a path  $\alpha$  from  $x$  to  $y$  in  $\Theta$  with  $\text{diam}(\alpha)$  bounded such that  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc of bounded diameter in  $\Theta'$ . Then  $f$  is universally sesquilipschitz.*

In other words, the lift,  $\tilde{f} : \tilde{\Theta} \rightarrow \tilde{\Theta}'$  is a quasi-isometry. The constants of quasi-isometry depend only on the constants of the hypotheses (though, again, we do not show this dependence to be computable). Of course, the fact that  $f$  itself is sesquilipschitz (i.e. a quasi-isometry) is an immediate consequence of the hypotheses.

Note that, since  $f$  is a homotopy equivalence, the conclusion means that  $\alpha$  must lie in a particular homotopy class relative to its endpoints  $x$  and  $y$ . We refer to this as the “right homotopy class”. The hypotheses tell us that, in particular, there is path of bounded diameter in the right class, and we need to find one of bounded length.

**Proof :** For the purposes of the proof (rescaling the metric on  $\Theta$  or  $\Theta'$  if necessary, and modifying the bounded geometry constants) we can assume, for notational convenience, that  $f$  is 1-lipschitz. We can also take the same constant  $\eta_0$  for both  $\Theta$  and  $\Theta'$ . Fix some  $\epsilon \leq \eta_0/6$ .

Let  $R$  be the bound on the length of  $\alpha$  in  $\Theta$  and let  $R'$  be the bound on the diameter of the disc in  $\Theta'$ . We assume  $R \leq R'$ . Let  $Q = N(x, R) \subseteq \Theta$  and let  $Q' = N(f(x), R' + 6\epsilon) \subseteq \Theta'$ . [Check  $R, R'$  etc.] Let  $P_0$  and  $P'_0$  be  $\epsilon$ -nets in  $Q$  and  $Q'$  respectively. Let  $P = P_0 \cup \{x, y\}$ , and let  $P' = P'_0 \cup f(P) \subseteq Q'$ . Thus,  $P$  and  $P'$  are  $\epsilon$ -dense in  $Q$  and  $Q'$  respectively. Note that since the diameters of  $Q$  and  $Q'$  are bounded,  $|P_0|$  and  $|P'_0|$  and hence  $|P|$  and  $|P'|$  are bounded.

Let  $\Upsilon = \Upsilon_\epsilon(P)$  and  $\Upsilon' = \Upsilon_\epsilon(P')$ , and write  $\theta : \Upsilon \rightarrow \Theta$  and  $\theta' : \Upsilon' \rightarrow \Theta'$  for the corresponding maps as constructed earlier. Note that the map  $f : V(\Upsilon) \rightarrow V(\Upsilon')$  extends to a simplicial map  $g : \Upsilon \rightarrow \Upsilon'$ . We see that  $f \circ \theta$  and  $\theta' \circ g$  are  $3\epsilon$ -close on the 1-skeleton of  $\Upsilon$ .

Let  $\alpha$  be the path connecting  $x$  to  $y$  as given by the hypotheses. Now  $\alpha \subseteq Q$ , so there is a path  $\bar{\alpha}$  from  $x$  to  $y$  in  $\Upsilon$  such that  $\theta \circ \bar{\alpha}$  is  $3\epsilon$ -close to  $\alpha$ . Thus  $f \circ \alpha$  is  $3\epsilon$ -close to  $f \circ \theta \circ \bar{\alpha}$  and hence  $6\epsilon$ -close to  $\theta' \circ g \circ \bar{\alpha}$ . Now  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc  $D \subseteq N(f(x), R')$ , and so  $\theta' \circ g \circ \bar{\alpha} \cup [f(x), f(y)]$  bound a disc in  $Q' = N(f(x), R' + 6\epsilon)$ . This pulls back to a disc in  $\Upsilon'$  bounding  $(g \circ \bar{\alpha}) \cup e$ , where  $e$  is the edge connecting  $f(x)$  to  $f(y)$  in  $\Upsilon'$ .

In summary, we have two simplicial 2-complexes,  $\Upsilon, \Upsilon'$  with  $|V(\Upsilon)|$  and  $|V(\Upsilon')|$  bounded, a simplicial map  $g : \Upsilon \rightarrow \Upsilon'$  and vertices  $x, y \in V(\Upsilon)$ , with the property that  $f(x)$  and  $f(y)$  are connected by some edge  $e$  in  $\Gamma'$ , and such  $x$  and  $y$  are connected by some path whose image under  $f$  together with  $e$  bounds a disc in  $\Upsilon'$ . Now there are boundedly many combinatorial possibilities for  $\Upsilon, \Upsilon', g, x, y$ . For each such  $\Upsilon, \Upsilon', g, x, y$ , we choose some path, say  $\beta$  so that  $(g \circ \beta) \cup e$  bounds a disc. Since there are only finitely many cases, there is some upper bound for the length of any such  $\beta$  depending only on the bounds on  $|V(\Upsilon)|$  and  $|V(\Upsilon')|$ . Let  $\gamma = \theta \circ \beta$ . This has bounded length in  $\Theta$ , and since  $(g \circ \beta) \cup e$  bounds a disc in  $\Upsilon$ , we see that  $(\theta' \circ g \circ \beta) \cup [f(x), f(y)]$  bounds a disc in  $\Theta'$  (of bounded diameter). Now  $\theta' \circ g \circ \beta$  is  $3\epsilon$ -close to  $f \circ \theta \circ \beta = f \circ \gamma$ , so  $(f \circ \gamma) \cup [f(x), f(y)]$  bounds a disc in  $\Theta'$ .

In other words, we can find a path from  $x$  to  $y$  of bounded length in the right homotopy class. The rest of the argument is now fairly standard.

Reinterpreting in terms of universal covers, we have a lift  $\tilde{f} : \tilde{\Theta} \rightarrow \tilde{\Theta}'$ . If  $a, b \in \tilde{\Theta}$  with  $d(\tilde{f}(a), \tilde{f}(b)) \leq \epsilon$ , then  $d(a, b)$  is bounded by some constant, say  $k$ . Given any  $a, b \in \Theta$ , we can connect  $a$  to  $b$  by a geodesic  $\sigma$  in  $\Theta'$  from  $\tilde{f}(a)$  to  $\tilde{f}(b)$ . Choose points  $\tilde{f}(a) = c_0, c_1, \dots, c_n = \tilde{f}(b)$  along  $\sigma$  so that  $d'(c_i, c_{i+1}) \leq \epsilon$  for all  $i$ , and so that  $n \leq d'(f(a), f(b))/\epsilon + 1$ . Since  $f$  is assumed surjective,  $\tilde{f}$  is also surjective, so we can find points  $a = a_0, a_1, \dots, a_n = b$  in  $\Theta$  with  $\tilde{f}(a_i) = c_i$ . Thus,  $d(a_i, a_{i+1}) \leq k$  for all  $i$ , so that  $d(a, b)$  is bounded above by a linear function of  $d'(f(a), f(b))$ .

Since  $f$  is lipschitz, so is  $\tilde{f}$ , and so  $\tilde{f}$  is a quasi-isometry as required.  $\diamond$

We finish this section with the observation that if a group  $\Gamma$  acts freely properly discontinuously on a riemannian manifold  $\Theta$ , then  $\Theta/\Gamma$  has bounded geometry if and only if  $\Theta$  has bounded geometry and the orbits of  $\Gamma$  are uniformly separated sets.

### 13. Lower bounds.

We have constructed, in Section 11, a lipschitz map between a model space and our 3-manifold. In this section, we begin the project of showing that there is also a linear lower bound on distortion of distances. We will do this in a series of steps. First we shall restrict our attention to the electric pseudometrics,  $\rho = \rho_P$  and  $\rho' = \rho_M$  as defined at the end of Section 8. We will need a result about 1-dimensional quasi-isometries. In what follows we will write  $[t, u] = [u, t]$  for the interval between  $t, u \in \mathbf{R}$ , regardless of the order of  $t$  and  $u$ .

Suppose that  $I \subseteq \mathbf{R}$  is an interval and that  $\sigma : I \rightarrow \mathbf{R}$  is a continuous map. We shall say that  $\sigma$  is *quasi-isometric* if it is a quasi-isometry to its range,  $\sigma(I)$ . Writing  $I = [\partial_- I, \partial_+ I]$ , it necessarily follows that  $\sigma(I)$  lies in a bounded neighbourhood of  $[\sigma(\partial_- I), \sigma(\partial_+ I)]$ . We shall allow the possibility that  $\partial_- I = -\infty$  and  $\partial_+ I = \infty$ . In this case,  $\sigma$  is a self quasi-isometry of  $\mathbf{R}$ .

We list the following properties of a continuous map  $\sigma : I \rightarrow \mathbf{R}$  which together will imply that it is quasi-isometric. Let  $Q$  be a closed subset of  $I$ . We suppose:

(Q1)  $(\forall k)(\exists K_0(k))$  if  $t, u \in I$  with  $|t - u| \leq k$  then  $|\sigma(t) - \sigma(u)| \leq K_0(k)$ .

(Q2)  $(\forall k)(\exists K_1(k))$  if  $t, u \in Q$  and  $|\sigma(t) - \sigma(u)| \leq k$  then  $|t - u| \leq K_1(k)$ .

(Q3)  $(\forall k)(\exists K_2(k))$  if  $t, u \in I$  and  $\text{diam}(\sigma[t, u]) \leq k$ , then  $|t - u| \leq K_2(k)$ .

(Q4)  $(\exists k_3)(\forall k)(\exists K_3(k))$  if  $t, u \in I$  and  $N([\sigma(t), \sigma(u)], k_3) \cap \sigma(Q) = \emptyset$  and  $|\sigma(t) - \sigma(u)| \leq k$  then  $|t - u| \leq K_3(k)$ .

(Q5)  $(\exists k_4)$  if  $t, u \in I$ ,  $[t, u] \cap Q = \emptyset$ , then  $[\sigma(t), \sigma(u)] \cap \sigma(Q) \subseteq N(\{\sigma(t), \sigma(u)\}, k_4)$ .

We can paraphrase the above conditions informally as follows. (Q1) gives an upper bound on distortion, and (Q2) gives a lower bound restricted to  $Q$ . (Q3) tells us that no long interval can get sent into a short interval. (Q4) gives a lower bound on distortion, so long as we stay away from  $\sigma(Q)$ . Finally (Q5) tells us that intervals in the complement of  $Q$  can not fold too deeply over  $Q$ . Note that we always apply (Q5) to a subinterval  $[t', u'] \subseteq [t, u]$  with  $\sigma([t', u']) = [\sigma(t'), \sigma(u')] = \sigma([t, u])$ .

**Lemma 13.1 :** *Let  $\sigma : I \rightarrow \mathbf{R}$  be a continuous map, with  $Q \subseteq I$  satisfying (Q1)–(Q5) above. Then  $\sigma$  is a quasi-isometry, and the constants of quasi-isometry depend only on the constants of the hypotheses.*

**Proof :** Let  $Q' = \sigma(Q)$  and let  $R' = \{t \in \mathbf{R} \mid N(t, 2k_3) \cap Q = \emptyset\}$ , and let  $R = \sigma^{-1}R'$ . Then  $R \subseteq I$  is closed. We first claim that each component of  $I \setminus (R \cup Q)$  has bounded length.

Note that each component of  $\mathbf{R} \cap (R' \cup Q')$  has length at most  $k_3$ . Suppose that  $J \subseteq I$  is an interval with  $J \cap (R \cup Q) = \emptyset$ . Now  $\sigma(J) \cap R' = \emptyset$ , and property (Q5) bounds the extent to which  $\sigma(J)$  can cross  $Q'$ . In fact, we get  $\text{diam}(\sigma(J)) \leq 2k_4 \cup 2k_3$ . It now follows by (Q3) that  $\text{diam}(J)$  is bounded (by  $K_2(2k_4 + 2k_3)$ ). This proves the claim.

Now fix some  $k_0 < k_3$ , and suppose that  $t < u \in I$  with  $|\sigma(t) - \sigma(u)| \leq k_0$ . We now claim that  $u - t$  is bounded. If  $t \in R$ , then  $\sigma(u) \in N(R', k_0) \subseteq N(R', k_3)$  so  $N([\sigma(t), \sigma(u)], k_3) \cap Q' = \emptyset$ , and  $u - t$  is bounded by (Q4). Similarly if  $u \in R$ . In order to prove the second claim, we can therefore assume that  $t, u \notin R$ .

Now if  $[t, u] \cap (R \cup Q) = \emptyset$ , then  $u - t$  bounded by our earlier claim. If not, let  $t_0$  and  $u_0$  be, respectively, the minimal and maximal points of  $[t, u] \cap (R \cup Q)$ . Again,  $u - u_0$  and  $t_0 - t$  are bounded. It in turn follows that  $|\sigma(t_0) - \sigma(u_0)|$  is bounded (by (Q1)). If  $t_0, u_0 \in Q$ , then  $u_0 - t_0$  is bounded by (Q2), and so  $u - t$  is bounded. If  $t_0 \in R$ , then let  $t_1$  be the maximal point of  $[t, u] \cap \sigma^{-1}\sigma(t_0)$ . Since  $\sigma(t_0) = \sigma(t_1) \in R'$ ,  $t_1 - t_0$  is bounded by (Q4), and so it is enough to consider the interval  $[t_1, u_0]$ . Similarly, if  $u_0 \in R$  let  $u_1$  be the minimal point of  $[t, u] \cap \sigma^{-1}\sigma(u_0)$ , we see that  $u_0 - u_1$  is bounded. But if  $\sigma|[t, u]$  enters any component of  $R'$  it must eventually leave by the same point. Thus the above observations allow us to reduce to the case where  $(t, u) \cap R = \emptyset$ , and so again we get  $u - t$  bounded, as before.

This proves the second claim. The fact that  $\sigma$  is quasi-isometric is now standard.  $\diamond$

We shall be applying this to spaces quasi-isometric to intervals, and we will need some general observations concerning such spaces.

Suppose that  $\Psi$  is a locally compact locally connected and connected space with two ends, deemed “positive” and “negative”. (In practice  $\Psi \cong \Sigma \times \mathbf{R}$ .) By an *end-separating* set  $Q$ , we mean a compact connected subset which separates the two ends of  $\Psi$ . We write  $C_+(Q)$ ,  $C_-(Q)$  for the components of  $\Psi \setminus Q$  containing the positive and negative ends of  $\Psi$  respectively, i.e. we adjoin all relatively compact components of the complement. (Note that  $C_0(C_0(Q)) = C_0(Q)$ .) We write  $C_0(Q) = \Psi \setminus (C_-(Q) \cup C_+(Q))$ . If  $Q'$  is another end separating set, we write  $Q < Q'$  to mean that  $Q \subseteq C_-(Q')$ . One can verify that this is equivalent to stating that  $Q' \subseteq C_+(Q)$ , and that  $<$  is a total order on any locally finite pairwise disjoint collection of end-separating sets. If  $Q < Q'$  we write  $[Q, Q'] = [Q', Q]$  for the closure of  $C_+(Q) \cap C_-(Q')$ . This is the compact region of  $\Psi$  between  $Q$  and  $Q'$ . We also note that if  $P \subseteq Q$  is end-separating, then  $C_-(Q) \subseteq C_-(P)$  and  $C_+(Q) \subseteq C_+(P)$ .

**Definition :** By a *quasiline* we mean a locally compact connected path-metric space  $(\Psi, \rho)$ , such that every point of  $\Psi$  lies in some end-separating set of  $\rho$ -diameter at most  $l$ , where  $l \geq 0$  is some constant.

Suppose now that  $\Psi$  is a quasiline. We choose such a set  $Q(x)$ . Note that if  $y \in C_0(Q(x))$ , then  $Q(y) \cap Q(x) \neq \emptyset$ , and so  $\text{diam } C_0(Q(x)) \leq 3l$ . In particular,  $C_0(Q(x))$  is another end-separating set of bounded diameter containing  $x$ .

Let  $\pi : \mathbf{R} \rightarrow \Psi$  be a bi-infinite end-respecting geodesic, i.e.  $\pi|_{(-\infty, 0]}$  goes out the negative end, and  $\pi|_{[0, \infty)}$  goes out the positive end of  $\Psi$ . Clearly any end-separating set must meet  $\pi(\mathbf{R})$  and so  $\Psi = N(\pi(\mathbf{R}), l)$ . In particular,  $\pi$  is a quasi-isometry. We see that any quasiline is quasi-isometric to  $\mathbf{R}$ . (Indeed we could equivalently define a quasiline as a locally compact path metric space quasi-isometric to  $\mathbf{R}$ .) We can see in fact, that any two geodesics in  $\Psi$  with the same endpoints must remain a bounded distance (in fact  $l$ ) apart. The same remains true if the respective endpoints are bounded distance apart. If  $x, y \in \Psi$  with  $Q(x) \cap Q(y) = \emptyset$ , then  $[Q(x), Q(y)]$  is a bounded Hausdorff distance from any geodesic from  $x$  to  $y$ . If  $t, u \in \mathbf{R}$  with  $u > t + 2l$ , then  $Q(\pi(t)) \cap Q(\pi(u)) = \emptyset$ , and  $[Q(\pi(t)), Q(\pi(u))]$  is a bounded Hausdorff distance from  $\pi([t, u])$ .

We shall be applying this in the case where  $\Psi = \Sigma \times \mathbf{R}$ , and every point of  $\Psi$  lies in the image of a homotopy equivalence,  $\psi$ , of  $\Sigma$  into  $\Psi$  of bounded diameter in  $\Psi$ . (Here all homotopy equivalences are assumed to be relative to the boundaries  $\partial\Sigma$  and  $\partial\Psi$ . By [FHS] we can find an embedded surface,  $Z$ , in an arbitrarily small neighbourhood of  $\psi(\Sigma)$ . This is a “fibre” of  $\Psi$  in the sense discussed in Section 3. If we have two such  $\psi$  and  $\psi'$ , with the  $\psi(\Sigma) \cap \psi'(\Sigma) = \emptyset$  and  $Z, Z'$  are nearby fibres, then  $[Z, Z']$  is a band in  $\Psi$  (with base surface  $\Sigma$ ) and  $[Z, Z']$  is a bounded Hausdorff distance from  $[\psi(\Sigma), \psi'(\Sigma)]$ .

Suppose that  $(\Psi, \rho)$  and  $(\Psi', \rho')$  are two such product spaces, and  $f : \Psi \rightarrow \Psi'$  is a proper lipschitz end-preserving map. Let  $\pi : \mathbf{R} \rightarrow \Psi$  and  $\pi' : \mathbf{R} \rightarrow \Psi'$ . We can find a map  $\sigma : \mathbf{R} \rightarrow \mathbf{R}$  such that for all  $t \in \mathbf{R}$ ,  $\rho'(\pi'(\sigma(t)), f(\pi(t)))$  is uniformly bounded.

Let us now focus on the case of interest where  $f : \Psi(P) \rightarrow \Psi(M)$  is the map between the non-cuspidal parts of our model space  $P$  and hyperbolic 3-manifold  $M$ . These have riemannian metrics,  $d$  and  $d'$  respectively. Let  $\mathcal{T}(P)$  and  $\mathcal{T}(M)$  be the sets of

Margulis tubes in  $\Psi(P)$  and  $\Psi(M)$  respectively, and let  $\Theta(P) = \Psi(P) \setminus \text{int} \bigcup \mathcal{T}(P)$  and  $\Theta(M) = \Psi(M) \setminus \text{int} \bigcup \mathcal{T}(P)$  be the respective thick parts. Now  $f$  maps  $\Theta(P)$  onto  $\Theta(M)$  and is lipschitz with respect to the metrics  $d$  and  $d'$ . We can define electric riemannian pseudometrics  $\rho$  and  $\rho'$  on  $\Psi(P)$  and  $\Psi(M)$  respectively, agreeing with  $d$  or  $d'$  on the thick parts, and equal to zero on each Margulis tube. The map  $f : (\Psi(P), \rho) \longrightarrow (\Psi(M), \rho')$  is then lipschitz. In what follows, all distances and diameters etc. refer to the metrics  $\rho$  or  $\rho'$  unless otherwise specified. As observed in Section 8, we can foliate  $\Psi(P)$  with fibres of bounded diameter. We write  $S(x)$  for the fibre containing  $x$ , so that  $S(x)$  varies continuously in the Hausdorff topology. We are thus in the situation described above with  $\Psi = \Psi(P)$  and  $\Psi' = \Psi(M)$ . We have geodesics  $\pi$  and  $\pi'$  and a map  $\sigma : \mathbf{R} \longrightarrow \mathbf{R}$ .

**Lemma 13.2 :** *The map  $\sigma$  arising as above (from the map  $f : \Psi(P) \longrightarrow \Psi(M)$ ) is a self-quasi-isometry of  $\mathbf{R}$ .*

To prove this, we shall apply Lemma 13.1. We set  $Q = \{t \in \mathbf{R} \mid S(\pi(t)) \cap \bigcup \mathcal{T}(P) \neq \emptyset\}$ . Thus  $Q$  is a closed subset of  $\mathbf{R}$ . Note that property (Q1) of Lemma 13.1, is an immediate consequence of the construction and the fact that  $f$  is lipschitz. We now set about verifying properties (Q2)–(Q5).

(Q2): This argument is based on a similar construction due to Bromberg, which is discussed in [Brocb]. (We are using images of fibres under  $f$ , in place of interpolations of pleated surfaces used by Bromberg.) Indeed the proof of (Q2) given here could be shortened by quoting Bromberg’s result (namely that two Margulis tubes a bounded distance apart in  $M$  are a bounded distance apart in the curve complex). However, we will need some variants of this construction later.

Suppose  $t, u \in Q$  with  $|\sigma(t) - \sigma(u)| \leq k$ . By the definition of  $Q$ ,  $S(\pi(t))$  and  $S(\pi(u))$  meet Margulis tubes  $T_0, T_1 \in \mathcal{T}(P)$  respectively. Let  $T'_i = f(T_i) \in \mathcal{T}(M)$ . We want to show that  $\rho(T_0, T_1)$  is bounded, since it then follows that  $\rho(\pi(t), \pi(u))$  is bounded, and so, since  $\pi$  is geodesic,  $|t - u|$  is bounded, as required. Clearly, we can assume that  $T_0 \neq T_1$ , or there is nothing to prove.

Now  $f(S(\pi(t)))$  meets  $T'_0$  and has bounded diameter. Also  $f(\pi(t)) \in f(S(\pi(t)))$  is a bounded distance from  $\pi'(\sigma(t))$  (by definition of  $\sigma$ ) and so  $\rho'(\pi'(\sigma(t)), T'_0)$  is bounded. Similarly,  $\rho'(\pi'(\sigma(u)), T'_1)$  is bounded. Also  $\rho'(\sigma(t), \sigma(u)) = |\sigma(t) - \sigma(u)| \leq k$ , and so  $\rho(T'_0, T'_1)$  is bounded (in terms of  $k$ ). In other words we can connect  $T'_0$  to  $T'_1$  by a path  $\beta$  of bounded  $\rho'$ -length. Indeed we can assume that  $\beta \cap \Theta(M)$  consists of a bounded number of paths  $\beta_1, \dots, \beta_n$  of bounded  $d$ -length connecting different Margulis tubes.

Since  $f : \Psi(P) \longrightarrow \Psi(M)$  is a homotopy equivalence and  $f^{-1}(T'_0) = T_0$  and  $f^{-1}(T'_1) = T_1$ , there is a path  $\alpha \subseteq f^{-1}\beta$  connecting  $T_0$  to  $T_1$  in  $\Psi(P)$ . For each  $x \in \alpha$ , there is a loop  $\gamma_x$  in  $\Psi(P)$  based at  $x$  of bounded  $d$ -length, with the properties (W2)–(W4) described in Section 4. In particular,  $\gamma_x$ , is freely homotopic to a curve,  $[\gamma_x] \in X(\Sigma)$  (via the natural homotopy equivalence). We can also take  $\gamma_x$  to lie either in  $\Theta(P)$  or else inside a Margulis tube and hence freely homotopic to the core of that tube. If  $x, y$  are sufficiently close then  $d_X([\gamma_x], [\gamma_y])$  is bounded: less than  $2r \geq 0$  say. (“Sufficiently” can be taken to imply some uniform positive constant, but uniformity is not needed here.)

Let  $Y = \{[\gamma_x] \mid x \in \alpha\} \subseteq X(\Sigma)$ . Since  $\alpha$  is connected, by the above observation, the  $r$ -neighbourhood of  $Y$  in  $\mathcal{G}(\Sigma)$  is connected. We claim that  $|Y|$  is bounded. This will place a bound on the diameter of  $Y$  in  $X(\Sigma)$ .

To see that  $|Y|$  is bounded, note that if  $x \in \alpha$ , then  $f(\gamma_x)$  is a loop in  $\Psi(M)$  based at  $f(x) \in \beta$ . Either  $f(x)$  lies in one of the segments  $\beta_i$  or else it is freely homotopic to the core of one of (the bounded number of) Margulis tubes passed through by  $\beta$ . But now for each  $\beta_i$ , there can be only boundedly many possibilities for the free homotopy classes of  $\beta_i$ . This follows from the standard fact of hyperbolic 3-manifolds that there are a bounded number of (based) homotopy classes of curves of bounded length based at any point of the 3-manifold (at least if we rule out multiples of a very short curve, which cannot happen here). Alternatively, this is a general statement about bounded geometry manifolds, given that we are working in the thick part. We see that there is bound on the number of possible  $[\gamma_x]$ . This bounds  $|Y|$  as required.

But now the core curves of  $T_0$  and  $T_1$  lie in  $Y$  and so we have bounded the distance between these curves in  $\mathcal{G}(\Sigma)$ . By the tautness assumption (see (W4) Section 4), it follows that  $\rho(T_0, T_1)$  is bounded as required, and so (Q2) follows.

(Q3): This is essentially a variation on the same argument. Suppose  $t, u \in \mathbf{R}$  with  $\text{diam} \sigma([t, u])$  bounded. Let  $R \subseteq \Psi$  be the region between  $S(\pi(t))$  and  $S(\pi(u))$ . Again, since  $f: \Psi(P) \rightarrow \Psi(M)$  is a homotopy equivalence, there is a path  $\alpha \subseteq R \cap f^{-1}(\pi'(\mathbf{R}))$  connecting  $S(\pi(t))$  to  $S(\pi(u))$ . Now  $\alpha$  will be a bounded Hausdorff distance from any geodesic also connecting  $S(\pi(t))$  to  $S(\pi(u))$  in  $\Psi(P)$ , in particular, the geodesic  $\pi([t, u])$ . Thus,  $f(\alpha)$  is a bounded Hausdorff distance from  $f(\pi([t, u]))$ , which, by the definition of  $\sigma$ , is in turn a bounded Hausdorff distance from  $\pi'(\sigma([t, u]))$  in  $\Psi(M)$ . By assumption,  $\text{diam}(\sigma([t, u]))$  is bounded. It therefore follows that  $\text{diam}(f(\alpha))$  is bounded. We see that  $f(\alpha) \cap \Theta(M)$  consists of a bounded number of segments  $\beta_1, \dots, \beta_n$ , each of bounded  $d'$ -length, and we proceed exactly as in (Q2) to show that  $|t - u|$  is bounded.

(Q4): Let us suppose that  $t, u \in \mathbf{R}$  and that  $N([\sigma(t), \sigma(u)], k_3) \cap \sigma(Q) = \emptyset$  for some constant  $k_3$  to be determined shortly, and suppose that  $|\sigma(t) - \sigma(u)| \leq k$ . Let  $x = \sigma(t)$  and  $y = \sigma(u)$ . We want to bound  $|t - u|$ , which is the same as bounding  $\rho(x, y)$ . Since  $t, u \notin Q$ , we have  $S(x), S(y) \subseteq \Theta(P)$ . In particular,  $x, y \in \Theta(P)$  and so the paths  $\gamma_x$  and  $\gamma_y$  lie in  $\Theta(P)$ . Since  $|\sigma(t) - \sigma(u)|$  is bounded,  $\rho'(f(x), f(y))$  is bounded. Also  $f(\gamma_x)$  and  $f(\gamma_y)$  have bounded  $d'$ -length. We want to apply Lemma 8.3. This means finding a band  $B \subseteq \Theta(M) \subseteq \Psi(M)$ , with base surface  $\Sigma$ , of bounded diameter, containing  $\gamma_x$  and  $\gamma_y$ , and which is  $r$ -convex for some uniform  $r > 0$ . (See the definition in Section 12.) For the latter, it is enough to find another band  $A \subseteq \Theta(M)$  containing a uniform neighbourhood of  $B$ .

Let  $h_1 > h_0 > 0$  be sufficiently large constants, to be determined shortly. Let  $t' = \sigma(t)$  and  $u' = \sigma(u)$ . We can suppose that  $t' \leq u'$ . Let  $t_0 = t' - h_0$ ,  $t_1 = t' - h_1$ ,  $u_0 = u' + h_0$ ,  $u_1 = u' + h_1$ . Thus  $t_1 < t_0 < t' < u' < u_0 < u_1$ . For  $i = 0, 1$ , let  $Z_i^-, Z_i^+$  be fibres in a small neighbourhood of  $f(S(\pi(t_i)))$  and  $f(S(\pi(u_i)))$  respectively. Now if  $h_0$  and  $h_1 - h_0$  are sufficiently large, then we have  $Z_1^- < Z_0^- < Z_0^+ < Z_1^+$ . Let  $B = [Z_0^-, Z_0^+]$  and  $A = [Z_1^-, Z_1^+]$ , so that  $B \subseteq A$ . Note that  $B$  and  $A$  are a bounded Hausdorff distance from

$\pi'([t_0, u_0])$  and  $\pi'([t_1, u_1])$  respectively, and that  $f(x)$  and  $f(y)$  are a bounded distance from  $\pi'(t')$  and  $\pi'(u')$  respectively. Thus, again by choosing  $h_0$  and  $h_1 - h_0$  sufficiently large, we can assume that  $f(\gamma_x), f(\gamma_y) \subseteq B$  and that  $N(B, r) \subseteq A$ , for some uniform  $r > 0$  (as usual, with respect to the metric  $\rho'$ ).

We claim that, provided  $k_3 - h_1$  is sufficiently large,  $A \subseteq \Theta(M)$ . Suppose that  $T \cap A \neq \emptyset$  for some  $T \in \mathcal{T}(M)$ . Let  $s \in I$  be such that the fibre  $S(\pi(s))$  meets the corresponding Margulis tube in  $P$ . By definition,  $s \in Q$ . Thus  $f(S(\pi(s))) \cap T \neq \emptyset$ . Now  $\pi'(\sigma(s))$  is a bounded distance from  $f(\pi(s))$  which is a bounded distance from  $T$  and hence from  $\pi'([t_1, u_1])$ . Thus  $\sigma(s)$  is a bounded distance from  $[t_1, u_1]$ . But  $\sigma(s) \in \sigma(Q)$ , so we get a contradiction by taking  $k_3 - h_1$  large enough. This shows that  $A \subseteq \Theta(M)$  as claimed.

But now  $\gamma_x, \gamma_y, B$  satisfy the hypotheses of Lemma 12.3, which means that  $d_{\mathcal{G}(\Sigma)}([\gamma_x], [\gamma_y])$  is bounded. By (W4) of Section 8, it follows that  $\rho(x, y)$  is bounded, thereby giving a bound on  $|t - u|$  as required. ■

(Q5): Suppose  $t, u \in \mathbf{R}$  with  $[t, u] \cap Q = \emptyset$ . Let  $x = \pi(t)$  and  $y = \pi(u)$ . Let  $R = [S(x), S(y)]$  be the band bounded by  $S(x)$  and  $S(y)$ . Note that  $R \subseteq \bigcup_{v \in [t, u]} S(\pi(v))$ . Since  $[t, u] \cap Q = \emptyset$ , we have  $R \subseteq \Theta(P)$ . If  $f(S(x)) \cap f(S(y)) \neq \emptyset$ , then  $\rho'(f(x), f(y))$  is bounded, so  $\rho'(\pi'(\sigma(t)), \pi'(\sigma(u)))$  and hence  $|\sigma(t) - \sigma(u)|$  is bounded, and there is nothing to prove. If not, let  $R' = [f(S(x)), f(S(y))]$  be the compact region between  $f(S(x))$  and  $f(S(y))$ . Thus  $R'$  is a bounded Hausdorff distance from the geodesic segment  $\pi'[\sigma(t), \sigma(t')]$ . Now, for homological reasons,  $f|f^{-1}R'$  must have degree 1, and so  $R' \subseteq f(R)$ . Since  $f(\Theta(P)) = \Theta(M)$ , we see that  $R' \subseteq \Theta(M)$ . Suppose now that  $v \in [\sigma(t), \sigma(u)] \cap \sigma(Q)$ . Let  $v = \sigma(s)$  for  $s \in Q$ . Thus  $S(\pi(s))$  meets some Margulis tube  $T \in \mathcal{T}(P)$ . Now  $f(S(\pi(s)))$  has bounded diameter, and is a bounded distance from  $\pi'(v)$  and from  $f(T)$ . But  $R' \subseteq \Theta(M)$ , so  $f(T) \cap R' = \emptyset$ . It follows that  $\pi'(v)$  must be a bounded distance from an endpoint of the segment  $\pi'([\sigma(t), \sigma(u)])$ , and so  $v$  is a bounded distance from either  $\sigma(t)$  or  $\sigma(u)$  as required.

We have verified properties (Q1)–(Q5) for the map,  $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ , and so it is a quasi-isometry, proving Lemma 13.2.

We note the following immediate consequence:

**Proposition 13.3 :**

*The map  $f : (\Psi(P), \rho) \rightarrow (\Psi(M), \rho')$  is a quasi-isometry.* ◇

The constants only depend on  $\kappa(\Sigma)$  and the Margulis constant defining  $\Psi(M)$ .

We need a version of this result for bands. We can express this by passing to appropriate covers.

Let  $B \subseteq \Psi(P)$  be a band with base surface  $\Phi$ . Let  $\mathcal{T}_B(P) = \{T \in \mathcal{T}(P) \mid T \cap \partial_V B \neq \emptyset\}$ . Thus  $\partial_V B$  consists of a set of annuli in the boundaries of elements of  $\mathcal{T}_B(P)$  and components of  $\partial\Psi(P)$ . Let  $\Xi_B(P) = \Psi(P) \setminus \text{int} \bigcup \mathcal{T}_B(P)$ , and let  $\Psi_B(P)$  be the cover of  $\Xi_B(P)$  corresponding to  $B$ . Thus  $B$  lifts to a compact subset of  $\Psi_B(P)$ , which we also

denote by  $B$ . We note that  $B$  cuts  $\Psi_B(P)$  into two non-compact components bounded by  $\partial_- B$  and  $\partial_+ B$  respectively. The inclusion of  $B$  into  $\Psi_B(P)$  is a homotopy equivalence. Indeed, if we remove those boundary components of  $\Psi_B(P)$  that do not meet  $B$ , then the result is homeomorphic to  $\Phi \times \mathbf{R}$ .

We can perform the same construction in  $\Psi(M)$ . We let  $\mathcal{T}_B(M) = \{f(T) \mid T \in \mathcal{T}_B(P)\}$ , and  $\Xi_B(M) = \Psi(M) \setminus \text{int} \bigcup \mathcal{T}_B(M)$ , and by Lemma 3.7,  $f : \Xi_B(P) \rightarrow \Xi_B(M)$  is a homotopy equivalence. We let  $\Psi_B(M)$  be the cover of  $\Xi_B(M)$  corresponding to  $\Psi_B(P)$ , so that  $f$  lifts to a homotopy equivalence  $\tilde{f} : \Psi_B(P) \rightarrow \Psi_B(M)$ . Indeed,  $\Psi_B(M)$  is homeomorphic to  $\Phi \times \mathbf{R}$  after removing certain boundary components, so we are in the same topological situation as before. We write  $g : B \rightarrow \Psi_B(M)$  for its restriction to  $B$ .

We shall assume that  $B$  has positive height. Let  $\mathcal{T}_B^0(P) = \{T \in \Theta(P) \setminus \mathcal{T}_B(P) \mid T \cap B \neq \emptyset\}$ . If  $T \in \mathcal{T}_B^0(P)$ , then either  $T \subseteq B$ , or  $T \cap B$  is a half torus bounded by an annulus in  $\partial_H B \cap T$ . We denote the lifted riemannian metric  $d$  by  $d_B$ , and write  $\rho_B$  for the modified metric obtained by setting the metric equal to zero on each  $T \in \mathcal{T}_B^0(P)$ . Every point  $x \in B$  lies in a fibre  $F(x) \subseteq B$ , of bounded  $\rho_B$ -diameter. If  $x \in \partial_{\pm} B$ , we can take  $F(x) = \partial_{\pm} B$ . We can assume that such fibres foliate  $B$ . We similarly define a metric  $\rho'_B$  on  $\Psi_B(M)$ . In what follows, all distances are measured with respect to  $\rho_B$  or  $\rho'_B$ , unless otherwise specified. Note that  $g : (B, \rho_B) \rightarrow (\Psi_B(M), \rho'_B)$  is lipschitz.

Let  $\pi : [a, b] \rightarrow B \subseteq \Psi_B(P)$  be a shortest geodesic from  $\partial_- B$  to  $\partial_+ B$ . Each fibre,  $F(x)$ , of  $B$  meets  $\pi([a, b])$  and we see that  $B$  lies in a uniform neighbourhood of  $\pi([a, b])$ . Let  $\pi' : \mathbf{R} \rightarrow \Psi_B(M)$  be a bi-infinite geodesic, with  $\pi'|(-\infty, 0]$  and  $\pi'|[0, \infty)$  going out a negative and positive end of  $\Psi_B(M)$  respectively. If  $x \in B$ , then  $g(F(x))$  intersects  $\pi'(\mathbf{R})$ . This enables us to define a continuous map  $\sigma_B : [a, b] \rightarrow \mathbf{R}$  such that  $\rho'_B(\pi'(\sigma_B(t)), g(\pi(t)))$  is uniformly bounded.

**Lemma 13.4 :** *The map  $\sigma_B : [a, b] \rightarrow \mathbf{R}$  is uniformly quasi-isometric.*

**Proof :** We define  $Q = \{t \in [a, b] \mid F(\pi(t)) \cap \bigcup \mathcal{T}_B^0(P)\} \neq \emptyset$ . Property (Q1) is an immediate consequence of the fact that  $g$  is lipschitz. We need to verify (Q2)–(Q5). The argument is essentially the same as before. There are a few subtleties we should comment on.

(Q2): Here we use the loops  $\gamma_x^B \subseteq B$  instead of  $\gamma_x$ . This time property (W8) of Section 4 tells us that  $\rho_B(x, y)$  is bounded above in terms of the distance between  $[\gamma_x^B]$  and  $[\gamma_y^B]$  in the curve graph  $\mathcal{G}(\Phi)$  or the modified curve graph  $\mathcal{G}'(\Phi)$  if  $\kappa(\Phi) = 1$ .

There is a slight complication in that  $g^{-1}\beta \subseteq B$  might not connect  $T_0$  to  $T_1$ . We may therefore need to allow the path  $\alpha$  to have up to three components, possibly connecting  $T_0$  or  $T_1$  to  $\partial_H B$ , and maybe also  $\partial_- B$  with  $\partial_+ B$ . But this makes no essential difference to the argument, since if  $x, y$  both lie in  $\partial_- B$  or both in  $\partial_+ B$ , then  $[\gamma_x^B]$  and  $[\gamma_y^B]$  are equal or adjacent in the curve graph. It follows that a uniform  $r$ -neighbourhood of  $Y \subseteq X(\Phi)$  is connected and the argument proceeds as before.

(Q3): As before.

(Q4): Here we apply [FHS] as before to find embedded surfaces in  $\Psi_B(M)$  close to the surfaces  $g(F(x))$ . These surfaces will be fibres in  $\Psi_B(M)$  and any two disjoint fibres bound

a band.

There is, however, an added complication in that in order to find our surfaces  $Z_i^\pm$ , we will need that  $t_1$  and  $u_1$  lie in  $\sigma([a, b])$ . We therefore need that  $t$  and  $u$  are not too close to the boundary of  $\sigma([a, b])$ . We are saved by property (Q3) which we have already proven.

Let  $\sigma([a, b]) = [a', b']$ , and suppose that  $t, u \in I$  with  $\rho'_B(\sigma(t), \sigma(u)) \leq k$  and that  $N([\sigma(t), \sigma(u)], 2k_3) \cap \sigma(Q) = \emptyset$ . Suppose that  $\sigma(t) \in N(a', k_3)$ , say. If  $\sigma([t, u]) \subseteq N(a', k_3)$ , then  $|t - u|$  is bounded using (Q3). If not, let  $s \in [t, u]$  be the first time that  $\sigma|[t, u]$  leaves  $N(a', k_3)$ . Now  $\sigma([t, s]) \subseteq N(a', k_3)$ , and so by (Q3)  $|s - t|$  is bounded. We can now replace  $t$  by  $s$  and continue the argument. We can do the same for the other end  $u$ . We are then reduced to the case where  $t, u \in [a' + k_3, b' - k_3]$ , and the argument proceeds as before. This time, the constant  $2k_3$  becomes our “new”  $k_3$ .

(Q5): As before. ◇

This shows that  $\sigma_B : [a, b] \rightarrow \mathbf{R}$  is quasi-isometric, and it follows that  $\sigma_B([a, b])$  lies in a bounded neighbourhood of  $[\sigma_B(a), \sigma_B(b)]$ . The constants depend only on the various constants inputted. To simplify notation in what follows, we will assume that  $\sigma(a) < \sigma(b)$ . If not, we could interpret everything by reversing the order on the range. (It will turn out, retrospectively, that if  $b - a$  is sufficiently large then this is necessarily the case, though we won't formally need to worry about this. The issue of vertical orientation of bands will eventually be taken care of automatically by the topology of the situation.)

If  $b - a$  is sufficiently large, then we can find embedded fibres close to  $f(\partial_\pm B)$  in  $\Psi_B(M)$  which will bound a band. We would like to find an embedded band  $B'$  in  $\Psi(M)$ . It would be enough to show that the projection to  $\Xi_B(M)$  is injective far enough away from  $\partial_H B'$ . For this, we will need the following lemma. For the statement, we can interpret the term “band” to be a 3-submanifold,  $A$  of  $\Psi$ , homeomorphic to  $\Phi \times [0, 1]$  with  $\partial_V A \equiv \partial\Phi \times [0, 1]$  and  $\partial_H A = \partial_- A \sqcup \partial_+ A = \Phi \times \{0, 1\}$ . As usual, a “subband” is a subset bounded by disjoint fibres. We shall assume that  $A$  carries a metric  $\rho$ . This need not be a path metric. (For our application it will be the restriction of an ambient path metric.)

**Lemma 13.5 :** *Suppose that  $(A, \rho)$  is a band and that each point of  $A$  lies in a fibre of (extrinsic) diameter at most  $k$ . Let  $\Xi$  be a complete non-compact riemannian manifold, and suppose that  $\theta : A \rightarrow \Xi$  is a  $\pi_1$ -injective locally isometric map with  $\partial_V A = \theta^{-1}(\partial\Xi)$  (and so  $\theta$  is 1-lipschitz). Suppose that any fibre of  $A$  is homotopic to an embedded surface in  $\Xi$ . Then  $\theta|_{A \setminus N(\partial_H A, 13k)}$  is injective.*

**Proof :** First, we claim that if  $x \in A$  with  $\rho(x, \partial_H A) \geq 13k$ , then there is some sub-band,  $A' \subseteq A$  (with the same base surface), containing  $x$ , with  $\rho'(\theta(x), \theta(\partial_H A')) \geq 2k$ . To see this, we can argue as follows.

Let  $C_\pm$  be the set of  $p \in A$  such that there is an arc  $\tau_\pm(p)$  from  $p$  to  $\partial_\pm A$  such that  $\theta(\tau_\pm(p))$  is geodesic in  $\Xi$ . Clearly  $\partial_\pm A \subseteq C_\pm$  (set  $\tau_\pm(p) = \{p\}$ ). Also  $A = C_- \cup C_+$ , since if  $p \in A$ , we can find a geodesic ray  $\sigma$  in  $\Xi$  based at  $p$  in  $\Xi$  (since  $\Xi$  is non-compact). Some component of  $\theta^{-1}(\sigma)$  must connect  $p$  to  $\partial_H A$  in  $A$ , and so  $p \in C_- \cup C_+$ . But now  $A$  is connected, and  $C_-$  and  $C_+$  are closed, so there must be some  $p \in C_- \cap C_+$ . In other words, there are arcs,  $\tau_\pm$  from  $p$  to points  $a_\pm \in \partial_\pm A$  with  $\theta(\tau_\pm)$  geodesic in  $\Xi$ . Since

$\tau_+ \cup \tau_-$  connects  $\partial_- A$  to  $\partial_+ A$  we have  $A = N(\tau_- \cup \tau_+, k)$ . Suppose now that  $x \in A$  with  $\rho(x, \partial_H A) \geq 13k$ . Let  $y \in \tau_- \cup \tau_+$  with  $\rho(x, y) \leq k$ , so  $\rho'(\theta(x), \theta(y)) \leq k$ . We can assume that  $y \in \tau_+$ . Now,  $\rho(y, \tau_+) \geq 12k$ , and so, since  $\theta(\tau_+)$  is geodesic,  $\rho'(\theta(y), \theta(a_+)) \geq 12k$ . If  $\rho'(\theta(y), \theta(p)) \leq 4k$ , let  $F$  be a fibre of diameter at most  $k$  through  $p$ . Note that  $\rho'(\theta(x), \theta(F)) \geq 4k - 2k = 2k$ , so we can set  $A'$  to be the band between  $F$  and  $\partial_+ A$ . On the other hand, suppose  $\rho'(\theta(y), \theta(p)) \geq 4k$ . The total length of  $\tau_-$  together with the segment of  $\tau_+$  from  $p$  to  $y$  is at least  $\rho(y, a_+) \geq 12k$ . Since  $\theta(\tau_-)$  and  $\theta(\tau_+)$  are both geodesic, it follows that  $\rho'(\theta(y), \theta(a_-)) \geq 12k - 8k = 4k$ . This time, we can take  $A' = A$ . This proves the claim.

Now if  $x \in A$  with  $\rho(x, \partial_H A) \geq 13k$ , we claim there is a fibre  $Z = Z(x)$  through  $x$ , with  $\theta|Z$  injective. To see this, start with any fibre,  $F$ , through  $x$ . By hypothesis,  $\theta(F)$  is homotopic to an embedded surface in  $\Xi$ , and so by [FHS] we can find such a surface  $S$ , in an arbitrarily small neighbourhood of  $\theta(F)$ . We shall assume that  $\theta(x) \in S$ . Since  $\text{diam}(S) < 2k$ , we have  $S \cap \theta(\partial_H A') = \emptyset$ . Now  $\theta|A'$  is a local homeomorphism away from  $\partial_H A'$ , and so  $S$  lifts to an embedded surface,  $Z$  in  $A' \subseteq A$  with  $x \in Z$ . Since the inclusion of  $Z$  into  $A$  is  $\pi_1$ -injective, it follows that  $F$  must be a fibre of  $A$ . Thus, the map  $\theta|Z \rightarrow S$  is a homeomorphism. In particular,  $\theta|Z$  is injective as required.

Note also that, in the above construction, if  $\theta(x) = \theta(y)$  we could take the same surface  $S$  for both, and we get fibres  $Z(x)$  and  $Z(y)$  with  $\theta|Z(x)$  and  $\theta|Z(y)$  both homeomorphisms to  $S$ . If  $x \neq y$ , then we must have  $Z(x) \cap Z(y) = \emptyset$ .

Suppose finally for contradiction, that  $x, y \in A$  with  $x \neq y$ ,  $\rho(x, \partial_H A) \geq 13k$ ,  $\rho(y, \partial_H A) \geq 13k$  and  $\theta(x) = \theta(y)$ . Construct fibres  $Z(x)$  and  $Z(y)$  as above. Let  $C = [Z(x), Z(y)]$  be the band between  $Z(x)$  and  $Z(y)$ . We construct a closed manifold  $R$  by gluing together  $\partial_- C = Z(x)$  and  $\partial_+ C = Z(y)$  via the homeomorphism  $(\theta|Z(y))^{-1} \circ (\theta|Z(x))$ . Now  $\theta$  induces a map from  $R$  to  $\Xi$  which is a local homeomorphism away from  $Z(x) \equiv Z(y)$ , and hence, by orientation considerations, a local homeomorphism everywhere. It must therefore be a covering space, giving the contradiction that  $\Xi$  is compact.  $\diamond$

We now apply this in the situation of interest. We return to the set-up of Lemma 13.4. We will need to assume that  $b - a$  is sufficiently large. As described earlier, we will assume, for notational convenience that  $\sigma(a) < \sigma(b)$ . The first step is to note that  $\partial_- B = F(\pi(a))$  and so  $g(\partial_- B)$  is a bounded distance from  $\pi'(\sigma(a))$ . Similarly,  $\partial_+ B = F(\pi(b))$  is a bounded distance from  $\pi'(\sigma(b))$ . Thus if  $b - a$  and hence  $\sigma(b) - \sigma(a)$  is sufficiently large,  $g(\partial_- B) \cap g(\partial_+ B) = \emptyset$ . We can find disjoint fibres,  $Z_\pm$  in a small neighbourhood of  $g(\partial_\pm B)$ , and let  $A$  be the band between  $Z_-$  and  $Z_+$ . Let  $\theta : A \rightarrow \Xi_B(M)$  be the inclusion of  $A$  into  $\Psi_B(M)$  composed with the covering map  $\Psi_B(M) \rightarrow \Xi_B(M)$ . Now if  $l_0$ , and  $b - a - 2l_0$  are sufficiently large, then by Lemma 13.4, we can arrange that  $\sigma(a) < \sigma(a + l_0) < \sigma(b - l_0) < \sigma(b)$ . Moreover,  $g(F(\pi(a + l_0)))$  is bounded diameter and a bounded distance from  $\pi(\sigma(a + l_0))$  and we can find a fibre,  $Z'_-$ , close to  $g(F(\pi(a + l_0)))$ . We similarly find  $Z'_+$  close to  $g(F(\pi(b - l_0)))$ . If  $l_0$  and  $b - a - 2l_0$  are sufficiently large, then we will have  $Z_- < Z'_- < Z'_+ < Z_+$ . Moreover, by Lemma 13.5, we can assume that  $\theta|B'$  is injective, where  $B' = [Z'_-, Z'_+]$ . Now, if  $l_1 > 0$  is sufficiently large, we can assume a uniform neighbourhood  $\sigma_B([a + l_0 + l_1, b - l_0 - l_1])$  lies inside  $[\sigma_B(a + l_0), \sigma_B(b - l_0)]$ . If this neighbourhood is large enough, then  $g(B) \subseteq B'$ , where  $B_0 = [F(a + l_0 + l_1), F(b - l_0 - l_1)]$ . Note that the depth of  $B_0$  in  $B$  (measured in the metric  $\rho_B$ ) is equal to  $l_0 + l_1$  up to

an additive constant. Since  $B'$  embeds in  $\Psi_B(M)$ , we can project the whole picture to  $\Xi_B(P) \subseteq \Psi(P)$  and  $\Xi_B(M) \subseteq \Psi(M)$ .

In summary, we have shown:

**Lemma 13.6 :** *There is some  $l > 0$  such that if  $B \subseteq \Psi(P)$  is a band, and  $B_0$  is a sub-band of depth at least  $l$ , then there is a band  $B' \subseteq \Psi(M)$  with the same base surface such that  $f(B_0) \subseteq B'$ .  $\diamond$*

For the moment, we can just interpret this to mean that  $B' \cong \Phi \times [0, 1]$  with  $\partial_V B' = B' \cap \mathcal{T}_B(M)$ . (In Section 11, we will insist in addition that the horizontal boundaries of bands should lie in the thick part.) In fact, from our construction of  $B'$  we see that every point of  $B'$  lies inside some fibre of bounded (extrinsic) diameter. By [FHS] we can take such fibres to be embedded (but we do not claim that such fibres foliate  $B'$ ). We can also refine Lemma 13.6 in various ways. Note, in particular, that if  $r \geq 0$ , then by choosing  $l = l(r)$  sufficiently large, we can assume that an  $r$ -neighbourhood of  $f(B_0)$  (with respect to  $\rho'$ ) lies inside  $B'$ .

At this point, it is still conceivable that  $f$  might also send a point far away from  $B$  into  $B'$ . To rule this out, we need to bring (W9) of Section 8 into play.

Suppose now that  $B$  is a maximal band. There is some  $r \geq 0$  such that if  $x \in \Psi(P) \setminus B$ , there there is some loop,  $\delta_x^B \ni x$ , of  $d$ -length at most  $r$  and such that either  $\delta_x^B$  is freely homotopic into a Margulis tube  $T$  not meeting  $B$ , or else  $[\delta_x^B]$  is not freely homotopic into the base surface,  $\Phi$ , in  $\Sigma$ . We can assume that  $B'$  does not meet any of the images of Margulis tubes of the above type, and so there is a bound on how deeply the image  $f(\delta_x^B)$  can enter into  $B'$ . Using the refinement of Lemma 13.6 mentioned above, we see:

**Lemma 13.7 :** *There is some  $l' > 0$  such that if  $B \subseteq \Psi(P)$  is a maximal band and if  $B_0 \subseteq B$  is a parallel subband of depth at least  $l'$  in  $B$  (with respect to the metric  $\rho$ ) then there is a band,  $A \subseteq \Psi(M)$ , with the same base surface such that  $f(B_0) \subseteq A$  and  $f(\Psi(P) \setminus B) \cap A = \emptyset$ .  $\diamond$*

To finish this section, we describe how the constructions can be adapted to case of finite, or semi-infinite models. From this we can deduce Theorems 5.3 and 5.4, (though this is not directly relevant to the proof of the Ending Lamination Theorem).

Suppose that  $I \subseteq \mathbf{R}$  is any interval, and that  $W = \bigcup \mathcal{W}$  is an annulus system in  $\Sigma \times I$  with no annulus crossing  $\Sigma \times I$ . We assume that  $M$  satisfies the a-priori bounds condition (APB), that is the corresponding closed geodesics in  $M$  all have length at most  $L$ . We can construct a model  $\Psi(P)$  and lipschitz homotopy equivalence,  $f : \Psi(P) \rightarrow \Psi(M)$  as in Section 11.

We define an electric pseudometric,  $\rho = \rho_P$ , on  $\Psi(P)$  by forcing the preimage of each Margulis tube in  $M$  to have diameter 0 in  $(\Psi(P), \rho)$ . Such a preimage is either a Margulis tube in  $P$  or a boundary annulus, that is  $\Delta(\Omega)$  for some  $\Omega \in \mathcal{W}_\partial$  (though not all such sets arise in this way). We see that  $f : (\Psi(P), \rho_P) \rightarrow (\Psi(M), \rho_M)$  is also uniformly lipschitz. The argument of Lemma 13.3 now goes through (using Lemma.1) as before to show that  $f$  is quasi-isometric.

**Proof of Theorem 5.4 :** Note that  $\rho(\alpha_M, \alpha_M^*)$  is bounded above in terms of  $l$ , so we can assume that  $\alpha_M = \alpha_M^*$ . Similarly, we can assume that  $\beta_M = \beta_M^*$ . Write  $\alpha = [\alpha_M]$  and  $\beta = [\beta_M]$ .

(1) We can reduce to the case where  $d_{\mathcal{G}}(\alpha, \beta) = 1$ . If  $\kappa(\Sigma) = 1$ , then the statement follows directly from the construction of Section 11. If  $\kappa(\Sigma) \geq 2$ , then  $\alpha$  and  $\beta$  are disjoint, and so we can extend  $\alpha_M$  and  $\beta_M$  to a pleated surface in  $M$ , which necessarily has bounded  $\rho_M$ -diameter. (Note that this gives explicit  $k_1$  and  $k_2$  computable in terms of  $\kappa(\Sigma)$ .)

(2) We can suppose that  $d_{\mathcal{G}}(\alpha, \beta) \geq 3$ . We construct a model,  $\Psi(P)$ , with  $\alpha, \beta$  as boundary curves, using Lemma 6.1 and 8.2 as described at the end of Section 11. Let  $f : \Psi(P) \rightarrow \Psi(M)$  be the map described above. As observed, we have  $d_{\mathcal{G}}(\alpha, \beta)$  linearly bounded above in terms of  $\rho_M(\alpha, \beta)$ . The linear function depends only on  $\kappa(\Sigma)$  and  $l$ .  $\diamond$

We can now also prove Lemma 5.3. Note that we can choose a model  $\Psi(P)$  so that  $f(\Psi(P)) = \Psi(M) \cap \text{core}(M)$ . In the doubly degenerate case, this has already been done. For our purposes, it is enough that  $(\Psi(M) \cap \text{core}(M), \rho_M)$  lies a bounded distance from  $f(\Psi(P))$  in the electric pseudometric. For this, it is enough that  $\partial_H \Psi(P)$  be sent a bounded electric distance from  $\Psi(M) \cap \text{core}(M)$ , as can be seen by a degree argument.

If  $M$  is singly degenerate, note that  $\partial \text{core}(M)$  is intrinsically a finite area hyperbolic surface. We choose any curve  $\alpha$  in  $\partial \text{core}(M)$  whose length is bounded in terms of  $\kappa(\Sigma)$ . We can now construct an annulus system in  $\Sigma \times [0, \infty)$ , with  $\alpha$  as a boundary curve, and with all curves in the hierarchy tending to the end invariant,  $a \in \partial \mathcal{G}(\Sigma)$ . In particular, all curves lie a bounded distance from a geodesic ray,  $\pi$ , in  $\mathcal{G}(\Sigma)$  tending to  $a$ . (This is based on a diagonal sequence argument — cf. Proposition 15.4) In particular, (APB) is satisfied for some constant,  $L$ , depending only on  $\kappa(\Sigma)$ , and we get a model,  $\Psi(P)$ , and a map  $f : \Psi(P) \rightarrow \Psi(M)$  as before. (This construction will be used again — see Lemma 16.3.)

Suppose that  $\alpha_M$  is a curve of length at most  $l$  in  $M$ . This lies a bounded  $\rho_M$  distance from  $f(\Psi(P))$ . In other words there is some  $x \in \Psi(P)$  with  $\rho_M(\alpha_M, f(x))$  bounded. By the construction of  $\Psi(P)$ ,  $x$  lies in some curve,  $\gamma_P$ , of bounded length in  $(\Psi(P), d_P)$  and with  $d_{\mathcal{G}}(\pi, \gamma)$  bounded (both in terms of  $\kappa(\Sigma)$ ). Now  $\rho_M(\alpha_M, f(\gamma_P))$  is bounded, so it follows from Theorem 5.4 that  $d_{\mathcal{G}}(\alpha, \pi)$  is bounded in terms of  $\kappa(\Sigma)$  and  $l$ . In particular, this shows that  $X(M, l) \subseteq N(\pi, t)$  in  $\mathcal{G}(\Sigma)$ , where  $t$  depends only on  $\kappa(\Sigma)$  and  $l$ . This proves Theorem 5.3 in this case.

The geometrically finite (quasifuchsian) case can be dealt with in a similar manner using a finite model, starting with curves,  $\alpha_M, \beta_M \in M$  of bounded length in each of the two boundary components of  $\text{core}(M)$ . In the case where  $d_{\mathcal{G}}(\psi, \beta) \leq 2$ , we need to observe that all the relevant subsets of  $\mathcal{G}(\Sigma)$  have bounded diameter.

## 14. Controlling the map on thick parts.

In this section, we shall show that the map  $f : \Theta(P) \rightarrow \Theta(M)$  as defined in Section 10 is universally sesquipschitz (Proposition 14.11). To this end, we will take the results of Section 13, and get ourselves into a position to apply Lemma 12.4.

Before we begin, we need a few observations about surfaces in 3-manifolds. As usual,

to simplify the discussion, our surfaces and 3-manifolds will be assumed orientable. Let  $\Theta$  be an irreducible 3-manifold with boundary  $\partial\Theta$ . By a *proper surface*, in  $\Theta$  we mean a properly embedded  $\pi_1$ -injective surface,  $S \subseteq \Theta$ , with  $\partial S = S \cap \partial\Theta$ .

**Definition :** An *incompressible surface* in  $\Theta$  is a proper surface, which is not a 2-sphere or torus, and whose inclusion into  $\Theta$  is  $\pi_1$ -injective.

Our orientability assumption means that  $S$  is 2-sided, i.e. extends to an embedding of  $S \times [-1, 1]$  in  $\Theta$ , where  $S$  is identified with  $S \times \{0\}$ . We say that two compact proper surfaces,  $S$  and  $S'$  are *parallel* if they are disjoint and homotopic in  $\Theta$  relative to  $\partial\Theta$ . By Waldhausen's Cobordism Theorem (given as Theorem 3.2 here),  $S \cup S'$  bounds a product region homeomorphic to  $S \times [0, 1]$  in  $\Theta$  (meeting  $\partial\Theta$  in  $\partial S \times [0, 1]$ ).

Suppose that  $\mathcal{S}$  is a locally finite collection of pairwise disjoint proper surfaces. We construct the dual graph  $\sigma = \sigma(\mathcal{S})$  with edge set  $E(\sigma) = \mathcal{S}$  and vertex set the set of components of  $\Theta \setminus \bigcup \mathcal{S}$ , by deeming an edge to be incident on a vertex if the corresponding surface is a boundary component of the corresponding complementary region. (We allow loops and multiple edges.)

Let  $\tilde{\Theta}$  be the universal cover of  $\Theta$ . The set  $\mathcal{S}$  lifts to a set,  $\tilde{\mathcal{S}}$ , of proper surfaces in  $\tilde{\Theta}$ . We write  $\tau(\mathcal{S}) = \sigma(\tilde{\mathcal{S}})$ . In this case,  $\tau(\mathcal{S})$  is a tree, and  $G = \pi_1(\Theta)$  acts on  $\tilde{\mathcal{S}}$  with quotient  $\sigma(\mathcal{S})$ . Note also that if  $\mathcal{S}_0 \subseteq \mathcal{S}$  is a maximal collection of parallel compact surfaces in  $\mathcal{S}$ , then the edges corresponding to  $\mathcal{S}_0$  form an arc in  $\sigma(\mathcal{S})$ , and lifts to a collection of arcs in  $\tau(\mathcal{S})$ . We can view  $\tau(\mathcal{S})$  as obtained by subdividing edges of the tree formed by removing all but one of the surfaces of  $\mathcal{S}_0$ , for all such maximal collections,  $\mathcal{S}_0$  of parallel surfaces.

The above is a standard construction of Bass-Serre theory — the collection  $\mathcal{S}$  determines a splitting of  $G$  as a graph,  $\sigma(\mathcal{S})$ , of groups, and  $\tau(\mathcal{S})$  is the associated Bass-Serre tree.

**Lemma 14.1 :** *Suppose that  $\Theta$  and  $\Theta'$  are 3-manifolds and that  $f : \Theta \rightarrow \Theta'$  is a proper homotopy equivalence. Suppose that  $\mathcal{S}$  and  $\mathcal{S}'$  are locally finite collections of pairwise disjoint non-parallel compact proper non-sphere surfaces in  $\Theta$  and  $\Theta'$  respectively. Suppose that  $\bar{f} : \mathcal{S} \rightarrow \mathcal{S}'$  is a bijection so that  $\bar{f}(S)$  is homotopic to  $f(S)$  for all  $S \in \mathcal{S}$ . Then there is a graph-of-groups isomorphism  $g : \sigma(\mathcal{S}) \rightarrow \sigma(\mathcal{S}')$ , which agrees with  $\bar{f}$  on  $E(\sigma(\mathcal{S})) = \mathcal{S} \rightarrow E(\sigma(\mathcal{S}')) = \mathcal{S}'$ .*

Here we can interpret a “graph-of-groups isomorphism” to mean a  $G$ -equivariant isomorphism,  $\tilde{g} : \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{S}'}$ , where  $G = \pi_1(\Theta) \cong \pi_1(\Theta')$ , which projects to a graph isomorphism  $g : \sigma(\mathcal{S}) \rightarrow \sigma(\mathcal{S}')$ .

To prove this we need the following observation. Suppose that  $\Theta_0$  is a codimension-0 submanifold of  $\Theta$ , with relative boundary  $\partial_R\Theta_0$  in  $\Theta$ . Suppose that  $\partial_R\Theta_0$  is compact and incompressible (i.e. each component is an incompressible surface). One can easily see that  $\Theta_0$  also incompressible in  $\Theta$ , i.e. the inclusion is  $\pi_1$ -injective. Thus,  $\Theta_0$  also embeds in the cover,  $\hat{\Theta}_0$ , of  $\Theta$  corresponding to  $\pi_1(\Theta_0)$ , and the inclusion is a homotopy equivalence. In particular,  $\hat{\Theta}_0$  retracts onto  $\Theta_0$ . Similar observations also apply to submanifolds of  $\Theta'$ .

Suppose now that  $f|_{\partial_R\Theta_0}$  is homotopic to an embedding  $h : \partial_R\Theta_0 \rightarrow \Theta'$ . We claim

that  $h(\partial\Theta_0)$  bounds a codimension-0 submanifold,  $\Theta'_0 \subseteq \Theta'$ , with  $\pi_1(\Theta'_0) = f_*(\pi_1(\Theta_0))$ . To see this, we can argue as follows. Note that we can extend  $h$  to a proper map  $h : \Theta_0 \rightarrow \Theta'$  which is homotopic to  $f|_{\Theta_0}$ . We let  $\Theta'_0$  be the set of points onto which  $h$  maps with degree 1. Thus  $\partial_R\Theta'_0 = h(\partial_R\Theta_0)$ . To show that this is homotopic to  $f(\Theta_0)$ , let  $\hat{\Theta}_0$  be the cover of  $\Theta$  corresponding to  $\Theta_0$ , as described above. We lift  $h$  to a map  $\hat{h} : \hat{\Theta}_0 \rightarrow \hat{\Theta}'_0$  where  $\hat{\Theta}'_0$  is the cover of  $\Theta'$  corresponding to  $f_*(\pi_1(\Theta_0)) = h_*(\pi_1(\Theta_0))$ . Let  $\hat{\Theta}''_0$  be the set of points to which  $\hat{h}$  maps with degree 1. Thus  $\hat{\Theta}''_0$  is a compact submanifold with  $\partial_R\hat{\Theta}''_0 = h(\partial_R\Theta_0)$ . Note that  $\partial_R\hat{\Theta}''_0$  maps bijectively to  $\Theta'_0$  under the covering map. Now the covering map will be injective on  $\hat{\Theta}''_0$ . For if not, it must map with degree 1 to some component,  $S$ , of  $\partial_R\hat{\Theta}'_0$ . This lifts bijectively to  $\hat{\Theta}'_0$ , and so must be homotopic into  $\hat{\Theta}''_0$  in  $\hat{\Theta}'_0$ . Thus,  $S$  is homotopic in  $\Theta'_0$ , hence also in  $\Theta'$  to a boundary component, contradicting the assumption that no two distinct elements of  $\partial_R\Theta_0$  are parallel. It follows that  $\hat{\Theta}''_0$  maps injectively to  $\Theta_0$ . Its image must be  $\Theta'_0$  (since their boundaries are equal). In particular, we see that  $\Theta'_0$  is connected, and that  $\pi_1(\Theta'_0) = f_*(\pi_1(\Theta_0))$ . (Retrospectively, we see that  $\hat{\Theta}'_0$  is the cover of  $\Theta'$  corresponding to  $\Theta'_0$  and that  $\hat{\Theta}''_0$  can be identified with  $\Theta'_0$ .)

**Proof of Lemma 14.1 :** Let  $\tau = \tau(\mathcal{S})$  and  $\tau' = \tau(\mathcal{S}')$ , and  $G = \pi_1(\Theta) = \pi_1(\Theta')$  as above. We already have a bijection  $\tilde{g} : E(\tau) \equiv \tilde{\mathcal{S}} \rightarrow E(\tau') \equiv \tilde{\mathcal{S}}'$ , and we need to check that this preserves adjacency.

Let  $v$  be a vertex of  $\sigma(\mathcal{S})$  with group  $H \subseteq G$ . This corresponds to a component,  $\Theta_v$  of  $\Theta \setminus \bigcup \mathcal{S}$  with  $\pi_1(\Theta_v) \equiv H$ . By the above observation, there is a submanifold,  $\Theta'_v \subseteq \Theta'$  with  $\pi_1(\Theta'_v) = H$ , bounded by the subsurfaces  $\bar{f}(S)$  where  $S$  runs through the boundary components of  $\Theta_v$ . (It is possible that these manifolds may have pairs of boundary components that are parallel on the outside, and hence identified to a single surface of our collection.) We claim that  $\Theta'_v$  contains no other surface of  $\mathcal{S}$ . For suppose that  $S \in \mathcal{S}$  were such that  $\bar{f}(S)$  lies in the interior of  $\Theta'_v$ . We see that  $\pi_1(S) \subseteq H$ . We can lift  $\Theta_0$  and  $S$  to the cover  $\hat{\Theta} = \tilde{\Theta}/H$  corresponding to  $H$ . Note that  $\hat{\Theta}$  retracts onto  $\Theta_0$ . Now  $S$  lies outside  $\Theta_0$  but is homotopic into it. Since it is a properly embedded compact  $\pi_1$ -injective surface, one deduces that  $S$  is homotopic to a boundary component of  $\Theta_0$ . Back down in  $\Theta$  we see that  $S$  is parallel to one of the boundary components of  $\Theta_0$  contradicting our hypothesis that no two elements of  $\mathcal{S}$  are parallel.

This shows that the interiors of the manifolds  $\Theta'_v$  are all disjoint in  $\Theta'$ . We see that (at least on the level of fundamental groups) the combinatorial situation in  $\Theta'$  is identical to that in  $\Theta$ , and so we get a graph-of-groups isomorphism.  $\diamond$

Suppose that  $f : \Theta \rightarrow \Theta'$  is a proper homotopy equivalence. Suppose that  $\mathcal{S}$  is a finite disjoint collection of compact proper surfaces, and that  $f(S) \cap f(S') = \emptyset$  for all  $S, S' \in \mathcal{S}$ . Suppose that for all  $S \in \mathcal{S}$ ,  $f(S)$  is homotopic (relative to  $\partial\Theta'$ ) to an embedding in  $\Theta'$ . By [FHS], we can find an embedded surface,  $Z(S)$ , homotopic to  $f(S)$  in an arbitrarily small neighbourhood of  $f(S)$ . Let  $\mathcal{S}' = \{Z(S) \mid S \in \mathcal{S}\}$ . These surfaces are all disjoint.

If no two surfaces in  $\mathcal{S}$  are parallel, we are in a position to apply Lemma 14.1. This breaks down if we allow parallel surfaces since  $f$  might fold up the associated product region in  $\Theta$ . However, we can take care of this using the following criterion:

**Lemma 14.2 :** *Suppose that  $\mathcal{S}$  is a finite collection of disjoint compact proper non-sphere surfaces in  $\Theta$ , and that  $f : \Theta \rightarrow \Theta'$  is a homotopy equivalence, such that the surfaces  $f(S)$  for  $S \in \mathcal{S}$  are all disjoint, and each is homotopic to an embedded surface in  $\Theta'$ . Suppose that whenever  $S, S' \in \mathcal{S}$  are parallel in  $\Theta$  there is a properly embedded ray (i.e. semi-infinite path)  $\alpha \subseteq \Theta$  based in  $S$ , such that  $f(\alpha) \cap f(S') = \emptyset$ . Let  $\mathcal{S}' = \{Z(S) \mid S \in \mathcal{S}\}$  be constructed as above. Then there is a graph-of-groups isomorphism from  $\sigma(\mathcal{S})$  to  $\sigma(\mathcal{S}')$  sending an edge corresponding to  $S \in \mathcal{S}$  to the edge corresponding to  $Z(S) \in \mathcal{S}'$ .*

**Proof :** Let  $\mathcal{S}_0$  be a transversal to the parallel relation (i.e. contains exactly one element from each equivalence class) and let  $\mathcal{S}'_0 = \{Z(S) \mid S \in \mathcal{S}_0\}$ . By Lemma 14.1, there is a graph-of-groups isomorphism  $\sigma(\mathcal{S}_0) \rightarrow \sigma(\mathcal{S}'_0)$ . Now  $\sigma(\mathcal{S})$  is obtained from  $\sigma(\mathcal{S}_0)$  by subdividing edges by adding valence-2 vertices. We similarly get from  $\sigma(\mathcal{S}'_0)$  to  $\sigma(\mathcal{S}')$ . One easily checks that the above criterion ensures that the orders of these subdivision are the same on both sides.  $\diamond$

We now move on to the case of interest, namely where  $\Theta = \Theta(P)$  and  $\Theta' = \Theta(M)$ , and  $f : \Theta(P) \rightarrow \Theta(M)$  is the map constructed in Section 7. Recall that we have riemannian path metrics,  $d$  and  $d'$  on  $\Theta(P)$  and  $\Theta(M)$  respectively, and that  $\rho = \rho_P$  and  $\rho' = \rho_M$  are the electric pseudometrics obtained by forcing each Margulis tube to have diameter 0. The map  $f$  is uniformly lipschitz from  $(\Theta(P), d)$  to  $(\Theta(M), d')$ , and hence also from  $(\Theta(P), \rho)$  to  $(\Theta(M), \rho')$ . We write  $G = \pi_1(\Theta(P)) \cong \pi_1(\Theta(M))$ . We shall say that a subset of  $\Theta(P)$  (or  $\Theta(M)$ ) is *k-small* if its diameter in the metric  $\rho$  (or  $\rho'$ ) is at most  $k$ . In what follows, we speak of a set has being “uniformly small” to mean that it is *k-small* for some  $k$  depending only on  $\kappa(\Sigma)$ .

Given  $x \in \Theta(P)$ , we can find a uniformly small fibre  $S(x) \subseteq \Theta(P)$ . This can be achieved by taking a horizontal fibre, and then pushing it slightly off any Margulis tube. This may significantly increase its  $d$ -diameter, but only increases the  $\rho$ -diameter by an arbitrarily small amount. (Note that this is slightly different from the notion used in Section 13. Here we are assuming that  $S(x)$  lies in the thick part. This greatly simplifies the description of various topological operations. The cost is that we can no longer assume that  $S(x)$  varies continuously in  $x$ , but that will not matter to us in this section.) By Lemma 7.4, we can find a proper surface  $S'(x) \subseteq \Theta(M)$  in an arbitrarily small neighbourhood of  $f(S(x))$ , and homotopic to  $f(S(x))$  in  $\Theta(M)$ . Note that  $S'(x)$  is a fibre in the product space  $\Psi(M)$ . It is also uniformly small in  $\Theta(M)$ .

We now use the following construction of expanding bands in  $\Theta(P)$ . We fix a constant,  $h_0$ , to be defined shortly. Given  $x \in \Theta(P)$ , set  $R_x[0] = S(x)$ , and  $R'_x[0] = S'(x)$ . Let  $\pi : \mathbf{R} \rightarrow \Theta(P)$  be a bi-infinite geodesic respecting the ends of  $\Theta(P)$ . Since  $\pi$  must cross  $R_x[0]$ , we can assume that  $\pi(0) \in R_x[0]$ . Given  $n \in \mathbf{Z}$ , let  $S_n = S(\pi(nh_0))$  and  $S'_n = S'(\pi(nh_0))$ . If  $h_0$  is large enough, the surfaces  $S_n$  will all be disjoint and occur in the correct order in  $\Psi(P)$ . Let  $R_x[n] = \Theta(P) \cap [S_{-n}, S_n]$ , in other words, the compact region of  $\Theta(P)$  bounded by  $S_{-n}$  and  $S_n$ . This gives an increasing sequence,  $R_x[0] \subseteq R_x[1] \subseteq R_x[2] \subseteq \dots$  of bands that eventually exhaust  $\Theta(P)$ . Now applying Lemma 9.2, again if  $h_0$  is large enough, the surfaces  $S'_n$  are all disjoint and occur in the correct order in  $\Theta(M)$ , and we similarly construct bands  $R'_x[n] = \Theta(M) \cap [S'_{-n}, S'_n]$ . We write  $CR_x[n]$  and  $CR'_x[n]$  for the closures of  $\Theta(P) \setminus R_x[n]$  and  $\Theta(M) \setminus R'_x[n]$  respectively. We can assume that:

**Lemma 14.3 :** For all  $x \in \Theta(M)$  and an all  $n \in \mathbf{N}$  we have

- (1)  $f(R_x[n]) \cap CR'_x[n+1] = \emptyset$ ,
- (2)  $f(CR_x[n+1]) \cap R'_x[n] = \emptyset$ .

**Proof :** This is a simple consequence of the discussion of end-separating sets in Section 13, and the fact that  $\sigma$  is a quasi-isometry (Lemma 13.2).  $\diamond$

Note in particular, that  $f(R_x[n]) \cap f(CR_x[n+2]) = \emptyset$ . It will also be convenient to fix some  $\eta > 0$  smaller than the injectivity radii of  $\Theta(P)$  and  $\Theta(M)$ , and we can refine Lemma 14.3 slightly to say that  $d'(f(R_x[n]), CR'_x[n+1]) \geq \eta$  and  $d'(f(CR_x[n+1]), R'_x[n]) \geq \eta$ . We also may as well assume that  $d(f(x), f(R_x[1])) \geq \eta$ .

We can also make a stronger statement concerning the nesting of the regions  $R_x[n]$ . See Lemma 14.6 below.

We have a similar process of shrinking bands. Let  $B$  be a maximal band in  $\Psi(P)$ , with base surface  $\Phi$ . Let  $\rho_B$  be the electric pseudometric on  $B$  described in Section 13 (Recall that this is essentially obtained by taking the path metric induced by  $d$ , and then shrinking each Margulis tube whose interior meets  $B$  to have diameter 0. We may need to modify the metric near  $\partial_H B$  to take account of the fact that we cannot easily control the local geometry of the surfaces  $\partial_{\pm} B$ . It is formally defined by passing to the appropriate covering space.) In this case, we use “small” to refer to diameter with respect to the metric  $\rho_B$ . Now each  $x \in B$  lies in some uniformly small surface  $F(x)$  in  $B \cap \Theta(P)$  that is a fibre for  $B$ . Applying Lemma 11.4 again, we can find a surface  $F'(x)$  in an arbitrarily small neighbourhood of  $f(F(x))$  in  $\Theta(M)$ , and homotopic to  $f(F(x))$  in  $\Theta(M)$ .

Now let  $\pi_B : [a^-, a^+] \rightarrow B \cap \Theta(P)$  be a shortest geodesic from  $\partial_- B$  to  $\partial_+ B$  in  $B \cap \Theta(P)$ , with respect to the metric  $\rho_B$ . We write  $h(B) = a^+ - a^-$  for the length of this geodesic. We fix  $h_0$  and  $h_1$  as described below, and set  $h(n) = 2h_1 + (2n + 1)h_0$ . Suppose  $h(B) \geq h(m)$ . For each  $n = 0, 1, \dots, m$ , let  $F_{n,\pm} = F(\pi_B(a^{\pm} \pm (h_1 + nh_0)))$ , and let  $F'_{n,\pm} = F'(\pi_B(a^{\pm} \pm (h_1 + nh_0)))$ . If  $h_0$  is big enough then the surfaces  $F_{0,-}, F_{1,-}, \dots, F_{m,-}, F_{m,+}, \dots, F_{1,+}, F_{0,+}$  are all disjoint and occur in this order in  $B$ . Let  $B[n] = B_{\Phi}[n] = [F_{n,-}, F_{n,+}] \cap \Theta(M)$  be the compact region of  $\Theta(P)$  bounded by  $F_{n,-}$  and  $F_{n,+}$ . Thus  $B[n] \subseteq B$ . In fact, we have  $B[m] \subseteq \dots \subseteq B[1] \subseteq B[0] \subseteq B \cap \Theta(P)$ . By applying Lemma 13.6, if  $h_1$  is big enough we can assume that  $f(B[0])$  lies inside a band  $A$  in  $\Theta(M)$ , and that  $f(\Theta(P) \setminus B)$  does not enter  $A$ . We are thus effectively reduced to considering the map  $f|_{B[0]}$  into  $A$ . Applying Lemma 9.3, we can assume that the surfaces  $F'_{0,-}, F'_{1,-}, \dots, F'_{m,-}, F'_{m,+}, \dots, F'_{1,+}, F'_{0,+}$  are disjoint and occur in this order in  $B$ . We set  $B'[m] = B'_{\Phi}[m] = [F_{n,-}, F_{n,+}] \cap \Theta(M)$ , and so  $B'[m] \subseteq \dots \subseteq B'[1] \subseteq B'[0] \subseteq A \cap \Theta(M)$ . We write  $CB[n]$  and  $CB'[n]$  for the closures of  $\Theta(P) \setminus B[n]$  and  $\Theta(M) \setminus B'[n]$  respectively. Again, if  $h_0$  and  $h_1$  are large enough, we have:

**Lemma 14.4 :** For each maximal band, and for all  $n$ , we have:

- (1)  $f(B_x[n]) \cap CB'_x[n+1] = \emptyset$ ,
- (2)  $f(CB_x[n+1]) \cap B'_x[n] = \emptyset$ .

$\diamond$

The above bands are defined provided  $h(n+1) \leq h(B)$ . If  $h(n) > h(B)$ , we can set  $B[n] = \emptyset$ . This is consistent with Lemma 14.4.

We can also assume that the bands  $B[0]$  lies inside a 1-collared band,  $B_0 \subseteq B$ . This means that the collection of bands  $B[0]$  that we construct will have a nesting property (see Lemma 14.6(1)).

Finally, we can carry out the expanding band construction within a band. Suppose that  $B$  is a maximal band with base surface  $\Phi$ . Suppose that  $x \in B_\Phi[m]$ . We set  $R_{\Phi,x}[0]$  to be a uniformly small fibre containing  $x$ , which we can assume lies in  $B_\Phi[n]$ . As before, we construct increasing sequences of bands  $R_{\Phi,x}[0] \subseteq R_{\Phi,x}[1] \subseteq \cdots R_{\Phi,x}[n]$  in  $B \cap \Theta(P)$ , and  $R'_{\Phi,x}[0] \subseteq R'_{\Phi,x}[1] \subseteq \cdots R'_{\Phi,x}[n]$  in  $A \cap \Theta(M)$ . We can assume that  $R_{\Phi,x}[n] \subseteq B_\Phi[m-n]$ . As before, applying Lemma 13.3, we get:

**Lemma 14.5 :** *For each maximal band, and for all  $n$ , we have:*

- (1)  $f(R_{\Phi,x}[n]) \cap CR_{\Phi,x}[n+1] = \emptyset$ ,
- (2)  $f(CR_{\Phi,x}[n+1]) \cap R_{\Phi,x}[n] = \emptyset$ .

◇

Let  $\mathcal{F}$  be the set of all subsurfaces of  $\Sigma$ . Given  $\Phi \in \mathcal{F}$ , let  $B_\Phi \subseteq \Psi(P)$  be the (possibly empty) maximal band with base surface  $\Phi$ . Given  $n \in \mathbf{N}$ , let  $\mathcal{F}[n] = \{\Phi \in \mathcal{F} \mid B_\Phi \neq \emptyset, h(B_\Phi) \geq h(n)\}$ . Let  $\mathcal{B}[n] = \{B_\Phi[n] \mid \Phi \in \mathcal{F}[n]\}$ , and let  $\mathcal{B}'[n] = \{B'_\Phi[n] \mid \Phi \in \mathcal{F}[n]\}$ .

**Definition :** We refer to elements of  $\mathcal{B}[n]$  and  $\mathcal{B}'[n]$  as *level  $n$  bands* in  $\Theta(P)$  and  $\Theta(M)$  respectively.

Given  $\Phi \in \mathcal{F}$ , we write  $\mathcal{F}_\Phi \subseteq \mathcal{F}$  for the set of proper subsurfaces of  $\Phi$ . Let  $\mathcal{F}_\Phi[n] = \mathcal{F}_\Phi \cap \mathcal{F}[n]$ ,  $\mathcal{B}_\Phi[n] = \{B_{\Phi'}[n] \mid \Phi' \in \mathcal{F}_\Phi\}$  and  $\mathcal{B}'_\Phi[n] = \{B'_{\Phi'}[n] \mid \Phi' \in \mathcal{F}_\Phi\}$ .

If we choose  $h_0$  and  $h_1$  large enough we have the following:

**Lemma 14.6 :**

- (1) If  $A, B \in \mathcal{B}[0]$  are distinct, and  $A \cap B \neq \emptyset$ , the base surfaces  $\pi_\Sigma A$  and  $\pi_\Sigma B$  are nested (one is proper a subset of the other).
- (2) Suppose that  $x \in \Theta(P)$  and  $A \in \mathcal{B}[0]$ . If  $A \cap R_x[n] \neq \emptyset$  then  $A \cap CR_x[n+1] = \emptyset$ .
- (3) Suppose  $\Phi \in \mathcal{F}[n+1]$ ,  $x \in B_\Phi[n+1]$  and  $A \in \mathcal{B}_\Phi[0]$ . If  $A \cap R_{\Phi,x}[n] \neq \emptyset$  then  $A \cap CR_{\Phi,x}[n+1] = \emptyset$ .
- (4) Suppose  $\Phi \in \mathcal{F}[n+1]$  and  $A \in \mathcal{B}_\Phi[0]$ . If  $A \cap CB_\Phi[n] = \emptyset$  then  $A \cap CB_\Phi[n+1] = \emptyset$ . ◇

**Lemma 14.7 :** *The same statement holds in  $\Theta(M)$ , with  $R'_x[n]$  replacing  $R_x[n]$  and  $B'[n]$  replacing  $B[n]$  etc.* ◇

We also have:

**Lemma 14.8 :** *Suppose that  $p, q \in \mathbf{N}$ .*

- (1) If  $x \in \Theta(P)$ , the volume of  $R_x[p] \setminus \bigcup \mathcal{B}[q]$  is bounded above in terms of  $p$  and  $q$ .
- (2) If  $\Phi \in \mathcal{F}[p]$  and  $x \in B_\Phi[p]$ , then the volume  $R_{\Phi,x}[p] \setminus \bigcup \mathcal{B}_\Phi[q]$  is bounded above in terms of  $p$  and  $q$ .

**Proof :** The riemannian notions of distance and volume (with respect to  $\rho$ ) are linearly bounded in terms of the combinatorial notions used in Section 12. This is therefore a direct corollary of Lemma 12.1.  $\diamond$

**Lemma 14.9 :** *Suppose that  $p, q \in \mathbf{N}$ .*

(1) *If  $x \in \Theta(P)$ , the volume of  $R'_x[p] \setminus \bigcup \mathcal{B}'[q]$  is bounded above in terms of  $p$  and  $q$ .*

(2) *If  $\Phi \in \mathcal{F}[p]$  and  $x \in B_\Phi[q]$ , then the volume  $R'_{\Phi,x}[p] \setminus \bigcup \mathcal{B}'_\Phi[q]$  is bounded above in terms of  $p$  and  $q$ .*

**Proof :**

(1) By Lemma 14.3(1) and 14.4(2) and the fact that  $f$  is surjective, we have

$$R'_x[p] \setminus \bigcup \mathcal{B}'[q] \subseteq f(R'_x[p+1] \setminus \bigcup \mathcal{B}'[q+1]).$$

Since  $f$  is uniformly lipschitz, the volume of the right hand side is bounded by Lemma 14.8(1), and the result follows.

(2) By Lemma 14.3(1) and 14.4(2) we have:

$$R'_{\Phi,x}[p] \setminus \bigcup \mathcal{B}'_\Phi[q] \subseteq R'_{\Phi,x}[p+1] \setminus \bigcup \mathcal{B}'_\Phi[q+1]$$

and the result follows by Lemma 14.8(2).  $\diamond$

In fact, using Lemma 12.1, we see that we can also bound the volume of an  $\eta$ -neighbourhood of these sets in terms of  $\eta$ .

We can now set about verifying the hypotheses of Lemma 12.4. Fix some constant  $\eta$  less than the injectivity radius of  $\Theta(M)$ .

**Proposition 14.10 :** *Suppose  $x, y \in \Theta(P)$  and  $d'(f(x), f(y)) \leq \eta$ . Then there is a path,  $\alpha$ , in  $\Theta(P)$ , of bounded diameter with respect to the metric  $d$ , such that  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc in  $\Theta(M)$  of bounded diameter with respect to the metric  $d'$ .*

The conclusion of Proposition 14.10 determines a homotopy class of path from  $x$  to  $y$  in  $\Theta(P)$ , which we shall refer to as the *right homotopy class*.

The basic strategy is to start with any path  $\alpha$  from  $x$  to  $y$  in the right homotopy class. (Such a path exists, since  $f$  is a homotopy equivalence from  $\Theta(P)$  to  $\Theta(M)$ .) We first push this into a region of bounded depth about  $x$ , and then push it off all bands of a given bounded depth. Lemma 14.8 then gives a bound on the diameter of such a path in  $(\Theta(P), d)$ . By our choice of  $\alpha$ ,  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc in  $\Theta(M)$ . We now push this disc into a region of bounded depth, and then off bands of bounded depth. Lemma 14.9 then bounds the diameter of this disc in  $(\Theta(M), d')$ . In practice we will only need to construct bands up to depth 10. (The proof Lemma 14.9 takes us up to depth 11.)

Let us first deal with the case where  $x, y \notin \bigcup \mathcal{B}[6]$ . We connect  $x$  to  $y$  by a path,  $\alpha$ , in the right homotopy class. We write  $R[n] = R_x[n]$ . Now  $d'(f(x), f(CR[1])) \geq \eta$  and so  $y \in R[1]$ . We first claim that we can push  $\alpha$  into  $R[2]$ .

To see this we pass to the universal covers,  $\tilde{f} : \tilde{\Theta}(P) \rightarrow \tilde{\Theta}(M)$ . Let  $\mathcal{S}[i] = \{\partial_- R[i], \partial_+ R[i]\}$  and  $\mathcal{S}'[i] = \{\partial_- R'[i], \partial_+ R'[i]\}$ . Let  $\mathcal{S} = \mathcal{S}[1] \cup \mathcal{S}[2]$  and  $\mathcal{S}' = \mathcal{S}'[1] \cup \mathcal{S}'[2]$ . Write  $\tau = \tau(\mathcal{S})$  and  $\tau' = \tau(\mathcal{S}')$  for the corresponding Bass-Serre trees. Now  $R[1] \subseteq R[2] \subseteq R[3]$ , and so applying Lemma 14.3, given any two distinct surfaces,  $S, S' \in \mathcal{S}$ , it is easy to construct a ray  $\beta$  from  $S$  to infinity such that  $f(\beta) \cap f(S') = \emptyset$ . (We only really need to do this if  $S$  and  $S'$  are parallel in  $\Theta(P)$ .) By Lemma 14.2, it then follows that there is an equivariant isomorphism from  $\tau$  to  $\tau'$ .

Let  $\tilde{\alpha}$  be a lift of the path  $\alpha$  to  $\tilde{\Theta}(P)$  connecting  $\tilde{x}$  to  $\tilde{y}$ . A lift,  $\delta$ , of the short geodesic  $[f(x), f(y)]$  connects  $\tilde{f}(\tilde{x})$  to  $\tilde{f}(\tilde{y})$  in  $\tilde{\Theta}(M)$ . Now if  $\alpha \subseteq R[1]$ , there is nothing to prove. If not, let  $z, w \in \alpha$  be the first and last intersection of  $\alpha$  with  $\partial_H R[1]$  (the relative boundary of  $R[1]$  in  $\Theta(P)$ ). We have  $z \in S_0 \in \mathcal{S}[1]$  and  $w \in S_1 \in \mathcal{S}[1]$ . Let  $\beta, \gamma$  be the subpaths of  $\alpha$  from  $x$  to  $z$  and  $y$  to  $w$  respectively, and let  $\tilde{z}, \tilde{w}, \tilde{\beta}, \tilde{\gamma}, \tilde{S}_0$  and  $\tilde{S}_1$  be the lifts to  $\tilde{\Theta}(P)$ . Now  $\beta, \gamma \subseteq R[1]$  and so, by Lemma 14.3,  $f(\beta), f(\gamma) \subseteq R'[2]$ . Thus  $\tilde{f}(\tilde{\beta}) \cup \delta \cup \tilde{f}(\tilde{\gamma})$  is a path in  $\tilde{\Theta}(M)$  connecting  $\tilde{f}(\tilde{z})$  to  $\tilde{f}(\tilde{w})$  and not meeting  $\bigcup \tilde{\mathcal{S}}'[2]$ . Thus  $\tilde{f}(\tilde{S}_0)$  and  $\tilde{f}(\tilde{S}_1)$  are not separated by any element of  $\tilde{\mathcal{S}}'[2]$ . Since the corresponding surfaces  $\tilde{S}'_0$  and  $\tilde{S}'_1$  lie in arbitrarily small neighbourhoods of  $\tilde{f}(\tilde{S}_0)$  and  $\tilde{f}(\tilde{S}_1)$ , these are not separated by any element of  $\tilde{\mathcal{S}}'[2]$ . Since we have an isomorphism of Bass-Serre trees, the corresponding surfaces in  $\Theta(P)$  have the same separation properties, and so  $\tilde{S}_0$  and  $\tilde{S}_1$  are not separated by any element of  $\mathcal{S}[2]$ . We can thus connect  $\tilde{z}$  to  $\tilde{w}$  by a path in  $\tilde{\Theta}(P)$  not meeting  $\bigcup \mathcal{S}[2]$ . Together with the paths  $\tilde{\beta}$  and  $\tilde{\gamma}$ , this gives a path from  $\tilde{x}$  to  $\tilde{y}$ . Projecting back down to  $\Theta(P)$ , this gives a path from  $x$  to  $y$  in  $R[2]$  in the right homotopy class, as claimed.

In fact, we can refine the above observation slightly. Note that every time  $\tilde{\alpha}$  crosses some component of  $\mathcal{S}[2]$  it must eventually cross back again, and so we can replace the intervening path by a path in an arbitrarily small neighbourhood of this component. Projecting to  $\Theta(P)$ , we see that we can find a new path  $\alpha$  in the right homotopy class in  $R[2]$  and in an arbitrarily small neighbourhood of our original  $\alpha$  union  $\partial_H R[2]$ . We refer to this operation as “pushing  $\alpha$  into  $R[2]$ ”.

Our next job is to push  $\alpha$  off every level 7 band. Suppose that  $B[7] \in \mathcal{B}[7]$ . By our initial assumption,  $x, y \notin B[6]$ . We can now apply the above argument, with  $CB[6]$  playing the role of  $R[2]$  and  $CB[7]$  playing the role of  $R[3]$  to push  $\alpha$  off  $B[7]$ . In other words, we replace  $\alpha$  by another path in the right homotopy class in  $CB[7]$ , and in a small neighbourhood of our previous  $\alpha$  union  $\partial_H B[7]$ . Our new path might now leave  $R[2]$ , however, since the pushing operations took place inside  $B[6]$  and so certainly inside  $B[0]$ , Lemma 14.6(2) ensures that the resulting path lies inside  $R[3]$ .

We want to perform this construction for all level 7 bands, however there is a risk that the various “pushing” operations may interfere with each other. We therefore proceed by (reverse) induction on the complexity of the bands. By Lemma 14.6(1), any two level 0 bands of the same complexity are disjoint, and therefore the pushing operations on such bands can be performed simultaneously (or more precisely, in any order). We thus start with the level 7 bands of complexity  $\kappa(\Sigma) - 1$ , and then move onto those of complexity  $\kappa(\Sigma) - 2$  and continue all the way down to bands of complexity 1 (observing that there are no 3HS bands). The pushing operations of a given complexity may affect those already performed at a higher complexity, but Lemma 14.6 parts (1) and (4) ensure that we will never enter a level 8 band. Again, Lemma 14.6(2) ensures we remain inside  $R[3]$ . We thus

end up with a path  $\alpha \subseteq R[3] \setminus \bigcup \mathcal{B}[8]$  in the right homotopy class.

Now by Lemmas 14.3 and 14.4,  $f(\alpha) \subseteq R'[4] \setminus \bigcup \mathcal{B}'[9]$ . Since  $\alpha$  lies in the right homotopy class,  $f(\alpha) \cup [f(x), f(y)] \subseteq R'[4] \setminus \bigcup \mathcal{B}'[9]$  bounds (the continuous image of) a disc  $D$  in  $\Theta(M)$ . Now the boundaries,  $\partial_{\pm} R'[4]$  are incompressible in  $\Theta(M)$ , and so we can push  $D$  into  $R'[4]$ , so that the resulting disc lies in a small neighbourhood of our original disc union  $\partial_H R'[4]$ .

Next, we push  $D$  off all level 10 bands, by reverse induction on complexity as before. For this we only need to observe that the boundaries of bands are incompressible. By Lemma 14.7, we end up with a disc  $D$  lying in  $R'[5] \setminus \bigcup \mathcal{B}'[10]$ .

In summary, we have found  $\alpha \subseteq R[3] \setminus \bigcup \mathcal{B}[8]$  such that  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc in  $R'[5] \setminus \bigcup \mathcal{B}'[10]$ . Using Lemma 14.8, we see that the diameter of  $\alpha$  in  $(\Theta(P), d)$  is bounded. Using Lemma 14.9, and the subsequent remark about the  $\eta$ -neighbourhood, we see that the diameter of the disc is bounded in  $(\Theta(M), d')$ . This proves Proposition 14.10 in this case.

All the above was done under the assumption that  $x, y \notin \bigcup \mathcal{B}[6]$ .

We now move on the case where  $x$  or  $y$  lies in some level 6 band. Among all bands in  $\mathcal{B}[6]$  that meet  $\{x, y\}$  we choose one, say  $B[6]$ , of minimal complexity. We can assume that  $x \in B[6]$ . Let  $\Phi = \pi_{\Sigma} B[6]$  be the base surface. By the minimal complexity assumption, we see that  $x, y \notin \bigcup \mathcal{B}_{\Phi}[6]$ . Let  $R_{\Phi}[n] = R_{\Phi, x}[n]$ . Since  $x \in B[6]$ , we get that  $R[5]$  exists and lies inside  $B[0]$ .

We can now carry out the above construction, with  $R_{\Phi}[n]$  replacing  $R[n]$ , and with  $\mathcal{B}_{\Phi}[n]$  replacing  $\mathcal{B}[n]$ . In this way, we get a path  $\alpha \subseteq R_{\Phi}[3] \setminus \bigcup \mathcal{B}_{\Phi}[8]$  such that  $f(\alpha) \cup [f(x), f(y)]$  bounds a disc in  $R'_{\Phi}[5] \setminus \bigcup \mathcal{B}'_{\Phi}[10]$ . By Lemmas 14.8 and 14.9 again, we see that these have bounded diameter in  $(\Theta(P), d)$  and  $(\Theta(M), d')$  respectively.

This proves Proposition 14.10.

Finally, putting Proposition 14.10 together with Lemma 9.4, we get:

**Proposition 14.11 :** *The map  $f : (\Theta(P), d) \longrightarrow (\Theta(M), d')$  (constructed as in Section 7) is uniformly universally sesquipschitz.  $\diamond$*

## 15. The doubly degenerate case.

In this section, we gather our constructions together to show that two doubly degenerate hyperbolic 3-manifolds with the same pairs of end invariants are isometric (Theorem 15.12).

Let  $M$  be a doubly degenerate 3-manifold. In other words,  $\Psi(M)$  is homeomorphic to  $\Sigma \times \mathbf{R}$ , and both ends are degenerate. We write  $e_-$  and  $e_+$  for the positive and negative ends of  $\Psi(M)$  respectively. This gives us two end invariants,  $a(e_-), a(e_+) \in \partial \mathcal{G}$ .

**Lemma 15.1 :**  $a(e_-) \neq a(e_+)$ .

**Proof :** This follows from the fact [Bon] that  $\lambda(e_-) \neq \lambda(e_+)$ .  $\diamond$

One could also deduce Lemma 15.1 by a slightly convoluted argument using the “a-priori bounds” estimate (see Proposition 15.4). If the end invariants were equal, we could take sequence of curves of bounded length in each end and connect them by tight geodesics in  $\mathcal{G}$ . This would then give rise to a sequence of curves also tending to the end invariant, while remaining in a compact set in  $\Psi(M)$ . We shall not give the details of this argument here.

We are now in a position to construct our model space for  $M$ .

To begin, Theorem 7.1 associates to the pair  $a(e_-), a(e_+)$  a complete annulus system.  $W = \bigcup \mathcal{W} \subseteq \Psi = \Sigma \times \mathbf{R}$ . (There might, of course, be many annulus systems satisfying the conditions of Theorem 7.1. We can arbitrarily choose one of them.) The construction of Section 7 now gives us a riemannian manifold  $(\Psi(P), d)$  — first open up each annulus into a torus, and then glue in a Margulis tube. Thus,  $\Psi(P)$  is also homomeorphic to  $\Sigma \times \mathbf{R}$ , and there is a natural (topological) proper homotopy equivalence of  $\Psi(P)$  with  $\Psi(M)$ . Each component of  $\partial\Psi(P)$  is a bi-infinite cylinder isometric to  $S^1 \times \mathbf{R}$ , in the induced path metric. We now construct  $P$  by gluing in a standard  $\mathbf{Z}$ -cusp to each such boundary component (a quotient of a horoball in  $\mathbf{H}^3$  by a  $\mathbf{Z}$ -action). Thus  $P$  is a complete riemannian manifold with empty boundary.

To relate the geometry of  $P$  to the geometry of  $M$ , the following “a-priori bounds” estimate is key. Given  $\alpha \in \Sigma$ , we write  $\bar{\alpha}$  for the closed geodesic in  $M$  in this homotopy class, and write  $l_M(\alpha)$  for its length.

**Theorem 15.2 :** *Suppose that  $\Phi \subseteq \Sigma$  is a subsurface. Suppose that  $\alpha, \beta, \gamma \in X(\Phi)$  and that  $\gamma$  lies in a tight geodesic from  $\alpha$  to  $\beta$  in  $\mathcal{G}(\Phi)$ . Then  $l_M(\gamma)$  is bounded above in terms of  $\kappa(\Sigma)$ ,  $l_M(\alpha)$ ,  $l_M(\beta)$  and  $\max\{l_M(\delta)\}$  as  $\delta$  ranges over the boundary components  $\partial^\Sigma\Phi = \partial\Phi \setminus \partial\Sigma$ .  $\diamond$*

Here we are allowing the possibility that  $\Phi = \Sigma$ . We do not require that  $M$  is doubly degenerate for this result, only that  $\Psi(M)$  is homeomorphic to  $\Sigma \times \mathbf{R}$ , possibly with a number of cusps corresponding to accidental parabolics removed. If  $\alpha$  is homotopic to such a parabolic, we set  $l_M(\alpha) = 0$ . We need not worry about this in this section, though it cannot be avoided in general — see Section 15.

This (or a very similar) a-priori bounds estimate was proven by Minsky [Mi4]. Another proof is given in [Bow2].

By induction, we see that if  $\alpha, \beta, \gamma \in X(\Sigma)$  and  $\gamma \in Y^\infty(\{\alpha, \beta\})$  as defined in Section 5, then  $l_M(\gamma)$  is bounded above in terms of  $\kappa(\Sigma)$ ,  $l_M(\alpha)$  and  $l_M(\beta)$ .

Now in Section 7, we constructed the annulus system  $\mathcal{W}$  out of a sequence of sets of the form  $Y(\alpha_i, \beta_i)$  as  $\alpha_i \rightarrow a$  and  $\beta_i \rightarrow b$ . By the defining property of end invariants, we can choose  $\alpha_i$  and  $\beta_i$  so that  $l_M(\alpha_i)$  and  $l_M(\beta_i)$  remain bounded, and so by Theorem 15.2, all the curves we construct will have bounded length in  $M$ . However, the choice of these sequences might depend on  $M$ . To see that all the curves have bounded length for any choice of sequences, we use the following variation on Theorem 15.2 also proven in [Bow2].

**Theorem 15.3 :** *Given any  $r \geq 0$  there is some  $r' \geq 0$  such that if  $\alpha, \beta, \gamma \in X(\Sigma)$  and  $\gamma$  lies on a tight geodesic from  $\alpha$  to  $\beta$  and  $d(\alpha, \gamma) \geq r'$  and  $d(\beta, \gamma) \geq r'$ , then  $l_M(\gamma)$  is bounded above in terms of  $\kappa(\Sigma)$ ,  $\min\{l_M(\delta) \mid \delta \in N(\alpha, r)\}$  and  $\min\{l_M(\epsilon) \mid \epsilon \in N(\gamma, r)\}$ .*  $\diamond$

Putting these together with Proposition 15.1 and the fact that  $\mathcal{G}(\Sigma)$  is uniformly hyperbolic, we obtain:

**Proposition 15.4 :** *There is some constant  $L \geq 0$  such that if  $\Omega \in \mathcal{W}$ , then  $l_M(\bar{\Omega}) \leq L$ , where  $\bar{\Omega}$  is the closed geodesic in  $M$  in the homotopy class of  $\Omega$ .*  $\diamond$

This is precisely the hypothesis (APB) of Section 7 that allowed us to construct the map  $f : \Psi(P) \rightarrow \Psi(M)$ . In particular, Proposition 7.1 gives us a partition,  $\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1$  such that  $f : \Theta(P) \rightarrow \Theta(M)$  is a proper lipschitz homotopy equivalence of the thick parts. (We remark that the partition of  $\mathcal{W}$  and hence the definition of  $\Theta(P)$  might depend on  $M$ , but that does not affect the logic of the argument.)

**Lemma 15.5 :** *The map  $f$  sends the positive (negative) end of  $\Psi(P)$  to the positive (negative) end of  $\Psi(M)$ .*

**Proof :** Let  $\Omega_i$  be a sequence of annuli tending out the positive end of  $\Psi(P)$ . Now  $f$  sends  $\partial T(\Omega_i)$  either to the associated geodesic  $\bar{\Omega}_i$  in  $\Psi(M)$ , or else to the boundary of the Margulis tube about  $\bar{\Omega}_i$ . In any case, since  $f$  is proper, the sequence  $\bar{\Omega}_i$  must go out an end,  $e$ , of  $\Psi(M)$ . By construction of  $\mathcal{W}$ , the homotopy classes of  $\Omega_i$  tend to  $a(e_+)$  in  $\mathcal{G} \cup \partial \mathcal{G}$  and so by the definition of end invariant (Proposition 15.1), we see that  $a(e) = a(e_+)$ , and so, by Lemma 15.1,  $e = e_+$ , as required.  $\diamond$

This proves the end consistency assumption (EC) of Section 10, and so  $f(\Psi(P)) = \Psi(M)$ .

We remark that we have all that is needed to show that the collection of Margulis tubes in  $\Psi(M)$  is unlinked. (We only need the constructions of Sections 11 and 13 for this.)

**Proof of Theorem 5.1 :** By Proposition 11.2, the set of tubes  $T_0(\bar{\Omega})$  are unlinked in  $\Psi(M)$ . But this includes all Margulis tubes with core curves less than some constant  $\eta$  depending only on  $\kappa(\Sigma)$ .  $\diamond$

We remark that, unlike [O2], this does not give us  $\eta$  explicitly in terms of  $\kappa(\Sigma)$ .

Also, the fact that  $f$  has degree 1 was all that was needed to get us to Proposition 10.41, and so we see that the map  $f : \Theta(P) \rightarrow \Theta(M)$  is uniformly universally sesquilipschitz.

For the moment,  $f$  is only defined topologically on each of the Margulis tubes in  $\mathcal{T}(P)$ . If  $T \in \mathcal{T}(P)$ , then we have a lipschitz map  $f : \partial T \rightarrow \partial T'$ .

**Lemma 15.6 :** *If  $T \in \mathcal{T}(P)$ , then  $f$  extends to a uniformly universally sesquilipschitz map,  $f : T \rightarrow T'$ .*

In other words, the extension,  $f$  is uniformly lipschitz and its lift to the universal covers,  $\tilde{f} : \tilde{T} \rightarrow \tilde{T}'$  is a quasi-isometry.

**Proof :** Let  $\tilde{\Theta}(P)$  and  $\tilde{\Theta}(M)$  be the universal covers of  $\Theta(P)$  and  $\Theta(M)$ , and let  $\hat{\Theta}(P) = \tilde{\Theta}(P)/H$  and  $\hat{\Theta}(M) = \tilde{\Theta}(M)/H$  be the covers corresponding to the subgroup  $H$  of  $G = \pi_1(\Theta(P)) \cong \pi_1(\Theta(M))$  generated by the longitude of  $T$ . We can identify  $\partial\tilde{T}$  and  $\partial\tilde{T}'$  with boundary components of  $\tilde{\Theta}(P)$  and  $\tilde{\Theta}(M)$  respectively. In the induced path metrics, they are euclidean cylinders whose longitudes have length uniformly bounded above and below.

By Proposition 10.11, the map  $\tilde{f} : \tilde{\Theta}(P) \rightarrow \tilde{\Theta}(M)$  is a lipschitz quasi-isometry, and so therefore is its projection,  $\hat{f} : \hat{\Theta}(P) \rightarrow \hat{\Theta}(M)$ . By Lemma 4.8,  $\partial\tilde{T}$  is quasi-isometrically embedded in  $\tilde{\Theta}(P)$ , and so we can conclude that  $\hat{f}|_{\partial\tilde{T}}$  is a quasi-isometry from  $\partial\tilde{T}$  to  $\partial\tilde{T}'$  in the induced euclidean path metrics.

We are therefore in the situation described by Lemma 8.4 and the subsequent remark. In particular, there is a universally sesquilipschitz homotopy from  $f|_{\partial T} : \partial T \rightarrow \partial T'$  to a bilipschitz homeomorphism  $g : \partial T \rightarrow \partial T'$ . By Lemma 8.8, such a map  $g$  extends to a bilipschitz homeomorphism  $g : T \rightarrow T'$ .

Now we can carry out the sesquilipschitz homotopy between  $f|_{\partial T}$  and  $g|_{\partial T}$  in a uniformly small neighbourhood of  $\partial T$  in  $T$ , and then use  $g$  to extend over  $T$ . This way, we extend  $f$  to a universally sesquilipschitz map  $f : T \rightarrow T'$ .

Performing this for each tube  $T \in \mathcal{T}(P)$  we get a lipschitz map  $f : \Psi(P) \rightarrow \Psi(M)$ . One can show this to be universally sesquilipschitz (cf. Proposition 15.8), but we are really interested in a further extension of  $f$  to the whole model space  $P$ . For this we still need to deal with the cusps.

Let  $R$  be a cusp of  $P$ , i.e. the closure of a component of  $P \setminus \Psi(P)$ . We have a corresponding cusp in  $R'$  in  $M$ , the closure of a component of  $M \setminus \Psi(M)$ . We have a proper lipschitz map  $f|_{\partial R} : \partial R \rightarrow \partial R'$ , between bi-infinite euclidean cylinders.

**Lemma 15.7 :** *The map then  $f|_{\partial R}$  extends to a uniformly universally sesquilipschitz map,  $f : R \rightarrow R'$ .*

**Proof :** The argument is similar to that for Lemma 15.6. Using Lemma 7.8 as before, we see that  $f|_{\partial R}$  is a uniform quasi-isometry to  $\partial R'$ . (In this case, we lift  $R$  rather than the universal cover of  $R$ .) Note that  $\partial R$  and  $\partial R'$  are both uniformly quasi-isometric to the real line, under horizontal projection. We can apply Lemma 8.1 directly to see that there is a bounded homotopy to a bilipschitz homeomorphism of the real line. Thus (as in Lemma 8.4) we get a universally sesquilipschitz homotopy from  $f|_{\partial R}$  to a bilipschitz homeomorphism  $g : \partial R \rightarrow \partial R'$ . (We use that fact that a lipschitz map from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  which sends horizontal lines to horizontal lines and is bilipschitz in the vertical direction must, in fact, be bilipschitz.) The extension of  $g$  over  $R$  is now trivial — just send rays isometrically to rays.  $\diamond$

Performing this for each cusp we get a proper lipschitz homotopy equivalence,  $f : P \rightarrow M$ .

**Proposition 15.8 :** *The map  $f : P \longrightarrow M$  is uniformly universally sesquilipschitz.*

**Proof :** In other words, we claim that the lift  $\tilde{f} : \tilde{P} \longrightarrow \tilde{M}$  is a uniform quasi-isometry. Since  $\tilde{f}$  is surjective, it is enough to put an upper bound on  $d(x, y)$  whenever  $d'(\tilde{f}(x), \tilde{f}(y)) < \eta$  for some fixed  $\eta > 0$ . But  $\tilde{P}$  and  $\tilde{M}$  are equivariantly decomposed into pieces, namely the lifts of thick parts, Margulis tubes, and cusps. We have shown that  $\tilde{f}$  respects this decomposition and that  $\tilde{f}$  restricted to each of the pieces is a uniform quasi-isometry. Moreover, we can assume that any two distinct pieces are distance at least  $\eta$  apart in  $\tilde{M}$ . The result now follows.  $\diamond$

We can summarise what we have shown as follows:

**Theorem 15.9 :** *Given two distinct  $a, b \in \partial\mathcal{G}$ , we can construct a complete riemannian manifold,  $P$ , homeomorphic to  $\text{int}(\Sigma) \times \mathbf{R}$ , such that if  $M$  is a doubly degenerate hyperbolic 3-manifold with base surface  $\Sigma$  and end invariants  $a, b$ , then there is a uniformly universally sesquilipschitz map from  $P$  to  $M$ .*  $\diamond$

Here “uniform” means that the constants depend only on  $\kappa(\Sigma)$ . (I don’t know if this dependence is computable.)

As a consequence we have:

**Theorem 15.12 :** *Suppose  $M, M'$  are doubly degenerate hyperbolic 3-manifolds with the same base surface and end invariants. Then  $M$  and  $M'$  are isometric.*

**Proof :** The argument is now standard. We can use the same model space  $P$  for both  $M$  and  $M'$ . The universally sesquilipschitz maps,  $P \longrightarrow M$  and  $P \longrightarrow M'$  give us an equivariant quasi-isometry between  $\tilde{M}$  and  $\tilde{M}'$ , both isometric to  $\mathbf{H}^3$ . This extends to an equivariant quasiconformal map  $\partial\mathbf{H}^3 \longrightarrow \partial\mathbf{H}^3$ . The result of Sullivan [Su] now tells us that this is fact conformal. Thus the two actions on  $\mathbf{H}^3$  are conjugate by an isometry of  $\mathbf{H}^3$ , and so  $M$  is isometric to  $M'$ .  $\diamond$

In other words, we have proven the Ending Lamination Theorem (Theorem 2.3) for doubly degenerate manifolds.

## 16. The proof of the main theorem in the indecomposable case.

In this section we describe how earlier arguments can be adapted to construct a model space for an indecomposable (orientable) complete hyperbolic 3-manifold. This will enable us to complete the proof of the Ending Lamination Theorem in that case. With some further modifications, the decomposable case can also be dealt with similarly. We begin with a topological discussion.

**Definition :** A 3-manifold  $\Psi$  with boundary,  $\partial\Psi$ , is *topologically finite* if we can embed  $\Psi$  in a compact 3-manifold,  $\bar{\Psi}$ , with boundary,  $\partial\bar{\Psi}$ , so that  $\partial\Psi$  is a subsurface of  $\partial\bar{\Psi}$ , and  $\bar{\Psi} = \Psi \cup \partial\bar{\Psi}$ .

In other words, we can compactify  $\Psi$  by adjoining the “ideal” boundary  $\partial_I\Psi = \partial\bar{\Psi} \setminus \partial\Psi$ . In fact, the topology of  $(\bar{\Psi}, \Psi)$  is determined by  $\Psi$ , though here we can regard  $\bar{\Psi}$  as part of the structure associated to  $\Psi$ .

Suppose that  $\Psi$  is topologically finite. In this case the existence of a Scott Core is elementary, and (in the indecomposable case, as here) it is equivalent to the following definition:

**Definition :** By a *core* of  $\Psi$  we will mean a compact submanifold,  $\Psi_0 \subseteq \Psi$ , such that  $\Psi \setminus \Psi_0$  is homeomorphic to  $\partial_I\Psi \times \mathbf{R}$ .

We write  $\partial_H\Psi_0$  for the relative, or *horizontal* boundary of  $\Psi_0$  in  $\Psi$ , and  $\partial_V\Psi_0 = \Psi_0 \cap \partial\Psi$  for the *vertical* boundary. Thus  $\partial\Psi_0 = \partial_H\Psi_0 \cup \partial_V\Psi_0$ . The ends of  $\Psi$  are in bijective correspondence with the components of  $\partial_H\Psi_0$ , and are just products.

**Definition :** An end  $e \cong \Sigma \times [0, \infty)$  is *incompressible* in  $\Psi$  if its inclusion into  $\Psi$  is  $\pi_1$ -injective.

**Definition :** We say that  $\Psi$  is *indecomposable* if all its ends are incompressible.

Via Dehn’s lemma, this is equivalent to saying that there is no disc in  $\bar{\Psi}$  whose boundary lies in  $\partial_I\Psi$ .

Now suppose that  $M$  is a complete orientable hyperbolic 3-manifold, and that  $\pi_1(M)$  is finitely generated. We shall also assume that  $M$  is not elementary, i.e. that  $\pi_1(M)$  is not abelian. Let  $\Psi(M)$  be the non-cuspidal part of  $M$ . Here we will be assuming that  $\Psi(M)$  is indecomposable, and so  $\Psi(M)$  is topologically finite by [Bon].

Let  $\Psi_0$  be a core for  $\Psi(M)$ . We note that each component of  $\partial\Psi(M)$  is either a bi-infinite euclidean cylinder, meeting  $\Psi_0$  in a compact annular component of  $\partial_V\Psi_0$ , or else is a torus and a component of  $\partial_V\Psi_0$ . These components bound  $\mathbf{Z}$ -cusps and  $\mathbf{Z} \oplus \mathbf{Z}$ -cusps respectively in  $M$ . We also note that no component of  $\partial_H\Psi_0$  is a disc or annulus.

We write  $\mathcal{E}(M)$  for the set of ends of  $\Psi(M)$ , and recall the partition of  $\mathcal{E}(M)$  as  $\mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$ , as discussed in Section 2. Let  $C = C(M)$  be the convex core of  $M$ , and let  $C(r)$  be its  $r$ -neighbourhood. Thus if  $r > 0$ ,  $\partial C(r)$  is a  $C^1$ -submanifold of  $M$ . If we choose  $r$  sufficiently large, then  $\Psi(M) \setminus \text{int } C(r)$  will consist of a disjoint union of geometrically finite ends. We can thus fix some  $r > 0$  and assume that each  $e \in \mathcal{E}_F(M)$  has this form. Let  $\partial_{He}$  be the relative boundary of  $e$  in  $\Psi(M)$ , let  $S$  be the component of  $\partial C(r)$  containing  $\partial_{He}$ , and let  $E$  be the component of  $M \setminus \text{int } C(r)$  with boundary  $\partial E = S$ . Thus  $e = E \cap \Psi(M)$  and  $\partial_{He} = S \cap \Psi(M)$ . We write  $\Psi(E) = E \cap \Psi(M) = e$ ,  $\partial_H\Psi(E) = \partial_{He}$  and  $\partial_V\Psi(E) = \Psi(E) \cap \partial\Psi(M)$ . Each component of  $\partial_V\Psi(E)$  is a euclidean half-cylinder and an end of a component of  $\partial\Psi(M)$ .

By Ahlfors’s finiteness theorem, each geometrically finite end has associated to it

a Riemann surface of finite type, which can be thought of as a geometrically finite end invariant. For the moment, however we will not be using this structure.

We now describe the main aim of this section. Suppose that  $\Psi$  is a topologically finite 3-manifold, with a decomposition of the ends,  $\mathcal{E} = \mathcal{E}_F \sqcup \mathcal{E}_D$ , so that no base surface of an end is a disc, annulus, sphere or torus, and no base surface in  $\mathcal{E}_F$  is a 3HS. Suppose that to each  $e \in \mathcal{E}_D$  there is associated an element,  $a(e)$ , in the boundary of the corresponding curve graph. We will associate to  $\Psi$ ,  $(a(e))_{e \in \mathcal{E}_D}$ , a “model” manifold  $P$ . This will be a complete riemannian manifold together with a submanifold,  $\Psi(P)$ , homeomorphic to  $\Psi$ , such that each component of  $P \setminus \Psi(P)$  is either a “standard”  $\mathbf{Z}$ -cusp or “standard”  $\mathbf{Z} \oplus \mathbf{Z}$ -cusp. (We could also include into  $P$  information about geometrically finite end invariants, but we will not worry about that for the moment.)

We will show:

**Theorem 16.1 :** *Let  $M$  be a tame indecomposable hyperbolic 3-manifold with non-cuspidal part  $\Psi(M)$ . Let  $P$  be the model manifold referred to above, constructed from  $\Psi(M)$ , the partition of its ends,  $\mathcal{E}(M) = \mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$  into geometrically finite and degenerate, and the collection  $(a(e))_{e \in \mathcal{E}_D(M)}$  of degenerate end invariants. Then there is a universally sesquipschitz map from  $P$  into  $M$ .*

Note that here we are no longer claiming that the constants of our sesquipschitz map are uniform. They might depend on the geometry as well as the topology of  $M$ . We suspect that some uniform statement could be made, but one would need to take into account the geometrically finite end invariants when constructing the model space. In any case, this would considerably complicate the construction.

The doubly degenerate case of Section 15, is where  $\Psi(M) \cong \Sigma \times \mathbf{R}$  and  $\mathcal{E}_F(M) = \emptyset$ . In this case, we can take the constants to depend only on  $\kappa(\Sigma)$ .

The basic idea is to construct a model for each end of  $\Psi(M)$ , and then put any riemannian metric on the core  $\Psi_0$ . The only requirement of the latter is that it should match up with the metric we have already on the boundaries of the model ends.

Let  $\Sigma$  be a compact surface. We want to associate to  $\Sigma$  a geometrically finite model  $P_\Sigma$ . Here we just give a very crude model that only depends on the topological type of  $\Sigma$ . A more sophisticated model, which takes into account an end invariant (Riemann surface) is described in [Mi4].

Let us fix any finite-area hyperbolic structure on  $\text{int } \Sigma$ . This is given by the quotient,  $\mathbf{H}^2/H$ , of a properly discontinuous action of  $H = \pi_1(\Sigma)$  on  $\mathbf{H}^2$ . We embed  $\mathbf{H}^2$  as a totally geodesic subspace of  $\mathbf{H}^3$  and extend the action to  $\mathbf{H}^3$ . Let  $P_\Sigma$  be the quotient of one of the half-spaces bounded by  $\mathbf{H}^2$ . We write  $\Psi(P_\Sigma)$  for the non-cuspidal part of  $P_\Sigma$ . Thus each component of  $\partial\Psi(P_\Sigma)$  is a euclidean half-cylinder, which (at least for notational convenience) we can assume to be isometric to  $S^1 \times [0, \infty)$ .

Note that  $P_\Sigma$  has a product structure as  $\partial P_\Sigma \times [0, \infty)$ , where the first co-ordinate of  $x \in P_\Sigma$  is the nearest point to  $x$  in  $\partial P_\Sigma$ , and the second co-ordinate,  $t = t(x)$ , is the distance of  $x$  from  $\partial P_\Sigma$ . Thus  $P_\Sigma$  is a warped riemannian product where the distances in the horizontal (constant  $t$ ) direction are expanded by a factor of  $\cosh t$ .

A geometrically finite end of  $M$  has qualitatively similar geometry. This is well un-

derstood. We only give an outline here.

Let  $e \in \mathcal{E}_F(M)$ , and let  $E, \Psi(E)$ , etc. be as defined above. Note that  $\partial E$  is component of  $\partial C(r)$ . Now  $E$  has a product structure  $E \cong \partial E \times [0, \infty)$  defined exactly as with  $P_\Sigma$ . In this case, the horizontal expansion at distance  $t$  from  $\partial E$  need not be constant, but will be bounded between two constants, namely,  $k_-(t) = \cosh(t+r)/\cosh(r)$  and  $k_+(t) = \sinh(t+r)/\sinh(r)$ . We note that the ratios of both  $k_-(t)$  and  $k_+(t)$  with  $\cosh t$  are bounded above and below (in terms of  $r$ ).

Now  $\partial E$  meets each  $\mathbf{Z}$ -cusp in a constant curvature cusp and so we can find a bilipschitz homeomorphism  $g : \partial P_\Sigma \rightarrow \partial E$ . We can now extend, using the product structures, to a homeomorphism,  $g : P_\Sigma \rightarrow E$ , which, by the above observations will also be bilipschitz. Unfortunately, this need not send  $\Psi(P_\Sigma)$  to  $\Psi(E)$ , though it is not hard to modify it so that it does. One way to describe this is as follows.

We fix some positive  $k < 1$  as described below, and choose  $g$  so that for each  $s \leq 1$  it sends any horocycle of length  $s$  in  $\partial P_\Sigma$  to a horocycle of length  $ks$  in  $\partial E$ . Now given any  $t \geq 0$ , the level  $t$  surfaces in  $P_\Sigma$  and  $E$  meet the  $\mathbf{Z}$ -cusps in cusps of constant curvature determined by  $t$  and  $r$ . Under the above construction,  $g$  will send a horocycle of length  $s \leq 1$  in such a surface in  $P_\Sigma$  to a horocycle in the corresponding surface in  $E$ , whose length is bounded above and below by fixed multiples of  $ks$ . By choosing  $k$  sufficiently small, we can assume that this length is always less than the Margulis constant. Thus,  $g$  sends each  $\mathbf{Z}$ -cusp in  $P_\Sigma$  to the corresponding  $\mathbf{Z}$ -cusp in  $E$ . We can now modify  $g$  by post-composing with projection of such a cusp in  $E$  to its boundary, using nearest point projection in the level surfaces in  $E$ . This projection will have bounded expansion on  $g(\Psi(E))$ . The resulting map  $f : \Psi(P_\Sigma) \rightarrow \Psi(E)$  is universally sesquilipschitz.

We have shown:

**Lemma 16.2 :** *If  $e \in \Psi(E)$  is a geometrically finite end of  $\Psi(M)$  with base surface  $\Sigma$ , then there is a universally sesquilipschitz map  $f : \Psi(P_\Sigma) \rightarrow \Psi(E)$ .*  $\diamond$

We next want to construct models for simply degenerate ends. We can use the following variation on Theorem 8.1. The proof is essentially the same, indeed a more direct application of Lemma 8.3. Given a compact surface  $\Sigma$ , write  $\Psi_+ = \Sigma \times [0, \infty)$  and  $\partial_H \Psi_+ = \Sigma \times \{0\}$ .

**Lemma 16.3 :** *Given a complete multicurve,  $\alpha$ , and some  $a \in \partial \mathcal{G}(\Sigma)$  we can find a complete annulus system  $W = \bigcup \mathcal{W} \subseteq \Psi_+$  with  $\pi_\Sigma(W \cap \partial_H \Psi_+) = \alpha$ , and satisfying the conditions (P1)–(P4) of Theorem 8.1.*  $\diamond$

We need to interpret condition (P1) which said that  $X(\mathcal{W}) \subseteq \bar{Y}^\infty(Y)$ . Here we can take  $Y$  to be the limit of the sets  $Y^\infty(X(\alpha) \cup \{\beta_i\})$  where  $\beta_i \in X(\Sigma)$  is some sequence converging to  $a$ . Here we are using the local finiteness properties of hierarchies (see Lemma 6.2), as we did in Section 15 (cf. Theorem 15.2).

Now let  $\mathcal{W}_I = \{\Omega \in \mathcal{W} \mid \Omega \cap \partial_H \Psi_+ = \emptyset\}$  and  $\mathcal{W}_\partial = \mathcal{W} \setminus \mathcal{W}_I$ . Let  $\Lambda(\mathcal{W})$  be the space obtained by opening out each annulus of  $\mathcal{W}$  as before. We have a natural map,  $p : \Lambda(\mathcal{W}) \rightarrow \Psi_+$ . Each  $\Omega \in \mathcal{W}_I$  gives us a solid torus,  $\Delta(\Omega)$ , and each  $\Omega \in \mathcal{W}_\partial$  gives us an annulus,  $A(\Omega)$  with boundary  $p^{-1}(\Omega \cap \partial_H \Psi_+)$ .

Now let  $\Psi(P_e) = \Lambda(\mathcal{W}, \mathcal{W}_I)$  be the space obtained by gluing in a solid torus,  $T(\Omega)$ , to each  $\Delta(\Omega)$  for  $\Omega \in \mathcal{W}_I$ . (We won't need to define a space  $P_e$ , but will write  $\Psi(P_e)$  for the sake of maintaining consistent notation.) We write  $\partial_H \Psi(P_e) = p^{-1}(\partial_H \Psi_+) \cup \bigcup_{\Omega \in \mathcal{W}_0} A(\Omega)$ . In other words, it consists of all the (3HS) components of  $\partial_H \Psi_+ \setminus \alpha$  connected by annuli  $A(\Omega)$ , so as to recover  $\Sigma$  up to homeomorphism. In fact,  $(\Psi(P_e), \partial_H \Psi(P_e)) \cong (\Sigma \times [0, \infty), \Sigma \times \{0\})$ .

We can now put a riemannian metric,  $d$ , on  $\Psi(P_e)$ , exactly as we did with  $\Psi(P)$ , by giving each  $T(\Omega)$  the structure of a Margulis tube. It also has a pseudometric,  $\rho$ , obtained by deeming each  $T(\Omega)$  to have diameter 0. Near the boundary,  $\partial_H \Psi(P_e)$ , these metrics may be a bit of a mess, but in a neighbourhood of the end of  $\Psi(P_e)$  they will have all of the properties, (W1)–(W9) laid out in Section 7.

We are now in a position to describe the model space,  $P$ . The only data we need is the topology of  $\Psi(M)$ , the partition of its ends as  $\mathcal{E}(M) = \mathcal{E}_F(M) \sqcup \mathcal{E}_D(M)$ , and the assignment of degenerate end invariants,  $(a(e))_{e \in \mathcal{E}_D(M)}$ .

Let  $\Psi_0(P)$  be a homeomorphic copy of the core,  $\Psi_0(M)$ , of  $\Psi(M)$ . We have a decomposition of its boundary into the horizontal and vertical parts,  $\partial \Psi_0(P) = \partial_H \Psi_0(P) \cup \partial_V \Psi_0(P)$ . For each  $e \in \mathcal{E}_F(M)$ , we take a copy  $\Psi(P_e) = \Psi(P_{\Sigma(e)})$  of the geometrically finite model, for the base surface  $\Sigma(e)$ , and glue  $\partial_H \Psi(P_e)$  to the corresponding component of  $\partial_H \Psi_0(P)$ . If  $e \in \mathcal{E}_D(M)$ , we take a copy of the degenerate model,  $\Psi(P_e) = \Psi(P_{a(e)})$  and again glue  $\partial_H \Psi(P_e)$  to the corresponding component of  $\partial_H \Psi_0(P)$ . This case involves making a choice of multicurve,  $\alpha \subseteq \Sigma(e)$ , to construct  $\Psi(P_{a(e)})$ . In principle we could take any multicurve, but to avoid some technical complications, we could take it so that no component of  $\alpha$  is homotopic in  $\Psi_0(P)$  into the vertical boundary,  $\partial_V \Psi_0(P)$  (i.e. so that no curve of  $\alpha$  ends up being an accidental parabolic). In this way, we have constructed a topological copy  $\Psi(P)$ , of  $\Psi(M)$ . We have already some riemannian metric on  $\partial_H \Psi_0(P)$ . The model ends were such that the boundary curves of  $\partial_H \Psi_0(P)$  all have unit length. Each component of  $\partial_V \Psi_0(P)$  is either an annulus bounded by two such curves, which we can take to be isometric to  $S(1) \times [0, 1]$ ; or else a torus, which we can take to be a unit square euclidean torus,  $S(1) \times S(1)$  (with any marking). This gives a riemannian structure to  $\partial \Psi_0(P)$ , which we extend to a riemannian metric on  $\Psi_0(P)$ . We can choose the metric in a neighbourhood of the boundary curves of  $\partial_H \Psi_0(P)$  so that the boundary,  $\partial \Psi(P)$ , is smoothly embedded in  $\Psi(P)$ .

Finally, to construct  $P$ , we note that each component of  $\partial \Psi_0(P)$  is either a square torus, in which case, we glue in a standard  $\mathbf{Z} \oplus \mathbf{Z}$ -cusp, or else a bi-infinite cylinder isometric to  $S(1) \times \mathbf{R}$  (made up from an annular component of  $\partial_V \Psi_0(P)$  together with the vertical boundary components of two model ends), in which case we glue in a standard  $\mathbf{Z}$ -cusp.

This gives us our model space,  $P$ . We now define a map  $f : P \rightarrow M$ , in a series of steps as follows.

First, for each  $e \in \mathcal{E}_F(M)$ , Lemma 16.2 gives us a universally sesquilipschitz map  $f : \Psi(P_e) \rightarrow \Psi(E) = e$ .

Now suppose that  $e \in \mathcal{E}_D(M)$ . We want to construct a map  $f : \Psi(P_e) \rightarrow \Psi(M)$ . This is best done by passing to the cover,  $\hat{\Psi}(M)$  of  $\Psi(M)$  corresponding to the end,  $e$ . Note that  $\hat{\Psi}(M) \subseteq \Psi(\hat{M})$ , where  $\hat{M}$  is the cover of  $M$  corresponding to  $e$ . (These need not be equal, since a cusp of  $M$  may open out in  $\hat{M}$ .) Now  $\hat{M}$  is a product manifold

with base surface  $\Sigma(e)$ , so that  $\Psi(\hat{M})$  is homomorphic to  $\Sigma \times \mathbf{R}$ , possibly with accidental parabolic cusps removed. In any case, the a-priori bounds theorem (Theorem 15.4) applies in this case. This means that if  $\Omega \in \mathcal{W}$  then  $l_M(\Omega)$  is bounded above in terms of  $\kappa(\Sigma)$ ,  $\max\{l_M(\delta) \mid \delta \in X(\alpha)\}$ , and the length bound in the definition of a simply degenerate end (Proposition 15.1). Here  $l_M$  denotes the length of the homotopic closed geodesic in  $M$ , or equivalently, in  $\hat{M}$ . If this happens to be parabolic, we set it equal to 0.

We are now in a position to apply the construction of Sections 6 and 7. This gives us a partition of  $\mathcal{W}_I$  as  $\mathcal{W}_0 \sqcup \mathcal{W}_1$ , and a map  $f : \Psi(P_e) \rightarrow \hat{M}$  which is lipschitz on the “thick part”,  $\Theta(P_e) = \Psi(P_e) \setminus \bigcup_{\Omega \in \mathcal{W}_0} \text{int } T(\Omega)$  but only, for the moment, defined topologically on the thin part — the union of the Margulis tubes  $T(\Omega)$  for  $\Omega \in \mathcal{W}_0$ .

There are a couple of minor complications in this procedure, which are most simply resolved by observing that we only need to have  $f$  defined geometrically on a neighbourhood of the end of  $\Psi(P_e)$  — any lipschitz extension to the remainder of  $\Psi(P_e)$  will do. The first complication is that some of the annuli in  $\mathcal{W}_I$  may correspond to accidental parabolics. The construction will still work in this case, but in any case, there are only finitely many such  $\Omega \in \mathcal{W}_I$ . Secondly we note that, by construction,  $f$  maps each component of  $\partial_V \Psi(P_e)$  the corresponding component of  $\Psi(\hat{M})$ , but it is still conceivable that  $f(\Psi(P_e))$  might enter other components of  $\hat{M} \setminus \hat{\Psi}(M)$ . As before,  $f$  is proper, and sends  $P_e$  out an end of  $\hat{\Psi}(M)$ , and this end cannot contain any such regions. This problem can therefore only arise in a compact subset of  $\Psi(P_e)$  and so can be fixed by the earlier observation. We now end up with a map to  $\hat{\Psi}(M)$ , which descends to a map  $f : \Psi(P_e) \rightarrow \Psi(M)$ .

Since  $f : \Psi(P_e) \rightarrow \Psi(M)$  is proper, it must send  $\Psi(P_e)$  out some end  $e'$  of  $\Psi(M)$ . If  $e \neq e'$ , then the corresponding base surfaces must be homotopic in  $\Psi(M)$ . Thus, applying Waldhausen’s cobordism theorem, we see that, in fact,  $\Psi(M)$  is just a product  $\Sigma \times \mathbf{R}$ , and so we are in the doubly degenerate situation dealt with in Section 15. We saw there that  $e \neq e'$  giving a contradiction. In other words, we have shown  $\Psi(P_e)$  must get sent out the corresponding end of  $\Psi(M)$ .

We now have  $f$  defined on each of the ends of  $\Psi(P)$  and hence on all of  $\partial_H \Psi_0(P)$ . We now extend to any lipschitz map of  $\Psi_0(P)$  into  $\Psi(M)$ , in the right homotopy class, such that each component of  $\partial_V \Psi_0(P)$  gets sent to the corresponding component of  $\Psi(M)$ .

This gives us a proper, end-respecting, homotopy equivalence  $f : \Psi(P) \rightarrow \Psi(M)$ , for the moment only defined topologically on the margulis tubes of the degenerate ends. In particular,  $f$  is surjective.

Suppose  $e \in \mathcal{E}_D(M)$ . Since  $f$  is proper, we can find a neighbourhood  $\Psi(M_e) \cong \Sigma(e) \times [0, \infty)$  of this end in  $\Psi(M)$  such that  $f^{-1}\Psi(M_e) \subseteq \Psi(P_e)$ . We can also find a neighbourhood,  $\Psi_1(P_e) \subseteq \Psi(P_e)$  so that  $f(\Psi_1(P_e)) \subseteq \Psi(M_e)$ . We can also assume that all the properties (W1)–(W9) of Section 4 hold in  $\Psi_1(P_e)$ . To understand this end, we are thus effectively reduced to considering the map  $f|_{\Psi_1(P_e)}$  into  $\Psi(M_e)$ . Since we only need to control the geometry of the map in some neighbourhood of the end, we can deem any finite set of Margulis tubes in  $\Psi(M_e)$ , and their preimages in  $\Psi_1(P_e)$ , to lie in the the respective “thick parts”. In particular, we can assume that  $\partial_H \Psi(M_e)$  and  $\partial_H \Psi_1(P_e)$  lie in the thick part, and that  $f(\partial_H \Psi_1(P_e))$  is homotopic in  $\Psi(M_e)$  to  $\partial_H \Psi(M_e)$ . Now all the arguments of Section 9 and 10 go through as before. For the pushing argument of Section 10, we need to assume that our points lie sufficiently far out the end, in order to

push our path into a band, but we only need to verify the sesquilipschitz property on some neighbourhood of the end.

We can thus extend  $f$  to a uniformly lipschitz map on each of the Margulis tubes in  $\Psi_1(P_e)$ , and we deduce that  $f|_{\Psi_1(P_e)}$  is universally sesquilipschitz to  $\Psi(M_e)$ . We can take  $f$  to be any lipschitz map in the right homotopy class in the remaining Margulis tubes in  $\Psi(P_e)$ .

We are now ready to show:

**Lemma 16.4 :** *The map  $f : \Psi(P) \longrightarrow \Psi(M)$  is universally sesquilipschitz.*

**Proof :** By construction,  $f$  is a proper lipschitz map. For each end  $e \in \mathcal{E}(M)$  we can choose any product neighbourhood,  $\Psi_1(M_e)$ , so any two distinct  $\Psi_1(M_e)$  are distance  $\eta > 0$  apart for some constant  $\eta > 0$ . If  $e \in \mathcal{E}_D(M)$  we can also take  $\Psi_1(M_e) \subseteq \Psi(M_e)$  as defined above. Let  $\Psi_1(P) \subseteq \Psi(P)$  be a core containing the preimage of an  $\eta$ -neighbourhood of  $\Psi(M) \setminus \bigcup_{e \in \mathcal{E}(M)} \Psi_1(M_e)$ . Since  $\Psi_1(M)$  is compact, the map  $f|_{\Psi_1(P)}$  is sesquilipschitz onto its range.

We want to show that the lift of  $f$  to the universal covers of  $\Psi(P)$  and  $\Psi(M)$  is a quasi-isometry. It is sufficient to bound the distance between two points in the domain that get sent to points at most  $\eta$  apart in the range. But this is now easy given that there are such bounds on each component of the lifts of  $\Psi_1(P)$  and each  $\Psi_1(P_e)$ .  $\diamond$

We finally need to define  $f : P \longrightarrow M$ . In other words, we need to extend  $f$  over each cusp  $R$  of  $P$ . Let  $R'$  be the corresponding cusp in  $M$ .

If  $R$  is a  $\mathbf{Z} \oplus \mathbf{Z}$ -cusp, we simply extend the bilipschitz homeomorphism,  $f|_{\partial R} : \partial R \longrightarrow \partial R'$  to a bilipschitz homeomorphism  $f : R \longrightarrow R'$  by sending each geodesic ray to a geodesic ray.

Suppose  $R$  is a  $\mathbf{Z}$ -cusp. Thus  $\partial R$  is a bi-infinite cylinder, and each of its ends is a vertical boundary components of a model end. For a geometrically finite model end, such a boundary component will be geodesically embedded. To deal with the general situation, we need to pass to the covers of  $\Psi(P)$  and  $\Psi(M)$  corresponding to  $\partial R$ . In a degenerate end, the same argument as Lemma 4.8 shows that its intersection with the lift of  $\partial R$  is quasi-isometrically embed in the lift of the end. Since the two ends lift to disjoint sets, it now follows that  $\partial R$  is quasi-isometrically embedded in the cover of  $\Psi(P)$ . It now follows that the map  $f|_{\partial R} : \partial R \longrightarrow \partial R'$  is a quasi-isometry with respect to the induced euclidean path metrics. We can thus extend  $f|_{\partial R}$  to a universally sesquilipschitz map  $f : R \longrightarrow R'$  exactly as in Section 15.

We have now defined  $f : P \longrightarrow M$ .

**Proof of Theorem 16.1 :** We know that  $f : P \longrightarrow M$  is a proper lipschitz homotopy equivalence, that it respects the decompositions of  $P$  and  $M$  into non-cuspidal parts and cusps, and that it is universally sesquilipschitz between the non-cuspidal parts and between corresponding cusps. It now follows easily that  $f$  is itself universally sesquilipschitz.  $\diamond$

As a consequence, we immediately get:

**Proposition 16.5 :** *Suppose that  $M, M'$  are complete indecomposable hyperbolic 3-manifolds and that there is a homeomorphism from  $M$  to  $M'$  that sends cusps of  $M$  into cusps of  $M'$  and conversely. Suppose that the induced map between the non-cuspidal parts sends each geometrically finite end to a geometrically finite end and each degenerate end to a degenerate end. Suppose that (under the induced homeomorphisms of base surfaces) the end invariants of corresponding pairs of degenerate ends are equal. Then there is an equivariant quasi-isometry between the universal covers of  $M$  and  $M'$ .*

**Proof :** We can use the same model manifold  $P$  for both  $M$  and  $M'$ . Theorem 16.1 tells us that there are universally sesquiplipschitz maps  $P \rightarrow M$  and  $P \rightarrow M'$ . The lifts then give us equivariant quasi-isometries between the universal covers.  $\diamond$

Note that if there are no geometrically finite ends, then Sullivan's theorem tells us that  $M$  and  $M'$  are isometric, exactly as in Theorem 15.12.

In general we need to take account of the geometrically finite end invariants:

**Theorem 16.6 :** *Let  $M, M'$  be as in Proposition 16.5, and assume, in addition that the corresponding geometrically finite end invariants are also equal. Then the homeomorphism between  $M$  and  $M'$  is homotopic to an isometry.*

One way to prove Theorem 16.6 would be to construct a model space using geometrically finite model ends that take account of the end invariants as in [Mi4]. In this case, one would show that the quasiconformal extension of the quasi-isometry given by Proposition 16.5 would be conformal. This is the approach taken in [BroCM1].

Given Proposition 16.5 as stated, one can also proceed as follows.

Write  $M = \mathbf{H}^3/\Gamma$  and  $M' = \mathbf{H}^3/\Gamma'$ , where  $\Gamma \cong \pi_1(M) \cong \pi_1(M') \cong \Gamma'$ , and write  $D(\Gamma)$  and  $D(\Gamma')$  for the discontinuity domains. By Ahlfors's finiteness theorem,  $D(\Gamma)/\Gamma$  and  $D(\Gamma')/\Gamma'$  are (possibly disconnected) Riemann surfaces of finite type. Our indecomposability assumption tells us that each component of either discontinuity domain is a disc.

Now Proposition 16.5 gives us a quasi-isometry from  $\mathbf{H}^3$  to itself which is equivariant with respect to these actions. This extends to an equivariant quasiconformal map,  $f : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$ . This maps  $D(\Gamma)$  to  $D(\Gamma')$  and descends to a quasiconformal map  $\bar{f} : D(\Gamma)/\Gamma \rightarrow D(\Gamma')/\Gamma'$ . Since the geometrically finite end invariants are equal,  $\bar{f}$  is homotopic to a conformal map  $\bar{g} : D(\Gamma)/\Gamma \rightarrow D(\Gamma')/\Gamma'$ . We can now lift  $\bar{g}$  to an equivariant conformal map  $g : D(\Gamma) \rightarrow D(\Gamma')$ . We set  $g$  to be equal to  $f$  on the limit sets. We thus get an equivariant bijection  $g : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$ , which is conformal on the discontinuity domains and a homeomorphism of limit sets.

If we knew that  $g$  were quasiconformal, we would see that it was conformal, and hence be finished. However, it is not immediately clear even that  $g$  is continuous. We are saved by the following:

**Lemma 16.7 :** *Suppose that  $U \subseteq \mathbf{C}$  is a proper simply connected domain. Suppose that  $f : U \rightarrow U$  moves each point a bounded distance with respect to the Poincaré*

metric. Then the extension of  $f$  by the identity of  $\mathbf{C} \setminus U$  is continuous. Moreover, if  $f|_U$  is quasiconformal, then its extension is uniformly quasiconformal.

Here of course, the ‘‘Poincaré metric’’ refers to the unique complete curvature  $-1$  metric in the conformal class.

**Proof :** Write  $d_e$  for the euclidean metric. Suppose that  $z \in U$  and  $d(z, \partial U) = r$ . Using the Koebe quarter theorem to compare with the Poincaré metric on the disc of radius  $r$  centred at  $z$ , we get the well known estimate  $|ds| \geq \frac{2}{r}|dz|$ , where  $|ds|$  is the infinitesimal Poincaré metric on  $U$ . Integrating, we deduce that if  $z, w \in U$  are distance at most  $k$  apart in the Poincaré metric, then  $d_e(z, w) \leq (e^{2k} - 1) \max\{d_e(z, \partial U), d_e(w, \partial U)\}$ . Continuity of  $g$  at  $\partial U$  now follows easily. The fact that it is quasiconformal follows, for example, using the above estimate to control the metric quasiconformal distortion of  $g$  on  $\partial U$ .  $\diamond$

**Proof of Theorem 16.6 :** Let  $h = g^{-1} \circ f : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$ . By construction,  $h$  is  $\Gamma$ -equivariant, and quasiconformal on  $D(\Gamma)$  and the identity on the limit set. Let  $U$  be a component of  $D(\Gamma)$ , conformally a disc. Now it is well-known that a quasiconformal map of the disc is a quasi-isometry of the Poincaré metric. (This is based on the fact that the modulus of any embedded annulus that separates two points from infinity is bounded above in terms of the hyperbolic distance between them.) It thus extends to a homeomorphism of the ideal boundary. Since the map  $h$  is equivariant with respect to a finite co-area action, it follows that it must be the identity on the ideal boundary, and hence moves every point a bounded distance in the Poincaré metric. Thus by Lemma 16.7 it extends to a continuous map that is the identity on the boundary  $\partial U$  of  $U$  in  $\partial\mathbf{H}^3$ . Now since this holds for every such component, Lemma 16.7 tells us that  $h$  is continuous and quasiconformal on  $\partial\mathbf{H}^3$ . Since  $f$  is quasiconformal, it follows that  $g$  is quasiconformal. But it is conformal on  $D(\Gamma)$  and hence, applying Sullivan’s result [Su] (see Theorem 3.8) it is conformal everywhere. Thus there is a hyperbolic isometry conjugating the  $\Gamma$  action to the  $\Gamma'$  action.  $\diamond$

We finish with a few remarks. A consequence of tameness is the Ahlfors conjecture which says that either the limit set is all of  $\partial\mathbf{H}^3$  or else has measure 0 (see [Can]). In the latter case (namely where there is at least one geometrically finite end) Sullivan’s theorem is redundant. We are only using the fact that a quasiconformal map that is conformal almost everywhere is conformal everywhere. Also, it is conjectured, but not proven in general, that the limit set is locally connected. In this case,  $\partial U$ , is naturally the continuous image of its ideal boundary, which simplifies the argument somewhat.

## 17. Compressible ends.

We now move on to the general case of the Ending Lamination Theorem.

There are several new issues to be addressed, though most are relatively straightforward. First, the issue of marking of end invariants in this case is more subtle. We have already discussed this in Section 2. Also there is some adjustment to be made when ap-

plying Sullivan's Theorem to complete the argument. This will be discussed in Section 24.

The main issue, however, arises from the fact that we cannot simply reduce to surface groups by lifting to an appropriate cover. As a consequence, many results we used before cannot be quoted directly. Instead we have to find a geometric means of "isolating" the ends of our manifold. We can then observe that the relevant techniques can be generalised. (In fact, our discussion here applies equally well to the incompressible case.)

We collect together here a few facts which aim to prove, as well as Lemma 17.3 which will be used later.

Let  $M$  be any complete hyperbolic 3-manifold with  $\pi_1(M)$  finitely generated, so that  $\Psi(M)$  is topologically finite. Let  $e \in \mathcal{E}(M)$ . The geometrically finite case will be very similar to that dealt with earlier. So we assume for the moment that  $e \in \mathcal{E}_D(M)$ . Write  $\Sigma = \Sigma(e)$ .

We know that every 3HS end is geometrically finite, so  $\kappa(\Sigma) \geq 1$ . If  $\kappa(\Sigma) = 1$ , then  $\Sigma$  is a 4HS or 1HT. In this case,  $e$  is necessarily incompressible. This is a fairly simple observation, given that the peripheral curves correspond to cusps. For this case we can therefore appeal to earlier results. Henceforth, we assume that  $\kappa(\Sigma) \geq 2$ .

By the definition we gave, every neighbourhood of  $e$  meets some closed geodesic in  $M$ . In fact, it is a consequence of tameness that every neighbourhood  $E \cong \Sigma \times [0, \infty)$  of  $e$  contains a closed geodesic of  $M$ , which is homotopic to a simple closed curve of  $\Sigma$ . (See [Th,Bon,Cana] for example.) We refer to such curves as "simple".

Based on this (or one of a number of equivalent statements) we will show:

**Proposition 17.1 :** *There is some constant,  $L_0 = L_0(\kappa(\Sigma))$  depending only on  $\kappa(\Sigma)$  such that there is a geodesic ray  $(\gamma_i)_{i \in \mathbf{N}}$  in  $\mathcal{G}(e)$  such that for all  $i$ ,  $\gamma_i$  is represented by a closed curve in  $e$  of length at most  $L_0$ .*

In fact, as we shall see, we can take each of these representatives to be a closed geodesic in  $M$ .

We fix some such constant  $L_0 = L_0(\kappa)$ , and make the following (somewhat artificial) definition:

**Notation :** Let  $\mathbf{a}(e) \subseteq \partial\mathcal{G}(e)$  be the set of points  $a \in \partial\mathcal{G}(e)$  such that there is some geodesic sequence  $(\gamma_i)_i$  in  $\mathcal{G}(e)$  satisfying the condition of Lemma 17.1 for some fixed  $L_0 = L_0(\kappa(\Sigma))$  and tending to  $a$ .

Thus Proposition 17.1 tells us that  $\mathbf{a}(e)$  is non-empty. We will later that  $\mathbf{a}(e)$  is a singleton (Proposition 23.3), and give some other, more natural, descriptions of this element.

To begin with, we will show the following:

**Proposition 17.2 :** *There is some constant,  $L = L(\kappa(\Sigma))$  depending only on  $\kappa(\Sigma)$  such that if  $(\gamma_i)_{i \in \mathbf{N}}$  is any tight geodesic ray in  $\mathcal{G}(e)$  tending to some element of  $\mathbf{a}(M, e)$ , then, for all sufficiently large  $i$ ,  $\gamma_i$  is represented by a curve of length at most  $L$  in  $e$  (which can*

be taken to be geodesic in  $M$ ).

These results will be proven in Section 22. In the incompressible case, Proposition 17.2 is a variant of the “a-priori bounds” theorem of Minsky [Mi2], reproven in this form, in [Bow2]. We shall explain how the arguments of [Bow2] can be adapted to this situation. This result, or rather its refinement described at the end of Section 7, are key to establishing that the map from the model space to the hyperbolic 3-manifold is lipschitz. From that point on, only a few minor modifications of the proof given on [Bow3] are required. These are discussed in Section 23.

In order to reduce to the incompressible case, we will need a purely topological observation, which we can give at this point.

**Lemma 17.3 :** *Suppose that  $N$  is a 3-manifold with boundary,  $\partial N$ , and that  $F \subseteq \partial N$  is a compact subsurface such that no boundary component of  $F$  is homotopically trivial in  $N$ . Suppose that  $F$  can be homotoped in  $N$  into some closed subset  $K \subseteq N \setminus F$ . Then  $F$  is  $\pi_1$ -injective in  $N \setminus K$ .*

**Proof :** We begin with a preliminary observation. Suppose that  $S \subseteq \partial N$  is a compact subsurface homotopic to a point in  $N$ . Then  $S$  is planar (i.e. genus 0). This can be shown by using Dehn’s Lemma [He] to cut  $S$  into 3-holed spheres glued along boundary curves which bound disjoint embedded discs in  $N$ . These must be connected in a treelike fashion, since any closed cycle of adjacencies would give rise to curve in  $S$  that is non-trivial in  $N$ . In other words,  $S$  is planar.

Now suppose, for contradiction, that  $\pi_1(F)$  does not inject into  $\pi_1(N \setminus K)$ . After lifting to the cover of  $N$  corresponding to  $F$ , we can assume that  $F$  carries all of  $\pi_1(N)$ . (Note that the homotopy of  $F$  into  $K$  also lifts, as does any disc spanning a non-trivial curve in  $F$ .)

By Dehn’s Lemma, there is an embedded disc  $D \subseteq N \setminus K$  with  $\partial D = D \cap \partial N = D \cap F$  a non-trivial curve in  $F$ . Let  $U$  be a small open product neighbourhood of  $D$  in  $N \setminus K$  so that  $U \cap \partial N = U \cap F$  is a small annular neighbourhood of  $\partial D$  in  $F$ . Let  $P = N \setminus U$ .

If  $P$  is connected then it determines a splitting of  $\pi_1(N)$  as a free product  $\pi_1(N) \cong \pi_1(P) * \mathbf{Z}$ . But  $F$  is homotopic into  $K \subseteq P$ , and so  $\pi_1(N)$  is conjugate into the  $\pi_1(P)$  factor giving a contradiction.

Thus  $P$  has two components,  $P_0$  and  $P_1$ . We can suppose that  $K \subseteq P_1$ . Now  $\pi_1(N) \cong \pi_1(P_0) * \pi_1(P_1)$  and as above,  $\pi_1(N)$  is conjugate into  $\pi_1(P_1)$ . Thus  $\pi_1(P_0)$  is trivial, and so in particular  $P_0 \cap F$  is homotopic to a point in  $N$ . It follows by the observation of the first paragraph, that  $P_0 \cap F$  is planar. Since  $\partial D$  is non-trivial in  $F$ ,  $P_0 \cap F$  cannot be a disc. It must therefore have another boundary component, say  $\beta \subseteq P_0 \cap \partial F$ . But, by hypothesis,  $\beta$  is non-trivial in  $N$ , giving a contradiction.  $\diamond$

We shall apply Lemma 17.3 in the following form. Suppose that  $e \cong \Sigma \times [0, \infty)$  is an end of  $\Psi(M)$ . We shall write  $\partial_H E = \Sigma \times \{0\}$  for the relative or *horizontal* boundary of  $E$  in  $\Psi(M)$ . This determines a proper homotopy class of proper maps of  $\text{int } \Sigma$  into  $M$ , called the *ambient fibre class* — “ambient” referring to homotopies that take place in  $M$ . Suppose that  $\phi : \text{int } \Sigma \rightarrow M$  is a proper map in this class, and suppose moreover that

$\phi^{-1}\Psi(M)$  is compact. We can shrink  $E$  (i.e. replace it by  $\Sigma \times [t, \infty)$  for some  $t \geq 0$ ) so that  $e \cap \phi(\text{int}(\Sigma)) = \emptyset$ . Let  $U$  be the component of  $M \setminus \phi(\text{int}(\Sigma))$  containing  $e$ . Then  $e$  is  $\pi_1$ -injective in  $U$ .

This can be seen, for example, by taking  $N = M \setminus \text{int}(e)$ , so that  $\partial N = \partial E = \partial_H E \cup \partial_V E$ , and setting  $F = \partial_H E$ . Note that  $M$  retracts onto  $N$ , and so  $F$  is homotopic into  $K = \phi(\text{int} \Sigma)$  in  $N$ . We now apply Lemma 17.3, to see that  $F$  and hence  $e$  is  $\pi_1$ -injective in  $U$  as claimed.

## 18. Pleating surfaces.

The original notion of a “pleated surface” surface goes back to Thurston [Th] (see [CanaEG] for a more detailed discussion). The notion is quite robust in that many of the basic properties survive some tinkering with the definition. We shall find it convenient to use more than one formulation in this paper. In this section, we restrict attention to maps into a hyperbolic 3-manifold, though much of it can be interpreted more generally as we explain in Section 19. Since our definition does not quite coincide with the standard one, we use the term “pleating surface” instead.

Let  $(M, d)$  be a complete hyperbolic 3-manifold. We fix a Margulis constant and let  $\Psi(M)$  be the non-cuspidal part. Let  $\Theta(M) \subseteq \Psi(M)$  be the *thick part* of  $M$ , i.e.  $\Psi(M)$  with open Margulis tubes removed. (For some applications, it is convenient to allow the Margulis constants defining different tubes to vary between two suitably chosen positive constants, as in [Bow3] for example. This makes no essential difference to what we have to say, and would only confuse the exposition here.)

Given  $x, y \in \Psi(M)$ , let  $\rho(x, y)$  be the electric pseudometric on  $\Psi(M)$ , that is, the minimum length of  $\beta \cap \Theta(M)$  as  $\beta$  varies over all paths from  $x$  to  $y$  in  $\Psi(M)$ .

Let  $\Sigma$  be a compact surface and let  $S = \text{int} \Sigma$ . We say that a homotopy class of maps from  $\Sigma$  into  $M$  (or equivalently  $S$ ) into  $M$  is *type-preserving* if it sends each boundary component of  $\Sigma$  (or equivalently each end of  $S$ ) homotopically to a generator of a  $\mathbf{Z}$ -cusp of  $M$ . We refer to this as the *associated cusp*. We remark that if two proper type preserving maps of  $S$  are homotopic then they are, in fact, properly homotopic.

**Definition :** A *pleating surface* is a uniformly lipschitz type preserving map  $\phi : (S, \sigma) \longrightarrow (M, d)$ , where  $\sigma$  is a finite area hyperbolic metric on  $S$ .

By “uniformly lipschitz” we mean  $\mu$ -lipschitz for some  $\mu \geq 0$ . In many situations, as in the traditional notion of pleated surface, one can take  $\mu = 1$ , though we want to allow for a larger constant depending only on  $\kappa(\Sigma)$ . The metric  $\sigma = \sigma_\phi$  is regarded as part of the data of the pleating surface. Note that  $\phi$  is necessarily proper.

In practice, the pleating surfaces we deal with will all have the property that any ray going out a cusp of  $S$  will, from a certain point on, get sent to a ray going out the associated cusp of  $M$ .

Given some  $\alpha \in X(\Sigma)$ , we write  $\alpha_S$  for its geodesic realisation in  $(S, \sigma)$ . Similarly, we write  $\beta_M$  for the geodesic realisation of a homotopy class,  $\beta$ , of closed curves in  $M$  —

that is assuming  $\beta$  is non-trivial and non-parabolic. If we fix a homotopy class of type-preserving maps  $\phi : S \rightarrow M$ , we write  $\alpha^* = (\phi(\alpha))^*$ . The notation is taken to imply that this exists.

**Definition :** A pleating surface,  $\phi : S \rightarrow M$  is said to *realise*  $\alpha \in X(\Sigma)$  if  $\phi|_{\alpha_S}$  maps  $\alpha_S$  locally isometrically and with degree  $\pm 1$  to  $\alpha^*$ .

By a *multicurve* in  $\Sigma$  we mean a non-empty (finite) set of elements of  $X(\Sigma)$  which can be realised disjointly in  $\Sigma$ . We say that a pleating surface *realises* a multicurve if it realises every component thereof.

**Lemma 18.1 :** *Suppose we are given a type-preserving homotopy class,  $\phi$ , from  $S$  into  $M$ , and a multicurve,  $\gamma$ , in  $\Sigma$ , such that the  $\phi$ -image of component of  $\gamma$  is non-trivial and non-parabolic in  $M$ . Then there is a pleating surface in this class realising  $\gamma$ .  $\diamond$*

This is a standard construction, due to Thurston, which we shall discuss shortly. First, we shall consider some consequences, and related results.

The following are now fairly routine observations. Suppose that  $\phi : S \rightarrow M$  is a  $\pi_1$ -injective pleating surface. Then  $S$  with the preimages of the associated cusps removed has a *principal component*,  $F$ , which carries all of  $\pi_1(\Sigma)$ . In fact, by choosing the Margulis constant of  $M$  sufficiently small in relation to  $\kappa(\Sigma)$ , we can assume that  $F$  contains each horocycle of length 1. Now it is well known that any simple geodesic in  $S$  cannot cross any such horocycle, and so must lie inside  $F$ . As an immediate consequence, applying Lemma 18.1, we have:

**Lemma 18.2 :** *Suppose we have a  $\pi_1$ -injective homotopy class,  $\phi : S \rightarrow M$ , and  $\alpha \in X(\Sigma)$  such that  $\phi(\alpha)$  is non-parabolic in  $M$ . Then the closed geodesic in  $M$  in the class of  $\phi(\alpha)$  cannot enter any of the cusps of  $M$  associated to  $\phi$ .  $\diamond$*

We will be applying this principle in a slightly different form — see Lemma 20.4.

Returning to our  $\pi_1$ -injective pleating surface  $\phi$ , another observation is that each component of  $\phi^{-1}\Theta(M)$  has bounded diameter. It follows easily that the  $\rho$ -diameter of  $\phi(F)$  is bounded. In fact:

**Lemma 18.3 :** *There is some  $r = r(\kappa(\Sigma))$  such that if  $\phi : S \rightarrow M$  is a  $\pi_1$ -injective pleating surface whose image,  $\phi(S)$ , only meets the associated cusps. Then the  $\rho$ -diameter of  $\phi(S) \cap \Psi(M)$  is at most  $r$ .*

**Proof :** In this case, the principal component,  $F$ , is a component of  $\phi^{-1}\Psi(M)$ , and we have seen that  $\phi(F)$  has bounded  $\rho$ -diameter. Any other components will be homotopic into an associated cusp. These are easily dealt with, but in practice we won't need to worry about them.  $\diamond$

Pleated surfaces in general this form are somewhat awkward to deal with. We can perform a “tidying up” operation. At the cost of increasing the Lipschitz constant by a bounded amount, we can assume that the principal component,  $F$ , is the same as  $\phi^{-1}\Psi(M)$

and that its boundary consists entirely of horocycles of length 1. From this point on, we are only interested in the metric restricted to  $F$ , which we can identify with the original surface  $\Sigma$ .

We will also require a somewhat deeper result concerning pleating surfaces, namely a version of the Uniform Injectivity Theorem, the original being due to Thurston.

Let  $E \rightarrow M$  be the projectivised tangent bundle of  $M$ . (We can think of this as the unit tangent bundle factored by the direction reversing involution.) If  $\phi : S \rightarrow M$  is a pleating surface realising a multicurve  $\gamma$ , we can lift  $\phi|_{\gamma_S}$  to a simple curve  $\gamma_E \subseteq E$ . We write  $\psi = \psi_\phi : \gamma_S \rightarrow \gamma_E$  for the lift of  $\phi|_{\gamma_S}$ . Thus,  $\psi$  is a homeomorphism and a local isometry with respect to the metrics  $\sigma$  and  $d_E$ .

**Lemma 18.4 :** *Given  $\kappa, \mu, \eta, \epsilon > 0$ , there is some  $\delta > 0$  with the following property. Suppose that  $S = \text{int } \Sigma$  is a surface with  $\kappa(\Sigma) = \kappa$ . Suppose that  $\phi : S \rightarrow M$  is a  $\mu$ -lipschitz pleating surface and let  $\psi : \gamma_S \rightarrow \gamma_E \subseteq E$  is the lift described above. Suppose that there is some  $\eta > 0$  such that the injectivity radius of  $M$  at each point of  $\gamma_M = \phi(\gamma_S)$  is at least  $\eta$ . Suppose moreover that there is a map  $\theta : N(\gamma_M, \eta) \rightarrow S$  such that  $\theta \circ \phi : N(\gamma, \eta/\mu) \rightarrow S$  is homotopic to the inclusion of  $N(\gamma, \eta/\mu)$  into  $S$ . If  $x, y \in \gamma_S$  with  $d_E(\psi(x), \psi(y)) \leq \delta$ , then  $\sigma(x, y) \leq \epsilon$ .*

Here  $N(\cdot, r)$  denotes the open  $r$ -neighbourhood. The map  $\theta$  need only be defined up to homotopy.

Lemma 18.4 is an immediate consequence of the statement for laminations given as Proposition 24.1, and we postpone the discussion until then.

We now go back to discuss some constructions of pleating surfaces that will be needed later.

Suppose that  $\phi : S \rightarrow M$  is a type-preserving homotopy class (not necessarily  $\pi_1$ -injective) and  $\alpha \in X(\Sigma)$  so that  $\phi(\alpha)$  is non-trivial and non-parabolic. We realise  $\alpha$  as some smooth curve, and choose any  $x \in \alpha$ , and extend this to an ideal triangulation of  $S$ , whose edges are loops based at  $x$  (including  $\alpha$ ) as well as properly embedded rays going out the cusps. We now chose any  $y \in \alpha^* = \phi(\alpha_S)$  and realise all these edges as geodesic loops or rays based at  $y$  (so that  $x$  gets sent to  $y$ , and  $\alpha$  to  $\alpha^*$ ). We then extend to  $S$  by sending 2-simplices homeomorphically to totally geodesic triangles in  $M$ . We pull back the metric to  $S$  to give a pseudometric  $\sigma$  on  $S$ . The realisation  $\phi : S \rightarrow M$  is then 1-lipschitz. This is the construction used by Bonahon in [Bon]. To obtain a hyperbolic metric on the domain, we need to adjust this somehow. One way is to use the ‘‘spinning’’ construction of Thurston. Note that there is a real line’s worth of possibilities for  $y$  (that is, taking account of the based homotopy class of our realisation  $(S, x) \rightarrow (M, y)$ ). By sending  $y$  off to infinity we converge on a pleated surface in the traditional sense. We note that, in fact, the same argument can be applied to any multicurve as in the hypotheses of Lemma 18.1.

Returning to Bonahon’s construction, we shall refer to a surface arising in this way as a *folding surface*. The essential point is that it is piecewise totally geodesic. This is sufficient for many purposes, but in order to ensure we get a metric on the domain, we need another assumption.

**Definition :** We say that a folding surface is *non-degenerate* if no 2-simplex in  $S$  gets collapsed to a geodesic (or point) in  $M$ .

In this case, the pull-back pseudometric is a metric. In fact it is hyperbolic apart from a singularity of angle at least  $2\pi$  at the vertex  $x$ .

To ensure that we can arrange that our folding surface is non-degenerate, we allow ourselves to move  $y$ . It is not hard to see that any given triangle can be degenerate for at most a discrete set of  $y$ , and so these points can all be avoided by a small perturbation.

In summary, we have shown the following variation on Lemma 18.1 (which we only need and state for a single curve).

**Lemma 18.5 :** *Suppose we have a type-preserving homotopy class of maps from  $S$  into  $M$  and that  $\alpha \in X(\Sigma)$  whose image in  $M$  is non-trivial and non-parabolic. Then there is an non-degenerate folding surface realising  $\alpha$ .  $\diamond$*

Let  $\Delta_y(M)$  be the unit tangent space to  $M$  at  $y$ , which we can identify with the unit sphere. Suppose we have a non-degenerate piecewise totally geodesic map  $\phi : S \rightarrow M$ . Then any point  $z \in S$  determines a closed polygonal path,  $\zeta(z)$ , in  $\Delta_{\phi(z)}(M)$ .

**Definition :** We say that  $\phi$  is *balanced* if for all  $x \in S$ ,  $\zeta(x)$  is not contained in any open hemisphere.

Note that this is only an issue at the vertices of the triangulation of  $\Sigma$ . At the vertex  $x$  of a folding surface,  $\zeta(x)$  contains two antipodal points coming from  $\alpha^*$ . Therefore:

**Lemma 18.6 :** *Any non-degenerate folding surface (of the type featuring in Lemma 18.5) is balanced.  $\diamond$*

## 19. Negatively curved spaces.

We will need to consider a notion of pleating surface in a broader context than that discussed in Section 18, in particular when the range is “negatively curved”, with upper curvature bound  $-1$ . Morally, having concentrated negative curvature can only work in our favour, though it introduces a number of technical issues that need to be addressed. Most of what need can be phrased in terms of locally  $\text{CAT}(-1)$  metrics, though in practice, all our metrics will be at least piecewise riemannian. In particular, we have a natural notion of area. We remark that  $\text{CAT}(-1)$  geometry has been used in a related context in [So], and also applied in [Bow7].

Let  $(R, d)$  be a metric space. A (*global*) *geodesic* in  $R$  can be defined as a path whose rectifiable length equals the distance between its endpoints. We say that  $(R, d)$  is a *geodesic space* if every pair of points can be connected by a geodesic. We can also define a *local geodesic* in the obvious manner, which for a riemannian metric coincides

(up to parameterisation) with the riemannian notion. We shall abbreviate “closed local geodesic” to “closed geodesic” since there can be no confusion in that case.

For any  $k \in \mathbf{R}$ , we have the notion of a “CAT( $k$ )” (or “locally CAT( $k$ )”) space which satisfies the CAT( $k$ ) comparison axiom globally (or locally). We refer to [BriH] for a detailed account of such spaces. The Cartan-Hadamard Theorem in this context says that if  $k \leq 0$ , then a locally CAT( $k$ ) space is globally CAT( $k$ ) if and only if it is simply connected. For a proper (complete locally compact) CAT( $-1$ ) space we have the usual classification of isometries into elliptic, parabolic and loxodromic. We are only interested here in discrete torsion-free groups, so there are no elliptics. This gives rise to the following “thick-thin” decomposition.

Let  $(R, d)$  be a proper locally CAT( $-1$ ) space and  $\eta > 0$ . Let  $\tau(R)$  be the set of  $x$  such that  $x$  lies in a homotopically non-trivial curve  $\gamma$  of length less than  $\eta$ . We write  $\tau_0(R) \subseteq \tau(R)$  for the set of  $x$  such that some such  $\gamma$  can be homotoped in  $R$  to be arbitrarily short. If  $\alpha$  is a closed geodesic, we write  $\tau(R, \alpha) \subseteq \tau(R)$  for the set of  $x$  such that some such  $\gamma$  can be homotoped to some multiple of  $\alpha$ . We write  $\tau_+(R)$  for the union of all  $\tau(R, \alpha)$  as  $\alpha$  varies over all closed geodesics in  $R$ . One can show that all of these sets are open, and that  $\tau(R) = \tau_0(R) \cup \tau_+(R)$ . (Without a lower curvature bound, these sets need not be disjoint.) We write  $\Psi(R) = \Psi_\eta(R) = R \setminus \tau_+(R)$  and  $\Theta(R) = \Theta_\eta(R) = R \setminus \tau(R)$ . (If  $R$  is a complete hyperbolic 3-manifolds, these sets agree with those already defined.)

Suppose that  $S = \text{int } \Sigma$  is a finite type surface with a complete locally CAT( $-1$ ) metric. We say that  $S$  has *finite area* if  $\Psi_\eta(S)$  is compact for some (hence any)  $\eta > 0$ .

An example of such a surface would be a riemannian metric of curvature at most  $-1$ , possibly with a discrete set of cone points of angles at least  $2\pi$ . More generally we can allow a piecewise riemannian metric which is riemannian outside an embedded 1-complex which has cone angles at the vertices and where the edges are smooth and outwardly curved with respect to the metric on each side. Note that in this situation, the notion of “finite area” coincides with the usual one.

We remark that we can also allow for  $S$  to have boundary components. In this case, there need be no constraint on the geometry at the boundary.

We will use the following means of constructing a locally CAT( $-1$ ) by cutting up a hyperbolic manifold.

Let  $M$  be a complete hyperbolic 3-manifold. We say that an open subset  $U \subseteq M$  is *polyhedral* if  $\partial U$  is an embedded locally finite 2-complex, where all the simplices are totally geodesic, and each edge is contained in at least one 2-simplex. Let  $\Pi$  be the metric completion of  $U$ , and set  $\partial\Pi = \Pi \setminus U$ . There is a natural map,  $\pi : \Pi \rightarrow M$  which is injective on  $U$ . Note that  $\Pi$  is locally compact. We write  $d_\Pi$  for the induced path metric on  $\Pi$  (which is the same as the completion of the induced path metric on  $U$ ). Thus,  $(\Pi, d_\Pi)$  is a geodesic space.

If  $x \in \Pi$ , we can define the unit tangent space  $\Delta_x(\Pi)$  of  $\Pi$  at  $x$  in the obvious sense. Removing those tangents in the boundary, we obtain  $\Delta_x(\Pi, U)$ . There is a natural map  $\pi : \Delta_x(\Pi) \rightarrow \Delta_{\pi(x)}(M)$ . This is injective on  $\Delta_x(\Pi, U)$ .

**Lemma 19.1 :** *Suppose that  $\Pi$  is a polyhedral set of the type described above. Then  $\Pi$  is locally CAT( $-1$ ) if and only if  $\Delta_x(\Pi)$  is globally CAT(1) for all  $x \in \partial\Pi$ .*

**Proof :** This is a standard fact about a polyhedral complex built out of hyperbolic polyhedra (see [BriH]). One can triangulate  $\Pi$  so that it has such a structure (though this is a somewhat artificial given that the relevant arguments apply directly to this situation).  $\diamond$

Now,  $\Delta_x(\Pi)$  is necessarily locally CAT(1), and the statement that it is globally CAT(1) is equivalent to asserting that there is no (intrinsic) closed geodesic of length less than  $2\pi$ .

Given  $y \in M$ , write  $\Delta_y(M, \partial U) \subseteq \Delta_y(M)$  for those tangent vectors lying in  $\partial U$ . This is a 1-complex in  $\Delta_y(M)$  with geodesic edges.

**Lemma 19.2 :** *Suppose that  $\Pi$  is a polyhedral complex as constructed above. If for all  $y \in M$ , no component of  $\Delta_y(M, \partial U)$  is contained in an open hemisphere of  $\Delta_y(M)$ , then  $\Pi$  is locally CAT(-1).*

**Proof :** Note that if  $x \in \partial\Pi$ , then  $\Delta_x(\Pi, U)$  maps injectively onto a component of  $\Delta_{\pi(x)}(M) \setminus \Delta_{\pi(x)}(M, \partial U)$ . Now any intrinsic geodesic of length less than  $2\pi$  in  $\Delta_x(\Pi)$  would map to a closed path in  $\Delta_{\pi(x)}(M)$  enclosing a component of  $\Delta_{\pi(x)}(M, \partial U)$  contained in an open hemisphere, contrary to the hypotheses.  $\diamond$

We also note that if for all  $y$ , each component of  $\Delta_y(M) \setminus \Delta_y(M, \partial U)$  is a topological disc, then  $U$  is  $\pi_1$ -injective in  $\Pi$ .

Now all the above criteria hold for the image of a non-degenerate balanced piecewise geodesic surface, as defined in Section 18. Thus:

**Lemma 19.3 :** *Suppose that  $M$  is a complete hyperbolic 3-manifold, and  $\phi : S \rightarrow M$  is a proper non-degenerate balance piecewise geodesic map, and let  $U$  be a component of  $M \setminus \phi(S)$ , and let  $\Pi$  be the metric completion of  $U$ . Then  $\Pi$  is locally CAT(-1). Moreover,  $U$  is  $\pi_1$ -injective in  $\Pi$ .*  $\diamond$

In particular, this applies to a folding map, as in Lemma 18.6.

Now suppose that  $(\Pi, d_\Pi)$  is a proper locally CAT(-1) space. Much of the discussion of pleating surfaces in Section 18 applies with  $M$  replaced by  $\Pi$ . We can define a pleating surface as a type preserving uniformly lipschitz map  $\phi : (S, \sigma) \rightarrow (\Pi, d_\Pi)$  where  $\sigma$  is a finite area locally CAT(-1) metric on the surface  $S$ . In practice, we only need to consider a polyhedral space  $\Pi$  of the type constructed above, and piecewise riemannian metrics (or pseudometrics) on  $S$ . Moreover, we can assume that the cusps of  $S$  are isometric to standard hyperbolic cusps in some neighbourhood of the end. In this setting we only consider  $\pi_1$ -injective pleating surfaces.

We have an exact analogue of Lemma 18.1. The map can be constructed via the folding construction. The only difficulty is that the triangles can no longer be assumed to lie in totally geodesic subspaces. We therefore need to use ruled surfaces: coning an edge over the opposite vertex. If one vertex is ideal, we cone over this vertex. This construction makes sense in any locally CAT(-1) space. It is a general fact that the pull-back (pseudo)metric on  $S$  is locally CAT(-1).

There are some technical issues involved here, but we need not worry about the details of the general case. In practice, our surface is built out of triangles obtained by coning a  $d_\Pi$ -

geodesic segment over a (possibly ideal) point. It is thus foliated by lines, the “generators” meeting at the vertex. It would be tedious to enumerate all kinds of local structure such a surface may have. However, it is not hard to see that such a triangle can be subdivided into cells. Each 1-cell of the cellulation either lies in a generator or is transverse to the generators. In the latter case, it maps to a (geodesic) 1-cell in the polyhedral complex  $\partial\Pi$ , or perhaps collapses to a point. The 2-cells are either totally geodesic, or have the form of a ruled surface joining two skew lines in hyperbolic 3-space. It is possible, in the former case, that the cell may collapse onto a geodesic segment. In all other cases, the pull-back metric to the cell is (non-singular) riemannian. Any edge that is not a generator will have non-negative outward curvature. If the metric does not degenerate at a particular vertex, then the angle sum around that vertex will be at least  $2\pi$ . This needs to be appropriately interpreted when edges of the cellulation collapse to points — the adjacent 2-cells do not contribute to the angle sum. In any case, the Gauss-Bonnet Theorem can be seen to apply to such a metric. (One can simply sum the contributions of the cells, making use of the angle sum inequality at the vertices, and the Euler formula.) In particular, this allows us to bound area in terms of the topological type, and hence diameters of the components of the thick part.

## 20. Isolating simply degenerate ends.

Suppose that  $e \in \mathcal{E}_D(M)$ . We want to focus our attention on the intrinsic geometry of  $e$ , so that the arguments of the incompressible case will apply equally well. We will do this by looking at the complement of a pleated surface. (A different trick of passing to branched covers was used by Canary [Can], and could probably be used here also. This has the advantage of producing a riemannian metric. However it introduces other technical complications.)

Let  $E$  be a neighbourhood of the end  $e$  homeomorphic to  $\Sigma \times [0, \infty)$ . Recall that  $\partial_H E = \Sigma \times \{0\}$  is the relative boundary of  $E$  in  $\Psi(M)$ . Let  $S = \text{int } \Sigma$ . Note that  $e$  determines a homotopy class of type-preserving maps from  $S$  into  $M$ , which we refer to as the *ambient fibre class*.

To get started we note:

**Lemma 20.1 :** *There is some  $\alpha \in X(e)$  which is homotopic in  $M$  to a closed geodesic  $\alpha^*$  in  $M$ .*

In other words, we need one curve that is neither trivial nor parabolic. This, of course, is an immediate consequence of the fact that there are a sequence of such geodesics tending out any simply degenerate end [Th,Bon,Cana] etc. It can alternatively be deduced by purely a topological argument, though we shall not elaborate on that here.

Now, by Lemma 18.5, there is a non-degenerate folding map  $\phi_0 : S \rightarrow M$ , in the ambient fibre class in  $M$ , realising  $\alpha$ . As in the discussion at the end of Section 2, we can assume that  $E \cap \phi_0(S) = \emptyset$ . Let  $U$  be the component of  $M \setminus \phi_0(S)$  containing  $E$ , and let  $\Pi$  be its metric completion. By Lemma 17.3, and the subsequent remarks, we know that

$E$  in  $\pi_1$ -injective in  $U$ . By Lemma 19.3, it is also  $\pi_1$ -injective in  $\Pi$ . By Lemmas 18.6 and 19.3, we see that  $\Pi$  is locally CAT(-1) in the induced path metric  $d_\Pi$ .

There is a canonical map,  $\pi : \Pi \rightarrow M$ , that is injective on  $U$ . For notational convenience in what follows, we will pretend that  $\pi$  is injective on  $\partial\Pi$ . (This is what one would expect generically.) We can thus identify  $\Pi$  as a subset of  $M$ . All our constructions have an obvious interpretation in the case where  $\pi$  might not be injective on  $\partial\Pi$ .

Recall, from Section 19, the definition of the thin part,  $\tau(\Pi) = \tau_0(\Pi) \cup \tau_+(\Pi)$  of  $\Pi$ . The set  $\tau_+(\Pi)$  is a union of  $\tau(\Pi, \alpha)$  as  $\alpha$  ranges over all closed  $d_\Pi$ -geodesics. If  $\alpha \cap \partial\Pi = \emptyset$ , then  $\alpha$  is also a closed geodesic in  $M$  (in particular non trivial in  $M$ ). If  $\tau(\Pi, \alpha) \cap \partial\Pi = \emptyset$ , then  $\tau(\Pi, \alpha)$  is a Margulis tube in  $M$  with respect to the hyperbolic metric. Suppose that  $\alpha, \beta$  are closed  $d_\Pi$ -geodesics with  $\tau(\Pi, \alpha) \cap \tau(\Pi, \beta) \cap U \neq \emptyset$ . Let  $x$  be a point of this intersection. By definition of the sets  $\tau(\Pi, \alpha)$  and  $\tau(\Pi, \beta)$ ,  $x$  lies in curves,  $\alpha'$  and  $\beta'$ , of length at most  $\eta$ , homotopic in  $\Pi$  respectively to powers of  $\alpha$  and  $\beta$ . Then the Margulis lemma (applied in  $M$ ) tells us that  $\alpha'$  and  $\beta'$  generate an abelian subgroup of  $\pi_1(M)$ . Such a group must be trivial, loxodromic or parabolic in  $M$ . Thus, if  $\alpha, \beta \subseteq U$ , then there are both hyperbolic geodesics and hence equal. If  $\alpha \neq \beta$ , we conclude that at least one of  $\alpha$  or  $\beta$  must meet  $\partial\Pi$ .

By similar reasoning, we see that if  $\alpha$  is a closed  $d_\Pi$ -geodesic and  $\tau(\Pi, \alpha) \cap \tau_0(\Pi) \cap U \neq \emptyset$ , then  $\alpha \cap \partial\Pi \neq \emptyset$ .

Now, by discreteness, only finitely many  $\tau(\Pi, \alpha)$  can meet  $\partial_H E$ , and these include all those meeting both  $e$  and  $\partial\Pi$ . Thus, after shrinking  $e$ , we can assume that if  $\tau(\Pi, \alpha) \cap e \neq \emptyset$ , then  $\tau(\Pi, \alpha) \subseteq U$ , and so it is a Margulis tube in  $M$ . We also assume that  $E$  only meets the associated cusps of  $M$ . From this, it follows easily that  $\partial_V E = E \cap \Psi(M) = E \cap \Psi(\Pi)$ . Thus  $e \cap \Theta(M) = e \cap \Theta(\Pi)$ .

Recall that, in Section 3, we defined a pseudometric,  $\rho = \rho_M$ , on  $M$  by collapsing the Margulis tubes. It will be more convenient, for the moment, to work with another metric,  $\rho_E$ , where we also collapse the complement of  $e \in \Psi(M)$ . In other words,  $\rho_E(x, y)$ , is the minimum length of  $\beta \cap e \cap \Theta(M)$  as  $\beta$  ranges over all paths from  $x$  to  $y$  in  $\Psi(M)$ . We can also view this as a pseudometric on  $\Psi(\Pi)$ . Clearly  $\rho_E \leq \rho_M$ .

Given  $x \in e$ , we write  $D(x) = \rho_M(x, \partial_H E) = \rho_E(x, \partial_H E)$  for the *depth* of  $x$  in  $e$ . We note that if  $x, y \in e$  have depth greater than the  $\rho_M$ -diameter of  $\partial_H E$ , then  $\rho_M(x, y) = \rho_E(x, y)$ . Given a subset  $Q \subseteq E$ , write  $D(Q) = \rho_E(Q, \partial_H E) = \inf\{D(x) \mid x \in Q\}$ . We write  $\rho_E\text{-diam}(Q)$  for the  $\rho_E$  diameter of  $Q$ .

There is natural type-preserving  $\pi_1$ -injective homotopy class of maps  $S \rightarrow \Pi$ , determined by  $e$ , which we refer to as the *fibre class*.

**Lemma 20.2 :** *There is some  $h_0 = h_0(\kappa(\sigma))$  such that if  $\phi : S \rightarrow \Pi$  is a pleating surface in the fibre class, then  $\rho_E\text{-diam}(\phi(S) \cap \Psi(\Pi)) < h_0$ .*

**Proof :** In applications, we only need this where the  $(S, \sigma_\phi)$  is piecewise riemannian. In this case we can apply the Gauss-Bonnet Theorem, so the argument follows as with Lemma 18.3.  $\diamond$

In particular, we get:

**Lemma 20.3 :** *If  $\phi : S \rightarrow \Pi$  is a pleating surface in the fibre class, and  $\phi(S) \cap E$  contains some point of depth at least  $h_0$  in  $E$ , then  $\phi(S) \cap \Psi(\Pi) \subseteq e$ .  $\diamond$*

Given  $\alpha \in X(e) \equiv X(\Sigma)$ , we write  $\alpha_\Pi$  for its realisation as a closed geodesic in  $\Pi$ . Thus if  $\alpha_\Pi \cap \partial\Pi = \emptyset$ , then  $\alpha_\Pi = \alpha_M$  is also a closed geodesic in  $M$ .

**Lemma 20.4 :** *Suppose that  $\alpha \in X(e)$  and  $\alpha_\Pi \cap e$  contains some point of depth at least  $h_0$ . Then  $\alpha_\Pi \subseteq e$ ,  $\alpha_\Pi = \alpha_M$ , and  $\rho_E\text{-diam}(\alpha_\Pi) < h_0$ .*

**Proof :** Let  $\phi : S \rightarrow \Pi$  be a pleating surface in the fibre class realising  $\alpha_\Pi$ . By Lemma 20.3,  $\phi(S) \cap \Psi(M) \subseteq e$ . Let  $F$  be the principal component of  $\phi^{-1}(\Psi(\Pi)) = \phi^{-1}(e) \subseteq S$ . A similar argument to that of Lemma 3.2 now applies. Assuming that we have chosen the Margulis constant,  $\eta$ , sufficiently small (depending on  $\kappa(\Sigma)$ ), any simple geodesic in  $S$  will lie in  $F$ . (One way to see this is to note that any point in the non-cuspidal part of  $S$  lies in a non-trivial non-peripheral curve of bounded length in  $S$ . This maps to a curve of bounded length in  $\Pi$ , which will lie outside all Margulis cusps for sufficiently small Margulis constant. We have only used the fact that the curvature on  $S$  is at most  $-1$ .) In particular,  $\alpha_S \subseteq F$ , and so  $\alpha_\Pi = \phi(\alpha_S) \subseteq E$ . Since  $e \cap \partial\Pi = \emptyset$ , it follows that  $\alpha_\Pi = \alpha_M$ . Finally,  $\rho_E\text{-diam}(\alpha_\Pi) \leq \rho_E\text{-diam}(\phi(S) \cap \Psi(\Pi)) < h_0$ .  $\diamond$

**Lemma 20.5 :** *Suppose that  $\alpha, \beta \in X(e)$  are adjacent in  $\mathcal{G}(e)$  and  $\alpha_\Pi \cap E$  contains some point of depth at least  $h_0$ . Then  $\beta_\Pi = \beta_M \subseteq e$ , and  $\rho_E\text{-diam}(\alpha_\Pi \cup \beta_\Pi) < h_0$ .*

**Proof :** The proof follows that of Lemma 20.4. This time we take  $\phi$  to realise the multicurve  $\{\alpha, \beta\}$ .  $\diamond$

We remark that, in retrospect, we see that the pleating surfaces arising in the proofs of Lemmas 20.3 and 20.4 do not meet  $\partial\Pi$ . If they were constructed by the folding procedure described in Section 3, then all the 2-simplices would be totally geodesic in  $M$ . We can then proceed to spin around the curves to give us pleated surfaces whose domains are hyperbolic surfaces.

**Lemma 20.6 :** *For all  $l \geq 0$ , there is some  $h_1(l)$  such that if  $\alpha \in X(e)$  is realised by a curve,  $\alpha_0$ , in  $e$  of length at most  $l$  and containing a point of depth at least  $h_1(l)$ , then  $\alpha_\Pi = \alpha_M \subseteq e$  and  $\rho_E(\alpha_0 \cup \alpha_\Pi) < h_1(l)$ .  $\diamond$*

**Proof :** This is now a fairly standard argument (cf. [Bon]). We can realise the homotopy between  $\alpha_0$  and  $\alpha_\Pi$  in  $\Pi$  by a ruled surface,  $\phi : A \rightarrow \Pi$ , where  $A$  is an annulus whose boundary components get mapped to  $\alpha_0$  and  $\alpha_\Pi$ . This can be constructed as for pleated surfaces. The pull-back (pseudo)metric on  $A$  is locally CAT( $-1$ ), and can be assumed piecewise riemannian. (In practice, we can approximate  $\alpha_0$  by a piecewise geodesic curve, which that the surface will have the form described at the end of Section 19.) As with Lemma 20.2, we see that the  $\rho_E\text{-diam}(\phi(A) \cap \Psi(\Pi))$  is bounded in terms of the area of  $A$ . (Note that there must be a path in  $\phi^{-1}(\Psi(\Pi))$  connecting the two boundary components. Applying Gauss-Bonnet, this is in turn bounded in terms of the length of  $\partial A$  which is at

most  $2l$ . In other words, there is some constant  $h' = h'(l)$  such that  $\rho_E \text{-diam}(\alpha_0 \cup \alpha_\Pi) \leq \rho_E \text{-diam}(\phi(A)) \leq h'$ . We can now set  $h_1 = h_0 + h'(l)$ . It then follows that  $D(\alpha_\Pi) > h_0$ , and so by Lemma 20.4,  $\alpha_\Pi = \alpha_M \subseteq e$  and  $\rho_E \text{-diam}(\alpha_\Pi) < h_0$ , and so the result follows.  $\diamond$

We have shown the essential properties we need. In order to restrict our attention to the end,  $e$ , we can perform a few “tidying up” operations on pleating surfaces. We will only be interested in pleating surfaces realising multicurves of depth at least  $h_0$ . As observed after Lemma 20.5, such a surface can be assumed to have domain a hyperbolic surface. Then, as discussed in Section 19, at the cost of increasing the lipschitz constant, we can assume that the preimage of the non-cuspidal part is a core bounded by horocycles of some fixed length. We can assume that these horocycles get mapped to euclidean geodesics in  $\partial_V E$ . From this point on, the remainder of the manifold is of little interest to us. We can reinterpret a “pleating surface” accordingly, as we set out at the beginning of the next section.

## 21. Quasiprojections and a-priori bounds.

The aim of this section is to give a generalisation of the a-priori bounds theorem to the compressible case. Specifically, we are aiming at Lemmas 21.5 and 21.6, which together are variations of Theorems 15.2 and 15.3. To this end, we will describe a projection map which associates to any curve in  $\Sigma$ , another curve which as bounded length representative in the end. A similar projection map was a key ingredient in the argument presented in [Bow4]. Once its basic properties are established, the remainder of the argument is almost identical to that presented there.

Let  $e \in \mathcal{E}_D(M)$ , and let  $\Sigma = \Sigma(e)$ . Note that  $\kappa(\Sigma) \geq 1$ . We have already observed that if  $\kappa(\Sigma) = 1$ , then  $e$  is incompressible, so we can assume, in fact, that  $\kappa \geq 2$ . Let  $E \cong \Sigma \times [0, \infty)$  be a neighbourhood of  $e$  in  $\Psi(M)$ .

Following on from the last section, we shall define a *pleating surface* in  $E$  to mean a uniformly lipschitz map  $\phi : (\Sigma, \sigma) \rightarrow (E, d)$ , where  $\sigma$  is some hyperbolic structure on  $\Sigma$  with each boundary component horocyclic of fixed length. (Geodesic of fixed length would do just as well.) We assume that each boundary component gets mapped to a euclidean geodesic in  $\partial_V E$ . All our pleated surfaces will be in the fibre class, i.e. identifying  $e$  topologically with  $\Sigma \times [0, \infty)$ , it is homotopic to the inclusion of  $\Sigma$  as the first co-ordinate. We note that Lemmas 20.2 and 20.3 apply equally well for a pleating surface interpreted in this way. We also note that Lemma 18.4 (uniform injectivity) applies in this situation: if an  $\eta$ -neighbourhood of the curve lies in  $E$ , then projection of  $E$  to  $\Sigma$  to this neighbourhood gives us our map  $\theta$ . For this we should insist that the curve has depth at least  $\eta$ , but after shrinking  $E$  slightly, we can forget this detail.

Let  $J \subseteq X(\Sigma)$  be the set of curves that are realised by a closed geodesic in the interior of  $E$ , which we shall henceforth denote by  $\alpha^*$ . This is also a closed geodesic in  $M$ . We write  $l_M(\alpha)$  for the length of  $\alpha^*$ , and write  $D(\alpha) = D(\alpha^*) = \rho_E(\alpha)$  for the depth of  $\alpha^*$  in  $e$ . We write  $J(h, l) = \{\alpha \in J \mid D(\alpha) > h, l_M(\alpha) \leq l\}$ .

Let  $h_0$  be the constant of Lemma 20.5. An immediate consequence of this lemma applied inductively is:

**Lemma 21.1 :** *If  $k \in \mathbf{N}$  and  $h \geq kh_0$ , then  $N(J(h, \infty), k) \subseteq J(h - kh_0, \infty)$ .*  $\diamond$

Here  $N(Q, k)$  denotes the  $k$ -neighbourhood of  $Q \subseteq X(\Sigma)$  with respect to the combinatorial metric,  $d_{\mathcal{G}}$ , on  $\mathcal{G}(\Sigma)$ .

Another consequence of Lemma 20.5 is that  $J(0, l)$  is a locally finite subset of  $\mathcal{G}(\Sigma)$  for any  $l \in (0, \infty)$  (that is, any bounded set meets  $J(0, l)$  in a finite set).

Now suppose that  $\alpha \in J(h_0, \infty)$ , and let  $(\Sigma, \sigma) \rightarrow e$  be a pleating surface in  $e$  realising  $\alpha$ , as given by Lemma 20.3. Let  $J_\sigma(l) \subseteq X(\Sigma)$  be the set of curves whose geodesic realisations in  $(\Sigma, \sigma)$  have length at most  $l$ . It is a standard and relatively straightforward fact that the diameter of  $J_\sigma(l)$  in  $X(\Sigma)$  is bounded in terms of  $l$ . Moreover, if  $l \geq l_0 = l_0(\kappa(\Sigma))$ , then it is non-empty. We fix  $l_0$  and abbreviate  $J_\sigma = J_\sigma(l_0)$ . Suppose that  $\phi' : (\Sigma, \sigma') \rightarrow e$  is another such pleating surface realising  $\alpha$ . A consequence of uniform injectivity (Lemma 18.4) is that the diameter of  $J_\sigma \cup J_{\sigma'}$  is bounded in terms of  $\kappa(\Sigma)$ . We can therefore choose some  $\gamma \in J_\sigma$  and set  $\text{proj}(\alpha) = \gamma$ . This is then well defined up to bounded distance in  $X(\Sigma)$ . If  $\alpha \in J(h_0, l_0)$  we can set  $\text{proj}(\alpha) = \alpha$ . Note that if  $\alpha$  and  $\beta$  are adjacent in  $X(\Sigma)$ , then we can choose the same pleating surface for both, showing that  $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta))$  is bounded. These arguments are essentially the same as in the incompressible case (see for example [Bow4]) and we refer there for details.

Suppose  $h \geq h_0 + h_1(l_0)$  where  $h_1$  is the function of Lemma 20.6. It follows that if  $\alpha \in J(h, \infty)$  then  $\text{proj}(\alpha) \in J(h - (h_0 + h_1(l_0)), r_0)$ .

Let us summarise what we have shown. For notational simplicity, we increase our original choice of constant  $h_0$  to  $h_0 + h_1(l_0)$ . In this way, it will serve for all the above purposes.

**Lemma 21.2 :** *There is some  $h_0 = h_0(\kappa(\Sigma))$ ,  $l_0 = l_0(\kappa(\Sigma))$ ,  $k_0 = k_0(\kappa(\Sigma)) \in \mathbf{N}$ , and a map  $\text{proj} : J(h_0, \infty) \rightarrow J(0, l_0)$  with the following properties:*

(P1) *If  $\alpha \in J(h, \infty)$  for some  $h \geq h_0$ , then  $\text{proj}(\alpha) \in J(h - h_0, l_0)$ .*

(P2) *If  $\alpha \in J(h_0, l_0)$ , then  $\text{proj}(\alpha) = \alpha$ .*

(P3) *If  $\alpha, \beta \in J(h_0, \infty)$  are adjacent in  $X(\Sigma)$ , then  $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta)) \leq k_0$ .*  $\diamond$

To deduce more about the map  $\text{proj}$ , we bring the hyperbolicity of the  $\mathcal{G}(\Sigma)$  into play. We are specifically aiming at Lemma 21.4. This depends on the following general observation about quasiprojections in a hyperbolic graph.

**Lemma 21.3 :** *For all  $k, s, t \geq 0$  there is some  $u \geq 0$  with the following property. Suppose that  $(\mathcal{G}, d)$  is a  $k$ -hyperbolic graph,  $A, B \subseteq V(\mathcal{G})$  and  $\omega : A \rightarrow B$  is a map satisfying  $\omega(x) = x$  for all  $x \in A \cap B$  and  $d(\omega(x), \omega(y)) \leq t$  whenever  $x, y \in A$  are adjacent. Suppose that  $x_0 x_1 \cdots x_n$  is a geodesic in  $\mathcal{G}$  with  $d(x_0, B) \leq s$ ,  $d(x_n, B) \leq s$  and  $N(x_i, u) \subseteq A$  for all  $i$ . Then for all  $i$ ,  $d(x_i, \omega(x_i)) \leq tu$ .*

In the case where  $A = X$ , this follows from the fact that the image of a quasiprojection (here  $\omega(X)$ ) is quasiconvex. (This is discussed in [Bow2] for example.) Here we have to take account of the fact the quasiprojection is only defined on a certain subset.

**Proof :** We first make the observation that if  $x \in A$  and  $d(x, A \cap B) \leq v \leq u$  (however  $u$  is defined) then  $d(x, \omega(x)) \leq tv$ . This follows by straightforward induction.

Now fix  $v \geq l$  to be determined shortly. Let  $\{p+1, p+2, \dots, q-1\}$  be a maximal set of consecutive indices such that  $d(x_i, B) > v$  for all  $p+1 \leq i \leq q-1$ . In other words,  $d(x_p, B) \leq v$ ,  $d(x_q, B) \leq v$  and  $d(x_i, B) > v$  whenever  $p < i < q$ . We can assume that  $u \geq v$  so that in particular,  $d(x_p, A \cap B) \leq v$  and  $d(x_q, A \cap B) \leq v$ . Let  $y_i = \omega(x_i)$ . Thus,  $d(x_p, y_p) \leq tv$  and  $d(x_q, y_q) \leq tv$  and  $d(y_i, y_{i+1}) \leq t$  for all  $i$ . Now the piecewise geodesic path  $\zeta = [y_p, y_{p+1}] \cup \dots \cup [y_{q-1}, y_q]$  has length at most  $t(q-p)$  and lies outside a  $(v-t/2)$ -neighbourhood of the geodesic segment  $x_p x_{p+1} \dots x_q$ . Given any  $a \geq 0$ , we can choose  $v$  sufficiently large in relation to  $a$ ,  $t$  and  $k$  so that the length of  $\zeta$  is at least  $ad(y_p, y_q) - c$ , where  $c$  depends on  $a$ ,  $t$ ,  $k$  and  $v$ . This is the standard fact, stated as Theorem 3.11 here, that distances outside a sufficiently large uniform neighbourhood of a geodesic segment in a hyperbolic space grow faster than any given linear function. Now set  $a = 2t$ , and fix some such  $v$ . Thus  $c$  depends only on  $t$  and  $k$ . Now  $2td(y_p, y_q) - c \leq \text{length}(\zeta) \leq t(q-p)$  and so  $2d(y_p, y_q) \leq (q-p) + c/t$ . Also  $d(y_p, y_q) \geq d(x_p, x_q) - 2tv = (q-p) - 2tv$  and so  $2((q-p) - 2tv) \leq (q-p) + c/t$ , and so  $q-p \leq 5tv + c/t$ , which is bounded in terms of  $k$  and  $t$ . We now set  $u = (4tv + c/t)$ . It follows that for each  $i$ ,  $d(x_i, y_p) \leq d(x_i, x_p) + d(x_p, y_p) \leq (4tv + c/t) + tv = u$ . Since this applies to any such segment,  $x_p x_{p+1} \dots x_q$ , we conclude that  $d(x_i, B) \leq u$  for all  $i$ .

Now by the first paragraph again, we see that for all  $i$ ,  $d(x_i, \omega(x_i)) \leq tu$  as required.  $\diamond$

We apply this as follows:

**Lemma 21.4 :** *Given  $s \geq 0$ , there exist  $h_2, k_2 \geq 0$ , depending only on  $s$  and  $\kappa(\Sigma)$  such that if  $\alpha_0, \alpha_1, \dots, \alpha_n$  is a geodesic in  $\mathcal{G}(\Sigma)$  with  $d_{\mathcal{G}}(\alpha_0, J(0, l_0)) \leq s$ ,  $d_{\mathcal{G}}(\alpha_n, J(0, l_0)) \leq s$  and with  $\alpha_i \in J(h_2, \infty)$  for all  $i$ , then for all  $i$ ,  $d_{\mathcal{G}}(\alpha_i, \text{proj}(\alpha_i)) \leq k_2$ .*

Here  $l_0$  is the constant involved in the definition of  $\text{proj} : J(h_0, \infty) \rightarrow J(0, l_0)$ . We can certainly choose  $h_2 \geq h_0$  so that  $\text{proj}(\alpha_i)$  is defined.

**Proof :** Let  $A = J(h_0, \infty)$  and  $B = J(0, l_0)$ , so that  $A \cap B = J(h_0, l_0)$ . We now apply Lemma 21.3, with  $\mathcal{G} = \mathcal{G}(\Sigma)$  and  $\omega = \text{proj} : A \rightarrow B$ . Let  $k = k(\kappa(\Sigma))$  be the hyperbolicity constant of  $\mathcal{G}(\Sigma)$ , and let  $t = k_0$  be the constant of Lemma 21.2. Let  $u$  be the constant of Lemma 21.3 and set  $k_2 = tu$  and  $h_2 = (k_2 + 1)h_0$ . Then, by Lemma 21.1,  $N(J(h_2, \infty), k_2) \subseteq J(h_0, \infty)$ . In particular, by hypothesis,  $N(\alpha_i, k_2) \subseteq J(h_0, \infty) = A$  for all  $i$ . Thus, applying Lemma 21.3, we see that  $d_{\mathcal{G}}(\alpha_i, \text{proj}(\alpha_i)) \leq tu = k_2$  for all  $i$ , as required.  $\diamond$

We now have all the ingredients to apply the arguments of [Bow2] to deduce the following version of the that a-priori bounds theorem to the compressible case.

**Lemma 21.5 :**

(1)  $(\exists h)(\forall l)(\exists l')$  such that if  $(\gamma_i)_{i=0}^n$  is a tight geodesic in  $\mathcal{G}(\Sigma)$  with  $\gamma_0, \gamma_n \in J(h, l)$  and  $\gamma_i \in J(h, \infty)$  for all  $i$ , then  $\gamma_i \in J(0, l')$  for all  $i$ .

(2)  $(\exists h)(\forall k, l)(\exists k', l')$  such that if  $(\gamma_i)_{i=0}^n$  is a tight geodesic in  $\mathcal{G}(\Sigma)$  with  $\gamma_0, \gamma_n \in N(J(h, l), k)$  and  $\gamma_i \in J(h, \infty)$  for all  $i$ , then  $\gamma_i \in J(h, l')$  for all  $i$  with  $k' \leq i \leq n + k'$ . ■

(Here  $l'$  just depends on  $l$  and  $\kappa(\Sigma)$  and in part (2),  $k'$  just depends on  $k$  and  $\kappa(\Sigma)$ .)

Note that, in the incompressible case, Lemma 21.5(1) and 21.5(2) are respectively implied by Theorems 15.2 and 15.3. Indeed, if we neglect the requirement about depths of curves, they are just rephrasing of those statements.

The proof of Lemma 21.5 follows exactly that of [Bow2], where the lower bound,  $h$ , on the depths of curves was redundant. Here it is needed to ensure that none of our constructions take us outside  $e$ . The argument is by contradiction. If the conclusion fails, we can find sequences,  $(\gamma_i^n)_i$  of tight geodesics in ends,  $e^n$ , of thick parts,  $\Psi(M^n)$ , of complete hyperbolic 3-manifolds,  $M^n$ , that satisfy the hypotheses of the lemma, but for which certain  $\gamma_i^n$  get arbitrarily long as geodesics in  $M^n$ . To derive the same contradiction as in [Bow2] it is sufficient to have a homotopy equivalence,  $e^n \rightarrow \Sigma$ , defined on each end  $e^n$ . The remainder of the manifold  $\Psi(M^n)$  is then irrelevant.

Lemma 21.5 is all we need to prove Propositions 17.1 and 17.2, as we will do in the next section. However, for their application we will also need a relative version of this, or at least of part (1) of the statement. Let  $\Phi$  be a connected proper  $\pi_1$ -injective subsurface of  $\Sigma$ . We shall assume here that any boundary component of  $\Phi$  that is homotopic to a boundary component of  $\Sigma$  equals that boundary component. We write  $\partial_\Sigma \Phi$  for the relative boundary of  $\Phi$  in  $\Sigma$ , and write  $X(\partial_\Sigma \Phi) \subseteq X(\Sigma)$  for the set of components of  $\partial_\Sigma \Phi$ . (It is possible that two such components might get identified in  $X(\Sigma)$ .) We can also identify  $X(\Phi)$  as a subset of  $X(\Sigma)$ . Note that, by Lemma 21.1, if for some  $h \geq h_0$ ,  $X(\partial_\Sigma \Phi) \cap J(h, \infty) \neq \emptyset$ , then  $X(\Phi) \subseteq J(h - h_0, \infty)$ .

The following is a generalisation of Theorem 11.4 to the compressible case.

**Lemma 21.6 :**  $(\exists h)(\forall l)(\exists l')$  such that if  $\Phi \subseteq \Sigma$  is a proper subsurface with  $X(\partial_\Sigma \Phi) \subseteq J(h, l)$  and  $(\gamma_i)_{i=0}^n$  is a tight geodesic in  $\mathcal{G}(\Phi)$  with  $\gamma_0, \gamma_n \in J(0, l)$ , then  $\gamma_i \in J(0, l')$  for all  $i$ .

Again this follows exactly as in [Bow2] (modulo restricting the homotopy equivalence to the ends). This time, the control on depth is automatic, given the assumption on  $\partial_\Sigma \Psi$ , and the observation preceding the lemma.

## 22. Short geodesics in ends.

In this section, we give proofs of Propositions 17.1 and 17.2, and describe a variant of the “a-priori bounds” result for our version of hierarchies.

As observed in Section 17 we can restrict attention to the case where  $\kappa(\Sigma) \geq 2$ .

Let  $e \in \mathcal{E}_D(M)$ . Choose a neighbourhood  $E \cong [0, \infty)$  of  $e$  in  $\Psi(M)$ , and let  $D : E \rightarrow [0, \infty)$  be the depth function as defined in Section 20. Let  $\text{proj} : J(h_0, \infty) \rightarrow J(0, l_0)$  be the quasiprojection defined in Section 21. We note that for all  $\alpha \in J(h_0, \infty)$ ,  $|D(\alpha) - D(\text{proj}(\alpha))| \leq h_0$ . Also, if  $\alpha, \beta \in V(\mathcal{G}(\Sigma))$  are adjacent and  $\alpha \in J(h_0, \infty)$ , then  $\beta \in J$  and

$|D(\alpha) - D(\beta)| \leq h_0$ ,  $|D(\text{proj}(\alpha)) - D(\text{proj}(\beta))| \leq h_0$  and  $d_{\mathcal{G}}(\text{proj}(\alpha), \text{proj}(\beta)) \leq k_0$ .

We start the discussion with the assumption that there is a sequence  $(\gamma_i)_{i \in \mathbf{N}}$  of distinct elements of  $J$  with  $D(\gamma_i) \rightarrow \infty$ . In other words, the geodesics  $\gamma_i^*$  go out the end  $e$ . It follows that  $D(\text{proj}(\gamma_i)) \rightarrow \infty$ , so in retrospect, we could have assumed that each  $\gamma_i$  lies in  $J(0, l_0)$ . (Also, using Lemma 19.6, we could alternatively have started from a sequence of curves each of which has some representative of bounded length in  $E$ , not a-priori geodesic.) In any case, we get a sequence of curves  $\gamma_i \in J(0, l_0)$  going out the end  $e$ . Given that  $J(0, l_0)$  is locally finite in  $\mathcal{G}(\Sigma)$  this is equivalent to saying that  $J(0, l_0)$  is infinite, or that it is unbounded. We now refine this statement in a series of steps, to arrive at Proposition 17.1.

**Lemma 22.1 :** *There exist  $k_0 = k_0(\kappa(\Sigma))$  and  $l_0 = l_0(\kappa(\Sigma))$  such that there is a sequence  $(\alpha_i)_{i \in \mathbf{N}}$  in  $J(0, l_0)$  with  $d_{\mathcal{G}}(\alpha_i, \alpha_{i+1}) \leq k_0$  for all  $i$  with  $D(\alpha_i) \rightarrow \infty$ .*

We can take  $l_0$  and  $h_0$  to be the same as those above.

**Proof :** From the observation above, we know that  $J(0, l_0)$  is infinite.

Now choose any  $\beta_0 \in J(0, l_0)$  with  $D_0 = D(\beta_0) \geq h_0$ . Let  $\mathcal{H}$  be the graph with vertex set  $V(\mathcal{H}) = J(D_0, l_0)$  and with  $\alpha, \beta \in V(\mathcal{H})$  deemed adjacent if  $d_{\mathcal{G}}(\alpha, \beta) \leq k_0$ . Note that  $\mathcal{H}$  is locally finite.

Let  $B = \{\beta \in V(\mathcal{H}) \mid D(\beta) \leq D_0 + 3h_0\}$ . Thus,  $B \subseteq V(\mathcal{H})$  is finite. Now each point of  $V(\mathcal{H})$  can be connected to  $B$  by some path in  $\mathcal{H}$ . To see this, suppose  $\beta \in V(\mathcal{H})$  and let  $\beta_0, \beta_1, \dots, \beta_n = \beta$  be any path in  $X(\Sigma)$  from  $\beta_0$  to  $\beta$ . Thus  $|D(\beta_i) - D(\beta_{i+1})| \leq h_0$  for all  $i$ . If  $\beta \notin B$ , there is some  $p$  so that  $D(\beta_i) \geq D_0 + h_0$  for all  $i \geq p$  and with  $D(\beta_p) \leq D_0 + 2h_0$ . Let  $\delta_i = \text{proj}(\beta_i)$  for all  $i \geq p$ . Thus  $\delta_p \in B$  and  $\delta_n = \beta$ . We see that  $\delta_p, \delta_{p+1}, \dots, \delta_n$  connects  $B$  to  $\beta$  in  $\mathcal{H}$  as claimed.

In summary,  $\mathcal{H}$  is infinite, locally finite, and has finitely many components. It thus has a component of infinite diameter, which then contains an infinite arc  $(\alpha_i)_{i \in \mathbf{N}}$ . It follows that  $D(\alpha_i) \rightarrow \infty$  as required.  $\diamond$

We can now prove the statement of Section 16, starting with Proposition 16.1. In the above notation, this says that, for some  $L \geq 0$  there is a sequence of elements of  $J(0, L)$  that form a geodesic in  $V(\mathcal{G}(\Sigma))$ .

**Proof of Proposition 17.1 :** We begin with the sequence,  $(\alpha_i)_{i \in \mathbf{N}}$ , given by Lemma 22.1. We want to replace this by a geodesic  $(\gamma_i)_{i \in \mathbf{N}}$ , perhaps at the cost of increasing the length bound.

For each pair,  $i, j \in \mathbf{N}$ , let  $\pi(i, j) \subseteq X(\Sigma)$  be a tight geodesic in  $\mathcal{G}$  from  $\alpha_i$  to  $\alpha_j$ . Let  $m(i, j) = \min\{D(\delta) \mid \delta \in \pi(i, j)\}$  (where  $m(i, j) = 0$  if some curve of  $\pi(i, j)$  lies outside  $J(0, \infty)$ ). Note that  $\pi(i, i) = \{\alpha_i\}$ , so  $m(i, i) \rightarrow \infty$  as  $i \rightarrow \infty$ . By Lemma 21.5(1), there is some  $h \geq 0$  such that if  $m(i, j) \geq h$ , then  $\pi(i, j) \subseteq J(0, L)$ . Here  $h$  and  $L$  depend only on  $\kappa(\Sigma)$ .

By construction,  $d_{\mathcal{G}}(\alpha_j, \alpha_{j+1}) \leq k_0$ , and so, by the hyperbolicity of  $\mathcal{G}(\Sigma)$ , the geodesics  $\pi(i, j)$  and  $\pi(i, j+1)$  lie a bounded distance apart (depending only on  $\kappa(\Sigma)$ ) for each  $i$  and  $j$ .

From this it follows that  $|m(i, j) - m(i, j + 1)|$  is bounded by some constant  $m_0 = m_0(\kappa(\Sigma))$  for all  $i, j$ .

We distinguish two cases:

Case (1):  $(\exists i)(\forall j \geq i)(m(i, j) \geq h)$ .

We fix some such  $i$ . Then,  $\pi(i, j) \subseteq J(0, L)$  for all  $j \geq i$ . Let  $\gamma_k^j$  be the  $k$ th curve in  $\pi(i, j)$ . Then  $\gamma_0^j = \alpha_i$ . Now  $D(\gamma_k^j)$  is bounded above in terms of  $D(\alpha_i)$  and  $k$ , and so, as  $j$  varies, there are only finitely many possibilities for any given  $k$ . After passing to a diagonal subsequence, we can therefore suppose that for each  $k$ ,  $\gamma_k^j$  eventually stabilises on some curve  $\gamma_k \in J(0, L)$  as  $j \rightarrow \infty$ . Thus,  $(\gamma_k)_k$  gives us our required geodesic.

Case (2):  $(\forall i)(\exists j \geq i)(m(i, j) < h)$ .

Now  $m(i, i) \rightarrow \infty$ , and we can assume that  $m(i, i) \geq h + m_0$  for all  $i$ . Since  $|m(i, j) - m(i, j + 1)| \leq m_0$  for all  $j$ , there is some  $j(i)$  such that  $h \leq m(i, j(i)) \leq h + m_0$ . Note that  $\pi(i, j(i)) \subseteq J(0, L)$ . Now let  $\delta_0^i \in \pi(i, j(i))$  be such that  $D(\delta_0^i) \leq h + m_0$ . Let  $(\gamma_k^i)_{k=0}^{p_i}$  be the subpath of  $\pi(i, j(i))$  going backwards from  $\delta_0^i$  to  $\delta_k^i = \alpha_i$ . As in Case (1), we see that there are only finitely many possibilities for the curve  $\delta_k^i$  as  $i \rightarrow \infty$ , so we can suppose that  $\delta_k^i$  stabilises on some curve  $\gamma_k \in J(0, L)$ . Thus  $(\gamma_k)_k$  is the required geodesics.  $\diamond$

**Proof of Proposition 17.2 :** Let  $(\gamma_i)_i$  be a tight geodesic converging on some point,  $a \in \mathbf{a}(M, e) \in \partial \mathcal{G}(\Sigma)$ . By definition, there is a geodesic  $(\gamma'_i)_i$  lying in  $J(0, L_0)$  and converging to  $a$ . By hyperbolicity,  $(\gamma_i)_i$  and  $(\gamma'_i)_i$  eventually remain a uniformly bounded distance apart. In other words, up to shifting the indices, we can assume that  $d_{\mathcal{G}}(\gamma_i, \gamma'_i) \leq k_1 = k_1(\kappa(\Sigma))$  for all  $i$ . Since  $\gamma'_i \in J(0, L)$  for all  $i$ , we can apply Lemma 21.5(2) to deduce that  $\gamma_i \in J(0, L')$  for all  $i$ , where  $L'$  depends only on  $L, k$  and  $\kappa(\Sigma)$ , and hence ultimately only on  $\kappa(\Sigma)$ .  $\diamond$

A combination of Propositions 21.5 and 21.6 now gives:

**Proposition 22.2 :** *For all  $h$  there is some  $h'$  and for all  $l$  there is some  $r'$  such that if  $Q \subseteq J(h', l)$ , then  $Y(Q) \subseteq J(h, l')$ .*  $\diamond$

If we iterate this procedure  $\nu = 2\kappa(\Sigma)$  times, then we arrive a set containing  $Y^\infty(Q)$ , as defined in Section 6. In particular, we see:

**Theorem 22.3 :** *There is some  $h_1$ , such that if  $Q \subseteq J(h, l)$ , then  $Y^\infty(Q) \subseteq J(0, l')$ , where  $l'$  depends only on  $r$  and  $\kappa(\Sigma)$ .*  $\diamond$

This means that all curves in the heirarchy can be realised as closed geodesics of bounded length in  $M$  which tend out the end. In particular they are non-trivial and non-peripheral in  $M$ .

In particular, this means that all curves in the heirarchy can be realised a closed geodesics of bounded length in  $M$ , which go out the end  $e$ . In particular, they are all non-trivial and non-peripheral.

We can now proceed to the final step, described in the next section.

### 23. The model space of a degenerate end.

Let  $M$  be a complete hyperbolic 3-manifold, with  $\pi_1(M)$  finitely generated, and let  $e \in \mathcal{E}_D(M)$ . Let  $\Sigma = \Sigma(e)$ . We can assume that  $\kappa = \kappa(\Sigma) \geq 2$ .

We describe a model,  $\Psi(P_e) \cong \Sigma \times [0, \infty)$  and a proper map,  $f_e : \Psi(P_e) \rightarrow \Psi(M)$ , which sends  $\Psi(P_e)$  out the end  $e$ . It will be a proper homotopy equivalence into a neighbourhood,  $E \cong \Sigma \times [0, \infty)$  of  $e$  in  $\Psi(M)$ . It follows that  $f_e$  maps surjectively to another neighbourhood  $E_0 \subseteq \Psi(M)$  of  $e$ .

Only the initial discussion, as far as Proposition 23.1, is directly relevant to the proof of the Ending Lamination Theorem, completed in Section 24. The remainder establishes that the end invariant is well defined, and that it has a number of natural descriptions.

Let  $E \cong \Sigma \times [0, \infty)$  be a neighbourhood of  $e$  in  $\Psi(M)$ . Let  $\rho_E$  be the electric pseudometric on  $E$ , as defined in Section 20. We recall Proposition 17.1, namely that there is a geodesic ray in  $\mathcal{G}(e)$  each of whose elements have representatives of length at most  $L_0 = L_0(\kappa)$  in  $E$ , and which go out the end  $e$ . In Section 17, we defined  $\mathbf{a}(e) \subseteq \partial\mathcal{G}(e)$  as the set of endpoints of such geodesic rays. Proposition 17.2 tells us that any the vertices of a tight geodesic tends to a point of  $\mathbf{a}(e)$  also have bounded length representatives in  $M$ .

We can continue, as in Section 16, to construct an annulus system,  $W = \bigcup \mathcal{W}$ , in  $\Sigma \times [0, \infty)$ . In particular, the curves corresponding to  $\Omega$  are all a bounded distance (in fact 1) from a geodesic ray,  $\pi$ , in  $\mathcal{G}(e)$ . This can be assumed to converge on any given  $a \in \mathbf{a}(e)$ . Moreover, for all  $\Omega \in \mathcal{W}$ , we have  $\Omega^* \subseteq E$  and  $\text{length}(\Omega^*) \leq L = L(\kappa)$ . (For this, we should note that any subsurface,  $\Phi$ , featuring in the construction of the heirarchy defining  $\mathcal{W}$  can be assumed incompressible in  $M$  — note that any curve disjoint from  $\partial\Phi$  can be realised as a closed geodesic in  $E$ , using Lemma 20.5, and is therefore non-trivial  $M$ . The fact that  $\Phi$  is incompressible then follows by Dehn's Lemma. This allows us to apply the a-priori bounds theorem for subsurfaces, see Theorem 15.2, as in Section 15.) In fact, we can also assume that  $D(\Omega^*) = \rho_E(\Omega^*, \partial E) \geq t$  for an arbitrarily large constant  $t \geq 0$  (chosen as below).

Now let  $\tilde{\Psi}(P_e)$  be the universal cover of  $\Psi(P_e)$ . There a cover  $\hat{E}$  of  $E$ , that naturally embedded in  $\mathbf{H}^3$ , namely the preimage of  $E$  under the covering map  $\mathbf{H}^3 \rightarrow M$ . Let  $\hat{\Psi}(P_e)$  be the corresponding cover of  $\Psi(P_e)$ . That is,  $\tilde{\Psi}(P_e)$  is triangulated by the truncated complex,  $R(\Pi)$ . We will construct an equivariant map  $\hat{f}_e : \hat{\Psi}(P_e) \rightarrow \hat{E}$  and then project to give us the map  $f_e : \Psi(P_e) \rightarrow \Psi(M_e)$ . For this we use the simplicial complex  $\Pi$  described as in Section 10, associated to the universal cover of  $\Psi(P_e)$ . That is,  $\tilde{\Psi}(P_e)$  is triangulated by the truncated complex,  $R(\Pi)$ . There is a quotient,  $\hat{\Pi}$  corresponding to  $\hat{\Psi}(P_e)$ . This satisfies the conditions (C1)–(C7) laid out in Section 10, so we can construct  $\hat{f}_e : \hat{\Psi}(P_e) \rightarrow M$  as in Section 11. In fact, this maps to the preimage of  $\Psi(M)$  in  $\mathbf{H}^3$ . Its image is a bounded distance from the union of the preimages of closed geodesics  $\bar{\Omega}$  and Margulis tubes  $\Delta(\Omega)$  for  $\Omega \in \mathcal{W}$ . Thus, by taking  $t$  large in relation to this bound, we will have  $\hat{f}_e(\hat{\Psi}(P_e)) \subseteq \hat{E}$ . Projecting, we get a proper map,  $f_e : \Psi(P_e) \rightarrow E$ . Let  $E_0 \subseteq \Psi(M_e)$  be a neighbourhood of the end, homeomorphic to  $\Sigma(e) \times [0, \infty)$ , contained in  $f_e(\Psi(P_e))$ .

Now, by Proposition 7.7, as in Section 16, the collection of Margulis tubes in  $E$  is unlinked. Since this includes all tubes with sufficiently short core curves, we note as a consequence:

**Proposition 23.1 :** *There is some  $\eta(\kappa) > 0$  depending only on  $\kappa = \kappa(\Sigma)$  such that if  $e \in \mathcal{E}_D$  is any degenerate end of  $\Psi(M)$ , then there is a neighbourhood,  $E$ , of  $e$  in  $\Psi(M)$ , with  $E \cong \Sigma(e) \times [0, \infty)$  such that the set of all closed geodesics in  $M$  of length at most  $\eta(\kappa)$  and lying in  $E$  are unlinked in  $E$ .*

We can go on to show that the end invariant of  $e$  is well defined:

**Proposition 23.2 :** *If  $e \in \mathcal{E}_D$ , then  $\mathbf{a}(e)$  is a singleton.*

**Proof :** Let  $a \in \mathbf{a}(e)$ . Let  $\Psi(P_e)$  be a model based on an annulus system  $W = \bigcup \mathcal{W}$  in  $\Sigma(e) \times [0, \infty)$ , where the core curves lie a bounded distance from a geodesic ray,  $\pi \subseteq \mathcal{G}(e)$  tending to  $a \in \partial\mathcal{G}(e)$ . As in Proposition 23.1, we see that every closed curve of length at most  $\eta(\kappa)$  in  $E$  corresponds to a Margulis tube in  $\Psi(P_e)$ , and so lies a bounded distance from  $\pi$  in  $\mathcal{G}(e)$ .

If  $a' \in \mathbf{a}(e)$ , then we can similarly construct  $\pi'$ ,  $W' = \bigcup \mathcal{W}'$ , a model  $\Psi(P'_e)$ , and a map  $f'_e : \Psi(P'_e) \rightarrow \Psi(M_e)$ .

Suppose there is a sequence of geodesics of length at most  $\eta(\kappa)$  going out the end  $e$  of  $\Psi(M)$ . The curves must lie a bounded distance from both  $\pi$  and  $\pi'$ . It follows that  $\pi$  and  $\pi'$  are parallel, and so  $a = a'$ .

Suppose not. Then we can assume that  $E$  contains no such geodesic. In other words, it has bounded geometry. Now every point  $x$  of  $E$  lies a bounded distance from a closed geodesic of the form  $\bar{\Omega}$  for  $\Omega \in \mathcal{W}$ . It also lies a bounded distance from another  $\bar{\Omega}'$  for  $\Omega' \in \mathcal{W}'$ . Applying Lemma 12.3, we see that the curves corresponding to  $\Omega$  and  $\Omega'$  are a bounded distance apart in  $\mathcal{G}(e)$ . Also, by the construction of the models, such curves lie a bounded distance respectively from  $\pi$  and  $\pi'$  in  $\mathcal{G}(e)$ . Letting  $x$  tend out the end  $e$ , we see again that  $\pi$  and  $\pi'$  are parallel in  $\mathcal{G}(e)$  and so  $a = a'$ .  $\diamond$

We can now define  $a(e)$  by setting  $\mathbf{a}(e) = \{a(e)\}$ .

Here is another, more natural way of describing the end invariant:

**Proposition 23.3 :** *Let  $\pi$  be any geodesic ray in  $\mathcal{G}(e)$  tending to  $a(e)$  in  $\partial\mathcal{G}(e)$ . Given any  $l \geq 0$ , there is some  $r$  depending only on  $\kappa(\Sigma(e))$  and  $l$ , and a neighbourhood  $E(l) \cong \Sigma \times [0, \infty)$ , of  $e$  in  $\Psi(M)$ , such that if  $\gamma_M$  is any simple curve in  $E(l)$  of length at most  $l$ , then  $d_{\mathcal{G}}(\pi, \gamma_M) \leq r$  in  $\mathcal{G}(e)$ , where  $\gamma = [\gamma_M] \in X(e)$ .*

It follows that any sequence of simple curves of bounded length in  $\Psi(M)$  going out the end  $e$  must converge on  $a(e)$  in  $\mathcal{G}(e)$ . This therefore gives another characterisation of  $a(e)$ . Note that, if  $\gamma = [\gamma_M]$ , then  $d_{\mathcal{G}}(\gamma, \text{proj } \gamma)$  is bounded in terms of  $\kappa(\Sigma)$ . Thus, in Proposition 23.3, there is no loss of generality in setting  $l = l_0$ , in which case,  $r$  will depend only on  $\kappa$ .

**Proof of Proposition 23.3 :**

This elaborates on the proof of Proposition 23.2.

We can take  $\pi$  to be any geodesic converging to  $a(e)$ . In particular, we can take it to be the geodesic featuring in the construction of the annulus system  $W = \bigcup \mathcal{W}$ , and

the resulting model,  $\Psi(P_e)$  described above. Thus,  $d_{\mathcal{G}}(\pi_V(\Omega), \pi)$  is bounded above in terms of  $\kappa(\Sigma)$  for all  $\Omega \in \mathcal{W}$ . We construct  $f_e : \Psi(P_e) \rightarrow \Psi(M)$  as before. We have neighbourhoods  $E, E_0 \cong \Sigma \times [0, \infty)$  with  $E_0 \subseteq f_e(\Psi(P_e)) \subseteq E$ .

Let  $\rho_E$  be the electric pseudometric on  $E$  (as defined in Section 20), and let  $D : E \rightarrow [0, \infty)$  be the depth function. Let  $\gamma^* \subseteq E$  be a closed geodesic in  $M$ , of length at most  $l_0$ , with  $\gamma = [\gamma^*] \in \mathcal{G}(e) \equiv \mathcal{G}(\Sigma)$ , and with  $D(\gamma^*)$  sufficiently large. We want to show that  $d_{\mathcal{G}}(\gamma, \pi)$  is bounded in terms of  $\kappa(\Sigma)$ .

We fix some geodesic  $\beta^* \subseteq E_0$  with  $\text{length}(\beta^*) \leq l_0$ , and with  $\beta = [\beta^*] \in \mathcal{G}(\Sigma)$ . We can take  $D_1 = D(\beta^*)$  to be arbitrarily large. We choose some  $D_2 > D_1$ , with  $D_2 - D_1$  sufficiently large, as determined below. We can assume that  $D(\gamma^*) \geq D_2$ .

We first claim that there is a sequence,  $\gamma = \gamma_0, \gamma_1, \dots, \gamma_p$ , in  $X(\Sigma)$  which is geodesic in  $\mathcal{G}(\Sigma)$ , and with  $\text{length}(\gamma_i^*) \leq l_1$ ,  $\gamma_i^* \subseteq E$ , and  $D(\gamma_i^*) \geq D_1$  for all  $i \in \{0, \dots, p-1\}$ , and  $D(\gamma_p) \leq D_1$ . Here  $l_1$  depends only on  $\Sigma(\Sigma)$ . Note that, necessarily  $\rho_E(\gamma_i^*, \gamma_{i+1}^*)$  is bounded above in terms of  $\kappa(\Sigma)$ . In particular,  $D_1 - D(\gamma_p)$  is bounded.

To prove the claim, first note that we can find a sequence  $\gamma = \beta_0, \beta_1, \dots, \beta_q$  with  $\text{length}(\beta_j^*) \leq l_0$ ,  $D(\beta_j^*) > D_1$  for  $j < q$ , and  $D(\beta_q) \leq D_1$ . For this, we apply the argument of Proposition 22.1. We now connect  $\beta_0$  to  $\beta_j$  by a geodesic  $\pi_j$  in  $\mathcal{G}(\Sigma)$ . We note that consecutive  $\pi_j$  remain a bounded distance apart in  $\mathcal{G}(\Sigma)$ . As in the proof of Proposition 17.1, we can obtain  $(\gamma_i)_{i=0}^p$  as an initial segment of  $\pi_j$ , for some  $j$ . The bound on  $\text{length}(\gamma_i^*)$  is a consequence of the a-priori bounds theorem (see Theorem 3.11).

We now proceed as in Lemma 8.2 to construct an annulus system,  $W_0 = \bigcup \mathcal{W}_0$  in  $\Sigma \times [-1, 1]$  with  $\gamma = \gamma_0 \subseteq X_-(\mathcal{W}_0)$  and  $\gamma_p \subseteq X_+(\mathcal{W}_0)$ . (Here  $X_{\pm}(\mathcal{W}_0) = \pi_V(W \cap (\Sigma \times \{\pm 1\}))$ .) From the construction, we assume that  $\gamma_i = \pi_V(\Omega_i)$  for some  $\Omega_i \in \mathcal{W}_0$ . From the a-priori bounds theorem, we have  $\text{length}(\Omega_i^*) \leq l$  for all  $\Omega \in \mathcal{W}_0$ , where  $l = l(\kappa(\Sigma))$  depends only on  $\kappa(\Sigma)$ . Also, taking  $D_1$  large enough, we can assume that  $D(\Omega^*)$  is greater than some arbitrarily large constant for  $\Omega \in \mathcal{W}_0$ . We now use  $\mathcal{W}$  to construct a model,  $\Psi(P)$ , where each annulus  $\Omega$  is replaced by a Margulis tube  $T(\Omega)$ , or an annulus  $\Delta(\Omega)$ , where  $\Omega$  meets  $\partial_H \Psi(P)$ .

Given that the  $\Omega^*$  all have bounded length and arbitrary depth in  $E$ , we can construct a Lipschitz homotopy equivalence  $f : \Psi(P) \rightarrow E$  as with  $\Psi(P_e)$ , using the construction of Section 11. Moreover, we can assume that  $f(\Psi(P)) \subseteq E_0$ . The map  $f$  sends the boundary curves  $\gamma_p$  and  $\gamma$  to  $\gamma_p^*$  and  $\gamma^*$  respectively.

Since  $D(\gamma^*) - D(\gamma_p^*) \leq D_2$  is arbitrarily large,  $\rho_E(\gamma_p^*, \gamma^*)$  is arbitrarily large. Now  $d_{\mathcal{G}}(\gamma_p, \gamma)$  is linearly bounded below in terms of  $\rho_E(\gamma_p^*, \gamma^*)$  and so we can assume that  $p = d_{\mathcal{G}}(\gamma_p, \gamma)$  to be as large as we want. We fix  $m \in \mathbf{N}$ , as determined below. We can assume that  $p \geq m$ , so that (after reparametrising the second coordinate)  $\Sigma \times \{0\}$  meets  $W$  in a curve,  $\alpha$ , with  $d_{\mathcal{G}}(\alpha, \gamma) = m$ . Let  $\Psi_0 \subseteq \Psi(P)$  be the subset of the model corresponding to  $\Sigma \times [-1, 1]$ . Thus  $\gamma$  and  $\alpha$  are boundary curves in  $\partial_H \Psi_0$ . Note that  $f(\alpha) = \alpha^*$ .

We distinguish two cases.

Suppose first that  $f(\Psi_0)$  is not a subset of the thick part,  $\Theta(M)$ . In other words, it meets some Margulis tube lying in  $E_0 \subseteq f_e(\Psi(P_e))$ . This Margulis tube must feature in both  $\Psi_0$  and  $\Psi(P_e)$ . In particular, its core curve corresponds to an annulus in both  $\mathcal{W}$  and  $\mathcal{W}_0$ . It is therefore a bounded distance from both  $\gamma$  and  $\pi$ . Thus,  $d_{\mathcal{G}}(\gamma, \pi)$  is bounded above in terms of  $m$  and hence in terms of  $\kappa(\Sigma)$ .

We can therefore assume that  $f(\Psi_0) \subseteq \Theta(M)$ . This implies that the injectivity radius in  $\Psi_0$  is also bounded below. The idea now is to find a sufficiently wide band  $B$  in  $E$ , lying in  $f(\Psi_0)$  and  $f(\Psi(P))$ . This will then contain bounded length (in fact geodesic) realisations of curves in the respective annulus systems. (As usual, by a “band”,  $B$ , we mean, a subset of  $E$  bounded by two disjoint horizontal fibres,  $\partial_+ B$  and  $\partial_- B$ .) We can then apply Lemma 12.3 to tell us that these curves are a bounded distance apart in  $\mathcal{G}(\Sigma)$ .

To this end, we know that  $f : \Psi(P) \rightarrow E$  is a quasi-isometry with respect to the electric pseudometrics. Thus, if  $m$  is sufficiently large, we have  $\rho_E(\partial_- \Psi_0, \partial_+ \Psi_0)$  large. In particular (using Theorem 3.1, as in the proof of (Q3) of Lemma 13.2), we can assume that  $f(\Psi_0)$  contains a band  $B$  with  $\partial_{\pm} B$  of bounded diameter, and with  $d_M(\partial_- B, \partial_+ B)$  large. Thus,  $B$  will be  $k$ -convex, where  $k$  depends only on the diameters of  $\partial_{\pm} B$ . Moreover, since  $B \subseteq f(\Psi_0) \cap f_e(\Psi(P_e))$ , we can assume that it contains closed geodesics,  $\Omega^*$  and  $\Omega_0^*$ , where  $\Omega \in \mathcal{W}$  and  $\Omega_0 \in \mathcal{W}_0^*$ . Writing  $\delta = [\partial_V \Omega]$ ,  $\delta_0 = [\partial_V \Omega_0] \in \mathcal{G}(\Sigma)$ , Lemma 12.3, tells us that  $d_{\mathcal{G}}(\delta, \delta_0)$  is bounded. In the above, the bounds depend only on  $\kappa(\Sigma)$ . We can therefore fix some  $m$  sufficiently large in relation to  $\kappa(\Sigma)$  to make the argument work. Now,  $d_{\mathcal{G}}(\delta, \pi)$  and  $d_{\mathcal{G}}(\delta_0, \gamma)$  are bounded in terms of  $\kappa(\Sigma)$ . It follows that  $d_{\mathcal{G}}(\gamma, \pi)$  is bounded, as required.  $\diamond$

## 24. Proof of the Ending Lamination Theorem in the general case.

In this section, we explain how the model space is constructed and give a proof of Proposition 4.1. We go on to explain how this implies the Ending Lamination Theorem.

We recall from Section 9, the notions of (universally) sesquipschitz maps.

Suppose  $\Psi$  is a topologically finite 3-manifold such that  $\partial\Psi$  is a disjoint union of tori and cylinders. Suppose we have a decomposition of its set of ends as  $\mathcal{E} = \mathcal{E}_F \sqcup \mathcal{E}_D$ , so that no base surface of any end is a disc, annulus, sphere or torus, and no base surface of an end of  $\mathcal{E}_D$  is a three-holed sphere. Suppose that to each end  $e \in \mathcal{E}_D$ , we have associated some  $a(e) \in \partial\mathcal{G}(e)$ . From this data, we construct a riemannian manifold,  $P$ , without boundary, with a submanifold,  $\Psi(P)$ , which is homeomorphic to  $\Psi$ . We show that it satisfies the conclusion of Proposition 4.1.

The construction of  $P$  will be essentially the same for the irreducible case. If  $e \in \mathcal{E}_F$ , we take the same model  $\Psi(P_e)$  as before. If  $e \in \mathcal{E}_D$ , we construct a model  $\Psi(P_e)$  as in Section 23. We now construct  $\Psi(P)$  by attaching the  $\Psi(P_e)$  to a core with an arbitrary riemannian metric extending that of the  $\Psi(P_e)$ , and such that the components of  $\partial\Psi(P)$  are all intrinsically euclidian cylinders or unit square tori. Finally, we construct  $P$  by gluing in cusps to these boundary components exactly as in the incompressible case.

We construct a map  $f : P \rightarrow M$  in stages. First, we define  $f_e : \Psi(P_e) \rightarrow \Psi(M)$  for each  $e \in \mathcal{E}$  and then extend arbitrarily over the compact core to give  $f : \Psi(P) \rightarrow \Psi(M)$ , and then over the cusps in the obvious way as before.

Suppose the  $e \in \mathcal{E}_F$ . Here the construction is as in the incompressible case. We made use of multiplicative bounds of distance distortion when projecting to neighbourhoods of the convex core, but the same bounds remain valid in this case.

Suppose  $e \in \mathcal{E}_D$ . We defined  $f_e : \Psi(P_e) \rightarrow \Psi(M)$  in Section 23. We can now proceed,

exactly as in Section 16 to show that  $f_e : \Psi(P_e) \rightarrow E$  is universally sesquilipshtiz, where  $E_e = E$  is the neighbourhood defined in Section 23.

The extension to the core,  $\Psi_0$ , and then to the cusps is essentially elementary. This gives us a lipschitz map  $f : P \rightarrow M$ , and by construction  $f|_{\Psi(P)}$  is proper, and each end of  $\Psi(P)$  goes to the corresponding end of  $\Psi(M)$ .

Since for all  $e \in \mathcal{E}$ , the map  $f_e : \Psi(P_e) \rightarrow \Psi(M_e)$  is universally sesquilipschitz, hence sesquilipschitz, the map  $\hat{f}_e : \hat{\Psi}(P_e) \rightarrow \hat{E}$  is sesquilipschitz. Note that  $\hat{\Psi}(P_e)$  and  $\hat{E}$  are subsets of the universal covers of  $\Psi(P)$  and  $\Psi(M)$  respectively. This is what we use when gluing the pieces together, to see, as in Section 16, that the lift between universal covers is a quasi-isometry.

This proves Proposition 4.1.

Now we can use the same model space for two homeomorphic hyperbolic 3-manifolds with the same degenerate end invariants to deduce:

**Proposition 24.1 :** *Suppose that  $M$  and  $M'$  are complete hyperbolic 3-manifolds and that there is a homeomorphism from  $M$  to  $M'$  that sends cusps of  $M$  into cusps of  $M'$  and conversely. Suppose that the induced map between the non-cuspidal parts sends each geometrically finite end to a geometrically finite end and each degenerate end to a degenerate end. Suppose that (under the induced homeomorphisms of base surfaces) the end invariants of corresponding pairs of degenerate ends are equal. Then there is an equivariant quasi-isometry between the universal covers of  $M$  and  $M'$ .  $\diamond$*

This is identical to Propostion 16.5 with the hypothesis of “indecomposable” omitted. The argument is the same, given Proposition 4.1.

In particular, we get an equivariant quasi-conformal extension,  $f : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$ . In the case where the geometrically finite end invariants are also equal, we can find another equivariant map,  $g : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$ , which agrees with  $f$  on the limit set, and is conformal on the discontinuity domain. We need to verify that  $g$  is also quasiconformal.

Let  $U$  be component of the domain of discontinuity of  $\Gamma \equiv \pi_1(M)$  acting on  $\partial\mathbf{H}^3$ . As in Section 16, we see that  $g^{-1} \circ f|_U$  moves every point a bounded distance in the Poincaré metric. To show that  $g^{-1} \circ f$ , and hence  $g$ , is conformal everywhere, it is enough to give a suitable bound on the euclidean metric,  $d_e$ , in term of the Poincaré metric (using an identification of  $\partial\mathbf{H}^3$  with  $\mathbf{C} \cup \{\infty\}$ ). In Section 16, the formula we used was:  $d_e(z, w) \leq (e^{2k} - 1) \max\{d_e(z, \partial U), d_e(w, \partial U)\}$ , under the assumption that  $U$  was simply connected. We want a variant of this when  $U$  might not be simply connected. Let  $\Delta$  be the the universal cover of  $U$ , which we identify with the Poincaré disc. We have a group,  $G$ , acting on  $\Delta$ , and a normal subgroup,  $H \triangleleft G$ , such that  $U = \Delta/H$  and  $R = \Delta/G$  is Riemann surface of finite type — here the corresponding geometrically finite end invariant. Note that  $H$  has no parabolic elements.

**Lemma 24.2 :** *If  $z, w \in U$  are a distance at most  $k$  apart in the Poincaré metric on  $U$ , then  $d_e(z, w) \leq (e^{\mu k} - 1) \max\{d_e(z, \partial U), d_e(w, \partial U)\}$ , where  $\mu > 0$  is a constant depending only on the Riemann surface  $R$ .*

**Proof :** Choose  $t > 0$  so that the shortest closed geodesic on  $R$  has length greater than  $2t$  in the Poincaré metric. Since  $H$  has no parabolics, any  $t$ -disc in  $R$  lifts to an embedded disc in  $U$ . Put another way, if  $z \in U$ , then the  $t$ -disc,  $D$ , about  $z$  in the Poincaré metric is embedded in  $U$ . We can lift this to the Poincaré disc,  $\Delta$ , centred at  $z$ . Here  $D$  will have euclidean radius  $\tanh t$ . Now consider the  $D \rightarrow U$  with respect the euclidean metric on both  $D$  and  $U$ . By the Koebe Quarter Theorem, the norm of the derivative at  $z$  is at most  $\frac{r}{4 \tanh t}$  at the centre. But a euclidean unit vector at the origin has Poincaré norm 2, and so we deduce that  $|ds| \geq \frac{1}{\mu r} |dz|$ , where  $\mu = 2/\tanh t$ , where  $|ds|$  and  $|dz|$  are the infinitesimal Poincaré and euclidean metrics, and  $r = d_e(z, \partial U)$ . We can now integrate this to derive the required inequality.  $\diamond$

Now since there are only finitely many geometrically finite ends we can take the same  $\mu$  for all components of the discontinuity domain. Thus, the inequality of Lemma 23.1 applies equally well when  $U$  is replaced by the whole discontinuity domain. This is sufficient to bound the metric quasiconformal distortion of  $g^{-1} \circ f$  on the limit set, showing that  $g^{-1} \circ f$  and hence  $f$  is quasiconformal.

The remainder of the argument is now standard. We see using the result of Sullivan, stated here as Theorem 3.8, that  $f$  must be conformal. It therefore gives rise to an isometric conjugacy between the actions of  $\pi_1(M)$  and  $\pi_1(M')$  on  $\mathbf{H}^3$ , showing that  $M$  and  $M'$  are isometric. The fact that this isometry satisfies the conditions laid out in Theorem 2.4 is now elementary.

## 25. The uniform injectivity theorem.

The original Uniform Injectivity Theorem for pleated surfaces goes back to Thurston [Th], and there have been several variations since. Most have made some assumption of incompressibility, which is sufficient for the indecomposable case of the Ending Lamination Conjecture, see for example [Mi1]. A variation for the pleating loci of pleated surfaces in handlebodies, is given in [N]. The argument there would apply to more general situations, where the surface is deep in the end of a 3-manifold, without assuming the end is incompressible. However the quantification means that the required depth may be dependent on other constants, and it is not clear that this result can be adapted to the argument given in [Bow2].

In this section, we give a version which depends just on a local incompressibility assumption. As with earlier versions, we argue by contradiction, passing to a limit. This means that the constants involved are not a-priori computable. We will make our statement for laminations, though we only apply it in this paper for multicurves. It will easily be seen to imply Lemma 18.4. To simplify notation we only deal with 1-lipschitz pleating surfaces, though the argument will be seen to apply equally well to uniformly lipschitz maps. The basic idea of constructing a partial covering space to derive a contradiction can be found in Thurston's original [Th], though since we have altered a number of definitions and hypotheses, we work things through from first principles.

Let  $(M, d)$  be a complete hyperbolic 3-manifold, with projectivised tangent bundle,  $E \rightarrow M$ . We write  $d_E$  for the metric on  $E$ . The map  $(E, d_E) \rightarrow (M, d)$  is then 1-

lipschitz. We shall define a *pleating surface* as a 1-lipschitz map  $\phi : (\Sigma, \sigma) \longrightarrow (M, d)$  where  $(\Sigma, \sigma)$  is a compact hyperbolic surface with (possibly empty) horocyclic boundary. By a *lamination*  $\lambda \subseteq \Sigma$ , we mean a geodesic lamination in the usual sense, see for example [CanEG]. We say that a pleating surface,  $\phi : \Sigma \longrightarrow M$  *realises*  $\lambda$  if it sends each leaf of  $\lambda$  locally isometrically to a geodesic in  $M$ . We write  $\psi = \psi_\phi : \lambda \longrightarrow E$  for the lift to  $E$ , and let  $\Lambda = \psi(\lambda)$ . (It will follow from subsequent hypotheses that  $\psi$  will be a homeomorphism to  $\Lambda$ , in which case, the notion coincides with that already defined for multicurves in Section 18.) Note that we are not assuming  $\lambda$  to be connected.

Here is our formulation of uniform injectivity:

**Proposition 25.1 :** *Given positive  $\kappa, \eta, \epsilon$ , there is some  $\delta > 0$  with the following property. Suppose that  $\Sigma$  is a compact surface with  $\kappa(\Sigma) = \kappa$ , and suppose that  $\phi : (\Sigma, \sigma) \longrightarrow (M, d)$  is a pleating surface to a complete hyperbolic 3-manifold  $M$ , realising a geodesic lamination  $\lambda \subseteq \Sigma$ . Suppose:*

(U1) *For all  $x \in \lambda$ , the injectivity radius of  $M$  at  $\phi(x)$  is at least  $\eta$ .*

(U2) *There is a map  $\theta : N(\phi(\lambda), \eta) \longrightarrow \Sigma$  such that the composition  $\theta \circ \phi|_{N(\lambda, \eta)} \longrightarrow \Sigma$  is homotopic to the inclusion of  $N(\lambda, \eta)$  in  $\Sigma$ .*

*Then for all  $x, y \in \lambda$ , if  $d_E(\psi_\phi(x), \psi_\phi(y)) \leq \delta$  then  $\sigma(x, y) \leq \epsilon$ .*

In (U1), we are demanding the  $f(\lambda)$  lie in the thick part of  $M$ . This is equivalent to putting a bound on the diameter of each component of  $f(\lambda)$  (and excluding cores of Margulis tubes, though such components could be easily dealt with explicitly).

In (U2),  $N(Q, r)$  denotes the open  $r$ -neighbourhood of  $Q$ . We are assuming that  $\phi$  is 1-lipschitz, and so  $\phi(N(\lambda, \eta)) \subseteq N(\phi(\lambda), \eta)$ . As remarked earlier, it is not hard to see that the hypotheses imply that  $\psi|_\lambda$  is injective (cf. Lemma 25.25). Thus,  $\psi$  is a homeomorphism to  $\Lambda$ . Note that  $\Lambda$  admits a decomposition into geodesic leaves, being invariant under a local geodesic flow. The map  $\theta$  need only be defined up to homotopy.

Although it is implicit in our earlier definitions that  $\Sigma$  is connected, this is not really required. Indeed for the proof, it is convenient to allow for a disconnected surface. This allows us to cut away the thin part of  $\Sigma$ , so that all of  $\Sigma$  maps into the thick part of  $M$ .

We can assume that the length of each boundary curve of  $\Sigma$  is bounded below by some positive constant. To see this, note that  $\lambda$  cannot cross any horocycle of length 1, we can simply cut away the remainder of  $\Sigma$ . Similarly, we can cut away the thin part of  $\Sigma$  along curves of length bounded below, and constant outward curvature bounded above. Thus, we can assume that the injectivity radius of  $\Sigma$  is bounded below, though at the possible cost of disconnecting the surface.

We finally note that it would be enough to assume that  $\phi$  is  $\mu$ -lipschitz for some  $\mu$ , in which case, of course,  $\delta$  will also be a function of  $\mu$ . The argument is unchanged modulo introducing various factors of  $\mu$  into the proceedings.

Before beginning with the proof, we recall some basic facts and introduce some notation relating to laminations.

Let  $\lambda \subseteq \Sigma$  be a lamination, and write  $\vec{\lambda}$  for the unit tangent bundle to  $\lambda$ . Thus  $\lambda$  is a quotient of  $\vec{\lambda}$  under the involution, denoted  $[\vec{a} \mapsto -\vec{a}]$ , that reverses direction. We write  $a \in \lambda$  for the “basepoint” of  $\vec{a}$  (or of  $-\vec{a}$ ). Given  $\vec{a} \in \vec{\lambda}$ , write  $\vec{a}_t \in \lambda$  for the vector obtained

by flowing a distance  $t$  in the direction of  $\vec{a}$ . If  $\vec{a}, \vec{b} \in \vec{\lambda}$ , we write  $\vec{a} \approx \vec{b}$  if  $\sigma(a_t, b_t) \rightarrow 0$ . This is an equivalence relation on  $\vec{\lambda}$ . Each equivalence class has at most two elements. (It identifies pairs in a finite set of non-closed directed boundary leaves.) If  $\vec{a}, \vec{b} \in \vec{\lambda}$  with  $\vec{a} \approx \vec{b}$  and with  $\sigma(a, b) \leq \eta$ , then  $\sigma(a_t, b_t)$  is monotonically decreasing for  $t \geq 0$ . If  $a, b \in \lambda$  lie in the same non-closed leaf,  $l$ , of  $\lambda$ , we write  $[a, b]$  for the interval of  $\lambda$  connecting them. Similarly, if  $a, b \in \Sigma$  with  $\sigma(a, b) < \eta$ , we write  $[a, b]$  for the unique shortest geodesic connecting them. These notations are consistent.

We write  $\Upsilon(\lambda)$  for the union of  $\lambda$  and all those intervals  $[a, b]$  for which  $\sigma(a, b) \leq \eta/8$  and  $\vec{a} \approx \vec{b}$  for some tangents,  $\vec{a}$  and  $\vec{b}$ . Thus, each component of  $\Upsilon(\lambda)$  is either a closed leaf of  $\lambda$ , or else a closed non-annular subsurface of  $\Sigma$ . Each component of  $\Upsilon(\lambda) \setminus \lambda$  is a “spike” between two asymptotic rays in  $\lambda$ .

We can now begin the proof of Proposition 25.1. Let us assume that it fails. In this case, we can find a sequence,  $\sigma_i$ , of hyperbolic metrics on  $\Sigma$ , geodesic laminations,  $\lambda_i \subseteq (\Sigma, \sigma_i)$ , pleating surfaces  $\phi_i : (\Sigma, \sigma_i) \rightarrow (M_i, d_i)$ , maps  $\theta_i : N(\phi_i(\lambda_i), \eta) \rightarrow \Sigma$ , and points,  $x_i, y_i \in \lambda_i$  so that  $\sigma_i(x_i, y_i) \geq \epsilon$ , but  $d_{E_i}(\psi_i(x_i), \psi_i(y_i)) \rightarrow 0$ , where  $\psi_i = \psi_{\phi_i}$ . For each  $i$ , the hypotheses of Proposition 25.1 are satisfied for fixed  $\eta, \epsilon > 0$ .

As observed earlier, we can assume that the lengths of the boundary curves of  $(\Sigma, \sigma_i)$  are all bounded below, and so the structures  $(\Sigma, \sigma_i)$  all lie in a compact subset of moduli space. We can thus pass to a subsequence so that these structures converge on some hyperbolic structure  $(\Sigma, \sigma)$ . This may involve precomposing the  $\phi_i$  and postcomposing the  $\theta_i$  with suitable inverse mapping classes of  $\Sigma$ . Indeed, after applying precomposing  $\phi_i$  suitable self homeomorphisms of  $\Sigma$ , we can suppose that the metrics  $\sigma_i$  converge to  $\sigma$ .

Passing to another subsequence we can assume that  $\lambda_i$  converges to a lamination  $\lambda \subseteq \Sigma$  in the Hausdorff topology. Note that  $N_i = N(\lambda_i, \eta/2)$  converges on  $N = N(\lambda, \eta/2)$ . Let  $O_i = N(\phi_i(\lambda_i), \eta/2)$ . Thus  $\phi_i(N_i) \subseteq O_i$ . We can again pass to a subsequence so that  $(O_i, d_i)$  converges on a space,  $(O, d)$ , in the Gromov-Hausdorff topology. The space  $(O, d)$  is an incomplete hyperbolic 3-manifold, in the sense of being locally isometric to  $\mathbf{H}^3$ . (The metric  $d$  need not be a path metric on  $O$ . Indeed  $O$  need not be connected.) We can also observe that the maps  $\phi_i : N_i \rightarrow O_i$  converge to a 1-lipschitz map  $\phi : N_i \rightarrow O_i$  (in the sense that their graphs converge in the Gromov-Hausdorff topology). Now let  $E_O \rightarrow O$  be the projectivised tangent bundle to  $O$ , and let  $\psi : \lambda \rightarrow E_O$  be the lift of  $\phi|_{\lambda}$  to  $E_O$ . We write  $\Lambda = \psi(\lambda)$ . As before,  $\Lambda$  is partitioned into leaves, which are images of leaves of  $\vec{\lambda}$  and  $\vec{\Lambda}$  for the tangent spaces of  $\lambda$  and  $\Lambda$  respectively. Note that if  $p, q \in \Lambda$  with  $d(p, q) \leq \eta/2$ , then there is a geodesic,  $[p, q]$ , of length  $d(p, q)$  connecting  $p$  and  $q$  in  $O$ .

Finally, we can pass to yet another subsequence so that  $x_i \rightarrow x \in \lambda$  and  $y_i \rightarrow y \in \lambda$ . Thus  $\sigma(x, y) > \eta$ , but  $\psi(x) = \psi(y)$ . In particular,  $\psi$  is not injective. (From this point on, we could focus our attention on one component of  $O$  where the restriction of  $\psi$  is not injective, though this is not logically necessary.)

**Lemma 25.2 :** *There is some  $k \geq 0$  such that if  $\pi$  is a path in  $N$  connecting two points,  $a, b \in \lambda$  with  $d(\phi(a), \phi(b)) \leq \eta/2$  such that  $\phi(\pi) \cup [\phi(a), \phi(b)]$  is homotopically trivial in  $O$ , then  $\pi$  is homotopic relative to  $a, b$  in  $\Sigma$  to a path in  $\Sigma$  of length at most  $k$ .*

**Proof :** We first note that if  $\tau$  is any closed curve in  $N$  such that  $\phi(\tau)$  is homotopically trivial in  $O$ , then  $\tau$  is homotopically trivial in  $\Sigma$ . To see this, note that for sufficiently large  $i$ ,  $\tau$  lies in  $N_i$  and  $\phi_i(\tau)$  is trivial in  $O_i$ . Thus,  $\theta_i \circ \phi_i(\tau)$  is trivial in  $\Sigma$ . But  $\theta_i \circ \phi_i$  is homotopic to the inclusion of  $N_i$  in  $\Sigma$ , and so  $\tau$  is trivial in  $\Sigma$  as claimed.

We now consider a lift  $\tilde{\phi} : \tilde{N} \rightarrow \tilde{O}$  to the universal cover,  $\tilde{O}$ , of  $O$  (or more precisely the appropriate connected component of  $O$ ), where  $\tilde{N}$  is some cover of  $N$ . Let  $\tilde{\pi}$  be a lift of  $\pi$  to  $\tilde{N}$ . The endpoints of  $\tilde{\phi}(\tau(\Pi))$  are  $(\eta/2)$ -close in  $\tilde{O}$ . It now follows (from the discreteness of the covering group on  $O$ , and its coboundedness  $\tilde{N}$ ) that the endpoints of  $\tilde{\pi}$  are connected by a path  $\tilde{\pi}'$  of bounded length in  $\tilde{N}$ . This projects to a path  $\pi'$  in  $N$ , with endpoints  $a, b$ . Now  $\phi(\pi \cup \pi')$  is homotopically trivial in  $O$ . (It lifts to the closed curve  $\tilde{\phi}(\tilde{\pi}) \cup \tilde{\phi}(\tilde{\pi}')$ ). Thus,  $\pi \cup \pi'$  is homotopically trivial in  $\Sigma$ . In other words,  $\pi'$  is homotopic to  $\pi$  relative to  $a, b$ , as claimed.  $\diamond$

Suppose  $\vec{a}, \vec{b} \in \vec{\lambda}$  with  $\vec{a} \approx \vec{b}$ . Then  $\sigma(a_t, b_t) \rightarrow 0$  and so  $d(\phi(a_t), \phi(b_t)) \rightarrow 0$ . We have the following converse:

**Lemma 25.3 :** *Suppose  $\vec{a}, \vec{b} \in \vec{\lambda}$  with  $\sigma(a, b) \leq \eta/2$  and with  $d(\phi(a_t), \phi(b_t)) \rightarrow 0$ , then  $\vec{a} \approx \vec{b}$ .*

**Proof :** Since  $d(\phi(a_t), \phi(b_t))$  is monotonically decreasing for  $t \geq 0$ , we have  $d(\phi(a_t), \phi(b_t)) \leq \eta/2$  for all  $t \geq 0$ . Let  $\pi_t$  be the path  $[a_t, a] \cup [a, b] \cup [b, b_t]$  from  $a$  to  $b$  in  $N$ . Now  $\phi(\pi_t) \cup [\phi(a_t), \phi(b_t)]$  is homotopically trivial in  $O$ , since  $\phi([a, b]) \cup [\phi(a), \phi(b)]$  has length less than  $\eta$  and  $\phi([a_t, a]) \cup [\phi(a), \phi(b)] \cup \phi([b, b_t]) \cup [\phi(b_t), \phi(a_t)]$  is spanned by the disc  $\bigcup_{u \in [0, t]} [\phi(a_u), \phi(b_u)]$ . It follows by Lemma 25.2 that  $\pi_t$  is homotopic in  $\Sigma$  to a path of bounded length  $k$ . This means that the half-leaves of  $\lambda$ , based at  $\vec{a}$  and  $\vec{b}$  are asymptotic, taking account of homotopy class. In other words, there is some  $s \in \mathbf{R}$  with  $\vec{c} \approx \vec{b}$ . Now  $\sigma(a, c) \leq \sigma(a, b) \leq \eta/2$ , so  $|s| \leq \eta/2$ . Note that  $\sigma(b_t, c_t) \rightarrow 0$  and  $d(\phi(a_t), \phi(c_t)) \leq d(\phi(a_t), \phi(b_t)) + d(\phi(b_t), \phi(c_t)) \rightarrow 0$ . Now some subsequence of  $\vec{a}_t$  must converge on some  $\vec{p} \in \vec{\lambda}$ . The corresponding  $\vec{c}_t$  converge on  $\vec{p}_s$ , and we have  $\phi(p) = \phi(p_s)$ . The loop  $\phi([p, p_s])$  in  $O$  has length at most  $\eta/2$  and so must be homotopically trivial. From this it follows that  $s = 0$ , and so  $\vec{b} \approx \vec{a}_s = \vec{a}$  as required.  $\diamond$

In particular, we see that if  $\vec{a}, \vec{b} \in \vec{\lambda}$  with  $\sigma(a, b) \leq \eta/2$ , then  $\vec{a} \approx \vec{b}$  if and only if  $d(\phi(a_t), \phi(b_t)) \rightarrow 0$  and if and only if  $d_E(\psi(a_t), \psi(b_t)) \rightarrow 0$ .

**Lemma 25.4 :** *If  $a, b \in \lambda$  with  $\psi(a) = \psi(b)$ , then either  $a = b$  or  $\sigma(a, b) > \eta/2$ .*

**Proof :** Suppose  $\sigma(a, b) \leq \eta/2$ . Let  $\vec{a}, \vec{b}$  be unit tangent vectors at  $a, b$  with  $\psi(\vec{a}) = \psi(\vec{b})$ . Then  $\psi(a_t) = \psi(b_t)$  for all  $t \in \mathbf{R}$ . Applying Lemma 25.3 for  $t \geq 0$  we get  $\vec{a} \approx \vec{b}$ , and for  $t \leq 0$ , we get  $-\vec{a} \approx -\vec{b}$ . It follows that  $a = b$ .  $\diamond$

Now consider the map  $\psi : \lambda \rightarrow \Lambda = \psi(\lambda)$ . Given  $n \in \mathbf{N}$ , let  $\Lambda(n) = \{c \in \Lambda \mid |\psi^{-1}(c)| \geq n\}$ , and let  $\lambda(n) = \psi^{-1}(\Lambda(n))$ . In view of Lemma 25.4,  $\Lambda(n)$  and hence  $\lambda(n)$  is closed. Moreover, these sets are invariant under flow along leaves. In particular,  $\lambda(n)$  is a sublamination of  $\lambda$ .

Now choose  $n$  maximal so that  $\lambda(n) \neq \emptyset$ . Since  $\psi$  is not injective,  $n \geq 2$ . Let  $\xi = \lambda(n)$  and let  $\Xi = \Lambda(n)$ . The map  $\psi : \xi \rightarrow \Xi$  is now everywhere  $n$  to 1. This is also the case for the lifted map  $\psi : \vec{\xi} \rightarrow \vec{\Xi}$ .

We define the closed equivalence relation,  $\sim$ , on  $\xi$  by writing  $a \sim b$  if  $\psi(a) = \psi(b)$ . We can thus identify  $\xi$  with  $\Xi/\sim$ . We similarly define  $\sim$  on  $\vec{\xi}$ , so that  $\vec{\Xi}$  gets identified with  $\vec{\xi}/\sim$ . We write  $[a]$  for the equivalence class of  $a$ .

**Lemma 25.5 :** *For all  $h \leq \eta/4$ , there is some  $h'$  such that if  $a, b, x \in \lambda$  with  $a \sim x$ ,  $\sigma(a, b) \leq h'$  and  $a \neq b$ . Then there is some  $y \in \lambda \setminus \{x\}$  with  $y \sim b$  and  $\sigma(x, y) < h$ .*

**Proof :** If this fails, there is a sequence  $b_i \rightarrow a \in \lambda$  with  $b_i \neq a$  and such that no point of  $N(x, h) \setminus \{x\}$  is equivalent to  $b_i$ . Now by Lemma 25.4, as  $z$  varies over  $[a] = [x]$ , the neighbourhoods  $N(z, h)$  are disjoint, and each can contain at most one element of  $[b_i]$ . Since all these classes have  $n$  elements, the pigeon-hole principle tells us that there is some  $c_i \sim b_i$  with  $\sigma(c_i, [a]) \geq h$ . Passing to a subsequence, we get  $c_i \rightarrow c \neq [a]$ . But  $b_i \rightarrow a$ , contradicting the fact that  $\sim$  is closed.  $\diamond$

**Lemma 25.6 :** *Suppose  $\vec{a}, \vec{b}, \vec{x}, \vec{y} \in \vec{\xi}$  with  $\sigma(a, b) < \eta/2$ ,  $\sigma(x, y) < \eta/2$ ,  $\vec{a} \sim \vec{x}$  and  $\vec{b} \sim \vec{y}$ . If  $\vec{a} \approx \vec{b}$ , then  $\vec{x} \approx \vec{y}$ .*

**Proof :** As observed before Lemma 25.3, we have  $d_E(\psi(a_t), \psi(b_t)) \rightarrow 0$ . But  $\psi(a_t) = \psi(x_t)$  and  $\psi(b_t) = \psi(y_t)$  for all  $t$ , and so  $d_E(\psi(x_t), \psi(y_t)) \rightarrow 0$ . By Lemma 25.3,  $\vec{x} \approx \vec{y}$  as claimed.  $\diamond$

**Lemma 25.7 :** *Suppose  $\vec{a}, \vec{b}, \vec{x}, \vec{y} \in \vec{\xi}$  with  $\sigma(a, b) < \eta/2$ ,  $\sigma(x, y) < \eta/2$ ,  $\vec{a} \approx \vec{b}$  and  $\vec{x} \approx \vec{y}$ . If  $\vec{a} \sim \vec{x}$ , then  $\vec{b} \sim \vec{y}$ .*

**Proof :** Let  $h' > 0$  be the constant of Lemma 25.5 given  $h = \eta/4$ . We can assume that  $h' \leq \eta/2$ . Choose  $t \geq 0$  so that  $\sigma(a_t, b_t) < h'$ . By Lemma 25.5, applied to  $a_t, b_t, x_t$ , there is some  $z \in \lambda \setminus \{x_t\}$  with  $z \sim b_t$  and  $\sigma(x_t, z) < \eta/2$ . Let  $\vec{z} \in \vec{\lambda}$  be the vector with  $\vec{z} \sim \vec{b}_t$ . Now  $\sigma(a_t, b_t) < \eta/2$  and  $\vec{a}_t \sim \vec{x}_t$ . Thus, applying Lemma 25.6 with  $\vec{a}_t, \vec{b}_t, \vec{x}_t, \vec{z}$ , we get  $\vec{x}_t \approx \vec{z}$ . But  $\vec{x}_t \approx \vec{y}_t$ , and  $\vec{z} \neq \vec{x}_t$ . Thus, since any  $\approx$ -class has at most two elements, it follows that  $\vec{z} = \vec{y}_t$ . In other words,  $\vec{b}_t \sim \vec{y}_t$ , and it follows that  $\vec{b} \sim \vec{y}$  as claimed.  $\diamond$

Lemmas 25.5, 25.6 and 25.7 are effectively telling us that the map  $\psi : \xi \rightarrow \Xi$  is a covering space. This can be made precise as follows. Let  $\Phi = \Upsilon(\xi) \subseteq \Sigma$  be the space obtained by filling in the spikes of  $\Sigma \setminus \xi$  as described earlier. Note that  $\Phi \subseteq N$ . We can extend  $\sim$  to a closed equivalence relation on  $\Phi$  as follows.

Suppose  $p \in \Phi \setminus \xi$ . Then  $p$  lies on a geodesic  $[a, b]$ , with  $a, b \in \xi$ ,  $\vec{a} \approx \vec{b}$  and  $\sigma(a, b) \leq \eta/8$ . Suppose  $q \in \Phi \setminus \xi$  similarly lies in  $[x, y]$ . Then  $\vec{a} \sim \vec{x}$  if and only if  $\vec{b} \sim \vec{y}$ . In this case, we write  $p \sim q$  if and only if  $p$  cuts  $[a, b]$  in the same ratio that  $q$  cuts  $[x, y]$ . It is now readily checked that  $\sim$  is a closed equivalence relation with  $n$  points in each class. Let  $\Omega = \Phi/\sim$ .

The quotient map  $\omega : \Phi \longrightarrow \Omega$  is an  $n$ -fold covering. Each component of  $\Omega$  is a circle or a closed surface.

Suppose  $p, q, a, b, x, y \in \Phi$  are as above. By assumption  $\sigma(a, b) \leq \eta/8$  and  $\sigma(x, y) \leq \eta/8$ . Since  $\phi : N \longrightarrow O$  is 1-lipschitz and  $\phi(a) = \phi(x)$  and  $\phi(b) = \phi(y)$ , we see that  $d(\phi(a), \phi(b)) \leq \eta/8$ . Put another way, if  $z \in \Omega$ , then  $\text{diam } \phi(\omega^{-1}(z)) \leq \eta/8$ . Note also that  $\phi(\Phi) \subseteq N(\phi(\lambda), \eta/8)$ .

Recall that  $\phi : N \longrightarrow O$  is a limit of the maps  $\phi_i : N_i \longrightarrow O_i$ . Moreover, by construction,  $\Phi \subseteq N_i$  for all sufficiently large  $i$ . We see that for all large  $i$ ,  $\text{diam } \phi(\omega^{-1}(z)) < \eta/4$ , say, for all  $z \in \Omega$ , and that  $\phi(\Phi) \subseteq N(\phi_i(\lambda_i), \eta/4)$ . Note that, by the condition of injectivity radius of  $\phi_i(\lambda_i)$  in  $O_i$ , we see that  $\phi_i(\omega^{-1}(z))$  lies in a hyperbolic  $(\eta/2)$ -ball embedded in  $O_i$ .

We now fix some such  $i$ , and set  $\mu(z) \in O_i$  to be the centre of  $\phi_i(\omega^{-1}(z))$ , in other words, the point so that  $\phi_i(\omega^{-1}(z))$  lies in the closed  $r$ -ball about  $\mu(z)$  for  $r$  minimal. Here  $r \leq \eta/8$  and the point is uniquely defined, given the fact that  $\phi_i(\omega^{-1}(z))$  lies in a hyperbolic  $(\eta/2)$ -ball embedded in  $O_i$ . Moreover, it gives us a continuous map,  $\mu : \Omega \longrightarrow O_i$ . Also, for each  $x \in \Phi$ ,  $d(\phi_i(x), \mu(\omega(x))) < \eta/4$ . So again by the condition of injectivity radius, we see that the maps  $\phi_i : \Phi \longrightarrow O_i$  and  $\mu \circ \omega : \Phi \longrightarrow O_i$  are homotopic (by linear homotopy along short geodesics).

By hypothesis,  $\theta_i \circ \phi_i : \Phi \longrightarrow \Sigma$  (being a restriction of  $\theta_i \circ \phi_i : N_i \longrightarrow \Sigma$ ) is homotopic to inclusion, and so therefore is  $\theta_i \circ \mu \circ \omega : \Phi \longrightarrow \Sigma$ . Writing  $f = \theta_i \circ \mu$ , we can summarise this as follows:

**Lemma 25.8 :** *We have an  $n$  to 1 covering map  $\omega : \Phi \longrightarrow \Omega$ , with  $n \geq 2$ , and a map  $f : \Omega \longrightarrow \Sigma$  such that  $f \circ \omega : \Phi \longrightarrow \Sigma$  is homotopic to inclusion.  $\diamond$*

In order to get a contradiction, we make the following purely topological observation.

**Lemma 25.25 :** *Suppose  $\Phi$  is a (not necessarily connected) subsurface of the compact surface  $\Sigma$ . Suppose that  $\omega : \Phi \longrightarrow \Omega$  is a  $n$ -fold covering map to a (not necessarily connected) surface  $\Omega$ . Suppose that there is a map  $f : \Omega \longrightarrow \Sigma$  such that  $f \circ \omega : \Phi \longrightarrow \Sigma$  is homotopic to inclusion. Then either  $n = 1$  or  $\Phi$  is homotopic into a 1-dimensional submanifold of  $\Sigma$  (a multicurve union the boundary of  $\Sigma$ ).*

**Proof :** It is clearly sufficient to prove the result when  $\Sigma$  is connected. We first note that we can also reduce to the case where  $\Phi$  is connected. For if  $\Phi_0$  is a component of  $\Phi$ , then  $\omega|_{\Phi_0}$  is an  $n_0$ -fold cover of a component,  $\Omega_0$ , of  $\Omega$ . Together with  $f|_{\Omega_0}$ , this satisfies the hypotheses of the lemma. Suppose that  $\Omega$  is not homotopic into a closed curve. The lemma then tells us that  $n_0 = 1$ . If  $n > 1$ , then some other component,  $\Phi_1$ , of  $\Phi$  also gets mapped homeomorphically to  $\Omega_0$ , so the inclusions of  $\Omega_0$  and  $\Omega_1$  into  $\Sigma$  are homotopic. In other words,  $\Omega_0$  can be homotoped to be disjoint from itself in  $\Sigma$ . But this is impossible since we assuming that it cannot be homotoped into a curve. We conclude that all components of  $\Omega$  are homotopic into curves. Moreover, it is easily seen that we can take these curves to be disjoint in  $\Sigma$  giving the result.

Let us therefore suppose that  $\Phi$  is connected. Suppose first that each (intrinsic) boundary component of  $\Phi$  is homotopically trivial in  $\Sigma$ . If  $\Phi$  is not homotopic to a point

in  $\Sigma$ , then  $\Sigma$  is closed, and each component of  $\Sigma \setminus \Phi$  is a disc. Let  $\Omega'$  be the closed surface obtained by gluing a disc to each boundary component of  $\Omega$ . We can now extend  $\omega$  to an  $n$ -fold branched cover  $\omega' : \Sigma \rightarrow \Omega'$ , and extend  $f$  to a map  $f' : \Omega \rightarrow \Sigma$ . The composition  $f' \circ \omega' : \Sigma \rightarrow \Sigma$  is homotopic to the identity, and it follows that  $n = 1$ .

Suppose that  $\alpha$  is a boundary curve of  $\Phi$  that is homotopically non-trivial in  $\Sigma$ . Its inclusion into  $\Sigma$  factors through the boundary curve,  $\omega(\alpha)$ , of  $\Omega$ . It follows that  $\omega|_{\alpha}$  is injective. If  $n > 1$ , then some other boundary curve,  $\beta$ , of  $\Phi$  also gets mapped homeomorphically to  $\omega(\alpha)$ . Now  $\alpha$  and  $\beta$  are homotopic in  $\Sigma$ , and hence bound an annulus. Unless this annulus contains  $\Phi$ , it must be a component of  $\Sigma \setminus \Phi$ . No other boundary component of  $\Phi$  can be homotopic to this annulus. We see that  $n = 2$ , and that the homotopically non-trivial boundary components of  $\Phi$  occur in pairs that bound annular components of  $\Sigma \setminus \Phi$ . It follows that  $\Sigma$  is closed and that each component of  $\Sigma \setminus \Phi$  is either such an annulus or a disc. Let  $\Omega''$  be the surface obtained by gluing a disc to each boundary curve of  $\Omega$  whose  $f$ -image is trivial in  $\Sigma$ . We now extend  $\omega$  to a map  $\omega'' : \Sigma \rightarrow \Omega''$  by collapsing the annular components of  $\Sigma \setminus \Phi$  to boundary curves, without twisting, and extending (anyhow) over the disc components. Let  $f'' : \Omega'' \rightarrow \Sigma$  be any extension of  $f$ . Thus,  $f'' \circ \omega'' : \Sigma \rightarrow \Sigma$  is homotopic to the identity. But it factors through the non-closed surface  $\Omega''$ , giving a contradiction.  $\diamond$

We can now apply this to the situation described by Lemma 25.8. Certain components of  $\Phi$  may be circles, as we have defined it, but these can be thickened up to annuli, so that makes no essential difference. From the construction, no two distinct components of  $\Omega$  can be homotoped into the same closed curve. Lemma 25.25 now gives the contradiction that  $n = 1$ , finally proving Proposition 25.1.

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