Manifolds MA3H5. Exercise Sheet 3

1: (Every odd dimensional sphere admits a nowhere vanishing vector field.) Given $x \in S^{2n-1} \subseteq \mathbb{R}^{2n}$, write

$$v(x) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1}).$$

Show that v is a vector field on S^{2n-1}.

2: Show that the normal bundle, $\nu(S^n, \mathbb{R}^{n+1})$, is diffeomorphic to $S^n \times \mathbb{R}$.

3: Show that the Möbius band is not orientable.

4: If M is a connected oriented manifold, show that every diffeomorphism from M to itself ether preserves or reverses orientation.

5: Let M and N be oriented manifolds of dimension m and n respectively. Give each of $M \times N$ and $N \times M$ the product orientation. Define $f : M \times N \longrightarrow N \times M$ by $f(x, y) = (y, x)$. Check that f is a diffeomorphism, and show that f preserves orientation if mn is even, and reverses orientation if mn is odd.

6: Let $f : S^n \longrightarrow S^n$ be the antipodal map on the n-sphere $S^n \subseteq \mathbb{R}^{n+1}$. (That is, $f(x) = -x$.) Show that f is orientation preserving if n is odd, and orientation reversing if n is even.

7: Let X be a vector field on M, and $f \in C^\infty(M)$. Define $Xf : M \longrightarrow \mathbb{R}$ pointwise (as in the lectures). Show that this map is smooth. In fact, writing $X = \sum_i \lambda_i \frac{\partial}{\partial x_i}$ in local coordinates (i.e. with respect to some chart) we have $Xf = \sum_i \lambda_i \frac{\partial f}{\partial x_i}$.

8: (Lie brackets) Let X, Y be vector fields on M. Given $f \in C^\infty_x(M)$, let $F = X(Yf) - Y(Xf)$ (so $F \in C^\infty(M)$, by the previous question). If $x \in M$, show that $F(x)$ depends only on the germ of f at x, and only requires f to be defined on a neighbourhood of x. Show that it defines a linear functional on the space of germs at x, and satisfies the Leibnitz condition ((L) in the lecture notes), and so gives rise to a vector $v(x) \in T_x M$. Show that $[x \mapsto v(x)]$ is a vector field on M. In fact, in local coordinates, if $X = \sum_i \lambda_i \frac{\partial}{\partial x_i}$ and if $Y = \sum_i \mu_i \frac{\partial}{\partial x_i}$, then

$$v = \sum_{i,j} \left(\lambda_i \frac{\partial \mu_j}{\partial x_i} - \mu_i \frac{\partial \lambda_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

We write $v = [X, Y]$, the "Lie bracket" of X and Y. Show that $[Y, X] = -[X, Y]$.

9: Suppose that X, Y are vector fields on M, and $f, g \in C^\infty(M)$. Show that $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

10: (Jacobi identity) Given vector fields, X, Y, Z show that

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$