# MA4H4 Geometric Group Theory

## Exercise sheet 6 - Solutions

If there are any corrections, comments or questions please email `alex@wendland.org.uk`.

**Question 1** Let $(X, d)$ be a length space. We shall write $xy$ for $d(x, y)$. Recall the Gromov product for $x, y, z \in X$, write
$$\langle x, y\rangle_z = \frac{1}{2}(xz + yz - xy).$$

**Question 1a** Check the Gromov product is always non-negative. When does it vanish? (i.e. how are $x$, $y$ and $z$ positioned relative to each other)

The product is non-negative from the triangle inequality i.e. $xy \leq xz + yz$. It vanishes when $z$ lies on a geodesic between $x$ and $y$.

**Question 1b** What does the Gromov product represent when $X$ is a tree?

It represents how far $z$ is from the geodesic connecting $x$ and $y$ i.e. $d(z, \overleftrightarrow{xy})$.

**Question 2** Let $\Delta$ be a geodesic triangle on vertices $x$, $y$ and $z$ in a length space $X$. Define a "tripod" $T(\Delta)$ this is a metric tree with one vertex of degree 3 and three vertices of degree 1, and whose edge lengths are $\langle x, y\rangle_z$, $\langle y, z\rangle_x$ and $\langle z, x\rangle_y$. We will allow for degenerate cases where some of the edg lengths are zero. Let $O_\Delta$ be the central vertex of $T(\Delta)$. (See lecture notes for a diagram).

**Question 2a** Show that there exists a map $\chi_\Delta : \Delta \to T(\Delta)$ which is an isometry when restricted to each side of $\Delta$. The map is unique modulo isometries of $T(\Delta)$ to itself.

Map $x$ to the vertex of degree one whos length from $O_\Delta$ is $\langle y, z\rangle_x$ and similarly for $y$ and $z$ then the geodesics between them continuously to the unique geodesic between the images of the end points. The length between $\chi_\Delta(x)$ and $\chi_\Delta(y)$ is
$$\begin{aligned}\langle y, z\rangle_x + \langle z, x\rangle_y &= \frac{1}{2}(xy + xz - yz + yz + xy - xz)\\ &= xy\end{aligned}$$
giving that the map on the sides of $\Delta$ is an isometry. The map is unique by looking at the longest side of $\Delta$ (if there are two or more with equal side length then there is an isometry of $T(\Delta)$ switching these) then there is a unique geodesic in $T(\Delta)$ with this length forcing the rest of the choices.

**Question 2b** Show that $X$ is $k$-hyperbolic for some $k$ if and only if there exists $k'$ such that for any geodesic triangle $\Delta$ in $X$,
$$\operatorname{diam}(\chi_\Delta^{-1}(O_\Delta)) \leq k'.$$
If $\operatorname{diam}(\chi_\Delta^{-1}(O_\Delta)) \leq k'$ then any point in $\operatorname{diam}(\chi_\Delta^{-1}(O_\Delta))$ is a $k'$-center, therefore $X$ is $k'$-hyperbolic.

Suppose $X$ is $k$-hyperbolic, then let $m$ be a $k$-center of $\Delta$. There exists $a \in [x, y]$ such that $d(a, m) \leq k$ therefore $a$ is a $2k$-center of $\Delta$. Choose $b \in [x, y]$ such that $d(b, x) = \langle y, z\rangle_x$, it follows from the equivalence of projections in section 6.3 of the notes that $d(a, b) \leq 2k$. Similar $a$ and $b$ exists for $a', b' \in [x, z]$ and $a'', b'' \in [y, z]$ however as $\chi_\Delta^{-1}(O_\Delta) = \{b, b', b''\}$ the triangle inequality gives us
$$\operatorname{diam}(\chi_\Delta^{-1}(O_\Delta)) \leq 6k.$$

**Question 2c** We call a triangle $\Delta$ $k'' - thin$ if $\text{diam}(\chi_\Delta^{-1}(p)) \leq k''$ for all $p \in T(\Delta)$. Show that the condition in the previous part is equivalent to the following: there exists a $k''$ such that all geodesic triangles in $X$ are $k''$-thin.

Clearly if a triangle is $k'' - thin$ then $\text{diam}(\chi_\Delta^{-1}(O_\Delta)) \leq k''$.

Suppose our space is $k$-hyperbolic and $\text{diam}(\chi_\Delta^{-1}(O_\Delta)) \leq k'$. Then let $\Delta$ be a geodesic triangle with vertices $x$, $y$ and $z$ and without loss of generality suppose $p \in T(\Delta)$ lies in the side of length $< y, z >_x$ i.e. there exists hyperbolic triangle $x$, $r$ and $r'$ with points $q \in [x, r]$ and $q' \in [x, r']$ such that $xr = xr' = \langle y, z \rangle_x$, $xq = xq'$ as $\chi_\Delta^{-1}(p) = \{q, q'\}$ and $rr' \leq k'$ as $r, r' \in \chi_\Delta^{-1}(O_\Delta)$. The path $\beta$ which follows the geodesic $xr$ then $rr'$ is a $k'$-taught as $xr + rr' \leq xr' + k'$. Let $\alpha$ be the geodesic $xr'$ then we know from Lemma 6.4 $\beta \subset N(\alpha, 4.5k')$ therefore there is a point $t \in \alpha$ such that $qt \leq 4.5k'$. Then observe from the triangle inequality we have $xq \leq xt + 4.5k'$ and $xt \leq xq + 4.5k'$ giving $|xq - xt| \leq 4.5k'$ however as $xq = xq'$ we have $|xq' - xt \leq 4.5k'$ so by the triangle inequality $qq' \leq qt + |xq' - xt| = 9k'$. Giving that $\Delta$ is $9k'$-thin.

Note: The last implication isn't true for any triangle, implicit within Lemma 6.4 is that the $[x, r, r']$ triangle must have a $k'$-center, which is what we used from the space being $k'$-hyperbolic.

**Question 3** A geodesic triangle in a length space $X$ is called $k - slim$ if each side is contained in the $k$-neighbourhood of the union of the other two sides.

**Question 3a** Show that any $k$-thin triangle is also $k$-slim.

Suppose $\Delta$ is $k$-thin, so $\text{diam}(\chi_\Delta^{-1}(p)) \leq k$ for all $p \in T(\Delta)$. Let $q \in [x, y]$ be in the interior and find $\chi_\Delta(q) = p$. Then $\chi_\Delta^{-1}(p)$ contains $q$ and atleast one other point $q'$ in either $[x, z]$ or $[y, z]$. As $\Delta$ is $k$-thin $d(q, q') \leq k$ hence $\Delta$ is $k$-slim.

**Question 3b** Show that any $k$-slim triangle is $k'$-thin, where $k'$ depends only on $k$.

These argument is similar to that in 2c, except the use of lemma 6.4 is replaced by that of slimness. Suppose we have $k$-slim triangle $\Delta$ with vertices $x$, $y$ and $z$. Let $c_x \in [y, z]$, $c_y \in [x, z]$ and $c_z \in [x, y]$ be such that $\chi_\Delta^{-1}(O_\Delta) = \{c_x, c_y, c_z\}$ then as $\Delta$ is $k$-slim without loss of generality there exists $t \in [x, y]$ such that $d(c_x, t) \leq k$ then by the triangle inequality $|yt - yc_z| \leq k$ (same argument as 2c) giving us that $c_x c_z \leq 2k$. Applying this same argument with $c_y$ in place of $c_x$ we get that without loss of generality $c_x c_y \leq 2k$ giving us that $c_y c_z \leq 4k$.

Let $p \in T(\Delta)$ where $p \neq O_\Delta$ and without loss of generality suppose $p$ lies in the edge of length $< y, z >_x$. Let $q \in [x, c_y]$ and $q' \in [x, c_z]$ such that $xq = xq'$ where $\chi_\Delta^{-1}(p) = \{q, q'\}$. From slimness there exists a point $t \in [x, y] \cup [y, z]$ such that $pt \leq k$. From the triangle inequality we have $|xq' - xt \leq k$ so $qq' \leq qt + |xq' - xt| = 2k$. Giving that $\Delta$ is $4k$-thin.