Manifolds MA3H5. Exercise Sheet 1

1: (i) Given real numbers $a < b$, find a diffeomorphism $(a, b) \to \mathbb{R}$.
(ii) Find a diffeomorphism $(0, \infty) \to \mathbb{R}$.
(iii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth map, and let graph(f) be the set $\{(x, y) \in \mathbb{R}^m \times \mathbb{R} \mid y = f(x)\}$. Show that the map $x \mapsto (x, f(x))$ is a diffeomorphism from $\mathbb{R}^m$ to graph(f).
(iv) Let C_0 be the cylinder $S^1 \times (0, \infty) \subseteq \mathbb{R}^3$ and let C be the cylinder $S^1 \times \mathbb{R}$. Find diffeomorphisms $C_0 \to \mathbb{R}^2 \setminus \{0\}$ and $C \longrightarrow \mathbb{R}^2 \setminus \{0\}$.
(iv) Let C_1 be the cylinder $S^1 \times [0, 1]$. Find a subset of $\mathbb{R}^2$ to which this is diffeomorphic.

2: Let $\mathcal{M}(n, \mathbb{R})$ be the set of real $n \times n$ matrices, identified with the real vector space $\mathbb{R}^{n^2}$, taking entries as coordinates. Find the derivative of the determinant map $\det : \mathcal{M}(n, \mathbb{R}) \longrightarrow \mathbb{R}$, as an $n^2 \times 1$ matrix (which you can rearrange into an $n \times n$ matrix).

3: Show that stereographic projection $S^2 \setminus \{p_N\} \longrightarrow \mathbb{R}^2$ is a diffemorphism (write down a formula for its inverse).
Show that $S^2 \subseteq \mathbb{R}^2$ is a 2-manifold.
Verify the formula for the transition map $\mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$ either geometrically, or by writing down formulae.

4: Show that if $U \subseteq \mathbb{R}^n$ is open, $f : U \longrightarrow \mathbb{R}^n$ is smooth, and $\det(d_x f) \neq 0$ for all $x \in U$, then $f(U)$ is open in $\mathbb{R}^n$.
Show that if $M \subseteq \mathbb{R}^n$ is an n-manifold, then M is an open subset of M.
Deduce that there is no compact n-manifold in $\mathbb{R}^n$.

5: Show that the dimension of a manifold is uniquely determined.

6: Show that the 2-manifolds, $T_{a,b} \subseteq \mathbb{R}^3$ and $T \subseteq \mathbb{R}^4$ (examples (E6) and (E7) in lectures) are diffeomorphic, for all $a > b > 0$.

7. Prove that if $M^m \subseteq \mathbb{R}^p$ and $N^n \subseteq \mathbb{R}^q$ are smooth manifolds then
(i) $M \times N \subseteq \mathbb{R}^{p+q}$ is a smooth manifold of dimension $m + n$, and
(ii) $T_{(x,y)}(M \times N) = T_x M \times T_y N$.

8: (Definiton of germs.) Let M be an m-manifold (in $\mathbb{R}^n$) Show that the relation $\sim$ defined on the set, $C^\infty_x(M)$, of local functions at x (as in lectures) is an equivalence relation.

Show that $\mathcal{G}_x(M) = C_x^\infty(M)/\!\sim$ has naturally the structure of a vector space.
Show that the derivative $v.f$ for $v \in T_xM$ and $f \in \mathcal{G}_x(M)$ is well defined.
Show that the map $[(v,f) \mapsto v.f] : T_xM \times \mathcal{G}_x(M) \longrightarrow \mathbb{R}$ is bilinear.
Show that, for all $f, g \in \mathcal{G}_x(M)$ we have $v.(fg) = f(x)(v.g) + g(x)(v.f)$.

9: (Stack of records theorem).
Suppose that M and N are smooth manifolds of the same dimension, n, with M compact. Suppose that $f : M \longrightarrow N$ is a smooth map, and that d_xf is invertible for all $x \in M$. Given any $y \in N$, show that $f^{-1}(y)$ is finite, and that there exists a neighbourhood V of y in N such that $f^{-1}(V)$ is a disjoint union of open sets of X, each of which is mapped diffeomorphically to V by f. Does either statement still hold if we drop the requirement that M be compact?
Show that f is surjective, and and indeed that any two points in N have the same number of preimages in M (again supposing that M is compact).

10: Let $\sim$ be the equivalence relation on $S^2 \subseteq \mathbb{R}^3$ which identifies antipodal points (i.e. $x \sim -x$). Let $P^2 = S^2/\!\sim$ with the quotient topology (the real procjective plane). Similarly, define an equivalence relation, $\sim$, on $\mathbb{R}^3 \setminus \{0\}$ by $x \sim y$ if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ with $y = \lambda x$. Again, we give $(\mathbb{R}^3 \setminus \{0\})/\!\sim$ the quotient topology.
Show that the natural map $P^2 \longrightarrow (\mathbb{R}^3 \setminus \{0\})/\!\sim$ is, in fact, a homeomorphism.

11: Let $M = \{(x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1) \mid x_1^2 + x_2^2 + x_3^2 = 1\} \subseteq \mathbb{R}^6$.
Show that M is a 2-manifold.
(For example, if $x_1 > 0$ and $x_3 > 0$, one can define smooth map neighbourhood of the correponding point of M, just by projecting to the 1st and 4th coordinates. This gives us local coordinates $t = x_1^2$, $u = x_1x_2$. From t, u we can recover x_1, x_2, x_3, locally by nice simple formulae: $x_1 = \sqrt{t}$, $x_2 = u/\sqrt{t}$ and $x_3 = \sqrt{1 - t - u^2/t}$. Use this to show that we have a chart in a neighbourhood of our point.
Note that at least one of x_1, x_2, x_3 must be non-zero. Up to symmetries, we see this deals all cases apart from the points where two of x_1, x_2, x_3 are 0, and the other is ± 1. For this, we need to project to different coordinates.)
Show that the projective plane, P^2, is homeomorphic to M (hence a 2-manifold).