Notes on Gromov's hyperbolicity criterion for path-metric spaces.

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0. Introduction.

In [G], Gromov describes the notion of a "δ-hyperbolic" (or what we shall call "almost-hyperbolic") path-metric space. With one simple axiom, (essentially property H1 described here,) he is able to capture a remarkable number of the global properties of a "negatively curved" space, or more specifically, a simply-connected Riemannian manifold, all of whose sectional curvatures are less than some negative constant.

In his paper, Gromov is primarily concerned with developing the properties of a "hyperbolic group", i.e. one with an almost-hyperbolic Cayley graph. In other words he deals primarily with almost-hyperbolic spaces which admit cocompact groups of isometries. However, the group structure is largely irrelevant to understanding the global geometry of these spaces. The aim of the present paper is to give a more detailed exposition of Gromov's criterion purely in the context of path-metric spaces. Of course much of Gromov's paper will not be touched upon here. Other expositions of Gromov's work are [ABCDFLMSS], [BGHHSST] and [CDP].

One of the main aims of this paper is to show the equivalence of a number of different characterisations of almost-hyperbolicity. This will be achieved in the first six chapters. We shall not always take the most direct route in this, but get involved in discussions of spanning trees, convexity, isoperimetric inequalities and pseudoisometries etc. All the arguments of this paper are elementary. In particular, we make no use of Riemannian geometry, except by way of example.

The structure of this paper, in outline, is as follows. Chapter 1 describes the main terms and conventions we will be using. In Chapter 2, we discuss in more detail our main definitions of almost hyperbolicity, H1–H5. In chapter 3, we describe the "treelike" nature of almost-hyperbolic spaces, and show the equivalence of the first two definitions. In Chapter 4, we define "almost-convex" sets, and develop a few of their properties. We give proofs of H1$\Rightarrow$H4 and H1$\Rightarrow$H5. We also show that almost-hyperbolicity is a pseudoisometric invariant. It is this fact that allows one to define the notion of an almost- hyperbolic group. Chapter 5 is devoted to developing a notion of "area" that seems appropriate to the context of path-metric spaces. That H3$\Rightarrow$H4 follows easily from this. In Chapter 6, we give the remaining proofs, H4$\Rightarrow$H2 and H5$\Rightarrow$H3. Chapter 7 explores further the treelike nature of almost-hyperbolic space. In Chapter 8, we give a proof that the property of almost-hyperbolicity "propagates", that is, a path-metric space which is simply connected (in some sense) and almost-hyperbolic on a large scale is globally almost-hyperbolic.

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CHAPTER 1 : Definitions.

1.1. Geodesic spaces.

Let $(\mathcal{S}, d)$ be a metric space. To say that d is a *path metric* means that, given any two points $X, Y \in \mathcal{S}$ and $\epsilon > 0$, there is a rectifiable path joining X and Y of length at most $d(X,Y) + \epsilon$. We then call $(\mathcal{S}, d)$ a *path-metric space*. Such spaces seem to be the natural context in which to speak of almost-hyperbolicity. However, to save ourselves a few unnecessary complications, we shall usually make an additional assumption.

Definition : A *geodesic space* is a complete, locally-compact path metric space.

It is an exercise to show that any closed uniform ball $\{X \in \mathcal{S} \mid d(X,Y) \leq r\}$ in a geodesic space $(\mathcal{S}, d)$ is compact. The following are easy consequences.

(1) If $X, Y \in \mathcal{S}$, then there is at least one path from X to Y of length equal to $d(X,Y)$. Such a path is called a *geodesic*.

(2) If $X \in \mathcal{S}$, and $Q \subseteq \mathcal{S}$ is any non-empty closed set, then there is some $Y \in Q$ with $d(X,Y) = d(X,Q)$, where $d(X,Q) = \inf_{Z \in Q} d(X,Z)$. We shall write $\mathrm{proj}_Q(X) = \{Y \in Q \mid d(X,Y) = d(X,Q)\}$. Each such Y is called a *projection* of X to Q.

Given any closed set $Q \subseteq \mathcal{S}$, we shall write $\breve{Q}$ for its topological interior, and $\partial Q = Q \setminus \breve{Q}$ for its topological boundary. We may check that if $X \notin \breve{Q}$, then $\mathrm{proj}_Q X \subseteq \partial Q$. If $r \geq 0$, we write $N_r(Q) = \{X \in \mathcal{S} \mid d(X,Q) \leq r\}$ for the uniform r-neighbourhood of Q. Note that $N_{r+s}(Q) = N_r(N_s(Q))$. We write $\breve{N}_r(Q)$ for the topological interior of $N_r(Q)$, and $\partial N_r(Q) = N_r(Q) \setminus \breve{N}_r(Q)$. Thus

$$\{X \in \mathcal{S} \mid d(X,Q) < r\} \subseteq \breve{N}_r(Q)$$

and

$$\partial N_r(Q) \subseteq \{X \in \mathcal{S} \mid d(X,Q) = r\}.$$

Any closed subset of a geodesic space will itself be a geodesic space in the induced path-metric, provided that we allow for the possibility that two points be an infinite distance apart if they cannot br joined by a rectifiable path. Given a closed subset Q of $(\mathcal{S}, d)$, we shall write $d_{0,Q}$ for the induced path-metric on $\mathcal{S} \setminus \breve{Q}$. More generally, we shall write $d_{r,Q}$ for the induced path-metric on $\mathcal{S} \setminus \breve{N}_r(Q)$.

Given two points X, Y in a geodesic space $(\mathcal{S}, d)$, we shall use $[X, Y]$ to denote some choice of geodesic from X to Y. We shall only be using this notation in the case where $(\mathcal{S}, d)$ is almost-hyperbolic. For such spaces, any two geodesics with the same endpoints remain a bounded distance apart. Once this has been established, we can afford to be a little careless in the use of this notation. For example, we will speak as though $[X, Y]$ were a well-defined object, even though it implies making a choice.

The above discussion applies only when $(\mathcal{S}, d)$ is a geodesic space. However all the results of this paper may be interpreted for general path-metric spaces, with simple modification. For example, we could interpret a "geodesic" from X to Y as a rectifiable path of length at most $d(X,Y)+1$, or $\mathrm{proj}_Q(X)$ as the set of points Y in Q satisfying $d(X,Y) \leq d(X,Q)+1$, and so on. Working with such things, however, would only confuse the exposition.

1.2. Almost-hyperbolicity.

We are now in a position to define the notion of almost-hyperbolicity. We give five definitions: H1–H5. For ease of reference, we collect together these definitions below. We shall discuss them in more detail in Chapter 2.

Let $(\mathcal{S}, d)$ be a geodesic space. Let $k_i, h_i \in [0, \infty)$.

Definition 1 : $(\mathcal{S}, d)$ is k_1-H1 if:
Given any four points $X, Y, Z, W \in \mathcal{S}$, at least one of $XY : ZW$, $XZ : YW$ or $XW : YZ$ holds, where $AB : CD$ is the statement that

$$\begin{aligned} d(A,B) + d(C,D) &\leq \max\big(d(A,C) + d(B,D), d(A,D) + d(B,C)\big) + k_1 \\ &\leq \min\big(d(A,C) + d(B,D), d(A,D) + d(B,C)\big) + 2k_1. \end{aligned}$$

Definition 2 : $(\mathcal{S}, d)$ is k_2-H2 if the following holds:
Suppose $X_1, X_2, X_3 \in \mathcal{S}$. Suppose α_i is any geodesic joining X_i to X_{i+1} (taking subscripts mod 3). Then,

$$N_{k_2}(\alpha_1) \cap N_{k_2}(\alpha_2) \cap N_{k_2}(\alpha_3) \neq \emptyset.$$

Definition 3 : $(\mathcal{S}, d)$ is (k_3, h_3)-H3 if the following holds:
Suppose that σ is a path-metric on the circle S^1, and that $\gamma : (S^1, \sigma) \longrightarrow (\mathcal{S}, d)$ is a distance non-increasing map. Then, there exist,
(i) a cellulation P of the unit disc D,
(ii) a path-metric ρ on the 1-skeleton Σ of P,
(iii) a distance non-increasing map $f : (\Sigma, \rho) \longrightarrow (\mathcal{S}, d)$, and
(iv) a distance non-increasing map $\partial f : (\partial D, \rho_{\partial D}) \longrightarrow (S^1, \sigma)$, where $\rho_{\partial D}$ is the path-metric on ∂D induced from ρ,
such that,
(i) $f | \partial D = \gamma \circ \partial f$,
(ii) ∂f has topological degree 1, and
(iii) we have

$$\sum_{c \in C_2(P)} (\rho(\partial c))^2 \leq k_3(\sigma(S^1) + h_3)$$

where $C_2(P)$ is the set of 2-cells of P, $\rho(\partial c)$ is the ρ-length of the boundary ∂c of c, and $\sigma(S^1)$ is the total σ-length of S^1.

Definition 4 : $(\mathcal{S}, d)$ is (k_4, h_4)-H4 if:

Given any geodesic segment $\alpha \subseteq \mathcal{S}$, and $X, Y \in \partial N_{k_4}(\alpha)$, we have

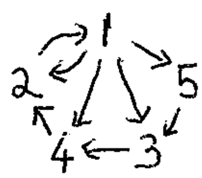
$$d_{k_4,\alpha}(X,Y) \geq 3d(X,Y) - h_4,$$

where $d_{k_4,\alpha}$ is the induced path-metric on $\mathcal{S}\backslash \check{N}_{k_4}(\alpha)$.

Definition 5 : $(\mathcal{S}, d)$ is (k_5, h_5)-H5 if the following holds:
Given any $A, X, Y \in \mathcal{S}$, with $d(A,X) = d(A,Y)$ and $d(X,Y) \geq k_5$, then $d_{r,A}(X,Y) \geq d(X,Y) + h_5$, where $r = d(A,X) = d(A,Y)$, and $d_{r,A}$ is the induced path-metric on $\mathcal{S}\backslash \check{N}_r(A)$.

We shall show that these definitions are equivalent in the following effective sense. Given $i, j \in \{1,2,3,4,5\}$, and $\underline{k} \in [0,\infty) \sqcup [0,\infty)^2$, there is some $\underline{k}' \in [0,\infty) \sqcup [0,\infty)^2$ such that if $(\mathcal{S}, d)$ is $\underline{k}$-H(i), then it is $\underline{k}'$-H(j).

The cycle of proofs will be:



We include H1 $\Rightarrow$ H2, H1 $\Rightarrow$ H3, and H1 $\Rightarrow$ H4, since they are much more direct than following the cycle.

Definition : We say that $(\mathcal{S}, d)$ is *almost-hyperbolic* if it is $\underline{k}$-H(i) for some $\underline{k}$ and i.

We shall call the constants k_1, k_2, k_3, h_3, k_4, h_4, k_5, h_5, appearing in the definitions, *parameters* of hyperbolicity. Note that all can be imagined as having the physical dimensions of length. Also, all except h_5 have the property that increasing them would weaken the definition. In fact, the quantities $-1/k_1^2$, $-1/k_2^2$, $-1/(\max(k_3,h_3))^2$ and $-1/(\max(k_4,h_4))^2$, can be thought of as a measure of the upper bound of the curvature, seen on a large scale. This can be made more precise for negatively curved Riemannian manifolds. (A space of "infinite negative curvature" is a metric tree, see Chapter 3.)

There is also a sense in which the parameters measure the "coarseness" of a space, where coarseness may be due to local concentrations of positive curvature, or to topological holes, etc. This is only an intuitive picture, and we make no attempt to formally isolate these notions. (It is possible to give a more clumsy version of H5 which fits into this scheme, see Chapter 2.)

1.3. Pseudoisometries.

From the definition H3, it is immediate that the notion of almost-hyperbolicity is invariant under bilipschitz equivalence. However, the notion of bilipschitz equivalence is is too strong for these kinds of spaces. It demands, for example that the spaces under consideration be homeomorphic, which is an unnatural constraint. We would like, for example, that an almost hyperbolic space admitting a properly-discontinuous, cocompact isometry group, should be equivalent to the Cayley graph of that group. A more appropriate notion, therefore, is that of a "pseudoisometry" (elsewhere known as a "coarse quasiisometry").

Definition : Let $(\mathcal{S}, d)$ and $(\mathcal{S}', d')$ be path-metric spaces, and $\lambda_1 \geq 1$, $\lambda_2 \geq 0$. A (λ_1, λ_2)-*pseudoisometry* between $\mathcal{S}$ and $\mathcal{S}'$ is a relation $R \subseteq \mathcal{S} \times \mathcal{S}'$ such that

$$\forall(x \in \mathcal{S})\, \exists(x' \in \mathcal{S}')\, xRx'$$
$$\forall(x' \in \mathcal{S}')\, \exists(x \in \mathcal{S})\, xRx'$$

and if xRx' and yRy', then

$$\begin{aligned} \frac{1}{\lambda_1}(d(x,y) - \lambda_2) &\leq d'(x', y') \\ &\leq \lambda_1 d(x,y) + \lambda_2. \end{aligned}$$

We say that a relation is a *pseudoisometry* if it is a (λ_1, λ_2)-pseudoisometry for some λ_1 and λ_2. Pseudoisometry is thus an equivalence relation on path-metric spaces.

We shall show (Proposition 4.10) that almost-hyperbolicity is invariant under pseudoisometry.

Note that if we take two finite generating sets for the same group, then the corresponding Cayley graphs are pseudoisometric. It therefore makes sense to define a finitely-generated group as being almost-hyperbolic if its Cayley graph is almost hyperbolic, irrespective of the choice of generators. We see in fact that a group is almost hyperbolic if and only if it acts as a properly discontinuous cocompact isometry group on some almost hyperbolic space.

Another point to note is that any path-metric $(\mathcal{S}, d)$ space is $(1+\epsilon, 2)$-pseudoisometric to a 1-complex in which each edge has length 1. The construction is as follows. For $\delta > 0$ sufficiently small, we take a maximal packing P of $\mathcal{S}$ by disjoint δ-balls. We form a metric 1-complex, (G, ρ) by taking the vertices to correspond to the balls of P, and joining two vertices by an edge of length 1, if the centres of the corresponding balls are distant at most 1 in $\mathcal{S}$. We define $R \subseteq \mathcal{S} \times G$ by $(x, y) \in R$ if and only if, for some vertex v of G, we have $d(x, p(v)) \leq 2\delta$ and $\rho(y, v) \leq \frac{1}{2}$, where $p(v) \in \mathcal{S}$ is the centre of the ball corresponding to v. We may check that this is a $(1+\epsilon, 2)$-pseudoisometry for $\epsilon = O(\delta)$. Since this construction applies to any path-metric space, we see that we see that our entire discussion of almost-hyperbolicity can, in principle, be given a purely combinatorial formulation.

1.4. Convention on inequalities.

We end this chapter by introducing a convention that will streamline our manipulation of inequalities, and, we hope, make our arguments conceptually simpler.

Suppose $K \geq 0$, and $x, y \in \mathbf{R}$. we shall write $x \simeq_K y$ to mean that $|x - y| \leq K$, and $x \preceq_K y$ to mean that $x \leq y + K$. Thus,

$$x \simeq_K y \simeq_K z \Rightarrow x \simeq_{2K} z$$

and

$$x \preceq_K y \preceq_K z \Rightarrow x \preceq_{2K} z.$$

Whenever we use this notation, K will be a funcion only of the parameters of hyperbolicity.

Usually, we shall drop the subscripts K, and behave as though the relations $\simeq$ and $\preceq$ were transitive. Thus, one should think of the constants involved as increasing with each application of the transitive law in an argument. We may always explicitly relate the constants produced at the end to the constants introduced at the beginning. Usually, however, keeping track in this way would only confuse the argument.

Given points X, Y in a path-metric space $(\mathcal{S}, d)$, we shall write $X \sim Y$ to mean that $d(X,Y) \simeq 0$. Again, we shall act as though $\sim$ were an equivalence relation.

In some places (mainly Chapter 3) we shall abbreviate $d(X,Y)$ to XY.

CHAPTER 2 : Five definitions of almost-hyperbolicity.

We discuss in more detail the definitions of Section 1.2.

2.1. Definition 1.

With the conventions introduced in Section 1.4, we may rewrite Definition 1 in the following way. Given any four points $X, Y, Z, W \in \mathcal{S}$, we may partition them into two sets of two elements, without loss of generality $\{\{X, Y\}, \{Z, W\}\}$, so that

$$d(X,Y) + d(Z,W) \preceq d(X,Z) + d(Y,W) \simeq d(Y,Z) + d(X,W).$$

We shall write $XY : ZW$ for this pair of inequalities. This definition is just a rephrasing of the central one given in [G].

Lemma 2.1.1, below, tells us that this hypothesis is equivalent to saying that the distances between X, Y, Z, W may be read off approximately (i.e. up to an additive constant) along a tree with a length assigned to each edge (Figure 2a).

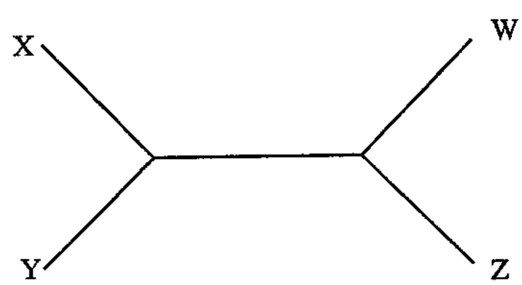


Figure 2a.

Note that there are three combinatorial possibilities for the tree, corresponding to the three possible partitions of $\{X, Y, Z, W\}$.

From this point of view, the axiom is analogous to the triangle inequalities of a metric space. Consider three points X, Y, Z in a metric space $(\mathcal{S}, d)$. The triangle inequalities tell us that we can find three non-negative numbers x, y, z such that

$$\begin{aligned} d(X,Y) &= x + y \\ d(Y,Z) &= y + z \\ d(Z,X) &= z + x, \end{aligned}$$

namely $x = \frac{1}{2}\big(d(Z,X) + d(X,Y) - d(Y,Z)\big)$ etc. In other words, the distances between X, Y and Z may be read off (precisely) from a tree with edges of length x, y and z. We shall write $(XYZ) \longleftrightarrow xyz$ to mean this (Figure 2b).

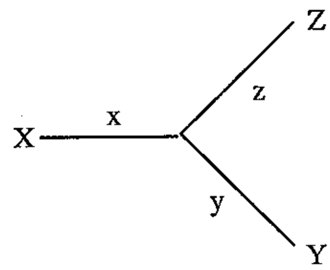


Figure 2b.

Lemma 2.1.1 : *Let X, Y, Z, W be four points in a metric space $(\mathcal{S}, d)$. Then, we have $XY : ZW$, that is*

$$d(X,Y) + d(Z,W) \preceq d(X,Z) + d(Y,W) \simeq d(Y,Z) + d(X,W),$$

if and only if there exist non-negative numbers x, y, z, w, u such that

$$\begin{aligned} d(X,Y) &\simeq x + y \\ d(Z,W) &\simeq z + w \\ d(X,W) &\simeq x + u + w \\ d(Y,Z) &\simeq y + u + z \\ d(X,Z) &\simeq x + u + z \\ d(Y,W) &\simeq y + u + w. \end{aligned}$$

(Figure 2c)

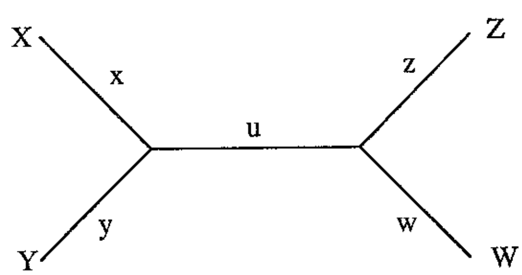


Figure 2c.

Proof : ($\Leftarrow$) is trivial. We prove ($\Rightarrow$).

Suppose we have $XY : ZW$. We find trees $(XYZ) \longleftrightarrow xya$ and $(YZW) \longleftrightarrow bzw$ as described above. (Figure 2d.)

Thus,

$$d(Y,Z) = y + a = b + z.$$

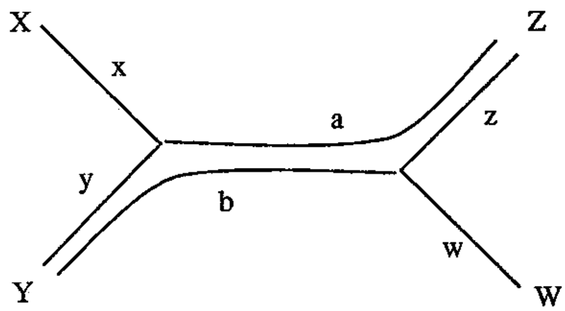


Figure 2d.

Now,

$$\begin{aligned} d(X,Z)+d(Y,W) &\succeq d(X,Y)+d(Z,W) \\ (x+a)+(w+b) &\succeq (x+y)+(z+w) \quad . \\ a+b &\succeq y+z \end{aligned}$$

But,

$$\begin{aligned} 2(a+y) &= (a+y)+(b+z) \\ &\succeq 2(y+z). \end{aligned}$$

Therefore,

$$a \succeq z.$$

Let $u = \max(a-z, 0)$. Then $u \simeq a - z = b - y$. Now,

$$\begin{aligned} d(X,W) &\simeq d(X,Z)+d(Y,W)-d(Y,Z) \\ &= (x+a)+(w+b)-(z+b) \\ &= x+(a-z)+w \\ &= x+u+w. \end{aligned}$$

◇

We shall write

$$XY : ZW \longleftrightarrow (xy)u(zw)$$

to express the situation described in Lemma 2.1.1. In fact, Lemma 3.1.7 tells us that, if $(\mathcal{S}, d)$ is almost-hyperbolic, then such a tree may be realised as a union of geodesic arcs in $\mathcal{S}$. (Though we may have to allow for a larger error of approximation than is obtained above.) If these geodesics are $[X,A]$, $[Y,A]$, $[A,B]$, $[B,Z]$, $[B,W]$ of lengths respectively x, y, u, z, w, then we shall write

$$(XY)AB(ZW) \longleftrightarrow (xy)u(zw).$$

In fact, this result may be extended to give spanning trees for any finite sets of points (see Section 3.3). Arguing further along these lines, one may show that any 0-H1 space is a metric tree, i.e. a path metric space which contains no topologically embedded circle (Section 3.4).

Finally, note that if we already know that the space $(\mathcal{S}, d)$ is H1, then the statement $XY : ZW$ becomes equivalent to $d(X,Z) + d(Y,W) \simeq d(Y,Z) + d(X,W)$.

2.2. Definition 2.

This is probably the conceptually simplest definition of almost-hyperbolicity. Given any triangle in $\mathcal{S}$, there is some point which lies within a bounded distance from all three edges. Such a point will be called a *centre* for the triangle. Lemma 3.1.5 shows that in an almost-hyperbolic space, any two centres for the same triangle are a bounded distance apart.

It should be stressed that in this definition, a "triangle" is taken to mean a set of three points, together with geodesic edges joining them. However, once it is known that a given space is almost-hyperbolic, it makes sense to speak of a centre of three points, without making an explicit choice of edges.

2.3. Definition 3.

This is the "isoperimetric inequality". Intuitively, it says that any closed curve (or *loop*) in our space "bounds a disc of area" at most a certain linear function of the curve's length. The main technical problem is in formulating what we mean by "bounding a disc" and "area" in an arbitrary path-metric space. The definition given in Section 1.2 gives one possibility, though the quantity representing the area in this case is perhaps more naturally termed "energy". We shall describe a few variations on these basic definitions below. Some will be more appropriate to certain contexts, for example when considering the Cayley graphs of almost-hyperbolic groups.

However we choose to define these terms, the essential property which we require of them may be summarised as follows.

Let $(\mathcal{S}, d)$ be a path-metric space. We define a *loop* in $\mathcal{S}$ as a distance non-increasing map $\gamma : (S^1, \sigma) \longrightarrow (\mathcal{S}, d)$, where σ is a path-metric on the circle S^1. Suppose we represent S^1 as the union of four closed intervals, L_1, L_2, L_3, L_4, cyclically ordered, and intersecting only in their boundaries. Let $\alpha_i = \gamma | L_i$. We write $\gamma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ for this, and we call $\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ a *rectangle* in $\mathcal{S}$. We shall want the following.

Rectangle Principle. (2.3.1) :

There are universal constants K_1 and K_2 such that the following holds.

Suppose that $(\mathcal{S}, d)$ is a path-metric space, and $\gamma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ is a rectangle in $\mathcal{S}$. Let $d_1 = d(\alpha_1, \alpha_3)$ and $d_2 = d(\alpha_2, \alpha_4)$. If γ "bounds a disc of area" A in $\mathcal{S}$, then

$$A \geq K_1 d_1 d_2 - K_2(d_1 + d_2 + 1).$$

The constants K_1 and K_2 will depend only on the definition of area we choose. Chapter 5 is devoted to a discussion of these matters.

We now give the definitions we shall use in this paper.

Let G be any finite graph. To save on words, we shall identify G (thought of combinatorially) with its realisation as a topological 1-complex. We write $C_0(G)$ for the set of vertices of G, and $C_1(G)$ for the set of edges of G. Any path metric on G is determined, up to isotopy (rel $C_0(G)$), by a map $C_1(G) \longrightarrow (0, \infty)$. Any path-metric ρ, in turn determines a natural parameterisation on each edge, and thus a measure on G. Given any closed subset $G_0 \subseteq G$, we shall write $\rho(G_0)$ for the measure of G_0, which we shall refer to as the *ρ-length*, or just *length* of G_0. (Note that the length $\rho(e)$ of an edge $e \in C_1(G)$ may be greater than the distance between its endpoints.) We shall write ρ_{G_0} for the path-metric induced on G_0 from ρ. Thus $\rho_{G_0}(x, y) \geq \rho(x, y)$ for all $x, y \in G_0$.

Let D be the closed (unit) disc in $\mathbf{R}^2$.

Definition : A *cellulation*, P, of the disc D, is a presentation of D as a CW-complex, such that each 0-cell meets at least three 1-cells, and the boundary, ∂c, of any 2-cell, c, is an embedded circle.

Note that the conditions imply that the endpoints of each 1-cell are distinct.

We write $\Sigma(P)$ for the 1-skeleton of P, and $C_i(P)$ for the set of i-cells of P, $i = 0, 1, 2$. (Thus $C_0(P) = C_0(\Sigma(P))$ and $C_1(P) = C_1(\Sigma(P))$.) We shall use $A(P) = |C_2(P)|$ to denote the number of 2-cells of P.

We are also interested in the following special case of a cellulation.

Definition : A *triangulation*, P, of the disc D, is a presentation of the disc as a simplicial complex.

Thus any two triangles of P meet along an edge, or at a vertex, or not at all.

In this case, we shall write $A_T(P) = |C_1(P)|$ for the number of 1-cells of P. Lemma 5.6 tells us that we can always subdivide a cellulation P to give a triangulation P' satisfying

$$A_T(P') \leq 54A(P).$$

.

Definition : A *metric cellulation (triangulation)*, (P, ρ) is a cellulation (triangulation), P, of the disc D, together with a path-metric ρ on the 1-skeleton $\Sigma(P)$ of P.

Given any metric cellulation (P, ρ), we define the *mesh* of (P, ρ) as

$$m(P, \rho) = \max_{c \in C_2(P)} (\rho(\partial c)).$$

We define the *energy* as

$$I(P, \rho) = \sum_{c \in C_2(P)} (\rho(\partial c))^2.$$

We see that

$$I(P,\rho) \leq A(P)m(P,\rho)^2.$$

In the special case where P is a triangulation, we may define

$$m_T(P,\rho) = \max_{e\in C_1(P)}(\rho(e))$$
$$I_T(P,\rho) = \sum_{e\in C_1(P)} (\rho(e))^2.$$

Again,

$$I_T(P,\rho) \leq A_T(P)m_T(P,\rho)^2.$$

Comparing these formulations for the same metric triangulation, we get

$$m_T \leq m \leq 3m_T$$
$$A_T \leq 3A \leq 3A_T$$
$$I_T \leq I \leq 6I_T.$$

The last of these follow from the inequalities $x^2+y^2+z^2 \leq (x+y+z)^2 \leq 3(x^2+y^2+z^2)$, for $x,y,z \geq 0$.

Now, let $(\mathcal{S},d)$ be any path-metric space.

Definition : A *cellular (simplicial) net*, (P,ρ,f), is a metric cellulation (triangulation), (P,ρ), of the disc D, together with a distance non-increasing map $f:(\Sigma(P),\rho)\longrightarrow(\mathcal{S},d)$.

Recall that a *loop*, (γ,σ), is a distance non-increasing map $\gamma:(S^1,\sigma)\longrightarrow(\mathcal{S},d)$. We refer to $\sigma(S^1)$ as the *length* of γ.

Definition : A net (P,ρ,f) is said to *bound* a loop (γ,σ), if $f|\partial D$ factors through a distance non-increasing map $\partial f:(\partial D,\rho_{\partial D})\longrightarrow(S^1,\sigma)$ of topological degree 1, i.e. $f|\partial D=\gamma\circ\partial f$.

One may mow give four versions of H3 as follows.

We may say that any loop (γ,σ), bounds a cellular (simplicial) net, (P,ρ,f), whose energy $I(P,\rho)$ ($I_T(P,\rho)$) is at most a certain linear function of $\sigma(S^1)$. This gives definitions H3ce (H3te).

Alternatively, we may say that any loop (γ,σ) bounds a cellular (simplicial) net whose mesh $m(P,\rho)$ ($m_T(P,\rho)$) is bounded by some fixed parameter, and for which the area $A(P)$ ($A_T(P)$) is at most a certain linear function of $\sigma(S^1)$. This gives definitions H3ca (H3ta).

Of these, the apparently weakest is H3ce, which we took as our main definition, and the apparently strongest is H3ta. That all these definitions amount to the same thing, should be apparent from the logical structure of Chapter 6. We prove, in fact, that H1 $\Rightarrow$ H3ta and that H3ce $\Rightarrow$ H4.

The formulations H3ca and H3ta seem best suited for combinatorial situations, for example where $\mathcal{S}$ is the Cayley graph of some finitely presented group. In this case, a loop is a word in the generators and their inverses, representing the trivial element. The area of a net bounding this loop is a measure of the number of applications of the relations we need to reduce this word to the trivial word.

The definition H3ta also makes the invariance of almost-hyperbolicity under pseudoisometries most apparent. Unfortunately, the details of the proof are a little messy, and instead, we give a different argument in Chapter 4 (Proposition 4.10).

Note that any metric on the 0-skeleton $C_0(P)$ of a cellulation P, determines (up to isotopy) a path-metric on the 1-skeleton $\Sigma(P)$. A metric cellulation (P, ρ) arises in this way precisely when the length of each edge equals the distance between its endpoints. If moreover, P is a triangulation, and the triangle inequalities for the three vertices of any triangle in $C_2(P)$ are strict, then (P, ρ) determines a singular-euclidean structure on the disc D, by gluing together euclidean triangles in the obvious way. This allows us to relate the energy $I_T(P, \rho)$ of such a triangulation to energy as defined in differential geometry as follows.

Recall that the energy of a map ϕ between two Riemannian manifolds is defined as

$$\frac{1}{2}\int \mathrm{trace}(D\phi)^T(D\phi)d\mu,$$

where $d\mu$ is the volume element of domain, and the derivative $D\phi$ is expressed in terms of orthonormal coordinates. In dimension 2, this quantity depends only on the conformal structure of the domain. Now, suppose we have a euclidean triangle Δ, with sides of length a, b, c. We may imagine Δ as the image of an equilateral triangle under an affine map. Simple trigonometry shows that the energy of this map equals $\frac{1}{4\sqrt{3}}(a^2 + b^2 + c^2)$. We may think of a triangulation of the disc D as determining a conformal structure on D, by taking each triangle to be euclidean-equilateral of the same size. Suppose now, we have a path-metric ρ on $\Sigma(P)$ which happens to satisfy strict triangle inequalities for each triangle in $C_2(P)$. This determines a singular euclidean structure on the disc, as described above. The energy of the identity map is then equal to $\frac{1}{2\sqrt{3}}(I_T(P, \rho) - B)$, where B is the boundary correction $\frac{1}{2}\sum\{(\rho(e))^2 \mid e \in C_1(P), e \subseteq \partial D\}$.

Finally, we remark that Lemmas 6.1.2, and 6.1.4 provide another version of H3.

2.4. Definition 4.

This criterion is a simple consequence of the isoperimetric inequality and rectangle principle (see Proposition 5.12), and is a weak form of the pseudogeodesic property (Proposition 4.9).

2.5. Definition 5.

This may be thought of as expressing, in a weak sense, the fact that spheres, on a large scale, have extrinsic curvatures bounded away from 0. Compare with the statement that in a negatively curved Riemannian manifold, all of whose sectional curvatures are bounded away from 0, we have that all the principle curvatures of spheres are bounded away from 0.

In Chapter 6, we shall see that we may weaken the definition of H5 given in Chapter 1, by choosing only those X, Y which satisfy $k_5 \leq d(X,Y) \leq 2k_5 + 2h_5$. From Proposition 4.7, we may also give the stronger formulation, that for some k_5', h_5', we have that $d(X,Y) \geq k_5'$ implies that $h_5' d_{r,A}(X,Y) \geq d(X,Y)^2$. Now, the parameters k_5', h_5' have an interpretation in terms of the curvature/coarseness of $\mathcal{S}$, as described in Section 1.2.

CHAPTER 3 : Centres and spanning trees.

In this chapter, we get the subject of almost-hyperbolicity off the ground, by showing the equivalence of definitions H1 and H2. Recall the notation $\simeq, \preceq, \sim$ introduced in Section 1.4. Also, in this chapter, we will find it useful to abbreviate $d(X, Y)$ to XY.

3.1. H1 $\Rightarrow$ H2.

Let $(\mathcal{S}, d)$ be a k_1-H1 geodesic space.

Lemma 3.3.1 : *Suppose that $X, Y, Z \in \mathcal{S}$, and that α is a geodesic joining X to Y. Then, there is some $A \in \alpha$ such that*

$$XZ \simeq XA + AZ$$
$$YZ \simeq YA + AZ.$$

(Figure 3a.)

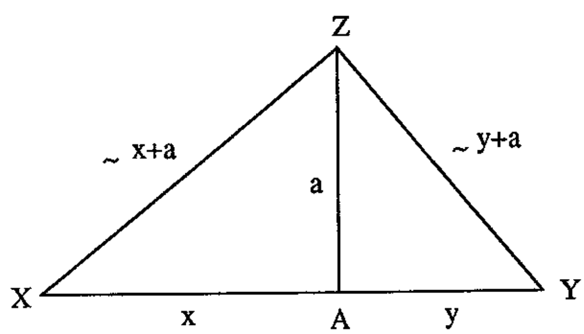


Figure 3a.

Proof : Let $XYZ \longleftrightarrow xyz$ so that $XY = x + y$, $YZ = y + z$ and $ZX = z + x$. Let $A \in \alpha$ be so that $XA = x$ and $AY = y$. Let $AZ = a$. (Figure 3b.)

Now

$$XZ \leq XA + AZ$$
$$x + z \leq x + a.$$

So,

$$z \leq a.$$

But,

$$XZ + AY = x + y + z$$
$$YZ + AY = x + y + z$$
$$XY + AZ = x + y + a.$$

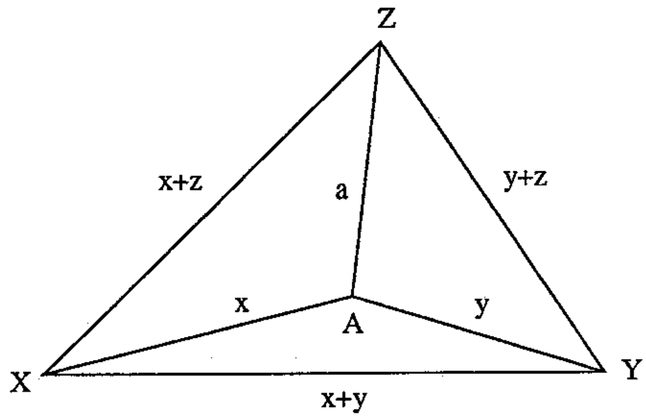


Figure 3b.

Applying the hypothesis k_1-H1 to $\{X, Y, Z, A\}$, we see that we must have $z \simeq a$.
$\diamond$

Lemma 3.1.2 : *There is some constant K ($= K(k_1)$) so that: if α is any geodesic joining X to Y, $X, Y \in S$, and β is any path from X to Y of length $\simeq d(X,Y)$, then $\beta \subseteq N_K(\alpha)$.*

Proof : Let $Z \in \beta$ be any point, and let $A \in \alpha$ be as in Lemma 3.1.1. Then,

$$\begin{aligned} XA + AY = XY &\simeq \text{length}\,\beta \\ &\geq XZ + ZY \\ &\simeq (XA + AZ) + (YA + AZ) \\ &= XA + AY + 2AZ. \end{aligned}$$

Thus

$$AZ \simeq 0,$$

i.e.

$$d(Z, [X,Y]) \simeq 0.$$

$\diamond$

In particular, this shows that any two geodesics joining the same pair of points remain a bounded distance apart.

Lemma 3.1.3 : *Suppose X, Y, Z are points of S, and $[X,Y]$, $[Y,Z]$, $[Z,X]$ are any geodesics joining them. Then, there exists $A \in S$ with*

$$\begin{aligned} d(A, [X,Y]) &\simeq 0 \\ d(A, [Y,Z]) &\simeq 0 \\ d(A, [Z,X]) &\simeq 0. \end{aligned}$$

Proof : Let $A \in [X,Y]$ be as in Lemma 3.1.1 (with $\alpha = [X,Y]$). Then $XA + AZ \simeq XY$. Applying Lemma 3.1.2, (with $\alpha = [X,Z]$ and $\beta = [X,A] \cup [A,Y]$), we see that $d(A,[Z,X]) \simeq 0$. Similarly, $d(A,[Y,Z]) \simeq 0$.
◇

Corollary 3.1.4, H1 ⇒ H2 : *$\forall k_1 \exists k_2$ such that if $(\mathcal{S},d)$ is k_1-H1, then it is k_2-H2.*

In fact, we see that k_2 is at most a fixed (universal) multiple of k_1. This is because it is derived from k_1 by a certain number of applications of the transitive law to our approximate inequalities. In particular, we see that a 0-H1 space is also 0-H2.

Definition : Suppose that $(\mathcal{S},d)$ is k_1-H1, and that $X,Y,Z \in \mathcal{S}$. We call $A \in \mathcal{S}$ a *centre* of XYZ if $d(A,[X,Y]) \simeq 0$, $d(A,[Y,Z]) \simeq 0$ and $d(A,[Z,X]) \simeq 0$.

In view of Lemma 3.1.2, this definition makes sense, irrespective of the choice of geodesics $[X,Y]$, $[Y,Z]$ and $[Z,X]$.

Note that we may choose a centre to lie on any one of these three geodesic edges.

Lemma 3.1.5 : *Suppose that C and D are both centres for XYZ in $\mathcal{S}$. Then $C \sim D$.*

Proof : Choose C', D' in $[X,Y]$ with $C \sim C'$ and $D \sim D'$. Without loss of generality, C' is nearer X. Let x,u,v,a,y be as in Figure 3c.

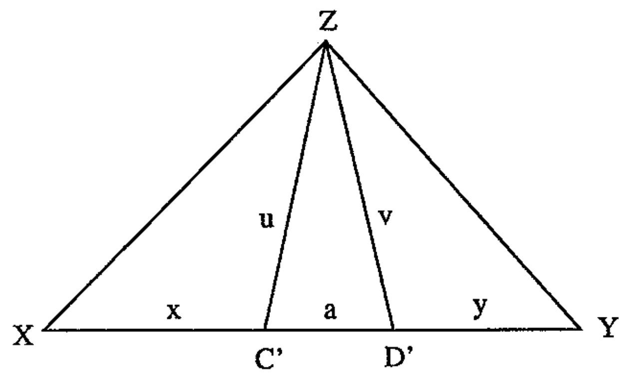


Figure 3c.

Now, $d(C',[X,Z]) \simeq 0$, and $d(D',[X,Z]) \simeq 0$. So,

$$XZ \simeq x + u$$

and

$$XZ \simeq x + a + v.$$

Thus,

$$u \simeq a + v.$$

Similarly.

$$v \simeq a + u.$$

Thus, $a \simeq 0$. So $C' \sim D'$ and $C \sim D$.

◇

Lemma 3.1.6 : *Given $X, Y, Z \in \mathcal{S}$, let $A \in \mathrm{proj}_{[X,Y]} Z$ (i.e. A is a nearest point on $[X, Y]$ to Z). Then, A is a centre of XYZ.*

Proof : Let C be a centre of XYZ on $[X, Y]$. Without loss of generality, $XC \leq XA$. Let x, b, y, c, a be as in Figure 3d.

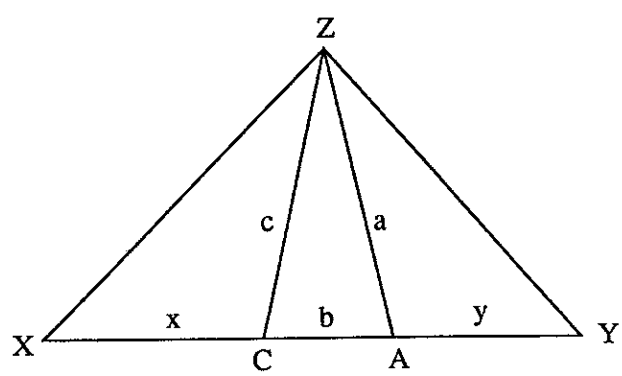


Figure 3d.

We have $a \leq c$, thus

$$\begin{aligned} c + b + y &\simeq YZ \\ &\leq a + y \\ &\leq c + y. \end{aligned}$$

Thus,

$$b \simeq 0.$$

In other words, $A \sim C$, and so A is a centre.

◇

We stated in Section 2.1, that, in a H1 space, the statement $XZ + YW \simeq XW + YZ$ implies $XY + ZW \preceq XZ + YW$ and thus is equivalent to $XY : ZW$.

Lemma 3.1.7 : *Let $X, Y, Z, W \in \mathcal{S}$. Suppose we have*

$$XZ + YW \simeq XW + YZ.$$

Then, there exist $C, D \in \mathcal{S}$, *with*

$$d(C,[X_1,X_2]) \simeq 0 \text{ provided } \{X_1,X_2\} \neq \{Z,W\}$$

and

$$d(D,[Y_1,Y_2]) \simeq 0 \text{ provided } \{Y_1,Y_2\} \neq \{X,Y\},$$

where $X_i, Y_i \in \{X,Y,Z,W\}$, $X_1 \neq X_2$ *and* $Y_1 \neq Y_2$.

Moreover, we have

$$Z_1Z_2 \simeq Z_1C + CD + DZ_2,$$

whenever $Z_1 \in \{X,Y\}$ *and* $Z_2 \in \{Z,W\}$.

Proof : Let A, B be centres of XYZ, XYW respectively on $[X,Y]$. Without loss of generality, A is nearer X. Let x, a, u, y, b be as in Figure 3e.

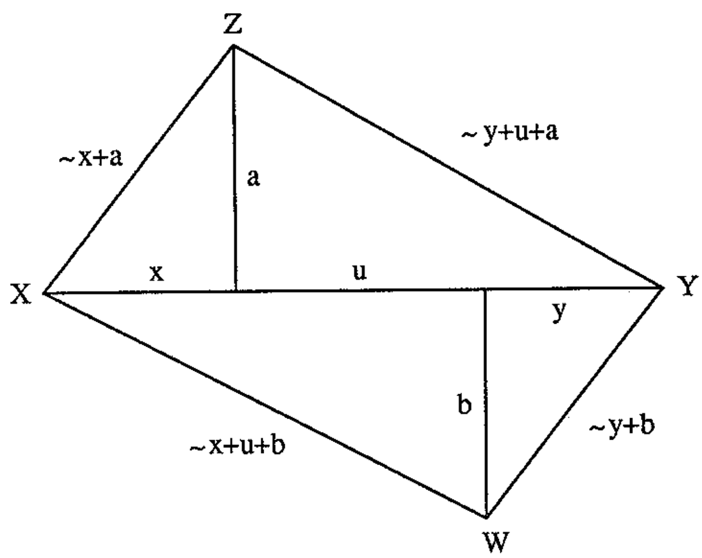


Figure 3e.

We have

$$XZ + YW \simeq XW + YZ$$
$$(x+a) + (y+b) \simeq (x+u+b) + (y+u+a).$$

Thus,

$$u \simeq 0.$$

Thus $A \sim B$, and so A is a centre for both XYZ and XYW. We take $C = A$.

Similarly, we find D on $[Z,W]$.

This proves the first part of the lemma.

Now, suppose that $Z_1 \in \{X,Y\}$ and $Z_2 \in \{Z,W\}$. We want to show that $Z_1Z_2 \simeq Z_1C + CD + DZ_2$. Without loss of generality, we can assume that $Z_1 = X$ and $Z_2 = Z$.

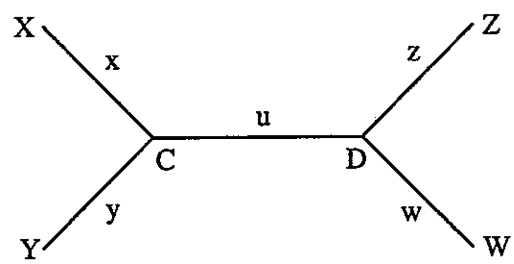


Figure 3f.

Now, there exist $C', D' \in [X, Z]$ with $C \sim C'$ and $D \sim D'$. If $XC' \leq XD'$, we have that

$$XZ = XC' + C'D' + D'Z \simeq XC + CD + DZ.$$

Thus, we may suppose that $XD' \preceq XC'$. Then,

$$XZ = XC' - C'D' + D'Z \simeq XC - CD + DZ.$$

But, $YW \leq YC + CD + DW$, and so

$$XZ + YW \simeq XC + YC + DZ + DW \simeq XY + ZW.$$

We noted, before the statement of the lemma that we must have $XZ + YW \succeq XY + ZW$, and so $XY + ZW \simeq XZ + YW$.

Applying the first half of the lemma to this case, we find a point $E \in S$ so that $d(E, [W_1, W_2]) \simeq 0$, provided that $W_1, W_2 \in \{X, Y, Z, W\}$ are distinct, and $\{W_1, W_2\} \neq \{X, W\}$. Now, C and E are both centres of XYZ, and D and E are both centres of YZW. Applying Lemma 3.1.5, we find that $C \sim E \sim D$. Thus, $CD \simeq 0$, and so

$$XZ \simeq XC - CD + DZ \simeq XC + CD + DZ.$$

◇

Given the points C, D of Lemma 3.1.7, we may construct the "spanning tree"

$$[X, C] \cup [Y, C] \cup [C, D] \cup [D, Z] \cup [D, W],$$

which we shall write as $(XY)CD(ZW)$. In other words, the notation $(XY)CD(ZW)$ is intended to define the points C and D. Now, all the distances between the points of $\{X, Y, Z, W\}$ may be read off, up to an additive constant, along the tree (c.f. Lemma 2.1.1). In fact, the corresponding geodesics run within a bounded distance of the tree (see Lemma 3.3.1 or Lemma 4.1). If $XC = x$, $YC = y$, $CD = u$, $DZ = z$ and $DW = w$, we shall write

$$(XY)CD(ZW) \longleftrightarrow (xy)u(zw).$$

(Figure 3f.)

The above discussion may be generalised to define spanning trees for arbitrary finite sets of points (see Section 3.3). First, however, we give a proof of H2 $\Rightarrow$ H1.

3.2. H2 $\Rightarrow$ H1.

Let $(\mathcal{S}, d)$ be k_2-H2.

Suppose that $X, Y, Z \in \mathcal{S}$, and that $[X, Y]$ is some geodesic from X to Y. If we choose geodesics α and β joining Z to X and Y respectively, then we may find some point $C \in [X, Y]$, with $d(C, \alpha) \leq 2k_2$ and $d(C, \beta) \leq 2k_2$. For all we know at the moment, the position of C on $[X, Y]$ might depend substantially on the choice of α and β. However, we must have

$$XZ \geq XC + CZ - 4k_2$$

and

$$YZ \geq YC + CZ - 4k_2.$$

Given X, Y, Z and $[X, Y]$ as above, we shall call any point $C \in [X, Y]$ a *near-projection* of Z to $[X, Y]$ if we have both $XZ \simeq XC + CZ$ and $YZ \simeq YC + CZ$. We shall deduce property H1 from the existence of such near-projections.

Lemma 3.2.1 : *Let $X, Y, Z \in \mathcal{S}$ and $[X, Y]$ be any geodesic from X to Y. Suppose that $A \in [X, Y]$. Then either*

$$XZ \simeq XA + AZ$$

or

$$YZ \simeq YA + AZ.$$

Proof : Let C be a near-projection of Z to $[X, Y]$. Suppose $AX \leq AC$. Let x, u, a, v, b be as in Figure 3g.

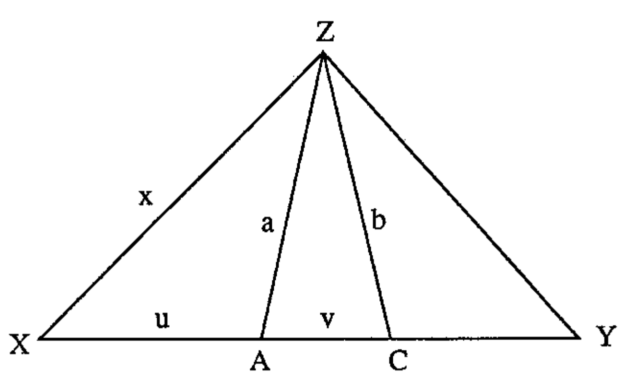


Figure 3g.

Thus $v + b \geq a$. Now,

$$XZ \simeq XC + CZ$$
$$x \simeq u + v + b$$
$$\geq u + a.$$

But $x \leq u + a$. Thus $x \simeq u + a$, i.e. $XZ \simeq XA + AZ$.

Similarly, if $AX \geq AC$, we have $YZ \simeq YA + AZ$.

◇

Lemma 3.2.2 : *Given any $X, Y, Z, W \in \mathcal{S}$, we have*

$$XY + ZW \preceq \max(XW + YZ, XZ + YW).$$

Proof : Choose any geodesic $[Z, W]$. Let C be any near-projection of X to $[Z, W]$ (Figure 3h).

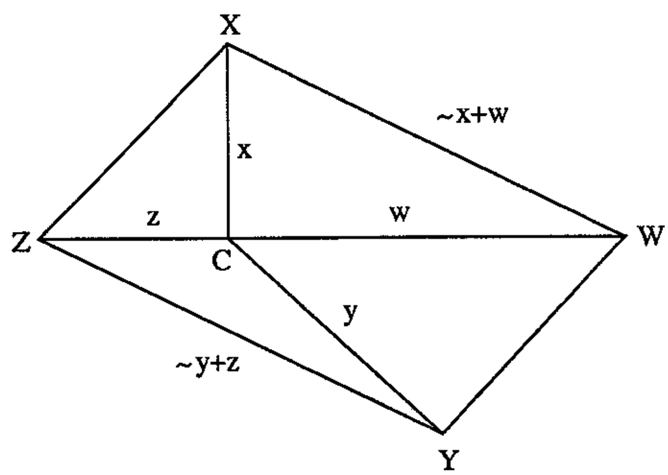


Figure 3h.

From Lemma 3.2.1, we have, without loss of generality,

$$ZY \simeq CZ + CY.$$

But,

$$XW \simeq CX + CW.$$

Thus,

$$XW + YZ \simeq (CX + CW) + (CZ + CY)$$
$$= (CX + CY) + (CZ + CW)$$
$$\geq XY + ZW.$$

$\diamondsuit$

Proposition 3.2.3 : *$\forall k_2 \, \exists k_1$ such that if $(\mathcal{S}, d)$ is k_2-H2, it is k_1-H1.*

Proof : Suppose that $(\mathcal{S}, d)$ is H2. Given any $X, Y, Z, W \in \mathcal{S}$, we have, without loss of generality,

$$XY + ZW \geq \max(XW + YZ, XZ + YW).$$

By Lemma 3.2.2,

$$XY + ZW \preceq \max(XW + YZ, XZ + YW).$$

Thus, without loss of generality,

$$XZ + YW \leq XW + YZ \simeq XY + ZW,$$

i.e. $XZ : YW$. So, $(\mathcal{S}, d)$ is H1.
$\diamondsuit$

Again k_1 is at most a certain fixed multiple of k_2. In particular, a 0-H2 space is also 0-H1.

3.3. Spanning trees.

Sections 3.3 and 3.4 are intended to make apparent the tree-like properties of almost-hyperbolic spaces.

In this section, we show that any finite set of points in an almost-hyperbolic space may be spanned by which allows us to measure distances up to an additive constant. We shall give a refinement of this result in Chapter 7.

We shall need the following stronger version of Lemma 3.1.2.

Suppose that $(\mathcal{S}, d)$ is k_1-H1.

Lemma 3.3.1 : *$\forall h \geq 0 \, \exists H = H(h, k_1)$ such that the following holds.*

If α is any geodesic joining X to Y in $\mathcal{S}$, and β is any path from X to Y of length at most $XY + h$, then we have both $\beta \subseteq N_H(\alpha)$ and $\alpha \subseteq N_H(\beta)$.

Proof : The statement $\beta \subseteq N_H(\alpha)$ is the same as Lemma 3.1.2, except that we are allowing h to be independent of k. This makes no essential difference to the proof.

The statement $\alpha \subseteq N_H(\beta)$ is new. There is a simple argument, see the proof of Proposition 4.9.
$\diamondsuit$

Proposition 3.3.2 : *$\forall n \in \mathbf{N} \, \exists K_n = K(n, k_1)$ such that the following holds.*

Let $V \subseteq \mathcal{S}$ be any set of $n+1$ points of $\mathcal{S}$. Then, there is an embedded tree $T_V \subseteq \mathcal{S}$ with geodesic edges, so that the distance XY between any two points $X, Y \in \mathcal{S}$ may be measured up to K_n along the arc $\beta(X,Y)$ in T_V joining X to Y, i.e.

$$XY \geq \text{length}\beta(X,Y) - K_n.$$

Proof : The proof is by induction on n. Suppose that V has $n+1$ elements, and that we have constructed T_V. Given $C, D \in T_V$, we write $\beta(C,D)$ for the arc in T_V joining C to D. For any $C, D \in T_V$, we may find $Z, W \in V$ such that $\beta(C,D) \subseteq \beta(Z,W)$. Since, by hypothesis, $\text{length}\beta(Z,W) \leq ZW + K_n$, we may deduce that also $\text{length}\beta(C,D) \leq CD + K_n$.

Let $X \in \mathcal{S}$ be any $(n+2)^{\text{th}}$ point. We want to span $V \cup \{X\}$. Let $A \in \text{proj}_{T_V} X$ (i.e. A is a nearest point in T_V to X). Let $T_{V \cup \{X\}} = T_V \cup [A, X]$.

Suppose $Y \in V$, we want to measure XY. Let B be a centre for AXY on $[A,Y]$. Now, $\text{length}\beta(A,Y) \leq AY + K_n$. So, by Lemma 3.3.1, we have $d(B, \beta(A,Y)) \leq H = H(K_n, k_1)$. Now,

$$XB + BA \simeq XA = d(X, T_V) \leq d(X, \beta(A,Y)) \leq XB + H.$$

Thus, $BA \preceq H$. (The meanings of $\simeq$ and $\preceq$ depend on k_1, but not on n.) But,

$$\begin{aligned} XY &\simeq XB + BY \\ &\succeq XA + AY - 2H \\ &\geq (\text{length}\beta(A,Y) - K_n) + AX - 2H \\ &= \text{length}([X,A] \cup \beta(A,Y)) - (K_n + 2H). \end{aligned}$$

We may therefore take $K_{n+1} \simeq K_n + 2H(K_n, k_1)$.
$\diamondsuit$

3.4. Metric trees.

The main purpose of this section is to show that 0-H1 path metric spaces are precisely what we shall call "metric trees" (elsewhere known as "**R**-trees").

Definition : A *metric tree* is a path-metric space which contains no embedded rectifiable circle.

(In fact, we shall see that a metric tree can contain no topologically embedded circle.)

Note that we are not making any assumptions of completeness or local-compactness. However, it is true that in any metric tree, $(\mathcal{S}, d)$, any two points may be joined by a (unique) geodesic. One may see this as follows.

First note that there is at most one rectifiable arc joining any two distinct points, $X, Y \in \mathcal{S}$. Suppose that α and β were two such arcs, with $\alpha \not\subseteq \beta$. Let β' be the closure of a component of $\beta \backslash \alpha$. Then β' meets α in two distinct points Z and W. If α' is the sub-arc of α lying between Z and W, then $\alpha' \cup \beta'$ is an embedded circle. This contradiction shows that $\alpha \subseteq \beta$. Similarly $\beta \subseteq \alpha$.

Now if γ is any rectifiable path joining X to Y, then the image of γ, being path-connected, contains an arc γ_0 with endpoints X and Y. (In this context, γ_0 may be obtained by applying Ascoli's theorem to a sequence of paths from X to Y lying in the image of γ, and whose lengths tend to the infinum for such paths.) Thus γ_0 is the unique rectifiable arc from X to Y.

Suppose that γ' is a path from X to Y of length at most $d(X,Y) + \epsilon$. Again, the image of γ' contains a rectifiable arc from X to Y which must therefore be γ_0. Thus $\text{length}\,\gamma_0 \leq \text{length}\,\gamma' \leq d(X,Y) + \epsilon$. Since ϵ is arbitrary, we have $\text{length}\,\gamma_0 = d(X,Y)$, i.e. γ_0 is a geodesic.

We have shown that any to points in a metric tree are joined by a unique rectifiable arc which is always a geodesic. (In fact, given that a metric tree contains no topological circle, we see that every closed arc in a metric tree is a geodesic.)

By a similar argument, we see that if we take any three points in a metric tree $(\mathcal{S}, d)$, then the three geodesics joining them meet in a single point. Thus, $(\mathcal{S}, d)$ is 0-H2.

Note that, in our discussion of geodesic spaces so far in this paper, we have made no use of the assumptions of local compactness or completeness, other than that there should exist a geodesic between any two points. (We have only used projection to compact sets.)

Proposition 3.4.1 : *Suppose that $(\mathcal{S}, d)$ is a path-metric space in which any pair of points are joined by at least one geodesic. Then $(\mathcal{S}, d)$ is 0-H1 if and only if it is 0-H2.*

Proof : See the remarks after Propositions 3.1.4 and 3.2.3.

$\diamondsuit$

Suppose that $(\mathcal{S}, d)$ is such a space (as described by Proposition 3.4.1), and that $V \subseteq \mathcal{S}$ is a finite set of points. Then, the construction of Proposition 3.3.2 gives an embedded tree $T_V \subseteq \mathcal{S}$ along which distances are measured precisely, i.e. $K(n, 0) = 0$ for all $n \in \mathbf{N}$. This is essentially because everywhere our approximate inequalities may be replaced by precise inequalities. Moreover, since the construction was inductive, we can take T_V to contain any previously chosen geodesic $[X, Y]$, with $X, Y \in V$. We leave the reader to check these statements.

Proposition 3.4.2 : *Let $(\mathcal{S}, d)$ be a path-metric space. The following are equivalent:*

(1) $(\mathcal{S}, d)$ is a metric tree,

(2) $(\mathcal{S}, d)$ contains no topologically embedded circle,

(3) $(\mathcal{S}, d)$ is 0-H1.

Proof : $(2) \Rightarrow (1)$ is trivial.

We argued above that any metric tree is 0-H2, and thus 0-H1 by Proposition 3.4.1. This proves $(1) \Rightarrow (3)$.

For (3) $\Rightarrow$ (2), suppose that $(\mathcal{S}, d)$ is a 0-H1 path-metric space. We will want to use the tree construction mentioned above, after Proposition 3.4.1. However, we do not yet know that every pair of points in $\mathcal{S}$ can be joined by a geodesic. We can get around this problem by taking the metric completion, $(\mathcal{S}_C, d)$, of $(\mathcal{S}, d)$. It is easily verified that $(\mathcal{S}_C, d)$ is also a 0-H1 path-metric space. (The extension of d to $\mathcal{S}_C$ is automatically a path-metric.) We claim that any pair of points, $A, B \in \mathcal{S}_C$, may be joined by a geodesic in $\mathcal{S}_C$. This may be seen as follows.

We can assume that $A \neq B$. Let $l = d(A, B)$, and suppose that $\alpha, \beta : [0, l] \longrightarrow \mathcal{S}_C$ are paths satisfying respectively $d(\alpha x, \alpha y) \leq |x - y| + \eta$ and $d(\beta x, \beta y) \leq |x - y| + \eta$ for all $x, y \in [0, l]$, and with $\alpha(0) = \beta(0) = A$ and $\alpha(l) = \beta(l) = B$. Given $x \in [0, l]$, let $E = \alpha x$ and $F = \beta x$. Thus, $d(A, E) \leq x + \eta$, $d(A, F) \leq x + \eta$, $d(B, E) \leq l - x + \eta$ and $d(B, F) \leq l - x + \eta$. Since $\mathcal{S}_C$ is 0-H1, we must have

$$\begin{aligned} d(A, B) + d(E, F) &\leq \max\big(d(A, E) + d(B, F), d(A, F) + d(B, E)\big) \\ &\leq l + 2\eta. \end{aligned}$$

Thus, $d(\alpha x, \beta x) \leq 2\eta$.

Now, for each $i \in \mathbf{N}$, choose a path $\gamma_i : [0, l] \longrightarrow \mathcal{S}_C$, with $\gamma_i(0) = A$, $\gamma_i(l) = B$, and $d(\gamma_i x, \gamma_i y) \leq |x - y| + 1/2^i$ for all $x, y \in [0, l]$. Thus if $x \in [0, d]$, and $j \geq i$, then $d(\gamma_i x, \gamma_j x) \leq 2/2^i$. Since $(\mathcal{S}_C, d)$ is complete, the paths γ_i converge uniformly to a path $\gamma : [0, d] \longrightarrow \mathcal{S}_C$. Clearly γ must be geodesic. This proves the claim.

Now, suppose (for contradiction) that $\Sigma \subseteq \mathcal{S} \subseteq \mathcal{S}_C$ is homeomorphic to a circle.

We need to find two points $Y, Z \in \Sigma$ with $[Y, Z] \not\subseteq \Sigma$. To do this, we take $W_1, W_2 \in \Sigma$ with $d(W_1, W_2)$ maximal. If $[W_1, W_2] \subseteq \Sigma$, then pick any $W_3 \in \Sigma \setminus [W_1, W_2]$. If $[W_1, W_3] \cup [W_3, W_2] \subseteq \Sigma$, then these three geodesics must cover Σ, and have disjoint interiors. Thus, $[W_1, W_2] \cap [W_2, W_3] \cap [W_3, W_1] = \emptyset$, contradicting the fact that $(\mathcal{S}_C, d)$ is 0-H1 and hence 0-H2.

Now, choose $C \in [Y, Z] \setminus \Sigma$, and let $\epsilon = d(C, \Sigma) > 0$. Since Σ is compact, we may find points $X_1, X_2, \ldots, X_p \in \Sigma$, cyclically ordered on Σ, so that $d(X_i, X_{i+1}) \leq \epsilon$. We can assume that $\{Y, Z\} \subseteq \{X_1, \ldots, X_p\} = V$. We may construct, in $\mathcal{S}_C$, a spanning tree T_V for V with $[Y, Z] \subseteq T_V$. Thus $C \in T_V$. Now C must must separate T_V, so we may write $T_V \setminus B = T_1 \cup T_2$, with T_i open in T_V. Let $V_i = V \cap T_i$. Thus $V = V_1 \cup V_2$ and $V_i \neq \emptyset$. We may find $D_1 \in V_1$ and $D_2 \in V_2$ adjacent on Σ. Thus $d(D_1, D_2) \leq \epsilon$. But the path β in T_V from D_1 to D_2 passes through C. Thus $d(D_1, D_2) = \text{length}\beta \geq 2\epsilon$.

We have contradicted the existence of Σ.

$\diamond$

CHAPTER 4. Convexity and pseudoisometries.

In this chapter, we define the notion of "almost convexity" for a subset of an almost-hyperbolic space $\mathcal{S}$. Examples, as we shall see, include "starlike" sets, and uniform neigbourhoods of other almost-convex sets. We define projections to almost-convex sets, and show that, in some sense, projections decrease distance by an exponential factor (Proposition 4.5). From this, we may deduce the exponential divergence of geodesic rays (Proposition 4.7), as well as the pseudogeodesic property (Proposition 4.9). We deduce properties H4 and H5, and conclude with a proof that almost-hyperbolicity is a pseudoisometric invariant (Proposition 4.10).

Let $(\mathcal{S}, d)$ be k_1-H1.

Lemma 4.1 : *There is some $h = h(k_1)$ such that for all $p \in \mathbf{N}$, the following holds.*

Suppose γ is a path in $\mathcal{S}$, consisting of at most p geodesic segments, which joins X to Y. Then, $[X,Y] \subseteq N_{ph}(\gamma)$, where $[X,Y]$ is any geodesic from X to Y.

Proof : Suppose $p = 2$, and $\gamma = [X,Z] \cup [Z,Y]$. Let A be a centre of XYZ on $[X,Y]$. Choose $B \in [X,Z]$ and $C \in [Y,Z]$ with $B \sim A$ and $C \sim A$. (Figure 4a.)

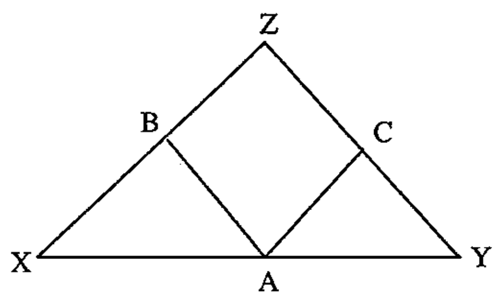


Figure 4a.

Lemma 3.1.2 now gives us a constant h such that $[X,A] \subseteq N_h([X,B])$ and $[A,Y] \subseteq N_h([Y,C])$. Thus, $[X,Y] \subseteq N_h([X,Z] \cup [Z,Y])$.

This also deals with the case $p = 1$: any two geodesics between the same points stay a distance at most h apart.

We now use induction on p. Suppose $\gamma = [X_0, X_1] \cup \cdots \cup [X_{p-1}, X_p]$. From the above,

$$\begin{aligned}[X_0, X_p] &\subseteq N_h\big([X_0, X_{p-1}] \cup [X_{p-1}, X_p]\big) \\ &\subseteq N_h\big(N_{(p-1)h}\big([X_0, X_1] \cup \cdots \cup [X_{p-2}, X_{p-1}]\big) \cup [X_{p-1}, X_p]\big) \\ &\subseteq N_{ph}(\gamma).\end{aligned}$$

$\diamondsuit$

Let $\lambda \geq 0$, and let $Q \subseteq \mathcal{S}$ be a closed set.

Definition : The set $Q \subseteq \mathcal{S}$ is λ-*convex* if for all $X, Y \in Q$, and all geodesics $[X, Y]$, we have $[X, Y] \subseteq N_\lambda(Q)$.

We shall call a set *almost-convex* if it is λ-convex for some $\lambda \geq 0$.

Lemma 4.1 provides us with many examples of almost-convex sets. Any set in which any two points can be joined by a piecewise geodesic path with a bounded number of geodesic segments is almost-convex. This includes all starlike sets, and any uniform neighbourhood of a geodesic segment. We also have:

Lemma 4.2 : *There is some $\lambda_0 = \lambda_0(k_1)$ such that if $Q \subseteq \mathcal{S}$ is λ-convex, and $r \geq 0$, then $N_r(Q)$ is $(\lambda_0 + \max(\lambda - r, 0))$-convex.*

Proof : Let $\lambda_0 = 3h$, where h comes from Lemma 4.1. Suppose $X, Y \in N_r(Q)$. Then there exist $W, Z \in Q$ with $d(X, Z) \leq r$ and $d(Y, W) \leq r$. Now, $[Z, W] \subseteq N_\lambda(Q)$ and $[X, Z] \cup [Y, W] \subseteq N_r(Q)$. So, by Lemma 4.1,

$$\begin{aligned}[X, Z] &\subseteq N_{3h}\big([X, Z] \cup [Z, W] \cup [W, Y]\big) \\ &\subseteq N_{3h}\big(N_{\max(\lambda, r)}(Q)\big) \\ &= N_{\lambda_0 + \max(\lambda - r, 0)}\big(N_r(Q)\big).\end{aligned}$$

$\diamondsuit$

Recall the definitions of $\check{Q}$, ∂Q, $\text{proj}_Q X$ and $d(r, Q)$ from Section 1.1. Given any $M \subseteq \mathcal{S}$, we shall write $\text{proj}_Q M = \bigcup_{X \in M} \text{proj}_Q X$. Also $\text{diam} M = \sup\{d(X, Y) | X, Y \in M\}$ is the diameter of M.

Suppose that $Q \subseteq \mathcal{S}$ is λ-convex. Then, for any $X \in \mathcal{S}$, the quantity $\text{diam}(\text{proj}_Q X)$ may be bounded in terms of λ and the hyperbolicity parameter, k_1. In fact, we may make the more general statement:

Lemma 4.3 : *Suppose that $Q \subseteq \mathcal{S}$ is λ-convex, and $M \subseteq \mathcal{S}$ satisfies* $\text{diam} M \leq 2d(Q, M)$, *then* $\text{diam}(\text{proj}_Q M) \leq J + 4\lambda$, *where J depends only on k_1.*

Proof : Let $\rho = d(Q, M)$. Suppose $X, Y \in M$, and $Z \in \text{proj}_Q X$ and $W \in \text{proj}_Q Y$ so that $d(X, Z) \leq 2\rho$, $d(X, Z) \geq \rho$ and $d(Y, W) \geq \rho$. We want to show that $d(Z, W) \preceq 4\lambda$.

Case (1), $XY : ZW$.

Let $(XY)AB(ZW)$ be a spanning tree, and let $(XY)AB(ZW) \longleftrightarrow (xy)u(zw)$. (Figure 4b. See the discussion after Lemma 3.1.7.)

Now, $d(B, [Z, W]) \simeq 0$. Thus $d(B, Q) \preceq \lambda$. Now,

$$\begin{aligned}d(X, Z) = d(X, Q) &\preceq d(X, B) + d(B, Q) \\ x + u + z &\preceq (x + u) + \lambda.\end{aligned}$$

Thus,

$$z \preceq \lambda.$$

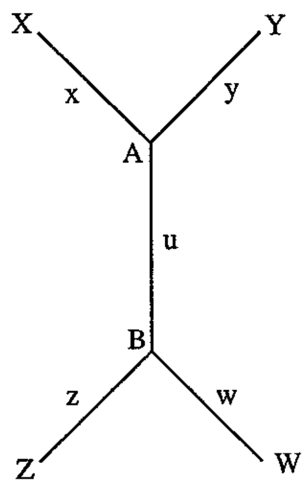


Figure 4b.

Similarly,

$$w \preceq \lambda.$$

Thus,

$$d(Z,W) \simeq z + w \preceq 2\lambda.$$

Case (2), $XZ : YW$.

Let $(XZ)CD(YW) \longleftrightarrow (xz)u(yw)$ be a spanning tree (Figure 4c).

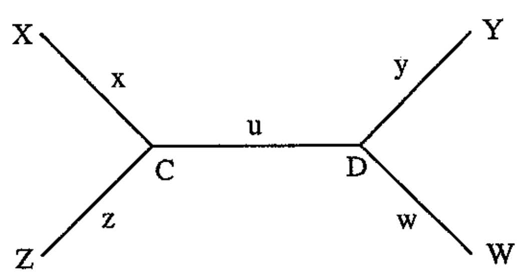


Figure 4c.

Similarly as in Case (1), we have $d(C,Q) \preceq \lambda$ and $d(D,Q) \preceq \lambda$. Thus,

$$d(X,Z) = d(X,Q) \preceq d(X,C) + \lambda$$
$$x + z \preceq x + \lambda.$$

Thus,

$$z \preceq \lambda.$$

Similarly,

$$w \preceq \lambda.$$

From the hypotheses, we have $x+z \succeq \rho$, $y+w \succeq \rho$ and $x+y+u \preceq 2\rho$. Thus,

$$\begin{aligned} 2\rho + u &\preceq (x+z)+(y+w)+u \\ &= (x+y+u)+(z+w) \\ &\preceq 2\rho + 2\lambda. \end{aligned}$$

Thus,

$$u \preceq 2\lambda,$$

and so,

$$d(X,Z) \simeq w+z+u \preceq \lambda+\lambda+2\lambda = 4\lambda.$$

Case (3), $XW : YZ$.

Let $(XW)FE(YZ) \longleftrightarrow (xw)u(yz)$ be a spanning tree (Figure 4d).

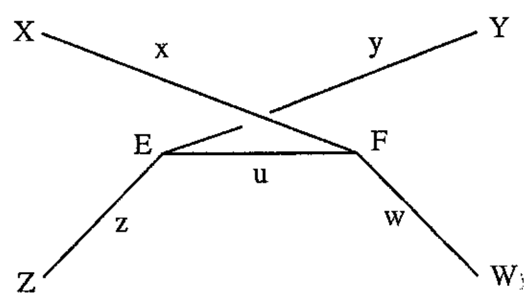


Figure 4d.

Now,

$$\begin{aligned} d(X,Z) = d(X,Q) &\leq d(X,E)+d(E,Q) \\ x+u+z &\preceq x+\lambda. \end{aligned}$$

Thus,

$$u+z \preceq \lambda.$$

Similarly,

$$u+w \preceq \lambda.$$

Thus,

$$d(Z,W) \simeq z+w+u \leq (z+u)+(w+u) \preceq 2\lambda.$$

$\diamondsuit$

Lemma 4.4 : *$\forall\lambda\,\exists r_0 = r_0(\lambda,k_1), K_0 = K_0(\lambda,k_1)$ such that the following holds.*

Suppose $Q \subseteq \mathcal{S}$ is λ-convex, and $r \geq r_0$. Suppose $X,Y \in \mathcal{S}\backslash \check{N}_r(Q)$, $Z \in \mathrm{proj}_Q X$ and $W \in \mathrm{proj}_Q Y$. Let $d_1 = d_{0,Q}(Z,W)$ and $d_2 = d_{r,Q}(X,Y)$. Then $d_2 \geq 3d_1 - K_0$.

(Note that if $d_1 = \infty$, then clearly $d_2 = \infty$.)

Proof : Let λ_0 be the constant from Lemma 4.2. Let $h = J + 4\lambda_0$, where J comes from Lemma 4.3. Let $r_1 = \max(h/2, \lambda)$ and $r_2 = 3h/2$. Let $r_0 = r_1 + r_2$.

Suppose that $r \geq r_0$, and $X, Y \in \mathcal{S}\backslash\breve{N}_r(Q)$. We join X to Y by a path α in $\mathcal{S}\backslash\breve{N}_r(Q)$ of length d_2. Let $X = X_0, X_1, \ldots, X_p, X_{p+1} = Y$ be points on α which divide the path into p segments of length $3h$, and a remaining segment of length at most $3h$. Thus, $ph \leq d_2/3$.

Let $N = N_{r_1}(Q)$. Since $r_1 \geq \lambda$, by Lemma 4.2, we see that N is λ_0-convex.

For each $i = 1, \ldots, p$, we choose $Z_i \in \text{proj}_N X_i$. Let Z_0 be the point of $[X, Z]$ with $d(Z_0, Z) = r_1$, and let Z_{p+1} be the point of $[Y, W]$ with $d(Z_{p+1}, W) = r_1$. (Figure 4e.)

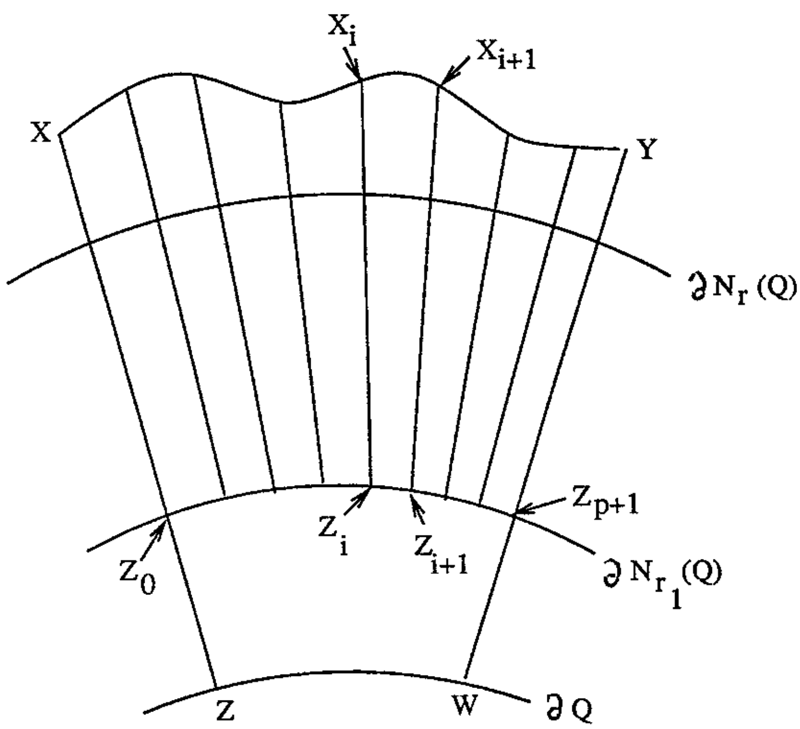


Figure 4e.

Thus, $Z_0 \in \text{proj}_N X$ and $Z_{p+1} \in \text{proj}_N Y$. Now, $d(X_i, X_{i+1}) \leq 3h$ and $d(X_i, N) \geq 3h/2$. Thus, by Lemma 4.3, $d(Z_i, Z_{i+1}) \leq J + 4\lambda_0 = h$ for each $i = 1, \ldots, p$. So, since $r_1 \geq h/2$, we have $[Z_i, Z_{i+1}] \cap \breve{Q} = \emptyset$ for each i. We can therefore join Z to W by a path

$$[Z, Z_0] \cup [Z_0, Z_1] \cup \cdots \cup [Z_p, Z_{p+1}] \cup [Z_{p+1}, W]$$

in $\mathcal{S}\backslash\breve{Q}$. Thus,

$$\begin{aligned} d_1 &\leq (p+1)h + 2r_1 \\ &= ph + (2r_1 + h) \\ &\leq \frac{d_2}{3} + (2r_1 + h). \end{aligned}$$

Thus,

$$d_2 \geq 3d_1 - (2r_1 + h).$$

$\diamondsuit$

Proposition 4.5 : it $\forall \lambda \geq 0 \, \exists \zeta = \zeta(\lambda, k_1), L = L(\lambda, k_1)$ such that the following holds.

Let $Q \subseteq \mathcal{S}$ be λ-convex, and $X, Y \in \mathcal{S} \backslash \check{N}_r(Q)$. Let $Z \in \operatorname{proj}_Q X$ and $W \in \operatorname{proj}_Q Y$. Set $d_1 = d_{0,Q}(Z, W)$ and $d_2 = d_{r,Q}(X, Y)$. If $r \geq L$ and $d_1 \geq L$, then

$$d_2 \geq e^{\zeta r} d_1.$$

Proof : By Lemma 4.2, there is some λ' so that any uniform neighbourhood of Q is λ'-convex. Let $r_0 = r_0(\lambda', k_1)$ and $K_0 = K_0(\lambda', k_1)$ be the constants given by Lemma 4.4. Let $L = \max(2r_0, K_0)$ and $\alpha = \frac{1}{2r_0} \log_e 2$.

Suppose that X, Y, Z, W, r, d_1, d_2 are as in the hypothesis. Let p be the integer part of (r/r_0). Let $Z = Z_0, Z_1, Z_2, \ldots, Z_p = X$ be points on $[Z, X]$ satisfying $d(Z_i, Z_{i+1}) = r_0$ for $i = 0, \ldots, p-2$. Let $W = W_0, W_1, W_2, \ldots, W_p = Y$ be similar points on $[W, Y]$. Thus, $Z_i \in \operatorname{proj}_{N(i)} X$ and $W_i \in \operatorname{proj}_{N(i)} Y$, where $N(i) = N_{ir_0}(Q)$. We write $\rho_i = d_{ir_0,Q}(Z_i, W_i)$. Thus, $d_1 = \rho_0$ and $d_2 = d_{r,Q}(X, Y) \geq d_{pr_0,Q}(X, Y) = \rho_p$, since $r \geq pr_0$.

By hypothesis, $\rho_0 = d_1 \geq L \geq K_0$, so by Lemma 4.4,

$$\rho_1 \geq 3\rho_0 - K_0 \geq 2\rho_0.$$

By induction,

$$\rho_i \geq 2\rho_{i-1} \geq 2^i \rho_0.$$

Thus,

$$d_2 \geq \rho_p \geq 2^p \rho_0 = 2^p d_1.$$

But $p \geq (r/r_0) - 1$, so

$$d_2 \geq (2^{1/r_0})^r \frac{d_1}{2} = \frac{1}{2} e^{2\zeta r} d_1.$$

Now, $r \geq L \geq 2r_0$, so $\zeta r \geq \log_e 2$. Thus,

$$d_2 \geq e^{\zeta r} d_1.$$

$\diamondsuit$

Suppose that Q is λ-convex, and $X, Y \in \partial Q$. It is a fairly easy consequence of Proposition 4.5 (c.f. Proposition 4.5 below), that any shortest path from X to Y in $\mathcal{S} \backslash \check{Q}$ lies within a bounded distance of Q, depending on λ.

Another point to note is that we can arrange for the rate of expansion ζ to be chosen independently of λ:

Proposition 4.6 : $\forall k_1 \, \exists \zeta_0, L_0$ *such that if* $Q, \lambda, r, X, Y, Z, W, d_1, d_2$ *are as in Proposition 4.5, with* $r \geq L_0 + \lambda$ *and* $d_1 \geq L_0 + 2\lambda$, *then*

$$d_2 \geq e^{\zeta_0 (r - \lambda)} (d_1 - 2\lambda).$$

Proof : Apply Proposition 4.5 to $N_\lambda(Q)$, which by Lemma 4.2 is λ_0-convex for fixed $\lambda_0 = \lambda_0(k_1)$.
$\diamondsuit$

We make two applications of Proposition 4.5. The first is to the case where Q is a uniform ball. In this case Proposition 4.5 can be interpreted as the statement that geodesic rays diverge exponentially (Proposition 4.7). From this we deduce property H5 (Corollary 4.8).

The second application is to the case where Q is a geodesic segment. From this, we get the pseudogeodesic property (Proposition 4.9), and a direct proof of H4. We show that almost-hyperbolicity is a pseudoisometric invariant (Proposition 4.10).

Proposition 4.7 : *$\forall k_1\ \exists \theta, M$ such that the following holds. Suppose $A, X, Y \in \mathcal{S}$ with $d(A,X) = d(A,Y) = r$. Set $d = d(X,Y)$ and $d' = d_{r,A}(X,Y)$. If $d \geq M$, then $d' \geq e^{\theta d}$.*

Proof : By Lemma 4.2 (or Lemma 4.1), every ball in $\mathcal{S}$ is λ_0-convex for $\lambda_0 = \lambda_0(k_1)$. Let ζ and L be as in Proposition 4.5 for $\lambda = \lambda_0$. We can suppose that $L \geq 1$. Let $M = 3L$ and $\theta = \zeta/3$.

Now, suppose that $A, X, Y \in \mathcal{S}$ with $d(A,X) = d(A,Y) = r$, and $d = d(X,Y) \geq M$. By continuity of the distance function, there exist $Z \in [A,X]$ and $W \in [A,Y]$, with $d(A,Z) = d(A,W)$ and $d(Z,W) = L$. Set $d(X,Z) = d(Y,W) = l$. Now, $3L = M \leq d \leq L + 2l$, so $l \geq L$. Let $Q = N_{r-l}(A)$, so that $Z \in \mathrm{proj}_Q X$ and $W \in \mathrm{proj}_Q Y$. Also, $d_{0,Q}(Z,W) \geq d(Z,W) = L \geq 1$. Now, Q is λ_0-convex, so applying Proposition 4.5, we get

$$d' = d_{r,A}(X,Y) = d_{l,Q}(X,Y) \geq e^{\zeta l} d_{0,Q}(Z,W) \geq e^{\zeta l}.$$

But $d \leq L + 2l \leq \frac{d}{3} + 2l$. Thus $\frac{d}{3} \leq l$, and so $d' \geq e^{\zeta d/3} \geq \theta l$.
$\diamondsuit$

Corollary 4.8 : *$\forall k_1, h_5\ \exists k_5$ such that if $\mathcal{S}$ is k_1-H1, then it is (k_5, h_5)-H5.*

Proof : Given θ and h_5, we have $e^{\theta d} \geq d + h_5$ for all sufficiently large d.
$\diamondsuit$

We now go on to consider pseudogeodesics. We may think of a path $\gamma : [0,t] \longrightarrow \mathcal{S}$ as a distance non-increasing map, with respect to the standard metric, $\sigma(x,y) = |x-y|$, on $[0,t]$. We shall frequently use γ to denote both the map, and its image in $\mathcal{S}$.

Definition : We call γ a (ν_1,ν_2)-*pseudogeodesic*, if for all $x, y \in [0,t]$, we have

$$\sigma(x,y) \leq \nu_1 d(\gamma x, \gamma y) + \nu_2.$$

Proposition 4.9 : *$\forall k_1, \nu_1, \nu_2\ \exists l$ such that the following holds.*

Suppose $\gamma : [0,t] \longrightarrow \mathcal{S}$ is a (ν_1,ν_2)-pseudogeodesic. Let $X = \gamma(0)$, $Y = \gamma(t)$, and suppose that $[X,Y]$ is any geodesic joining X to Y. Then, $\gamma \subseteq N_l([X,Y])$ and $[X,Y] \subseteq N_l(\gamma)$.

Proof : We will first prove γ lies inside a uniform neighbourhood of $[X,Y]$. For this, we intend to apply Proposition 4.5 to the λ_0-convex set $Q = [X,Y]$.

Let ζ and L be the constants given by Proposition 4.5, for $\lambda = \lambda_0$. Let

$$\begin{aligned} l_1 &= \max(L, \frac{1}{\zeta}\log_e(1+\nu_1)) \\ m &= \max(L, 2\nu_1 l_1 + \nu_2) \\ l_2 &= \frac{1}{2}(\nu_1(m+2l_1)+\nu_2) \\ l_0 &= l_1 + l_2. \end{aligned}$$

We claim that $\gamma \subseteq N_{l_0}([X,Y])$. If $\gamma \subseteq N_{l_1}([X,Y])$, we are done. If not, let β be a component of γ lying in $\mathcal{S} \setminus \mathring{N}_{l_1}([X,Y])$. Thus β is a (ν_1,ν_2)-pseudogeodesic joining two points A and B in $\partial N_{l_1}([X,Y])$. Let C and D be nearest points on $[X,Y]$ to A and B respectively. (Figure 4f.)

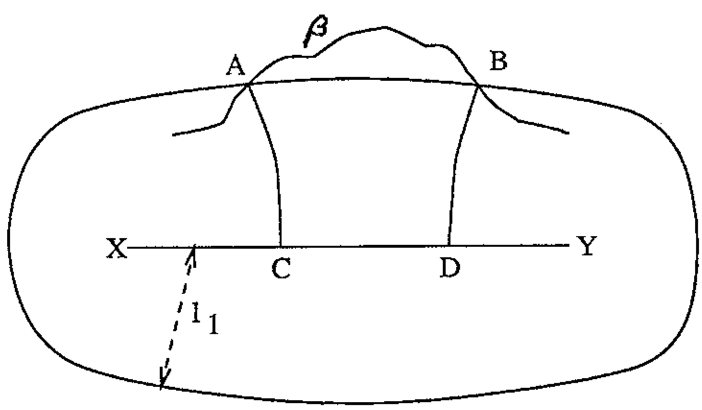


Figure 4f.

Set $d_1 = d(C,D)$ and $d_2 = d_{l_1,[X,Y]}(A,B)$. Thus $d_1 \leq d_{0,[X,Y]}(C,D)$. Now,

$$\begin{aligned} d_2 &\leq \text{length}\beta \leq \nu_1 d(A,B) + \nu_2 \\ &\leq \nu_1(d_1 + 2l_1) + \nu_2 \\ &= \nu_1 d_1 + (2l_1\nu_1 + \nu_2). \end{aligned}$$

Applying Proposition 4.5, we find that either

$$d_1 \leq L,$$

or

$$\begin{aligned} e^{\zeta l_1} &\leq d_2 \\ &\leq \nu_1 d_1 + (2l_1\nu_1 + \nu_2), \end{aligned}$$

so that

$$d_1 \le (e^{\zeta l_1} - \nu_1)d_1 \le 2l_1\nu_1 + \nu_2.$$

Either way,

$$d_1 \le m.$$

Thus,

$$\begin{aligned}\text{length}\beta &\le \nu_1(m + 2l_1) + \nu_2 \\ &= 2l_2\end{aligned}$$

and so, $\beta \subseteq N_{l_2}(N_{l_1}([X,Y])) = N_{l_0}([X,Y])$ and the claim follows.

It remains to show that $[X,Y]$ lies in a uniform neighbourhood of γ. Let $l = 2l_0 + \frac{1}{2}$. We claim that $[X,Y] \subseteq N_l(\gamma)$.

To see this, we choose points $X = X_1, X_2, \ldots, X_p = Y$ on γ, so that $d(X_i, X_{i+1}) \le 1$ for all i. Let $Y_i \in \text{proj}_{[X,Y]}X_i$, so that $Y_0 = X$ and $Y_p = Y$. Now, $d(Y_i, Y_{i+1}) \le l_0 + d(X_i, X_{i+1}) + l_0 \le 2l_0 + 1$. We conclude that each point of $[X,Y]$ lies at most a distance $\frac{1}{2}(2l_0 + 1)$ from some point Y_i. But $Y_i \in N_{l_0}(\gamma)$. Thus, $[X,Y] \subseteq N_{l_0+\frac{1}{2}}(N_{l_0}(\gamma)) = N_l(\gamma)$.
◇

A simple corollary of Proposition 4.9 is that H1 ⇒ H4. We shall deduce H3 ⇒ H4 in the next chapter.

Our main application of Proposition 4.9 is the following. Recall the definition of pseudoisometry from Section 1.3.

Proposition 4.10 : *$\forall k, \mu_1, \mu_2\ \exists k'$ such that the following holds.*

Suppose that $(\mathcal{S}, d)$ and $(\mathcal{S}', d')$ are (μ_1, μ_2)-pseudoisometric geodesic spaces, and suppose that $(\mathcal{S}, d)$ is k-H2, then $\mathcal{S}'$ is k'-H2.

Proof : Let $R \subseteq \mathcal{S} \times \mathcal{S}'$ be a (μ_1, μ_2)-pseudoisometry. Suppose $X', Y', Z' \in \mathcal{S}'$. Choose $X, Y, Z \in \mathcal{S}$ with XRX', YRY' and ZRZ'. Let A be a centre for XYZ in $\mathcal{S}$. Choose $A' \in \mathcal{S}'$ with ARA'. Let α' be any geodesic in $\mathcal{S}'$ joining X' to Y'. We aim to show that $d(A', \alpha')$ is bounded in terms of k, μ_1 and μ_2.

Let $X' = X'_0, X'_1, \ldots, X'_p = Y'$ be points on α' so that $d'(X'_i, X'_{i+1}) = 1$ for $i \le p - 2$, and $d'(X'_{p-1}, Y') \le 1$. For $i = 1, \ldots, p-1$, choose $X_i \in \mathcal{S}$ with $X_i R X'_i$. Let $X_0 = X$ and $X_p = Y$. Thus, $d(X_i, X_{i+1}) \le \mu_1 + \mu_2$. Let $\alpha \subseteq \mathcal{S}$ be the piecewise geodesic path

$$[X_0, X_1] \cup [X_1, X_2] \cup \cdots \cup [X_{p-1}, X_p]$$

from X to Y (parametrised by arc-length).

Suppose that $C, D \in \alpha$. Then, without loss of generality, $C \in [X_i, X_{i+1}]$ and $D \in [X_{j-1}, X_j]$ with $i < j$. Write ρ for the distance from C to D measured along α. Then,

$$\rho \le (j - i)(\mu_1 + \mu_2).$$

But

$$\begin{aligned}(j - i) &\le d'(X'_i, X'_j) + 1 \\ &\le (\mu_1 d(X_i, X_j) + \mu_2) + 1.\end{aligned}$$

Also,

$$d(X_i, X_j) \le d(C, D) + 2(\mu_1 + \mu_2),$$

and so

$$\rho \le (\mu_1 + \mu_2)\big(\mu_1 d(C, D) + (2\mu_1^2 + 2\mu_1\mu_2 + \mu_2 + 1)\big).$$

We see that α is a pseudogeodesic.

Now, Proposition 4.9 gives us a constant l such that $[X, Y] \subseteq N_l(\alpha)$. But A is a centre of XYZ, so there exists $B \in [X, Y]$ with $d(A, B) \le k$. Now, $B \in N_l(\alpha)$, so there exists $i \in \{0, 1, \ldots, p\}$ such that

$$d(B, X_i) \le l + \frac{1}{2}(\mu_1 + \mu_2).$$

Thus,

$$d(A, X_i) \le k + l + \frac{1}{2}(\mu_1 + \mu_2),$$

and so,

$$\begin{aligned} d'(A', X_i') &\le \mu_1\big(k + l + \frac{1}{2}(\mu_1 + \mu_2)\big) + \mu_2 \\ &= k'. \end{aligned}$$

We have shown that $d'(A', \alpha') \le k'$.

The same argument applies to any geodesics β' from Y' to Z', and γ' from Z' to X'. Thus $d'(A', \beta') \le k'$ and $d'(A', \gamma') \le k'$. We conclude that

$$N_{k'}(\alpha') \cap N_{k'}(\beta') \cap N_{k'}(\gamma') \ne \emptyset.$$

$\diamondsuit$

CHAPTER 5. Area.

The purpose of this chapter is to describe a few notions of area appropriate to the setting of path-metric spaces. We shall begin with a discussion of the notion of degree for maps of the circle. This will enable us to give a description which is intrinsic to the disc. This gives a number of different formulations of the "rectangle principle" (5.7, 5.8, 5.11), of which the central one for our purposes is 5.7. We conclude with a proof that the rectangle principle, together with the isoperimetric inequality imply property H4.

We shall identify the circle S^1 with $\mathbf{R}/\mathbf{Z}$. Let J_1, J_2, J_3, J_4 be the quotients, under the $\mathbf{Z}$-action, of the intervals $\tilde{J}_i = [\frac{i}{4} - \frac{3}{16}, \frac{i}{4} + \frac{3}{16}] \subseteq \mathbf{R}$ for $i = 1,2,3,4$. Thus, $\{J_1, J_2, J_3, J_4\}$ is a covering of S^1 by four overlapping intervals satisfying $J_1 \cap J_2 = \emptyset$ and $J_2 \cap J_4 = \emptyset$.

Definition : A 4-*link*, (F_1, F_2, F_3, F_4) is a (cyclically ordered) collection of four closed subsets F_1, F_2, F_3, F_4 of S^1 satisfying $F_1 \cup F_2 \cup F_3 \cup F_4 = S^1$, $F_1 \cap F_3 = \emptyset$ and $F_2 \cap F_4 = \emptyset$.

We shall take subscripts mod 4, so that $F_{i+4} = F_i$.

Definition : A continuous map $f : S^1 \longrightarrow S^1$ is *associated* to a 4-link (F_1, F_2, F_3, F_4), if $f(F_i) \subseteq J_i$ for each $i = 1,2,3,4$.

Lemma 5.1 : *Every 4-link has some map $f : S^1 \longrightarrow S^1$ associated to it. Moreover, any two maps associated to the same 4-link are homotopic.*

Proof : The second part is immediate: any two maps associated to the same 4-link are never antipodal, and are thus homotopic (by linear homotopy).

To prove the first part, let (F_1, F_2, F_3, F_4) be any 4-link. We can find another 4-link, (G_1, G_2, G_3, G_4), such that $F_i \subseteq G_i$ for each i, and $\{G_1, G_2, G_3, G_4\}$ is a collection of general-position 1-manifolds of S^1 (i.e. each G_i is a finite union of intervals, and $\partial G_i \cap \partial G_j = \emptyset$ if $i \neq j$). The sets G_i may be obtained by taking a small uniform neighbourhood of the F_i in the standard path-metric on S^1.

Now let $B_{ij} = G_i \cap \partial G_j$ for $i \neq j$, and $B = \bigcup_{i,j,i \neq j} B_{ij} = \bigcup_{i=1}^4 \partial G_i$. Note that $B_{ij} = \emptyset$ if $j \equiv i + 2$. Thus, the B_{ij} partition B into eight subsets indexed by $\{(i,j) \mid i - j \text{ is odd}\}$.

Each complementary region e of $S^1 \setminus B$ is a connected component of one of the sets $E_i = \breve{G}_i \setminus \bigcup_{j \neq i} G_j$ or $E_{ij} = \breve{G}_i \cap \breve{G}_j$. If e is bounded by $x \in B_{ij}$ and $y \in B_{kl}$, the possibilities are as follows:

(1) $i = k$, $j = l$, and either $e \subseteq E_i$ or $e \subseteq E_{ij}$,

(2) $i = k$, $j \equiv l + 2$, and $e \subseteq E_i$,

(3) $i = l$, $j = k$ and $e \subseteq E_{ij}$.

A similar discussion applies to the intervals J_i on S^1. This time, if $i - j$ is odd, then $J_i \cap \partial J_j$ consists of a single point x_{ij} ($= \frac{1}{16}(3i + j)$ if we take $0 \leq i, j \leq 3$).

We may now define $f : S^1 \longrightarrow S^1$ by sending each $v \in B_{ij}$ to x_{ij}, and mapping each complimentary region in linearly. From the above discussion, it is easily checked that $f(G_i) \subseteq J_i$. Thus $f(F_i) \subseteq J_i$.

◊

Lemma 5.1 allows us to define the degree $\deg(F_1, F_2, F_3, F_4)$ of any 4-link as the topological degree of any associated map. The following properties are easily deduced.

Proposition 5.2 : *Let (F_1, F_2, F_3, F_4) be any 4-link on S^1. Then,*
(1) $\deg(F_1, F_2, F_3, F_4) = \deg(F_2, F_3, F_4, F_1) = -\deg(F_4, F_3, F_2, F_1)$.
(2) If (F'_1, F'_2, F'_3, F'_4) is another 4-link with $F_i \subseteq F'_i$ for each i, then $\deg(F_1, F_2, F_3, F_4) = \deg(F'_1, F'_2, F'_3, F'_4)$.
(3) If $g : S^1 \longrightarrow S^1$ is any map, then $(g^{-1}F_1, g^{-1}F_2, g^{-1}F_3, g^{-1}F_4)$ is a 4-link, and

$$\deg(g^{-1}F_1, g^{-1}F_2, g^{-1}F_3, g^{-1}F_4) = \deg g \deg(F_1, F_2, F_3, F_4).$$

Proof :
(1) Compose an associated map with a rotation or reflection.
(2) Any map associated to (F'_1, F'_2, F'_3, F'_4) is also associated to (F_1, F_2, F_3, F_4).
(3) If the map f is associated to (F_1, F_2, F_3, F_4), then $f \circ g$ is associated to

$$(g^{-1}F_1, g^{-1}F_2, g^{-1}F_3, g^{-1}F_4).$$

◇

We will often abbreviate $\deg(F_1, F_2, F_3, F_4)$ to $\deg(\{F_i\})$.

Remark : We can clearly generalise the notion of 4-link to a "p-link" for any $p \geq 3$. The closed sets $F_1, \ldots, F_p$ form a p-link, if they cover S^1, and have the same intersection properties as p cyclically overlapping intervals on S^1, namely $F_i \cap F_j = \emptyset$ unless $j \equiv i-1, i, i+1 (\mathrm{mod} p)$ for $p \geq 4$, or $F_1 \cap F_2 \cap F_3 = \emptyset$ if $p = 3$.

We can also give a combinatorial formulation of degree as follows.

Suppose that G is a finite graph. We write $C_0(G)$ for the set of vertices of G, and $C_1(G)$ for the set of edges of G.

Definition : A *4-colouring*, (V_1, V_2, V_3, V_4) of G is a partition of $C_0(G)$ into four disjoint subsets, $C_0(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, such that no vertex in V_1 is adjacent to any vertex in V_3, and no vertex in V_2 is adjacent to any vertex in V_4.

Now, any finite set of points $V \subseteq S^1$ gives us a representation of S^1 as a 1-complex with vertex set V. Any 4-colouring of this complex, $V = V_1 \cup V_2 \cup V_3 \cup V_4$ gives rise to a map $g : S^1 \longrightarrow S^1$, where $g(v) = i/4$ for all $v \in V_i$, and each edge of the 1-complex is mapped either to a point of $V^0 = \{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\}$, or to a component of $S^1 \backslash V^0$. This map is well defined up to homotopy on the edges, so we may define the degree of (V_1, V_2, V_3, V_4) as $\deg(V_1, V_2, V_3, V_4) = \deg g$.

The following lemma relates the degree of a 4-colouring on S^1 to the degree of a 4-link.

Lemma 5.3 : *Suppose that σ is a path-metric on S^1, and that (F_1, F_2, F_3, F_4) is a 4-link. Let $d = \min(\sigma(F_1, F_3), \sigma(F_2, F_4))$. Suppose that $V \subseteq S^1$ is a finite set of points, such that the σ-length of any of any component of $S^1 \setminus V$ is strictly less than d. Suppose that $\{V_1, V_2, V_3, V_4\}$ is a partition of V into disjoint subsets with $V_i \subseteq F_i$ for each i. Then, (V_1, V_2, V_3, V_4) is a 4-colouring, and* $\deg(V_1, V_2, V_3, V_4) = \deg(F_1, F_2, F_3, F_4)$.

Proof : That (V_1, V_2, V_3, V_4) is a 4-colouring, follows from the intersection properties of the F_i.

We associate a 4-link to (V_1, V_2, V_3, V_4) by constructing the "Dirichlet domains" about the points of V as follows. Let V' be the set of σ-midpoints of components of $S^1 \setminus V$. Given $v \in V$, let $D(v)$ be the closure of the component of $S^1 \setminus V'$ containing v. Thus, $D(v) \subseteq N_\delta(v)$, where 2δ is the maximum σ-length of any component of $S^1 \setminus V$. Let $D_i = \bigcup_{v \in V_i} D(v)$. Then, (D_1, D_2, D_3, D_4) is a 4-link. We define $f : S^1 \longrightarrow S^1$ associated to (D_1, D_2, D_3, D_4) as follows. If $v \in V_i$, we set $f(v) = \frac{i}{4}$. If $v' \in V'$, and the adjacent vertices of V lie in V_i and V_j respectively, with $|i-j| \leq 1$, we set $f(v') = \frac{i+j}{8}$. We map in linearly all the components of $S^1 \setminus (V \cup V')$. From the construction, f also defines the degree of (V_1, V_2, V_3, V_4). Thus $\deg(\{V_i\}) = \deg(\{D_i\})$.

Now, $D_i \subseteq N_\delta(V_i) \subseteq N_\delta(F_i)$. But $\delta < d/2$, and so $(N_\delta(F_1), N_\delta(F_2), N_\delta(F_3), N_\delta(F_4))$ is a 4-link. Making two applications of Proposition 5.2 (2), we find that $\deg(\{D_i\}) = \deg(\{N_\delta(F_i)\}) = \deg(\{F_i\})$.

◊

Let G be a finite graph, and let $W \subseteq C_0(G)$ be a subset of the vertices. We shall write span(W) for the subgraph of G whose vertex set is W, and whose edge set comprises those edges both of whose endpoints lie in W. By a "path in W", we mean a path in span(W), i.e. a path, all of whose vertices lie in W. An "arc" is a path which passes through no vertex more than once. A "component" of W is the intersection of $C_0(G)$ with a connected component of span(W). Any two points in the same component of W may be joined by an arc in W.

Lemma 5.4 : *Suppose that P is a triangulation of the disc D, and that $C_0(P) = V_1 \cup V_2 \cup V_3 \cup V_4$ is a 4-colouring of the 1-skeleton $\Sigma(P)$. Write $V_i^\partial = V_i \cap \partial D$. Then,* $\deg(V_1^\partial, V_2^\partial, V_3^\partial, V_4^\partial) = 0$.

Proof : Note that every triangle $c \in C_2(P)$ must have at least two vertices lying in the same V_i. We define $f : D \longrightarrow S^1$, by sending $v \in V_i$ to $\frac{i}{4}$, and mapping linearly triangles and edges. (Each triangle gets sent either to a vertex of $V^0 = \{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\}$, or to a component of $S^1 \setminus V^0$.) Thus $\deg(\{V_i^\partial\}) = \deg(f|\partial D) = 0$.

◊

Lemma 5.5 : *Suppose that P is a triangulation of the disc D, and that $C_0(P) \cap \partial D = V_1^\partial \cup V_2^\partial \cup V_3^\partial \cup V_4^\partial$ is a 4-colouring of ∂D. Suppose that $C_0(P) = W_1 \cup W_2$, with $W_1 \cap W_2 = \emptyset$, and $W_1 \cap \partial D = V_1^\partial \cup V_3^\partial$ and $W_2 \cap \partial D = V_2^\partial \cup V_4^\partial$. If $\deg(V_1^\partial, V_2^\partial, V_3^\partial, V_4^\partial) \neq 0$, then there is either an arc from V_1^∂ to V_3^∂ in W_1, or an arc from V_2^∂ to V_4^∂ in W_2.*

Proof : Suppose neither kind of arc exists. For $i = 1, 2$, let V_i be the set of vertices of P connected to V_i^∂ in W_i. Let $V_3 = W_1 \setminus V_1$ and $V_4 = W_2 \setminus V_2$. Thus (V_1, V_2, V_3, V_4) is a 4-colouring of $\Sigma(P)$, and $V_i \cap \partial D = V_i^\partial$ for each i. By Lemma 5.4, we have $\deg(V_1^\partial, V_2^\partial, V_3^\partial, V_4^\partial) = 0$.
◇

The following lemma will enable us to construct triangulations of the disc from cellulations. We demand that every vertex of a cellulation should meet at least three edges.

Lemma 5.6 : *Suppose P is a cellulation of the disc. Then, we can subdivide P to give a triangulation P' with at most $54|C_2(P)|$ edges.*

Proof : We construct the triangulation in two stages. First, we take a point in each 2-cell of P, and subdivide the 2-cell as a cone about this point (Figure 5a).

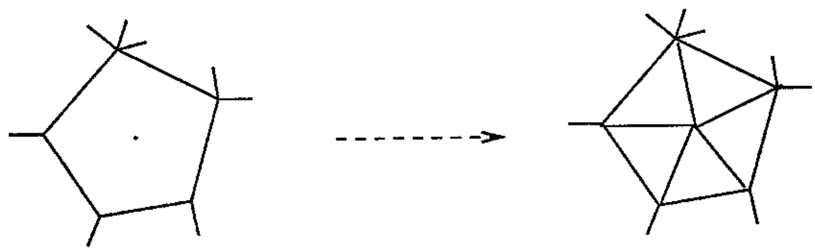


Figure 5a.

This gives a complex P^0 which might contain vertices of degree 2 (Figure 5b)

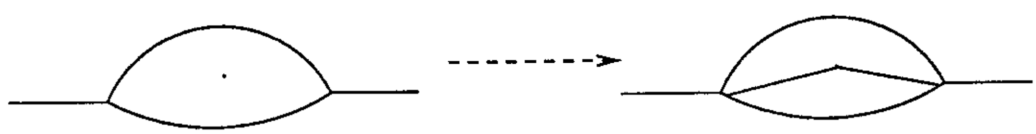


Figure 5b.

To deal with this, we take the first barycentric subdivision P' of P^0 (Figure 5c).

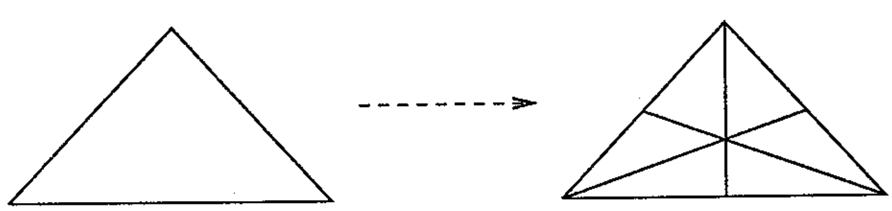


Figure 5c.

Let $t_i = |C_i(P)|$ be the number of i-cells of P. Similarly, let $t_i^0 = |C_i(P^0)|$ and $t_i' = |C_i(P')|$. We write $t_1 = t_1^I + t_1^B$, where $t_1^B = |\{e \in C_1(P) \mid e \subseteq \partial D\}|$ is the number of boundary edges of P, and t_1^I is the number of interior edges. Now $2t_1 \geq 3t_0$ since each vertex has degree 3, and $t_0 + t_2 - t_1 = 1$. Thus,

$$\begin{aligned} 3t_2 - t_1 &= 2t_1 + 3t_2 - 3t_1 \\ &\geq 3t_0 + 3t_2 - 3t_1 \geq 3, \end{aligned}$$

and so,

$$t_1 \leq 3t_2 - 3 \leq 3t_2.$$

Now,

$$t_2^0 = 2t_1^I + t_1^B \leq 2(t_1^I + t_1^B) = t_1 \leq 6t_2,$$

and

$$t_0 = t_1 + t_2^0 \leq 3t_2 + 6t_2 = 9t_2.$$

Thus,

$$\begin{aligned} t_1' &= 2t_1^0 + 6t_2^0 \\ &\leq 2(9t_2) + 6(6t_2) \\ &= 54t_2. \end{aligned}$$

$\diamond$

Now, recall the definitions of metric triangulations and cellulations from Section 2.3, as well as the notions of area, energy and mesh, A, A_T, I, I_T, m, m_T. We are now in a position to give the main formulation of the rectangle principle:

Proposition 5.7 : *There is some universal constant $\theta > 0$ such that the following holds.*
Suppose that (P, ρ) is a metric cellulation of the disc D, and suppose (F_1, F_2, F_3, F_4) is a 4-link on $S^1 = \partial D$. Let $d_1 = \rho(F_1, F_3) > 0$ and $d_2 = \rho(F_2, F_4) > 0$. Then,

$$I(P, \rho) \geq \theta d_1 d_2 |\deg(F_1, F_2, F_3, F_4)|.$$

Our proof will give $\theta = \frac{1}{216}$.

In fact, all we shall need in this paper is the result that, if $\deg(F_1, F_2, F_3, F_4) \neq 0$, then $I(P, \rho) \geq \theta d_1 d_2$. This much is easier to prove. However, for completeness, we shall outline the rest of the argument. Note that if P is a metric triangulation, then from the inequalities of Section 2.3, we get that

$$I_T(P, \rho) \geq \frac{1}{6} I(P, \rho) \geq (\frac{\theta}{6}) d_1 d_2 |\deg(F_1, F_2, F_3, F_4)|.$$

We shall begin by proving the analogous result for metric triangulations of bounded mesh. This would suffice for the corresponding formulation of H3 (H3ta, Section 2.3).

Proposition 5.8 : *Suppose that (P, ρ) is a metric triangulation, and that (F_1, F_2, F_3, F_4) is a 4-link on ∂D. Let $d_1 = \rho(F_1, F_3)$, $d_2 = \rho(F_2, F_4)$ and $n = \deg(F_1, F_2, F_3, F_4)$. Suppose that $m_T(P, \rho) < \min(d_1, d_2)$. Then,*

$$A_T(P)(m_T(P,\rho))^2 \geq |n|\big(d_1 d_2 - (m_T(P,\rho))\min(d_1, d_2)\big).$$

Proof : Write $C_0^\partial(P) = C_0(P) \cap \partial D$. Write $m_T = m_T(P, \rho)$ and $A_T = A_T(P)$. Let m be any number strictly greater than m_T. Thus, $A_T m \geq A_T m_T \geq \rho(\Sigma)$ (where $\rho(\Sigma)$ is the ρ-length of the 1-skeleton Σ). Let q be the integer part of d_1/m, so that $(q+1)m \geq d_1$, hence $qm \geq d_1 - m$.

For $r \in \{1, 2, \ldots, q\}$, let $G_1^r = N_{(r-1)m}(F_1) \subseteq \Sigma$ and $G_3^r = \Sigma \setminus \check{N}_{rm}(F_1)$, where N_t denotes the uniform t-neighbourhood in (Σ, ρ). Thus, $\rho(G_1^r, G_3^r) = m > m_T$. (Figure 5d.)

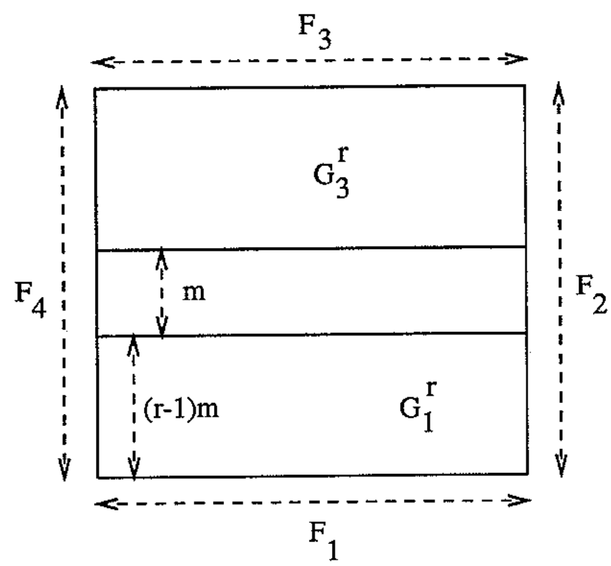


Figure 5d.

Let $F_1^r = G_1^r \cap \partial D$ and $F_3^r = G_3^r \cap \partial D$. Thus $F_1 \subseteq F_1^r$, and, since $rm \leq qm \leq d_1 \leq \rho(F_1, F_3)$, we have $F_3 \subseteq F_3^r$. We see that (F_1^r, F_2, F_3^r, F_4) is a 4-link on ∂D. By Proposition 5.2 (2), we have

$$\deg(F_1^r, F_2, F_3^r, F_4) = \deg(F_1, F_2, F_3, F_4) = n.$$

Now let

$$V_1^\partial = C_0^\partial(P) \cap F_1^r$$
$$V_3^\partial = C_0^\partial(P) \cap F_3^r$$
$$V_2^\partial = C_0^\partial(P) \cap F_2 \setminus (F_1^r \cup F_3^r)$$
$$V_4^\partial = C_0^\partial(P) \cap F_4 \setminus (F_1^r \cup F_3^r).$$

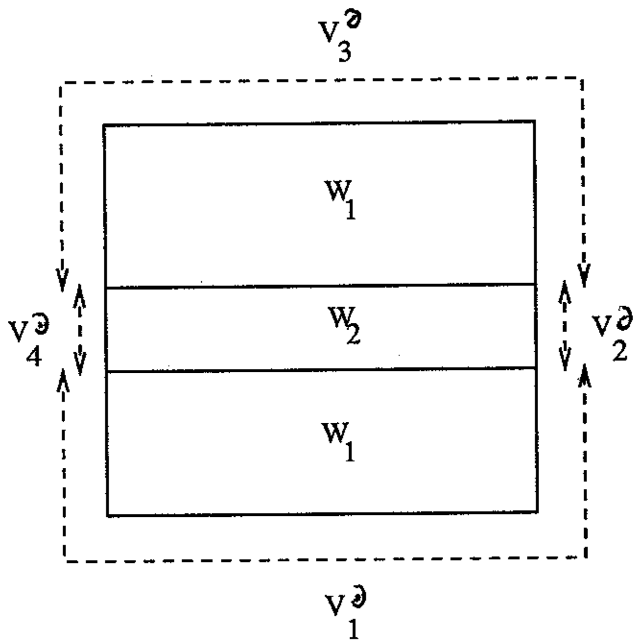


Figure 5e.

(Figure 5e.)

Since $m_T < \min\bigl(\rho(F_1^r, F_3^r), \rho(F_2, F_4)\bigr)$, Lemma 5.3 tells us that $(V_1^\partial, V_2^\partial, V_3^\partial, V_4^\partial)$ is a 4-colouring of ∂D, and that

$$\deg(V_1^\partial, V_2^\partial, V_3^\partial, V_4^\partial) = \deg(F_1^r, F_2, F_3^r, F_4) = n.$$

Let $W_1 = C_0^\partial(P) \cap (G_1^r \cup G_3^r)$ and $W_2 = C_0^\partial(P) \backslash W_1$, so that $W_1 \cap \partial D = V_1^\partial \cup V_2^\partial$ and $W_2 \cap \partial D = V_2^\partial \cup V_4^\partial$.

Since $m_T < \rho(G_1^r, G_3^r)$, there is no path from V_1^∂ to V_3^∂ in W_1.

Now, suppose that $n \neq 0$. Lemma 5.5 tells us that there must be an arc, α_r from V_2^∂ to V_4^∂ in W_2. Thus, α_r runs from F_2 to F_4, and lies entirely in $\Sigma \cap N_{rm}(F_1) \backslash N_{(r-1)m}(F_1)$. Such an arc must have ρ-length at least $\rho(F_2, F_4) = d_2$. Moreover, the α_r for $r = 1, 2, \ldots, q$ are all disjoint. Thus, $\rho(\Sigma) \geq \rho(\bigcup_{r=1}^q \alpha_r) \geq q d_2$, and so

$$\begin{aligned} A_T m^2 &\geq (\rho(\Sigma))m \\ &\geq (qd_2)m = d_2(qm) \\ &\geq d_2(d_1 - m) = d_1 d_2 - m d_2. \end{aligned}$$

Now, without loss of generality, we have $d_2 \leq d_1$. Letting $m \longrightarrow m_T$, we have shown that, for $n \neq 0$,

$$A_T m_T^2 \geq d_1 d_2 - m_T \min(d_1, d_2).$$

As remarked above, this is all we shall need for this paper. However, we see that the full statement of Proposition 5.8 would follow if, in the above argument, we could show

that there were $|n|$ edge-disjoint arcs from F_2 to F_4 lying in $\Sigma \cap N_{rm}(F_1) \setminus N_{(r-1)m}(F_1)$. This, in fact, is a consequence of a generalisation of Lemma 5.5, namely Lemma 5.9 below.
$\diamondsuit$

Lemma 5.9 : *Suppose P is a triangulation of the disc D, and that $C_0^\partial(P) = C_0(P) \cap \partial D = V_1^\partial \cup V_2^\partial \cup V_3^\partial \cup V_4^\partial$ is a 4-colouring of ∂D. Suppose that $C_0(P) = W_1 \cup W_2$, with $W_1 \cap W_2 = \emptyset$, $W_1 \cap \partial D = V_1^\partial \cup V_3^\partial$ and $W_2 \cap \partial D = V_2^\partial \cup V_4^\partial$. Let $n = \deg(V_1^\partial, V_2^\partial, V_3^\partial, V_4^\partial)$. If there is no arc from V_1^∂ to V_3^∂ in W_1, then there is a set of $|n|$ edge-disjoint arcs from V_2^∂ to V_4^∂ in W_2.*

Sublemma 5.10 : *Suppose P is a triangulation of the disc. Suppose $C_0(P) = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$, and suppose $x, y \in X_2 \cap \partial D$. If there is no path from x to y in X_2, then there is a an arc α in X_1, which meets ∂D only at its endpoints, and which separates x from y.*

Proof of 5.10 : Let J, K be the two components of $\partial D \setminus \{x, y\}$, so that J, K are open intervals. Let $Y_1^\partial = X_1 \cap J$ and $Y_3^\partial = X_1 \cap K$. Let Y_2^∂ be the intersection, with ∂D, of the component of X_2 containing x. Let $Y_4^\partial = (X_2 \cap \partial D) \setminus Y_2^\partial$. Thus $y \in Y_4^\partial$. We check that $|\deg(Y_1^\partial, Y_2^\partial, Y_3^\partial, Y_4^\partial)| = 1$. Thus, by Lemma 5.5, there is an arc from Y_1^∂ to Y_3^∂ in X_1. Let α be such an arc with a minimal number of edges.
$\diamondsuit$

Proof of 5.9 : (Sketch.) The proof is by induction over the number, c, of components of W_2.

Suppose first, that $c \geq 2$.

If there is some component X of W_2 with $X \cap \partial D = \emptyset$, then, by redefining $W_2' = W_2 \setminus X$ and $W_1' = W_1 \cup X$, we would reduce c. So, by induction, we could find $|n|$ edge-disjoint paths from V_2^∂ to V_4^∂ in $W_2' \subseteq W_2$.

Thus, we may assume that each component of W_2 meets ∂D. Let x, y lie in distinct components of W_2. By Sublemma 5.10, there is an arc α in W_1, which meets ∂D only in its endpoints w and z, and which separates x from y. Now, by hypothesis, there is no arc from V_1^∂ to V_3^∂ in W_1. Thus, without loss of generality, we have that $w, z \in V_1^\partial$.

Now, the arc α cuts D into two discs D^1 and D^2, with $\partial D^1 \cap \partial D^2 = \alpha$. For $j = 1, 2$, we may obtain 4-colourings $(V_1^{\partial(j)}, V_2^{\partial(j)}, V_3^{\partial(j)}, V_4^{\partial(j)})$ of ∂D^j, by $V_1^{\partial(j)} = (V_1^\partial \cap \partial D^j) \cup (\alpha \cap C_0(P))$ and $V_i^{\partial(j)} = V_i^\partial \cap \partial D^j$ for $i = 2, 3, 4$. In other words, we assign each vertex of α to $V_1^{\partial(j)}$. It is easily checked that

$$\deg(\{V_i^\partial\}) = \deg(\{V_i^{\partial(1)}\}) + \deg(\{V_i^{\partial(2)}\}).$$

For $i = 1, 2$ and $j = 1, 2$, define $W_i^j = W_i \cap D^j$. Now, each of W_2^1 and W_2^2 has fewer than c components. By induction, we may find $|n(j)|$ edge disjoint arcs from V_2^∂ to V_4^∂ in W_2^j, where $n(j) = \deg(\{V_i^{\partial(j)}\})$.

We are thus reduced to the case when $c = 0$ or 1, i.e. when W_2 is connected. We can assume (by interchanging the indices 1 and 3 if necessary) that $n = \deg(\{V_i^\partial\}) \geq 0$. Now, it is not difficult to find points $v_1, v_2, \ldots, v_{2n}$, cyclically arranged on ∂D, with $v_i \in V_2^\partial$ for i odd, and $v_i \in V_4^\partial$ for i even. (Note that if $n = 0$, there is nothing to prove.)

Let $T \subseteq \text{span}(W_2) \subseteq \Sigma$ be a minimal subgraph which connects all the points of $\{v_1, v_2, \ldots, v_{2n}\}$. Thus, T is a planar tree whose endpoints are alternately labelled 2 and 4. We claim that such a tree contains n edge-disjoint arcs, each joining a vertex labelled 2 to one labelled 4.

We prove this claim by induction on n. Let T' be the subtree of T spanned by the set of interior nodes (i.e. vertices of degree at least 3 in T). Any extreme point w of T' will be "connected to" (at least) two consecutive vertices v_i and v_{i+1} in T. That is to say, the arc β from v_i to v_{i+1} in T, meets T' only in w. We now let T'' be the subtree of T spanned by $\{v_j \mid j \neq i, i+1\}$, so that $T'' \subseteq (T \backslash \beta) \cup \{w\}$, and apply induction.

◊

We next give a version of Proposition 5.8 for metric cellulations.

Proposition 5.11 : *Suppose that (P, ρ) is a metric cellulation, and that (F_1, F_2, F_3, F_4) is a 4-link on ∂D. Let $d_1 = \rho(F_1, F_3)$, $d_2 = \rho(F_2, F_4)$ and $n = \deg(F_1, F_2, F_3, F_4)$. Suppose that $m(P, \rho) \leq \min(d_1, d_2)$. Then,*

$$A(P)(m(P,\rho))^2 \geq \frac{1}{54}|n|\big(d_1 d_2 - (m(P,\rho)) \min(d_1 d_2)\big).$$

Proof : By Lemma 5.6, we can subdivide P to give a triangulation P' with

$$A_T(P') \leq 54 A(P).$$

We put a path-metric ρ' on $\Sigma(P')$, by assigning each new edge a ρ'-length equal to $m = m(P, \rho)$. It is easily checked that ρ' restricted P agrees with ρ, i.e. we do not introduce any "short-cuts" across 2-cells of P. In particular, we have $\rho'(F_i, F_{i+2}) = \rho(F_i, F_{i+2}) = d_i$ for $i = 1, 2$. Also $m_T(P', \rho') = m(P, \rho) = m$. Applying Proposition 5.8, we get

$$A_T(P')m^2 \geq |n|\big(d_1 d_2 - m \min(d_1, d_2)\big),$$

and so

$$A(P)m^2 \geq |n| \frac{1}{54}\big(d_1 d_2 - m \min(d_1, d_2)\big).$$

◊

From this, we may finally deduce the inequality of 5.7, namely that

$$I(P, \rho) \geq \frac{1}{216}|n| d_1 d_2.$$

Proof of 5.7 : Let (P, ρ), (F_1, F_2, F_3, F_4), d_1, d_2 be as in the hypotheses of Proposition 5.7. Let $n = \deg(F_1, F_2, F_3, F_4)$, and let $\delta = \min\{\rho(\partial c) \mid c \in C_2(P)\}$.

We subdivide (P, ρ) to give a new metric cellulation, (P', ρ'), as follows. We imagine each 2-cell $c \in C_2(P)$ as a euclidean square of side-length $\frac{1}{4}\rho(\partial c)$. We subdivide c into a grid of much smaller squares, with a path metric on the 1-skeleton induced from the euclidean metric, i.e. we assign to each 1-cell of this grid, a length equal to $\frac{\rho(\partial c)}{4h(c)}$, where $(h(c))^2$ is the number of subsquares. We perform such a construction for each 2-cell of P, and take ρ' to be the induced path-metric on the whole of $\Sigma(P')$. By taking the mesh of this subdivision much smaller than δ, we can can arrange that all the subsquares are about the same size. More precisely, given any $\epsilon > 0$, we can arrange that $\rho'(\partial c') \geq (1-\epsilon)m'$ for all $c' \in C_2(P')$, where $m' = m(P', \rho')$. Thus,

$$I(P', \rho') \geq (1-\epsilon)^2 A(P')(m')^2.$$

Note also that if $c \in C_2(P)$, then

$$\rho(\partial c)^2 = \sum \{\rho'(\partial c')^2 \mid c' \in C_2(P'), c' \subseteq c\},$$

since both sides are equal to 16 times the euclidean area of the subdivided square c. Thus,

$$I(P, \rho) = I(P', \rho').$$

Now, if $x, y \in \Sigma(P)$ are any two points, it is easily seen that $\rho'(x,y) \geq \frac{1}{2}\rho(x,y)$. In other words, taking short-cuts across 2-cells of P will shorten any path from x to y in $\Sigma(P)$ by a factor of at most 2. In particular, we have $\rho'(F_1, F_3) \geq d_1/2$ and $\rho'(F_2, F_4) \geq d_2/2$.

We may suppose that we have taken $m' < \frac{1}{2}\min(d_1, d_2)$, so can apply Proposition 5.11 to (P', ρ') to get

$$\begin{aligned} A(P')(m')^2 &\geq \frac{1}{54}|n|\big((\frac{d_1}{2})(\frac{d_2}{2}) - \frac{1}{2}m' \min(d_1, d_2)\big) \\ &= \frac{1}{216}|n|\big(d_1 d_2 - 2m' \min(d_1, d_2)\big). \end{aligned}$$

Thus,

$$\begin{aligned} I(P, \rho) = I(P', \rho') &\geq (1-\epsilon)^2 A(P')(m')^2 \\ &\geq \frac{1}{216}(1-\epsilon)^2 |n| \big(d_1 d_2 - 2m' \min(d_1, d_2)\big). \end{aligned}$$

Now, let $\epsilon \longrightarrow 0$ and $m' \longrightarrow 0$. We conclude that

$$I(P, \rho) \geq \frac{1}{216}|n| d_1 d_2.$$

◇

Clearly, there is much room for improvement in the factor of $\frac{1}{216}$, particularly in the proof of Proposition 5.11.

We now use Proposition 5.7 to derive property H4 from H3.

Let $(\mathcal{S}, d)$ be a geodesic space. Suppose that $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ is a rectangle (as defined in Section 2.3). Thus, $\gamma : (S^1, \sigma) \longrightarrow (\mathcal{S}, d)$ is distance non-increasing, and we have

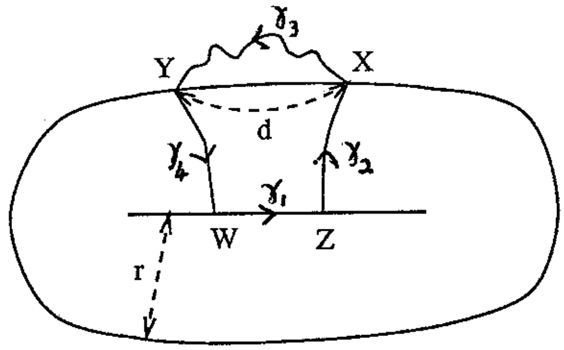


Figure 5f.

$S^1 = L_1 \cup L_2 \cup L_3 \cup L_4$, with $\gamma_i = \gamma|L_i$. Let $d_i = d(\gamma(L_i), \gamma(L_{i+2}))$ for $i = 1, 2$. Suppose that γ bounds a cellular net (P, ρ, f) with $\partial f : (\partial D, \rho_{\partial D}) \longrightarrow (S^1, \sigma)$. Let $F_i = (\partial f)^{-1} L_i \subseteq \partial D$. Thus, $\deg(F_1, F_2, F_3, F_4) = \deg\partial f = 1$, and $f(F_i) = \gamma \circ \partial f(F_i) = \gamma(L_i)$. Since $f : (\Sigma, \rho) \longrightarrow (\mathcal{S}, d)$ is distance non-increasing, we have $\rho(F_i, F_{i+2}) \geq d(\gamma(L_i), \gamma(L_{i+2})) = d_i$, for $i = 1, 2$. By Proposition 5.7, we get that

$$I(P, \rho) \geq \theta d_1 d_2.$$

Now, if $(\mathcal{S}, d)$ is (k_3, h_3)-H3, we can choose (P, ρ, f) so that

$$I(P, \rho) \leq k_3(\sigma(S^1) + h_3),$$

and so

$$\sigma(S^1) \geq (\frac{\theta}{k_3}) d_1 d_2 - h_3.$$

With the weaker version of the rectangle principle stated in Section 2.3, we get

$$k_3(\sigma(S^1) + h_3) \geq K_1 d_1 d_2 - K_2(d_1 + d_2 + 1).$$

This may be derived directly from Proposition 5.8 or 5.11 given the corresponding formulation of H3.

Lemma 5.12 : *$\forall k, l > 0\, \exists r, s$ such that the following holds.*

Suppose that $(\mathcal{S}, d)$ is a geodesic space, and suppose that for each rectangle $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, we have

$$\text{length}\gamma \geq k d_1 d_2 - l(d_1 + d_2 + 1)$$

where $d_1 = d(\gamma_1, \gamma_3)$ and $d_2 = d(\gamma_2, \gamma_4)$. Then $(\mathcal{S}, d)$ is (r, s)-H4.

Proof : Let $r = \frac{4+l}{k}$ and $s = 2kr^2 + lr + l + 4r$.

Suppose that $\beta \subseteq \mathcal{S}$ is a geodesic segment. Let $X, Y \in \partial N_r(\beta)$, and write $d = d(X, Y)$ and $d' = d_{r,\beta}(X, Y)$. We join X to Y by a path, γ_3, of length d' lying in $\mathcal{S} \backslash \breve{N}_r(\beta)$. Let Z, W be nearest points to X, Y respectively, on β. Thus $d(X, Z) = d(Y, W) = r$. (Figure 5f.)

Let $\gamma_1 = [W, Z]$, $\gamma_2 = [Z, X]$ and $\gamma_4 = [Y, W]$. Let γ be the rectangle $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$. We have $d_1 = d(\gamma_1, \gamma_3) = r$ and $d \geq d_2 = d(\gamma_2, \gamma_4) \geq d - 2r$. Also

$$\text{length}\gamma \leq (d + 2r) + r + d' + r = d + d' + 4r.$$

So, by hypothesis,

$$\begin{aligned} d + d' + 4r &\geq kd_1d_2 - l(d_1 + d_2 + 1) \\ &\geq kr(d - 2r) - l(r + d + 1) \\ &= (kr - l)d - (2kr^2 + lr + l). \end{aligned}$$

Thus,

$$\begin{aligned} d' &\geq (kr - l - 1)d - (2kr^2 + lr + l + 4r) \\ &= 3d - s. \end{aligned}$$

◇

Proposition 5.13 : $\forall k_3, h_3\ \exists k_4, h_4$ *such that if* $(\mathcal{S}, d)$ *is* (k_3, h_3)*-H3, then it is* (k_4, h_4)*-H4.*

Proof : From Lemma 5.12 and previous discussion.

◇

CHAPTER 6. Remaining implications.

It remains to give proofs of H5 $\Rightarrow$ H3, and H4 $\Rightarrow$ H2. We prove the first implication in Section 6.1. We also include a proof of H1 $\Rightarrow$ H3, since it is much more direct than following the cycle. The final argument, $H4 \Rightarrow H2$, is given in Section 6.2.

6.1. H5 $\Rightarrow$ H3.

The idea of the proof is to show that any sufficiently long loop may be shortened by a definite amount, by replacing a portion of the loop by a geodesic segment. A sequence of these "short-cut" operations will give us a cellular net. The existence of such short-cuts is shown, for H1 spaces, in Lemma 6.1.4, and for H5 spaces, in Lemma 6.1.6. First, we get the technicalities out of the way.

Recall the definitions of Section 2.3.

Lemma 6.1.1 : *Let $(\mathcal{S},d)$ be a path-metric space. Suppose that $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n$ is a sequence of loops in $\mathcal{S}$, where $\gamma_i : (S^1, \sigma_i) \longrightarrow (\mathcal{S}, d)$ is distance non-increasing. Suppose there are a sequence of arcs $J_i \subseteq S^1$, for $i = 1, \ldots, n-1$, such that γ_i agrees with γ_{i+1} on $S^1 \backslash J_i$ and σ_i agrees with σ_{i+1} on $S^1 \backslash J_i$. Suppose also that $\sigma_i(J_i) \leq K$, $\sigma_{i+1}(J_i) \leq K$, and $\sigma_n(S^1) \leq 2K$, where $K \in (0, \infty)$. Then, γ bounds a cellular net (P, ρ, f) with $A(P) = n$, and $m(P, \rho) \leq 2K$.*

Proof : We see (by induction on n) that the combinatorial structure of the arcs J_i determine a cellulation, P, of the disc D, with n 2-cells. The path-metrics, σ_i, determine a path-metric ρ on the 1-skeleton $\Sigma(P)$, so that for any 2-cell, $c \in C_2(P)$, we have $\rho(\partial c) \leq 2K$. The maps γ_i together give a distance non-increasing map $f : (\Sigma, \rho) \longrightarrow (\mathcal{S}, d)$, such that $f | \partial D = \gamma$. Thus, $\partial f : (\partial D, \rho_{\partial D}) \longrightarrow (S^1, \sigma)$ is an isometry, and so (P, ρ, f) bounds γ.

$\diamondsuit$

Lemma 6.1.2 : $\forall K, b \, \exists k_3, h_3$ *such that the following holds.*

Let $(\mathcal{S},d)$ be a geodesic space, and suppose for that any loop $\gamma : (S^1, \sigma) \longrightarrow (\mathcal{S}, d)$ with $\sigma(S^1) \geq 2K$, there is an arc $J \subseteq S^1$ satisfying $\sigma(J) \leq K$, and $d(\gamma(p), \gamma(q)) \leq \sigma(J) - b$, where $p, q \in S^1$ are the endpoints of J. Then, $(\mathcal{S}, d)$ is (k_3, h_3)-H3.

Proof : Let $\gamma : (S^1, \sigma) \longrightarrow (\mathcal{S}, d)$ be a loop with $\sigma(S^1) \geq 2K$. Let $J_1 = J \subseteq S^1$ be the arc given by the hypotheses. Rescale the metric σ on J by a factor of $\frac{\sigma(J)-b}{\sigma(J)}$ to give a new path-metric σ_2 on S^1, with $\sigma_2(S^1) = \sigma(S^1) - b$. Define a new loop $\gamma_2 : (S^1, \sigma_2) \longrightarrow (\mathcal{S}, d)$ by $\gamma_2 | (S^1 \backslash J) = \gamma | (S^1 \backslash J)$, and taking $\gamma_2 | J$ to map linearly along the geodesic $[\gamma(p), \gamma(q)]$. where $p, q \in S^1$ are the enpoints of J. Note that $\sigma_2(J) = \sigma(J) - b \geq d(\gamma(p), \gamma(q))$.

Continue by induction. After $n - 1 \leq \frac{\sigma(S^1)}{b}$ steps, we arrive at a loop $\gamma_n : (S^1, \sigma_n) \longrightarrow (\mathcal{S}, d)$ with $\sigma_n(S^1) \leq 2K$. Applying Lemma 6.1.1, we get a cellular net (P, ρ, f) with

$m(P,\rho) \leq 2K$, and $A(P) = n \leq \frac{1}{b}(\sigma(S^1) + b)$. This gives H3ca.

Now, the inequality $I(P,\rho) \leq A(P)(m(P,\rho))^2$ gives the main definition, H3ce. Finally, Lemma 6.1.3 below gives us the strongest definition H3ta.

◇

Lemma 6.1.3 : *Let $(\mathcal{S}, d)$ be a geodesic space. Suppose that is a loop in $\mathcal{S}$ bounding a cellular net (P,ρ,f). Then, γ bounds a simplicial net (P',ρ',f') with $m_T(P',\rho') \leq m(P,\rho)$, and $A_T(P') \leq 54A(P)$.*

Proof : Lemma 5.6 gives us P' combinatorially as a subdivision of P. We define ρ' by assigning, to each new edge $e \in C_1(P')$ with $e \not\subseteq \sigma(P)$, a length equal to $m(P,\rho)$.

Now, for each 2-cell $c \in C_2(P)$, we choose some $X_c \in f(\partial c)$, and map each vertex of $C_0(P') \cap (c \backslash \partial c)$ to X_c. We map in each edge $e \in C_1(P')$ with $e \subseteq c$ and $e \not\subseteq \partial c$, linearly along a geodesic segment. This defines $f' : (\Sigma(P'),\rho') \longrightarrow (\mathcal{S}, d)$. We check that f' is distance non-increasing.

◇

Lemma 6.1.4 : $\forall k_1, b\, \exists l$ *such that the following holds.*

Suppose $(\mathcal{S}, d)$ is k_1-H1, and that $\gamma : (S^1, \sigma) \longrightarrow (\mathcal{S}, d)$ is a loop with $\sigma(S^1) > l + b$. Then, there is an arc $J \subseteq S^1$ with $\sigma(J) = l + b$, and $d(\gamma(p), \gamma(p)) \leq l$ where p and q are the endpoints of J.

Proof : Lemma 2.1.1 tells us that the distances between any four points of $\mathcal{S}$ may be measured up to an additive constant $h = h(k_1)$ along a tree. Let $l = b + 8h$.

Suppose that γ is a loop with $\sigma(S^1) \geq l+b$. Choose any point $X \in \mathcal{S}$, and let $Y = \gamma(t)$ be a point in $\gamma(S^1)$ furthest from X. Let J be a closed interval of σ-length $l + b$ centred at t. Let p, q be the endpoints of J. Thus, $\sigma(p,t) = \sigma(q,t) = \frac{1}{2}(l + b)$. Let $Z = \gamma(p)$ and $W = \gamma(q)$. Thus, $d(Z,Y) \leq \frac{1}{2}(l + b)$ and $d(W,Y) \leq \frac{1}{2}(l+b)$. We claim that $d(Z,W) \leq l$.

Case (1) : $XY : ZW \longleftrightarrow (xy)u(zw)$. (Figure 6a.)

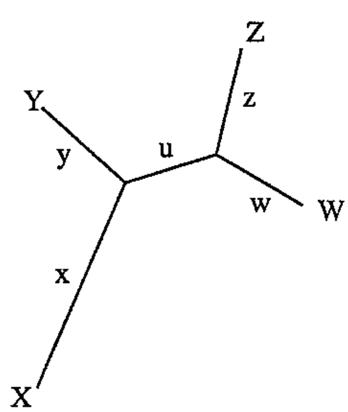


Figure 6a.

Now, $d(X, W) \leq d(X, Y)$, thus

$$x + u + w \preceq_{2h} x + y$$
$$w \preceq_{2h} y,$$

and so,

$$d(W, Z) \simeq_h w + z \preceq_{2h} y + u + z \simeq_h d(Y, Z) \leq \frac{1}{2}(l + b),$$

i.e.

$$d(W, Z) \leq \frac{1}{2}(l + b) + 4h$$
$$= l.$$

Case (2) : without loss of generality, $XZ : YW \longleftrightarrow (xz)u(yw)$. (Figure 6b.)

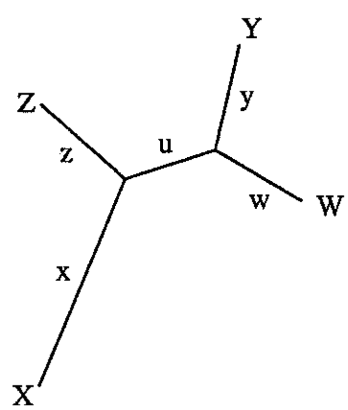


Figure 6b.

Again $d(X, W) \leq d(X, Y)$, thus

$$x + u + w \preceq_{2h} x + u + y$$
$$w \preceq_{2h} y,$$

and so

$$d(W, Z) \simeq_h w + u + z \preceq_{2h} y + u + z \preceq_h d(Y, Z) \leq \frac{1}{2}(l + b),$$

i.e.

$$d(W, Z) \leq \frac{1}{2}(l + b) + 4h$$
$$= l.$$

$\diamondsuit$

Corollary 6.1.5 : *$\forall k_1 \exists k_3, h_3$ such that if (S, d) is k_1-H1, it is (k_3, h_3)-H3.*

Proof : Set $b = 1$. Apply Lemmas 6.1.4 and 6.1.2.

◇

Lemma 6.1.6 : *Suppose that $(\mathcal{S}, d)$ is (a,b)-H5, and $\gamma : (S^1, \sigma) \longrightarrow (\mathcal{S}, d)$ is a loop in $\mathcal{S}$, with $\sigma(S^1) > 2(a+b)$. Then, there is an arc $J \subseteq S^1$ with $\sigma(J) \leq 2(a+b)$ and $d(\gamma(p), \gamma(q)) \leq \sigma(J) - b$, where $p, q \in S^1$ are the endpoints of J.*

Proof : Choose any point $A \in \mathcal{S}$. Given any $r \geq 0$, let $C(r) = \gamma^{-1}\big(\mathcal{S} \setminus \check{N}_r(A)\big)$. (Figure 6c.)

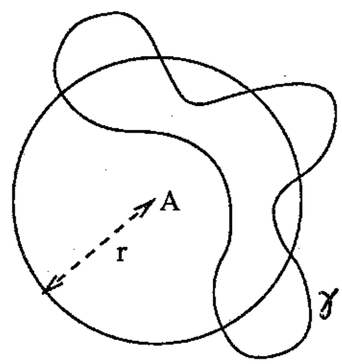


Figure 6c.

Thus, $C(r)$ is a closed subset of S^1, with $C(0) = S^1$ and $C(r_0) = \emptyset$ for some r_0. Define $g(r) \in [0, \infty)$ to be the largest σ-length of any connected component of $C(r)$. Now, $g : [0, r_0] \longrightarrow [0, \infty)$ in non-increasing and upper-semicontinuous in r. We have $g(0) = \sigma(S^1) > 2(a+b)$ and $g(r_0) = 0$. Let $R = \sup\{r > 0 \mid f(r) > a+b\}$. Thus $f(R) \geq a+b$, i.e. there is some component L of $C(R)$ with $\sigma(L) \geq a+b$. However, we see that each component of $\gamma^{-1}\big(\mathcal{S} \setminus N_R(A)\big)$ has σ-length at most $a+b$. From this, it is easy to find an arc $J \subseteq L$ with $a+b \leq \sigma(J) \leq 2(a+b)$, and with $\gamma(p) \in \partial N_R(A)$ and $\gamma(q) \in \partial N_R(A)$, where $p, q \in S^1$ are the endpoints of J. Write $d = d(\gamma(p), \gamma(q))$ and $d' = d_{r,A}(\gamma(p), \gamma(q))$. (Figure 6d.)

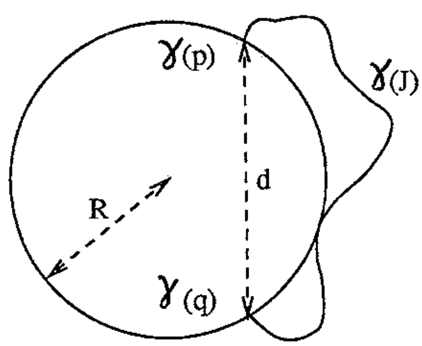


Figure 6d.

If $d \leq a$, then $\sigma(J) \geq a+b \geq d+b$.

If $d \geq a$, then, since $(\mathcal{S}, d)$ is (a,b)-H5, we have $d' \geq d+b$. Thus, $\sigma(J) \geq d' \geq d+b$. Either way, $\sigma(J) \geq d+b$, i.e. $d(\gamma(p), \gamma(q)) \leq \sigma(J) - b$.

◇

Proposition 6.1.7 : *$\forall k_5, h_5 \exists k_3, h_3$ such that if $(\mathcal{S}, d)$ is (k_5, h_5)-H5 it is (k_3, h_3)-H3.*

Proof : Lemmas 6.1.6 and 6.1.2.

◇

Remark : Note that the hypotheses of Lemma 6.1.2 serve as another definition of almost-hyperbolicity.

6.2. H4 ⇒ H2.

Let $(\mathcal{S}, d)$ be a geodesic space. Given $X, Y \in \mathcal{S}$, we write $[X, Y]$ for some choice of geodesic from X to Y. If Z, W in $[X, Y]$, we shall always take $[Z, W] \subseteq [X, Y]$.

Lemma 6.2.1 : *Suppose $(\mathcal{S}, d)$ is (r, L)-H4. Let $J = r + \frac{L}{4}$. Suppose $X, Y, Z \in \mathcal{S}$ with $d(Y, Z) \leq r$. Then $[X, Z] \subseteq N_J[X, Y]$ and $[X, Y] \subseteq N_J[X, Z]$, (that is, for any choice of geodesics $[X, Y]$ and $[X, Z]$).*

Proof : Let $[A, B] \subseteq [X, Z]$ be a component of $[X, Z] \backslash \check{N}_r[X, Y]$ so that $A, B \in \partial N_r[X, Y]$. (Figure 6e.)

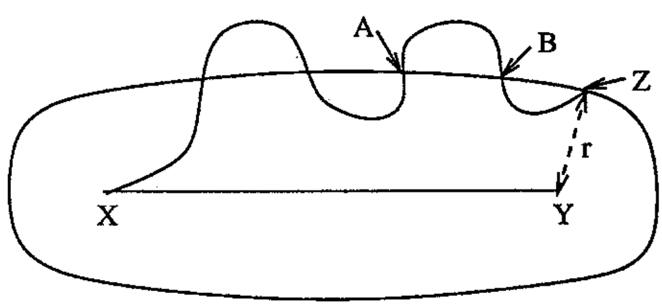


Figure 6e.

Since $(\mathcal{S}, d)$ is (r, L)-H4, we have $d(A, B) = d_{r,[X,Y]}(A, B) \geq 3d(A, B) - L$, and so $d = d(A, B) \leq \frac{L}{2}$.

Now, $[A, B] \subseteq N_{r+\frac{d}{2}}[X, Y]$, and so $[X, Z] \subseteq N_J[X, Y]$. Similarly $[X, Y] \subseteq N_J[X, Z]$.

◇

Lemma 6.2.2 : *Suppose $(\mathcal{S}, d)$ is (r, L)-H4. Let $R = J + 3r$ (where $J = r + \frac{L}{4}$ comes from Lemma 6.2.1). Suppose that $X, Y, Z \in \mathcal{S}$ and $[X, Y], [X, Z]$ are geodesics with $d(Y, [X, Z]) \geq R$ and $d(Z, [X, Y]) \geq R$. Then, there exist $A \in [X, Y]$ and $B \in [X, Z]$ such that*

$$d([A, Y], [X, Z]) \geq r$$

$$d([B, Z], [X, Y]) \geq r$$

$$[X,B] \subseteq N_R[X,A]$$

$$[X,A] \subseteq N_R[X,B].$$

(Note that this implies that $r \leq d(A,B) \leq R$.) (Figure 6f.)

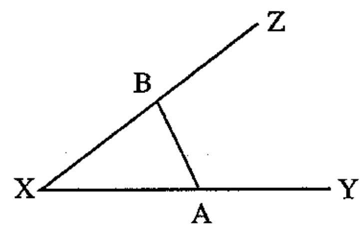


Figure 6f.

Proof: Let A be the point on $[X,Y] \cap N_r[X,Z]$ nearest Y. (By hypothesis, $Y \notin N_R[X,Z]$.) Let $C \in [X,Z]$ be a nearest point on $[X,Z]$ to A. Let $B \in [C,Z]$ be the point distant $J+2r$ from C. (B exists since $d(Z,[X,Y]) \geq R$. (Figure 6g.)

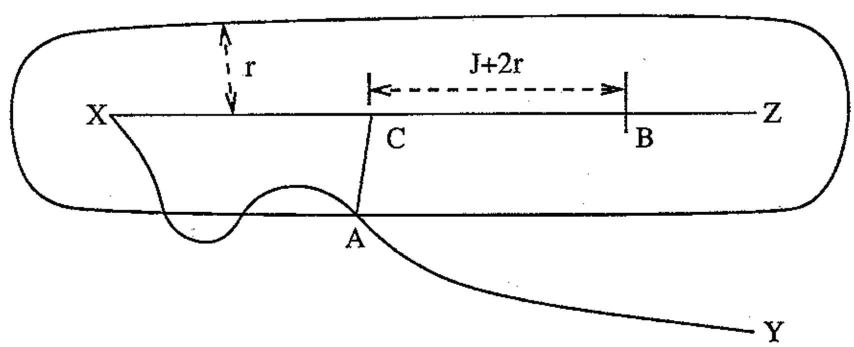


Figure 6g.

Now, $d(C,A) = r$, therefore by Lemma 6.2.1, $[X,A] \subseteq N_J[X,C]$, and so $[X,A] \subseteq N_R[X,B]$. Also by Lemma 6.2.1, $[X,C] \subseteq N_J[X,A]$. Since $[C,B] \subseteq N_{r+(J+2r)}(A)$, we have $[X,B] \subseteq N_R[X,A]$.

By construction, we have $d([A,Y],[X,Z]) \geq r$.

Finally, suppose for contradiction, that we could find $D \in [B,Z]$ and $E \in [X,Y]$ with $d(D,E) \leq r$. Since $d([A,Y],[X,Z]) \geq r$, we must have $E \in [X,A]$. Since $[X,A] \subseteq N_J[X,C]$, there is some $F \in [X,C]$ with $d(E,F) \leq J$. (Figure 6h.)

Now, $d(D,F) \leq d(D,E) + d(E,F) \leq J + r$. But $d(C,B) = J + 2r$, and $[C,B] \subseteq [F,D]$ so $d(D,F) \geq J + 2r$. We have contradicted the existence of D an E. Thus, $d([B,Z],[X,Y]) \geq r$.

$\diamondsuit$

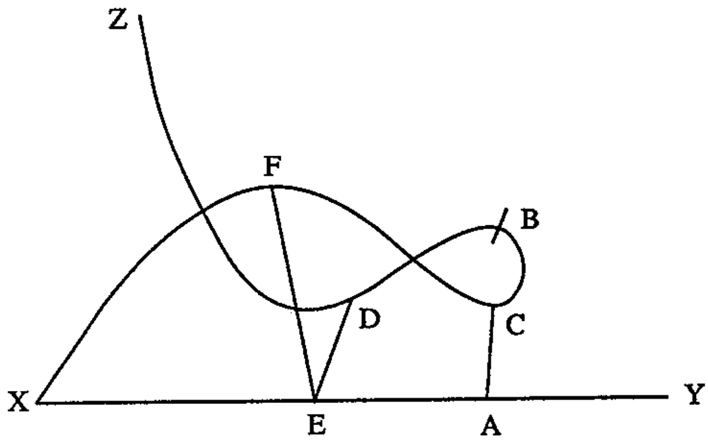


Figure 6h.

Lemma 6.2.3 : *Suppose $(\mathcal{S}, d)$ is (r, L)-H4. Let $M = 10R + L$ (where $R = 4r + \frac{L}{4}$ comes from Lemma 6.2.2). Then, given $X, Y, Z \in \mathcal{S}$, and geodesics $[X, Y]$, $[Y, Z]$ and $[Z, X]$, there is some $W \in \mathcal{S}$ such that*

$$d(W, [X, Y]) \leq M$$
$$d(W, [Y, Z]) \leq M$$
$$d(W, [Z, X]) \leq M.$$

We call such a point W, a "centre" for the triangle XYZ.

Proof : Suppose (for contradiction) that there is no centre for XYZ.

Then, in particular, $d(Y, [X, Z]) \geq M > R$ and $d(Z, [X, Y]) \geq M > R$. Let $A \in [X, Y]$ and $B \in [X, Z]$ be the points given by Lemma 6.2.2. Let C, D and E, F be similar points with respect to Y and Z respectively, so that $A, D \in [X, Y]$, $C, E \in [Y, Z]$ and $F, B \in [Z, X]$.

We claim that $A \in [X, D]$. Suppose (to contradict the claim) that $D \in [X, A]$. (Figure 6i.)

By hypothesis, (Lemma 6.2.2), $D \in N_R[X, B]$. Also $d(D, C) \leq R$. Thus, D is a centre for XYZ, contradicting the initial supposition. This proves the claim.

We can make similar statements about the order of points on $[Y, Z]$ and $[Z, X]$. Thus, the points are ordered cyclically $XADYCEZFBX$ about the triangle XYZ. (Figure 6j.)

Now without loss of generality, we can assume that

$$d(A, D) \geq \max\big(d(C, E), d(F, B)\big).$$

Let $d = d(A, D)$. We know that $d(A, B) \geq r$ and $d(D, C) \geq r$. Let G be the point on $[A, B] \cap N_r[X, Y]$ nearest to B. Let H be the point on $[C, D] \cap N_r[X, Y]$ nearest to C. Thus $G, H \in \partial N_r[X, Y]$. (Figure 6k.)

Let α be the path $[G, B] \cup [B, F] \cup [F, E] \cup [E, C] \cup [C, H]$, so that length$\alpha \leq R + d + R + d + R = 3R + 2d$.

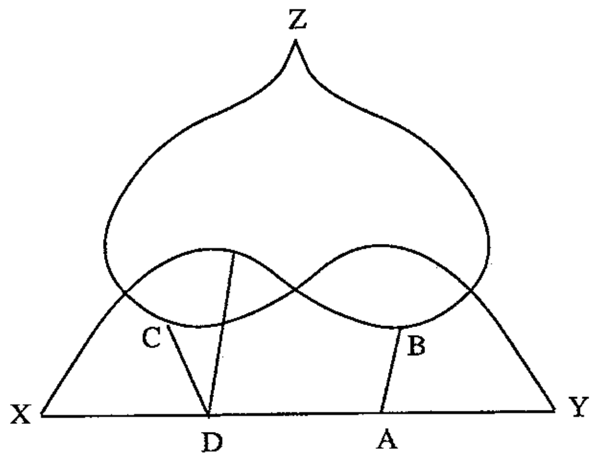


Figure 6i.

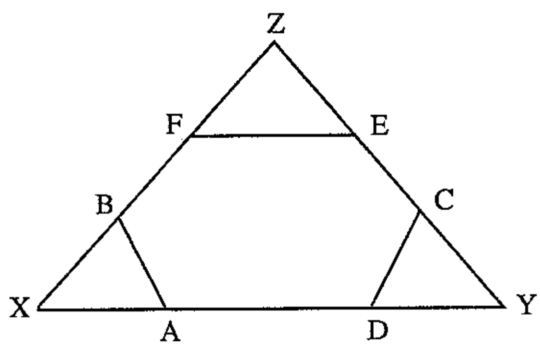


Figure 6j.

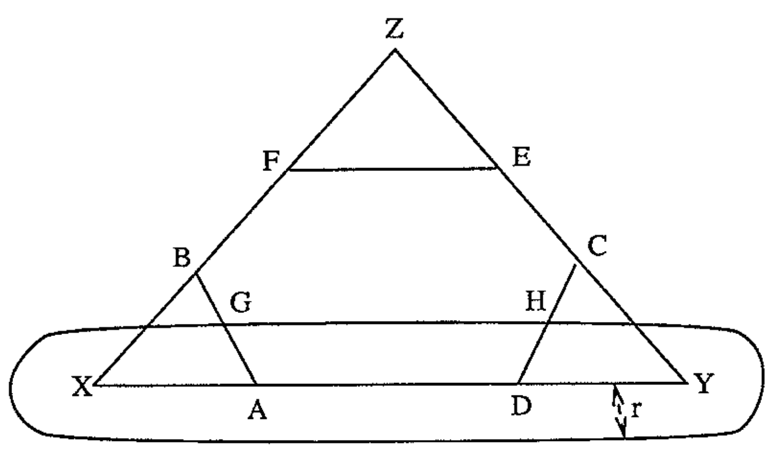


Figure 6k.

By hypothesis (Lemma 6.2.2), we have $d([B,F],[X,Y]) \geq r$ and $d([E,C],[X,Y]) \geq r$.

Since $d(F,E) \leq R$, any point of $[F,E] \cap N_r[X,Y]$ would be a centre for XYZ. We see that $d([F,E],[X,Y]) \geq r$. Thus $\alpha \cap \breve{N}_r[X,Y] = \emptyset$. Since $\mathcal{S}$ is (r,L)-H4, we get

$$\begin{aligned} 3R + 2d &\geq \text{length}\alpha \\ &\geq d_{r,[X,Y]}(G,H) \\ &\geq 3d(G,H) - L \\ &\geq 3(d-2R) - L = 3d - 6R - L. \end{aligned}$$

Thus,

$$d \leq 9R + L.$$

Now,

$$\begin{aligned} d(A,[Y,Z]) &\leq d(A,D) + d(D,C) \\ &\leq d + R \\ &\leq 10R + L = M. \end{aligned}$$

Thus, A is a centre for XYZ.

This contradicts our initial supposition, and so proves the lemma.

$\diamond$

Proposition 6.2.4 : *$\forall k_4, h_4 \, \exists k_2$ such that if $(\mathcal{S}, d)$ is (k_4, h_4)-H4, then it is k_2-H2.*

Proof : Lemma 6.2.3.

$\diamond$