Manifolds MA3H5. Exercise Sheet 4

1: Let $P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$ be real projective n-space. Given $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, write $[x_0, \dots, x_n] \in P^n$ for its $\sim$-class. Given $0 \le i \le n$, if $x_i \neq 0$, write $\phi_i([x_0, \dots, x_n]) = (x_0/x_i, x_1/x_i, \dots, x_n/x_i) \in \mathbb{R}^n$, where, on the right-hand side, we have omitted the term "x_i/x_i". Note that this is well defined.
Show that the collection of maps $\{\phi_i\}_{i=0}^n$, defined on suitable domains, gives rise to a smooth atlas for P^n.

2: Give a proof of Hadamard's lemma: If $U \subseteq \mathbb{R}^n$ is open $a \in U$, and $f : U \longrightarrow \mathbb{R}^n$ is smooth, then there are smooth funcitons $g_i = g_i^U : U \longrightarrow \mathbb{R}^n$ such that $f(x) = f(a) + \sum_{i=1}^m (x_i - a_i) g_i(x)$ for all $x \in U$.
Show, moreover, that the g_i^U can be chosen consistently, that is in such a way that $g_i|V$ depends only on $f|V$ where $V \subseteq U$ is any open set containing containing a.
Deduce that Hadamard's lemma also holds for germs: that is, if $f \in \mathcal{G}_a(M)$, then we can write $f = f(a) + \sum_{i=1}^m (\pi_i - a_i) g_i$ for $g_i \in \mathcal{G}_a(M)$, where $\pi_i \in \mathcal{G}_a(M)$ is the germ of the projection map $[x \mapsto x_i]$.

3: Suppose that $E \longrightarrow M$ and $F \longrightarrow N$ are vector bundles over manifolds M and N. Show that the direct product $E \times F \longrightarrow M \times N$ is a vector bundle. What is the fibre?

4: Suppose that $p : F \longrightarrow N$ is a vector bundle over N, and that $M \subseteq N$ is a submanifold. Let $E = p^{-1}M$. Show that $(p|E) : E \longrightarrow M$ is a bundle.

5: Show that the diagonal $\Delta = \{(x, x) \mid x \in M\}$ is a submanifold of $M \times M$. Show how (3) and (4) can be used to give an equivalent construction of the Whitney sum of two bundles over M.

6: If $E, F, G \longrightarrow M$ are vector bundles over M, show that
$(E \oplus F) \oplus G \equiv E \oplus (F \oplus G)$.

7: Show that for every $q \in \mathbb{N}$, $q > 0$, there is a non-trivial bundle over S^1 with fibre (isomorphic to) $\mathbb{R}^q$.

8: Let $E \longrightarrow M$ be a vector bundle. Show that there is a canonical isomorphism from E to E^{**}.

Show that E is trivial if and only if E^* is trivial.

9: Let $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$ be the 1-form on $\mathbb{R}^2$, with cartesian coordinates x, y. Calculate $\int_\gamma \omega$, where γ is the unit circle, $\gamma(t) = (\cos t, \sin t)$.

10: Let M be a manifold and $x \in M$. Let $J \subseteq \mathcal{G}_x(M)$ be the of germs $f \in \mathcal{G}_x(M)$ which vanish at x (i.e. $f(x) = 0$). Check that this is a subspace of $\mathcal{G}_x(M)$.
Let $K \subseteq J$ be the subspace spanned by $\{fg \mid f, g \in J\}$. Let $\theta : \mathcal{G}_x(M) \longrightarrow T_x^*M$ be the linear map given by $\theta(f) = df$ at x.
Show that θ is surjective.
Show that $\ker\theta = K$ (use Hadamard's Lemma for germs (Q1) above).
Deduce that T_x^*M is canonically isomorphic to J/K.
(This gives rise an equivalent, and direct, way of defining the cotangent space, T_x^*M. In this approach, one could retrospectively define the tangent space, T_xM, as the dual to T_x^*M.)