Manifolds MA3H5. Exercise Sheet 5

1: Given an example of a smooth map, $f : (0,1) \longrightarrow (0,1)$, which does not extend to any continuous function on any strictly larger domain. Given an example of an increasing smooth map, $f : (0,1) \longrightarrow (0,1)$ which does not extend to any smooth function on any larger domain. Must it extend to a continuous map on a large domain?

2: Let M be a connected manifold, and let $\mathcal{U}$ be a locally finite cover of M by locally compact sets. Show that $\mathcal{U}$ is countable. For example, start with any $U_0 \in \mathcal{U}$. Write $\{U \in \mathcal{U} \mid U \cap U_0 \neq \emptyset\} = \{U_0, U_1, U_2, \ldots, U_n\}$ (check that this is finite) Now enumerate the new sets in $\{U \in \mathcal{U} \mid U \cap U_1 \neq \emptyset\}$ as $U_{n+1} \ldots$ and then move on to U_2. This gives an infinite sequence $(U_n)_{n=0}^{\infty}$. Show that $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$.

3: Let M be a connected manifold. Show that there is a smooth proper map $f : M \longrightarrow [0,\infty) \subseteq \mathbb{R}$.

4: Let $A, B \subseteq M$ be disjoint closed sets. Show that there is an open cover, $\mathcal{U}$, of M such that no element of $\mathcal{U}$ meets both A and B.
Show that there is a smooth function, $f : M \longrightarrow [0,1]$, with $f|A \equiv 0$ and $f|B \equiv 1$.

5: Let $\langle ., . \rangle$ denote a riemannian metric (on each tangent space) of a smooth manifold, M. Show that in local coordinates, $x_1, \ldots, x_m$, this can be written in the form $\langle v, w \rangle = \sum_{i,j} g_{ij} \lambda_i \mu_j$, where $v = \sum_i \lambda_i \frac{\partial}{\partial x_i}$, $w = \sum_j \mu_j \frac{\partial}{\partial x_j}$, and where g_{ij} are local smooth real-valued functions with the matrix $(g_{ij}(x))_{ij}$ symmetric and positive definite for all x.