Manifolds MA3H5. Exercise Sheet 2

1: Show that $(\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})\subseteq\mathbb{R}^2$ is not a manifold. (Consider what the tangent space could be at the origin.)
Show that $(\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})\times\mathbb{R}\subseteq\mathbb{R}^3$ is not a manifold.
Show that the cone $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1^2+x_2^2=x_3^2\}$ is not a manifold.

2: Show that if M is a connected manifold of dimension at least 2, then $M\setminus\{x\}$ is connected for all $x\in M$. (This gives another argument for the cone is not a manifold.)

3: Suppose that $M\subseteq\mathbb{R}^n\times[0,\infty)\subseteq\mathbb{R}^{n+1}$ is a manifold (of some dimension) in $\mathbb{R}^{n+1}$. If $x\in M\cap(\mathbb{R}^n\times\{0\})$, show that $T_xM\subseteq\mathbb{R}^n\times\{0\}$. (So, if M has dimension n, then $T_xM=\mathbb{R}^n\times\{0\}$.)

5: Let $m,n\in\mathbb{N}$ and identify the set, $\mathcal{M}$, of $m\times n$ matrices with $\mathbb{R}^{mn}$. If $r\leq\min\{m,n\}$, prove that the set of $m\times n$ matrices of rank r is a manifold, and find its dimension.
Here is one way to proceed. First show that the set of matrices of rank at least r is open in $\mathcal{M}$. (Recall that this is equivalent to saying that a matrix has an $r\times r$ submatrix with non-zero determinant.) If $\mathrm{rank}(A)\geq r$, then after permuting rows and columns, we can write it in the form

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

where B is a non-singular $r\times r$ matrix. Postmultiplying by

$$\begin{pmatrix} I & -B^{-1}C \\ 0 & I \end{pmatrix}$$

show that $\mathrm{rank}(A)=r$ if and only if $E=DB^{-1}C$. Show that the derivative of the map $[A\mapsto E-DB^{-1}C]$, viewed as a map $\mathbb{R}^{mn}\longrightarrow\mathbb{R}^{(m-r)(n-r)}$, has maximal rank at A.

6 Let $M\subseteq\mathbb{R}^n$ is an embedded submanifold of dimension m, and $c\in M$. We aim to show that M "looks like" a graph, near c, up to rigid motion.
Note that, up to rigid motion (eucidean isometry) of $\mathbb{R}^n$, we can can assume that $c=0^n$ and that $T_cM=\mathbb{R}^m\times\{0^{n-m}\}$.
Show that there is an open set $U\subseteq\mathbb{R}^n$ containing c, and a smooth map, $f:U\cap\mathbb{R}^m\longrightarrow\mathbb{R}^{n-m}$, such that $U\cap M=\{(x,f(x))\in\mathbb{R}^m\times\mathbb{R}^{n-n}\equiv\mathbb{R}^n \mid x\in U\cap\mathbb{R}^m\}$. (Consider the inverse of a chart, postomposed with projection to $\mathbb{R}^m$, and apply the inverse function theorem.)