MA4H4 Geometric Group Theory

Exercise sheet 5 - Solutions

If there are any corrections, comments or questions please email `alex@wendland.org.uk`.

Question 1 Show that a f.g. virtually Abelian groups is virtually $\mathbb{Z}^n$.

Let Γ be f.g. virtually abelian group, then it has a $G \leq \Gamma$ such that G is abelian and $[G : \Gamma] < \infty$, however this implies that G is f.g. therefore by the classification theorem $G \cong \mathbb{Z}^n \oplus_{i\in I} C_i$ with C_i some finite cyclic group but then $\mathbb{Z}^n \leq G$ of finite index therefore $\mathbb{Z}^n \leq \Gamma$ of finite index and Γ is virtually $\mathbb{Z}^n$.

Question 2 Fill in the details in the proof of the following theorem given in lecture: A groups that is q.i. to $\mathbb{Z}$ is virtually $\mathbb{Z}$.

To complete the proof we have to prove the following two exercises.

Exercise 1 If $f : \mathbb{Z} \to \mathbb{R}$ such that

1. for all n, $|f(n) - f(n+1)|$ is bounded, and
2. for all $r \geq 0$ there exists $p \in \mathbb{N}$ such that if $|f(n) - f(m)| \leq r$ then $|m - n| \leq p$.

then the image of f is cobounded in $\mathbb{R}$.

Let d be the bound from part (1) of the definition, then we know $C := B(f(\mathbb{Z}), d) \subset \mathbb{R}$ is an open interval from property (1). If C was bounded then there would exists an r such that $|f(n) - f(m)| \leq r$ for all n, m contradicting property (2) therefore $C = (a, \infty)$, $C = (-\infty, a)$ or $C = \mathbb{R}$ if it is the last case we are done, so without loss of generality lets assume the first case. Look at $f(n)$ (similarly for $f(-n)$) suppose $f(n)$ doesn't tend to ∞, then there exists a convergent subsequence $f(b_n) \to b$. From definition there exists an N such that for all $n, m > N$ $|f(b_n) - f(b_m)| < 1$ clearly contradicting property (2) as $b_n \to \infty$. So we have that $f(n)$ and $f(-n)$ both tend to infinity, therefore for sufficiently large n there exists $m \in \mathbb{N}$ such that $|f(n) - f(-m)| < d$ from property (1) however this contradicts property (2) as $|n + m|$ can be arbitrarily large. Therefore $C = \mathbb{R}$ and the image of f is cobounded.

Exercise 2 If ψ is a quasi-isometry of $\mathbb{R}$ then $\psi([0, \infty))$ lies bounded distance from either $[\psi(0), \infty)$ or $(-\infty, \psi(0)]$ (each point in either set lies a bounded distance from a point in the other set).

Assume ψ has constants k_i so that

$$k_1|x - y| - k_2 \leq |\psi(x) - \psi(y)| \leq k_3|x - y| + k_4.$$

Then as ψ is a q.i. we have that $\psi([a-1, a+1]) \subset B(\psi(a), (k_3 + k_4))$ therefore $C' := \psi([0, \infty)]) \subset B(\psi(\mathbb{N}), (k_3 + k_4)) =: C$ so it suffices to show C is bounded distance from $[\psi(0), \infty)$ or $(-\infty, \psi(0)]$. As $|\psi(a) - \psi(a+1)| \leq k_3 + k_4$ we get that $C \subset \mathbb{R}$ is an open interval, which can't be bounded as ψ is a q.i. therefore $C = [b, \infty)$, $C = (-\infty, b]$ or $C = \mathbb{R}$, if it is either of the first cases we are done. Suppose $C = \mathbb{R}$ this implies there exists subsequences x_n and y_n of $\mathbb{N}$ such that $\psi(x_n) \to \infty$ and $\psi(y_n) \to -\infty$. As $C = \mathbb{R}$ choose $T \in \mathbb{N}$ such that $|\psi(T)| \leq k_3 + k_4$ now choose $N > T$ such that $k_1|N - T| - k_2 \geq 2(k_3 + k_4) + 1$ then there exists $N < x_n < y_m$ such that $\psi(x_n) > 0$ and $\psi(y_m) < 0$. However as $|\psi(a) - \psi(a+1)| \leq k_3 + k_4$ there exists $x_n \leq M \leq y_m$ such that $|\psi(M)| \leq k_3 + k_4$ implying $|\psi(M) - \psi(T)| \leq 2(k_3 + k_4)$ however $|\psi(M) - \psi(T) \geq k_1|M - T| - k_2 \geq 2(k_3 + k_4) + 1$, therefore $C \neq \mathbb{R}$ and we are done.

Question 3a Give some examples of subgroups of F_2 isomorphic to F_n.

We have $F_n \cong \langle b^k a b^{-k} | 1 \leq k \leq n \rangle \leq F_2$.

Question 3b Show that is K is a finite graph then $\pi_1(K) \cong F_n$ where $n = E - V + 1$.

Take a finite connected graph K then let $T \subset K$ be a spanning tree (subgraph with no cycles, but still includes every vertex) then as $\pi_1(T)$ is trivial so we have that $\pi_1(K) = \pi_1(K/T)$ however K/T is a wedge of circles so $\pi_1(K/T) \cong F_{|E(K/T)|}$ however one can show that $|E(T)| = V(K) - 1$ giving that $|E(K/T)| = E - V + 1$.

Question 3c Show that a finite index subgroup of F_n is isomorphic to F_m for some $m \geq n$.

Let R_n signify the graph which contains one vertex and n edges. Then $\pi_1(R_n) = F_n$, then let $G \leq F_n$ be a group of finite index. Therefore G corresponds to a $[G : F_n]$-cover of R_n called C. However as C is a finite cover of a finite graph, C is also a finite graph. Therefore $\pi_1(C) \cong F_m$ however by the correspondence we know that also $F_m \cong \pi_1(C) \cong G$.

Question 4 Let $F_2 = \langle a, b \rangle$. Let $S = \{b^n a b^{-n} | n \in \mathbb{Z}\}$. Show that $T = \langle S \rangle$ is the normal closure of $\{a\}$ in F_2.

Note that $N := \langle\langle a \rangle\rangle = \langle waw^{-1} | w \in F_n \rangle$ therefore we get $T \leq N$ as S is a subset of the generators of N. However notice that elements of N contain as many b's as b^{-1}'s, let $n \in N$ then

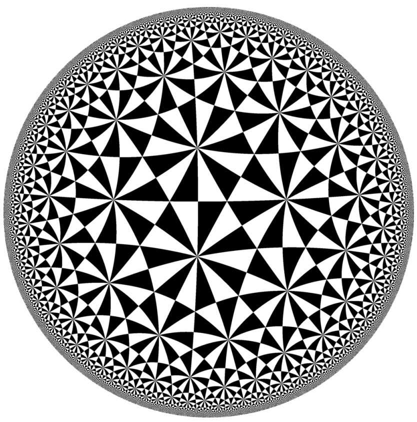
$$\begin{aligned}
n =& b^{b_1} a^{a_1} b^{b_2} a^{a_2} \ldots a^{a_{n-1}} b^{b_n} \\
=& (b^{b_1} a b^{-b_1}) b^{b_1} a^{a_1 - 1} \ldots a^{a_{n-1}} b^{b_n} \\
=& (b^{b_1} a b^{-b_1})(b^{b_1} a b^{-b_1}) b^{b_1} a^{a_1 - 2} \ldots a^{a_{n-1}} b^{b_n} \\
& \ldots \\
=& (b^{b_1} a b^{-b_1})^{a_1} b^{b_1 + b_2} a^{a_2} \ldots a^{a_{n-1}} b^{b_n} \\
& \ldots \\
=& (b^{b_1} a b^{-b_1})^{a_1} \ldots (b^{b_1 + \ldots + b_{n-1}} a b^{-(b_1 + \ldots + b_{n-1})})^{a_{n-1}} b^{b_1 + b_2 + \ldots + b_n}
\end{aligned}$$

however as there are as many b's as b^{-1}'s we get that $b_1 + b_2 + \ldots + b_n = 0$ therefore $n \in T$ and $N = T$.

Question 5 Recall the triangle group $\Delta(p, q, r)$ with presentation

$$\langle a, b, c | a^2, b^2, c^2, (ab)^p, (bc)^q, (ca)^r \rangle.$$

Question 5a Draw the Cayley graph of $\Delta(7, 3, 2)$.



Above is the tiling of the hyperbolic plane given by the group $\Delta(7,3,2)$, to realise the Cayley graph put a vertex in every triangle then connect two vertices who's relative triangles share an edge. Then notice that the corner of every triangle either meets 4, 6 or 14 triangles (including itself) in the Cayley graphs these relate to the relations $(ca)^4$, $(bc)^6$ and $(ab)^7$ respectively. Using these relations one can colour the Cayley graph uniquely.

Question 5b Decide for which triples (p,q,r) is $\delta(p,q,r)$ is q.i. to a point, a euclidean space, or a hyperbolic space. (Note this question may require some knowledge of basic hyperbolic geometry)

Answer the same to Sheet 2 Question 6b. If $1/p + 1/q + 1/r > 1$ we get it is q.i. to a point, as its Cayley graph is finite (embeds in a sphere) if $1/p + 1/q + 1/r = 1$ we get it is a tiling of the Euclidean space therefore q.i. to it and lastly $1/p + 1/q + 1/r < 1$ tiles the hyperbolic space.

Question 5c Describe a proper discontinuous action of $\Delta(p,q,r)$ by isometries on either the sphere, euclidean plane or hyperbolic plane, There should be examples where the fundamental domain is a triangle.

Answer the same to Sheet 2 Question 6c.