7. Isoperimetric functions.

7.1. Area:

Let (M, d) be a geodesic metric space.

We write Ω for the set of all closed curves in M.

Given a closed curve, $\gamma \in \Omega$ in M, we write $L(\gamma)$ for it (rectifiable) length in M.

We assign to each closed curve γ in M a "spanning area" $A(\gamma)$, which we assume satisfies certain axioms namely:

(A1) (Triangle inequality for theta curves):
If $\gamma_1, \gamma_2, \gamma_3 \in \Omega$ form a theta-curve, then

$$A(\gamma_3) \leq A(\gamma_1) + A(\gamma_2).$$

(A2) (Rectangle inequality):
Suppose $\gamma \in \Omega$ is split into four subpaths,

$$\gamma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4.$$

Then,

$$A(\gamma) \geq d_1 d_2,$$

where $d_1 = d(\alpha_1, \alpha_3)$ and $d_2 = d(\alpha_2, \alpha_4)$.

Examples.

(1) The general idea is that $A(\gamma)$ is the minimal area of a spanning disc of A.
This makes sense in a Riemannian manifold.
(To save worrying about existence, we can just take infimum.)
In fact, we could take any spanning surface.
Property (A2) is the Besicovitch inequality.

(2) Here is more abstract, but very general definition.
We consider a cellulation of the unit disc, D, and let σ be its 1-skeleton (so that $\partial D \subseteq \sigma$).
If $f : \sigma \longrightarrow M$ is any map we can define

$$A(\sigma, \lambda) = \sum \{(L(\lambda(\partial c)))^2\}$$

as c varies over the set of all 2-cells. We now define $A(\gamma)$ be the infimum of all $A(\sigma, \lambda)$ as σ and λ vary over all such cellulations and maps such that $\lambda | \partial D$ is just the curve γ.

Again this notion satisfies (A1) and (A2) above.

(3) In the above, we can insist that the length of $\lambda(\partial c)$ is bounded above by some constant, and then measure area of (σ, λ) the number of 2-cells of σ.

(4) In the case of a Cayley graph of a group, we can reinterpret this in terms of the number of applications of the relators needed to reduce a word representing the identiy element to the trivial word.
We discuss this in more detail later.

Given $x \in [0, \infty)$, let
$$f(x) = \sup\{A(\gamma) \mid \gamma \in \Omega, L(\gamma) \leq x\}.$$
Thus $f : [0, \infty) \longrightarrow [0, \infty)$ is the *isoperimetric function.*
Note that $f(x)$ is non-decreasing in x.

More generally we say that f is an *isoperimetric bound* if, for any curve γ, we have
$$A(\gamma) \leq f(L(\gamma)).$$

7.2. Linear isoperimetric bounds and hyperbolicity.

It turns out that a geodesics space, M, is hyperbolic if and only if has a linear isoperimetric bound.

We suppose that A is a spanning area satisfying (A1) and (A2) above.

Theorem 7.1 : *Let (M, d) be geodesic metric space.*
(1) If M is k-hyperbolic and there is some l_0 depending only on k such that each curve of length curve l_0 has area bounded by some A_0, then M has a linear isoperimetric bound, depending only on k and A_0.
(2) If M has a linear isoperimetric bound, f, then it is k-hyperbolic, where k depends only on f.

Remarks :
(1) In fact it is enough to assume a subquadratic isoperimetric bound.
In other words there are no isoperimetric functions strictly between linear

(Exercise: If $r \in [2, \infty)$ there is a complete riemannian plane with isoperimetric function asymptotic to $[x \mapsto x^r]$.
I.e. there no other gaps for general geodesic spaces.)

(2) If we assume hyperbolicity, then we can in fact, show that any curve spans a disc of bounded area in sense (3) above.

To show that hyperolicity implies isoperimetric function, we use the following:

Lemma 7.2 : *Suppose that M is k-hyperbolic. Then there are constants, $h > 0$ and l depending only on k such that if $\gamma \in \Omega$ has length at least l, then there is an arc in γ if length l such that the distance in M between its endpoints is at most $l - h$.*

Proof : Fix any basepoint, p, in M and choose $x \in \gamma$ to maximise $d(p, x)$.
Let $\alpha \subseteq \gamma$ be the arc of length l centred on x.
Let y, z be the endpoints of α.
Now:

$$\begin{aligned} d(p, y) &\leq d(p, x) \\ d(p, z) &\leq d(p, x) \\ d(x, y) &\leq l/2 \\ d(x, z) &\leq l/2. \end{aligned}$$

It follows by hyperbolicity that $d(y, z)$ is at most $l/2$ up to an additive constant (for example, since this is true without and additive constant in a tree).
If $h > 0$, then choosing l sufficiently large, we get $d(y, z) \leq l - h$, and the result follows. $\diamondsuit$

From this, we can deduce a linear isoperimetric bound.

Start with a curve γ as above, and find an arc α as given by Lemma 7.2.
Connecting its endpoints by a geodesic, β gives a theta curve, with components γ, $\alpha \cup \beta$, and γ_1 where γ_1 is obtained by taking γ and replacing α by β.
Now

$$L(\alpha \cup \beta) \leq 2l$$

and so $A(\alpha \cup \beta)$ is bounded, by some A_0.
Also

$$L(\gamma_1) \leq L(\gamma) - h.$$

We now carry out a similar argument with γ_1 replacing γ to give a sequence of at most $n \leq L(\gamma)/h + 1$ curves

$$\gamma, \gamma_1, \gamma_2, \ldots, \gamma_n,$$

terminating on a curve γ_n of length at most $2l$.
By (A2), we have

$$A(\gamma) \leq nA_0 \leq L(\gamma)A_0/h + A_0,$$

which is linear in $L(\gamma)$.

In fact, we have constructed a spanning disc of the type described in (3) above.

For the converse, let us assume that $[x \mapsto \lambda x + k]$ is an isoperimetric bound for M.

In other words, for all $\gamma \in \Omega$,

$$A(\gamma) \leq \lambda L(\gamma) + k.$$

Lemma 7.3 : *There is some $r > 0$, depending only on λ and k, such that if α is a geodesic, and β is a $(3, 0)$-quasigeodesic with the same endpoints, then $\beta \subseteq N(\alpha, r)$.*

Proof : Fix suitable $t > 4\lambda$ (see below).
If $\beta \subseteq N(\alpha, t)$, we are done.

If not, let γ_1 be a maximal segment of $\beta \setminus N(\alpha, t)$. It has endpoints, $x, y \in \partial N(\alpha, t)$.
Choose $z, w \in \alpha$ with $d(x, z) = d(y, w) = t$.
Let $\gamma_3 = [z, w] \subseteq \alpha$ and let $\gamma_2 = [x, z]$ and $\gamma_4 = [y, w]$.
Thus,

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$$

is a rectangle in M.

We want to bound length(γ_1).
Now let

$$u = \frac{1}{3} \operatorname{length} \gamma_1.$$

Since β is $(3, 0)$-quasigeodesic, we have $d(x, y) \geq u$.
It now follows that

$$d(\gamma_1, \gamma_3) \geq t$$

$$d(\gamma_2, \gamma_4) \geq u - 2t.$$

Thus, by (A2), we have

$$A(\gamma) \geq t(u - 2t).$$

Also,

$$L(\gamma) \leq 3u + t + t + (3u + 2t) = 6u + 4t.$$

So the isoperimetric bound gives:

$$t(u - 2t) \leq \lambda(6u + 4t) + k$$

so

$$(t - 6\lambda)u \leq 4\lambda t + 2t^2 + k.$$

We set $t = 7\lambda$, so this gives $u \leq u_0$, where $u_0 = (4\lambda t + 2t^2 + k)/\lambda = 126\lambda + k/\lambda$.

Now

$$\beta \subseteq N(\alpha, t + 3u/2)$$

so we may set $r = t + 3u_0/2$, which depends only on λ and k. $\diamondsuit$

From the conclusion of Lemma 7.3, we may deduce that every triangle has a centre as follows.

Let (α, β, γ) be a geodesic triangle in M.
Let z be the vertex opposite α, and let a be the nearest point to z on a.
Let x be the common vertex of α and β.

We claim that $[x, a] \cup [a, z]$ is $(3, 0)$-quasigeodesic.

To see this, suppose that $u \in [x, a]$ and $v \in [a, z]$.
Now $d(a, v) \leq d(u, v)$, and so

$$\begin{aligned} d(u, a) + d(a, v) &\leq (d(u, v) + d(v, a)) + d(a, v) \\ &= d(u, v) + 2d(a, v) \\ &\leq 3d(u, v) \end{aligned} .$$

In other words, the segment lying between u and v has length at most $3d(u, v)$.
But any other segment must lie in either of the geodesics $[x, a]$ or $[a, v]$, and so the claim follows.

But now, by Lemma 7.3, we see that a must lie a bounded distance from β.
Similarly, it is bounded distance from γ, and hence is a centre for (α, β, γ). By definition, it follows that M is hyperbolic.

This proves Theorem 7.1.

(The last part of the above argument comes from Masur and Minsky's paper.)