Manifolds MA3H5. Exercise Sheet 6

(Recall the "classical" notions of "divergence, and curl" denoted $\nabla . v$ and $\nabla \times v$, of a vector field v defined on $\mathbb{R}^3$. In x, y, z coordinates, if $v = (P, Q, R)$, then $\nabla . v = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ etc.)

1: Let V be a finite-dimensional vector space, and let $E = V^*$. Let $e_1, \ldots, e_m$ be a basis for E. Given $1 \leq i < j \leq m$, define $\phi_{ij} : V^2 \longrightarrow \mathbb{R}$ by $\phi_{ij}(v, w) = e_i(v)e_j(w) - e_i(w)e_j(v)$.
Show that ϕ is an alternating linear map.
Given any $\eta \in \Lambda^2 E$, write $\eta = \sum_{i<j} \lambda_{ij} e_i \wedge e_j$, and set $\eta(v, w) = \sum_{i<j} \lambda_{ij} \phi_{ij}(v, w)$.
If $e, f \in E$, show that $(e \wedge f)(v, w) = e(v)f(w) - e(w)f(v)$.
Show how this generalises to $\lambda^p E$. That is, for $\omega \in \Lambda^p E$, and $v_1, \ldots, v_p$, one can define $\omega(v_1, \ldots, v_p)$ to be an alternating multilinear map in the v_i.

2: Let M be a manifold, and let ω be a 1-form on M. Let X, Y be vector fields on M, and let $[X, Y]$ be the Lie bracket (as defined on Sheet 3).
Show that $d\omega(X, Y) = X(\omega Y) - Y(\omega X) - \omega[X, Y]$.
Here $d\omega(X, Y)$ for the 2-form, $d\omega$, is defined as in Q2.
(Note, we can assume that ω has the form udv for real functions u, v.)

3: If $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is compactly supported. Show that $\int_{\mathbb{R}^m} f\omega$ is just the usual intergral $\int_{\mathbb{R}^m} f \, dx_1, \ldots, dx_m$, where ω is the volume form on $\mathbb{R}^n$.

4: (Green's Theorem) Let $P, Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be smooth functions.
Let ω be the 1-form $\omega = Pdx + Qdy$. Calculate dw.
Suppose that $D \subseteq \mathbb{R}$ is a compact disc, with smooth boundary, C: a compact 1-manifold in $\mathbb{R}^2$. If t is a local parameter for C, show that the induced form ω on C is given by $\left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right) dt$.
Show that $\int_C \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right) dt = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \, dy$.

5: Suppose that v is a vector field in $\mathbb{R}^3$. Write $v = (P, Q, R)$, where P, Q, R are real-valued functions. Write $\omega = Pdx + Qdy + Rdz$.
Calculate $d\omega$.
Under the natural identification of $\Lambda^2\mathbb{R}^3$ with $\mathbb{R}^3$, show that the operation $[\omega \mapsto d\omega]$ correspond to taking the curl of the vector field

$[v \mapsto \nabla \times v]$.

6: (Divergence Theorem) Let $B \subseteq \mathbb{R}$ be a compact 3-submanifold with boundary ∂B (an embedded 2-manifold). Given a vector field v on $\mathbb{R}^3$, we aim to show that $\int_B (\nabla.v)\, dV = \int_{\partial B} (v.n)\, dA$, where "$dV$" informally denotes the volume form in $\mathbb{R}^3$, "dA" denotes the area (volume) form on ∂B, and where n denotes the unit outward normal.
For this, write $v = (P, Q, R)$ for real functions P, Q, R, and let ω be the 2-form $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$.
Show that $d\omega = (\nabla.v)dx \wedge dy \wedge dz$.
We claim that the induced from on ∂B can be written as $(v.n)\eta$, where η is the volume (area) form on ∂B.

To do this, let $p \in \partial B$. By linearity in v, it's enough to check this when $v(p) = (0, 0, 1)$. We can find local coordinates a, b, c in a neighbourhood, U, of p such that $S \cap U$ corresponds to $c = 0$, and such that $\frac{\partial}{\partial a}\, \frac{\partial}{\partial b}\, \frac{\partial}{\partial c}$ is an orthonormal frame in $T_p(\partial B)$. In this way, $n(p) = \frac{\partial}{\partial c}$ and $\eta(p) = da \wedge db$ at p. Note that since the Jacobian at p is orthogonal, we have $\frac{\partial c}{\partial z} = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}$. Thus, $dx \wedge dy = \frac{\partial c}{\partial z} = (0, 0, 1).n(p)$.