7.3. Back to group presentations.

Let Γ be a group with generating set S, and let $\Delta = \Delta(\Gamma, S)$ be the Cayley graph.
Recall that Δ/Γ is a finite wedge of circles — one for each generators.
Suppose that Γ is finitely presented, and that R is a finite set of relators for Γ with this generating set.
Now each element of R determines a closed path in Δ/Γ.
We construct a complex, $\Psi = \Psi(\Gamma, S, R)$ by gluing a topological disc to this path, for each element of R.
In other words, R is in bijective correspondence to the set of 2-cells.

Definition : We refer to Ψ as the associated *presentation complex*.

Let Φ be the universal cover of Ψ.
We claim that can identify $\Delta(\Gamma, S)$ with the 1-skeleton of Ψ.

In fact, an equivalent way of constructing Ψ is as follows.
Each relator gives us a closed path in Δ based at the identity, 1.
The set of Γ-images of these paths correspond to conjugates of the relators.
We attach a disc to each such curve, to give us a 2-complex Φ.
Now Γ acts on Φ, permuting these discs.
It is easily seen that the quotient Φ/G is precisely the presentation Φ.
To justify the above claim, it is enough to note that this construction gives us Φ simply connected.
But this follows directly from the definition of presentation — every word representing the identity is a product of conjugates of the relators, and so every closed path can be homotoped to a point, by pulling it accross the 2-cells.

Remark: to construct Φ we may need to add several discs to the same closed paths, since each congugate of a relator has a preferred basepoint.
(Consider for example the cyclic group, $\langle a; a^n \rangle$.)

If Ψ is a presentation 2-complex of Φ, then $\pi_1(\Psi) \cong \Gamma$.
In fact, Γ acts on Φ by covering transformations.

Examples

(1) Free groups

(2) $\mathbf{Z} \oplus \mathbf{Z}$

(3) Surface groups.

A combinatorial construction.

Suppose Δ is any connected locally finite graph, and $n \in \mathbf{N}$.

Let $\Phi_n(\Delta)$ be the 2-complex obtained by attaching a 2-cell to every cycle of length at most n to Δ.

Lemma 7.4 : *Suppose Γ is a group generated by a finite set, $S \subseteq \Gamma$, and let $\Delta = \Delta(\Gamma, S)$ be the Cayley graph. Then Γ is finitely presented if and only if there is some $n \in \mathbf{N}$ such that $\Phi_n(\Delta)$ is simply connected.*

Proof :

We first make the observation that, in the definition of Φ_n, it makes no difference to simple connectedness if we allow ourselves to add any finite number of 2-cells to each our cycles, rather than just one.

Suppose that $\Phi_n(\Delta)$ is simply connected.

Let R_n be the set of all words in $S \cup S^{-1}$ of length at most n representing the identity in Γ.

Now, any word, w, representing the identity corresponds to a closed path in Δ based at 1.

We can reduce this to the trivial path in a finite number of steps. At each step, we replace some subarc by another, so that the combined length is at most n.

This corresponds to multiplying by a conjugate of a relator in R_n, in the free group $F(S)$.

It follows that w is a product of such congugates in $F(S)$,

In other words, lies in the normal closure of R_n.

This shows that $\Gamma = \langle S, R_n \rangle$.

Conversely, suppose that $\Gamma = \langle S, R \rangle$ is a finite presentation.

Let n be the length of the longest element of R.

Let Φ be the universal cover of the presentation 2-complex.

Now Φ_n is obtained from Φ by identiying multiple 2-cells with the same boundary, and adding additional cells.

These operations preserve simple connectedness. $\diamondsuit$

Note that a consequence of the proof that hyperbolicity implies a linear isoperimetric inequality is:

Corollary 7.5 : *A hyperbolic group is finitely presented.*

Proof :

Let Δ be any cayley graph of a hyperbolic group.

The procedure for reducing a closed path to a point (in Theorem 7.3) shows that $\Phi_n(\Delta)$ is simply connected for sufficiently large n. $\diamondsuit$

Non-example :

We also note that Lemma 7.4 can be used to show that certain f.g. groups are not finitely presented, i.e. admit no finite presentation at all.

For example, the "lamplighter group", L, has two generators, x and t, with relators $x^2 = 1$ and $[x, t^n x t^{-n}] = 1$ for all $n \in \mathbf{N}$. (Here t corresponds to the lamplighter moving one step to the right, and x corresponds to lighting/extinguishing the lamp at the lamplighter's current position. Switching the lamp on/off has order 2. Also fiddling with the n'th lamp has no effect on any other, so these operations commute.)

Exercise : Let γ_n be the closed curve in the Cayley graph, Δ corresonding to the word $xt^n xt^{-n} xt^n xt^{-n}$. Show that there is a distance non-increasing map $f : \Delta \longrightarrow \gamma_n$ with $f|\gamma_n$ the identity. Deduce that $\Phi_m(\Delta)$ is not simply connected for any m. Deduce that L is not finitely presented.

Lemma 7.6 : *Suppose Δ and Δ' are locally finite quasi-isometric graphs. If there is some $n \in \mathbf{N}$ such that $\Phi_n(\Delta)$ is simply connected, then there is some $m \in \mathbf{N}$ such that $\Phi_m(\Delta')$ is simply connected.*

Proof : Exercise. $\diamondsuit$

Putting Lemmas 7.4 and 7.6 together we deduce:

Corollary 7.7 : *Finite presentablity of groups is a quasi-isometry invariant.*

We can now reinterpet isoperimetic inequalities in these terms.

Suppose Δ is graph Φ is a simpy connected 2-complex with 1-skeleton Δ.

Let Ω_0 be the set of cycles lying Δ.

We can define any area (restricted to Ω_0) by setting $A(\gamma)$ to be the minimal number of 2-cells a cellular disc spanning γ.

This satisfied (A1) and (A2).

We can define isoperimetric function and isoperimetric bound based on this notion of area.

Suppose Φ is the universal cover of a presentation complex of a f.p. group $\Gamma = \langle S, R \rangle$.

Let f be the isoperimetric function.

Then $f(n)$ is the maximum of the minimal number of conguates of relators we need to apply in to reduce a word of length n, representing the identity in Γ to the trivial word.

If Γ is hyperbolic, then this is at most linear for any presentation.
Conversely, if it is linear for some presentation, then Γ is hyperbolic.

Remark : In the definition of $\Phi_n(\Delta)$ we could have used circuits instead of cycles. This variant of $\Phi_n(\Delta)$ is simply connected, if and only if, the original was (exercise), and so it would not effect our discussion in any essential way. (Recall that that a "circuit" is an embedded cycle.)

Quasi-isometry invariance of isoperimetric functions.

Given

$$f, g : [0, \infty) \longrightarrow [0, \infty),$$

write

$$f \prec g$$

to mean that there are constants, $a, b, c, d \geq 0$ that for all t, we have

$$f(t) \leq ag(ct + d) + b.$$

We write

$$f \sim g$$

to mean that

$$f \prec g \quad \text{and} \quad g \prec f.$$

Finally we write

$$f \sim' g$$

to mean

$$f + id \sim g + id,$$

where id is the identity map on $[0, \infty)$.

In other words, $f \sim' g$ if and only if $f \sim g$ or else f and g are both bounded by linear functions.

Note, for example, that, for any $p \geq 1$, the property of asymptotic to $[t \mapsto t^p]$ is invariant under the relation $\sim'$.

If Δ is a graph, we say that Δ has an isoperimetric bound if there is some n such that $\Phi_n(\Delta)$ is simply connected, and if the isoperimetric function of $\Phi_n(\Delta)$ is always finite.

This is true, for example, of the Cayley graph of a finitely presented group.

Lemma 7.8 : *Suppose that Δ and Δ' are quasi-isometric locally finite graph, with respective isoperimetric functions f and f'. Then $f \sim' f'$.*

Proof : Exercise. $\diamondsuit$

Note this is interpreted to imply that if Δ has an isoperimetric bound then so does Δ'.

Thus, isoperimetric function of a finitely presented group is well defined up to this equivalence.

Definition : The equivalence class of the isoperimetric function is commonly called the *Dehn function* of the group.

Note that it makes sense to say that the Dehn function of a group is order t^p for $p \geq 1$, or is polynomial or exponential.

Examples

(1) The Dehn function of a hyperbolic group is linear class.

(2) The Dehn function of Z^n is quadratic for any $n \geq 2$.

(3) The Heisenberg group is cubic.

(4) The Dehn function of many soluble groups is exponential.

(5) If the Dehn function of a finitely presented group is asymptotic to t^p, then either $p = 1$ or $p \geq 2$.
However such Dehn functions exist for a dense set of p in $[2, \infty)$.
This was shown by Brady and Bridson.

(6) A fairly complete description of which functions arise as Dehn functions as been given by Birget, Olshanskii, Rips and Sapir.

7.4. The word problem again.

We can give another perspective on the discussion of Section 6.11.
Let Γ be a group with finite generating set S.
Let $W(S \cup S^{-1})$ be the set of words in S and their (formal) inverses.

Definition : We say that Γ has solvable word problem if there is an algorithm to decide if w represents the trivial word.

Note: this is independent of the choice of finite generating set S.

Remark: There are examples of f.p. groups that do not have solvable word problem.

Recall that a function $f : \mathbf{N} \longrightarrow \mathbf{N}$ is *computable* if it there is a Turing machine that outputs $f(n)$ with input n.

We say that a function f is *subcomputable* if it is bounded above by a computable function.

The following is a simple observation:

Lemma 7.9 : *If $f \sim g$ then f is subcomputable if and only if g is.*

The following relates these two notions.

Lemma 7.10 : *Let Γ be a finitely presented group. The following are equivalent.*
(1) Γ has solvable word problem.
(2) Its Dehn function is subcomputable.
(3) Its Dehn function for every presentation is computable.

Proof : : Let $\Gamma = \langle S, R \rangle$ and let f be the Dehn function.

(2) $\Rightarrow$ (1).
Suppose $f \leq g$ where g is computable.
Let w be a word of length n.
We search all possible ways of reducing w to 1 using at most $g(n)$ relators.
Then w represents 1 if and only if we find such a reduction.

We need to observe that at each step in the reduction to the trivial word, we can bound the length of the conjugating elements by the length of the word at that point, so that it is a finite search.

(1) $\Rightarrow$ (3).

Suppose Γ has solvable word problem.

Given, n, we:
List all words of length n,
Select those which represent 1.
For each such word, search all products of conjugates of relators until we find one that gives this word.
Now search for one of minimal length, by considering all possiblities for at most this number of relators.
This is $f(n)$.

To see that the last step above is computable, we need again the earlier observation about bounding conjugating elements.

Since each step is computable, f is computable.

(3) $\Rightarrow$ (2).
Trivial. $\diamondsuit$

From the equivalence of (1) and (2) we get:

Corollary 7.11 : *Having soluble word problem is a q.i. invariant among f.p. groups.*

Remark: It appears to be unknown whether the same statement is true for f.g. groups.

Note also we get another proof of Corollary 6.21:

A hyperbolic group has soluble word problem.

Remark :
The solubility of the word problem for surface groups was proven by Dehn in the 1920s.