MA4H4 Geometric Group Theory

Exercise sheet 4 - Solutions

If there are any corrections, comments or questions please email `alex@wendland.org.uk`.

Question 1 Show that there is exactly one n-regular tree up to isomorphism.

Suppose we have a n-regular tree, T. Choose random basepoints $x_\emptyset$ then we can label all other the vertices with a finite sequence of integers between 1 and n where there is an edge between two vertices if and only if the sequence are of the form $(a_i)_{1\leq i\leq n}$ and $(a_i)_{1\leq i\leq n+1}$ i.e one is longer by one entry and they agree on the earlier position. We assign marking inductively on the distance away from $x_\emptyset$. We know $x_\emptyset$ has n neighbours so choose an ordering on them and label then x_i with $1 \leq i \leq n$. Suppose we have a labelling of all vertices at distance k then to vertex $x_{(a_i)_{1\leq i\leq k}}$ we know it has $n-1$ neighbours without a label as it is a tree therefore choose an arbitary ordering on them and label that $x_{(a_i)_{1\leq i\leq k+1}}$ where $1 \leq a_{k+1} \leq n-1$. This labels all vertices distance $k+1$ away, so we get such a labelling as T is a tree.

However as T was an arbitrary n-tree if we have another n-tree T' we can construct another such labelling $x'_{(a_i)_{1\leq i\leq k}}$ and map one to another by $x_{(a_i)_{1\leq i\leq k}} \mapsto x'_{(a_i)_{1\leq i\leq k}}$ which preserving edges and who's inverse is defined by $x'_{(a_i)_{1\leq i\leq k}} \mapsto x_{(a_i)_{1\leq i\leq k}}$ giving us an isomorphism.

Question 2 Show that $T_n \sim T_m$ if $m, n \geq 3$. In fact, any tree all of whose vertices have degree between 3 and n for some $n \geq 3$, then $T \sim T_3$. What could happen if we don't impose an upper bound on the degree of vertices?

We show the later statement, suppose T is a tree who's degree is bounded between 3 and n. Use the labelling above on the tree T, so every vertex is denoted by a finite sequence of integers between 1 and n and for vertex $x_{(a_i)_{1\leq i\leq k}}$ let $d_{(a_i)_{1\leq i\leq k}}$ denote it's degree. Then we are going to inductively define a map from $f : T_3 \mapsto T$, first arbitarily choose a path of length $d_\emptyset - 3$ in T_3 then map the path to vertex $x_\emptyset$, note that this path has $d_\emptyset$ edges protruding from it, arbitrarily map them to the edges of $x_\emptyset$. Suppose we have done this procedure for all vertices of sequence length at most k and for sequences of length k they only have a partial definition (i.e. the edge leading into them is defined). For $x_{(a_i)_{1\leq i\leq k}}$ choose a path of length $d_{(a_i)_{1\leq i\leq k}} - 3$ starting at the partially defined vertex in T_3 and not using the edge that has already been mapped under f. Then map this path to $x_{(a_i)_{1\leq i\leq k}}$ and map the $d_{(a_i)_{1\leq i\leq k}}$ protruding edges arbitrarily to the edges of $x_{(a_i)_{1\leq i\leq k}}$. Note that as T_3 is a tree these choices are always possible and never overlap. Then f is a surjective map so has quasi-dense image and as the path we are contracting are of length at most $n-3$ we get

$$d_{T_3}(x, y)/(n-2) - (n-3) \leq d_T(f(x), f(y)) \leq d_{T_3}(x, y).$$

Question 3 Show that $\mathbb{R}$ is not q.i to T_3.

Label T_3 as above and pick 3 paths originating from $x_\emptyset$, labelled x_{a_i}, x_{b_i} and x_{c_i} such that $a_1 = (1)$, $b_1 = (2)$ and $c_1 = (3)$ and that a_{i+1} is a sequence of length 1 larger than a_i, similarly for b and c. Suppose we have q.i. $\phi : T_3 \to \mathbb{R}$, with

$$k_1 d_{T_3}(x, y) - k_2 \leq d_{\mathbb{R}}(\phi(x), \phi(y)) \leq k_3 d_{T_3}(x, y) + k_4$$

and $N(f(T_3), k_5) = \mathbb{R}$. Then note that $d_{T_3}(x_\emptyset, x_{a_n}) = d_{T_3}(x_\emptyset, x_{b_n}) = d_{T_3}(x_\emptyset, x_{c_n}) = n$ so we have

$$k_1 n - k_2 \leq d_{\mathbb{R}}(x, \phi(x_{a_n})) \leq k_3 n + k_4$$

though also we have that

$$k_1 - k_2 \leq d_{\mathbb{R}}(\phi(x_{a_n}), \phi(x_{a_{n+1}})) \leq k_3 + k_4$$

so pick N large enough such that $k_1 N - k_2 > k_3 + k_4$ then for all $n > N$ we have that $\phi(x_{a_n}) > x$ or $\phi(x_{a_n}) < x$, but we get the same statement for b and c so two of which must be mapped to the same component of $\mathbb{R}\backslash\{x\}$. Without loss of generality suppose x_{a_n} and x_{b_n} for $n > N$ are both mapped to $[x, \infty)$. However as $d_{\mathbb{R}}(\phi(x_{a_n}), \phi(x_{a_{n+1}})) \leq k_3 + k_4$ any point in $[x, \infty)$ is at most $(k_3 + k_4)/2$ from some point $\phi(x_{a_n})$. Moreover, this tells me for all $m > N$ there exists $n \in \mathbb{N}$ such that $|\phi(x_{b_m}) - \phi(x_{a_n})| \leq (k_3 + k_4)/2$ however if we pick m large enough such that $k_1 m - k_2 > (k_3 + k_4)/2$ then we know that $|\phi(x_{b_m}) - \phi(x_{a_n})| \geq k_1(m + n) - k_2 > (k_3 + k_4)/2$ contradicting the fact that ϕ is a q.i., so no such ϕ exists.

Question 4 Show that $\mathbb{R}^2$ is not q.i. T_3 (Hint: Suppose $\phi : \mathbb{R}^2 \to T_3$ is a q.i. consider the image of a large equilateral triangle.)

Lemma 0.1 (Bridson and Haefliger - Metric Spaces of Non-Positive Curvature; Lemma 1.11 'Taming Quasi-Geodesics' p 403)**.** *Let X be a geodesic space $c : [a, b] \to X$ a (λ, ϵ)-q.i. embedding, one can find a continuous (λ', ϵ') q.i. embedding $c' : [a, b] \to X$ such that:*

1. *$c(a) = c'(a)$ and $c(b) = c'(b)$;*
2. *$\epsilon' = 2(\lambda + \epsilon)$;*
3. *length$(c'|_{t,t'}) \leq k_1 d(c'(t), c'(t')) + k_2$, for all $t, t' \in [a, b]$, where $k_1 = \lambda(\lambda+\epsilon)$ and $k_2 = (\lambda\epsilon' + 3)(\lambda+\epsilon)$;*
4. *the Hausdorf distance between the images of c and c' is less than $(\lambda + \epsilon)$.*

Proof. Define c' to agree with c on $\Sigma := \{a, b\} \cup (\mathbb{Z} \cap [a, b])$. Then choose geodesic segments joining the images of successive points in Σ and define c' by concatinating linear representations of these geodesic segments. Note that the length of each geodesic segment is at most $(\lambda+\epsilon)$. So every point of $im(c) \cup im(c')$ lies in a $(\lambda + \epsilon)/2$ neighbourhood of $c(\Sigma)$, thus (4) and (1) hold from definition.

For the rest of the proof, see Bridson, Haefliger Metric Spaces of Non-Positive Curvature, we will not need it for what is done here. $\square$

Suppose $\phi : \mathbb{R}^2 \to T_3$ is a *q.i.* with constants k_i as above. Then consider an equilateral triangle in $\mathbb{R}^2$ with corners A, B and C and geodesics $\alpha = [A, B]$, $\beta = [B, C]$ and $\gamma = [C, A]$ connecting them. Consider $\alpha_\phi := \phi \circ \alpha : I \to T_3$ by the lemma above we know there exists continuous map $\alpha' : I \to T_3$ such that $\alpha_\phi \subset N(\alpha', r)$ where r only depends on the constants in the q.i. ϕ, similarly for β and γ. One can see that any paths connecting three points in the tree must have at least one common point in the image, m. However as $\alpha_\phi \subset N(\alpha', r)$ there exists a point in the image of $\phi \circ \alpha$ with distance at most r from m, similarly for β and γ. However as ϕ is a q.i. there exists points on the image of α, β and γ such that

$$\begin{aligned} k_1 d_{\mathbb{R}}(x, y) - k_2 \leq& d_{T_3}(\phi(x), \phi(y)) && \text{as } \phi \text{ is a q.i.} \\ d_{\mathbb{R}}(x, y) \leq& \frac{2r}{k_1} + \frac{k_2}{k_1} && \text{as } d_{T_3}(\phi(x), \phi(y)) \leq d_{T_3}(\phi(x), m) + d_{T_3}(\phi(y), m) \leq 2r \end{aligned}$$

However r just relies on the constants of ϕ so just pick a equilateral triangle large enough such that the sides of the triangle has no such points.

Question 5 Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a proper continuous map. ("proper" means that $f^{-1}(K)$ is compact for all compact K.) Suppose there is some $k \geq 0$ such that for all $x \in \mathbb{R}^n$, diam$(f^{-1}(x)) \leq k$. Then f is surjective. (The idea of the proof is to extend f to a continuous map between the onepoint compactifications $f : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$, and using appropriate identifications of $\mathbb{R} \cup \{\infty\}$ with the sphere, S^n, we can apply the Borsuk-Ulam theorem to get a contradiction.)

Theorem 0.2 (BorsukUlam theorem)**.** *If $\phi : S^n \to \mathbb{R}^n$ is continuous then there exists $x \in S^n$ such that $\phi(x) = \phi(-x)$.*

Consider the one-point compactfication $\mathbb{R}^n \cup \{\infty\}$ where the open sets are U open in $\mathbb{R}^n$ and $(\mathbb{R}^n \backslash K) \cup \{\infty\}$ such that $K \subset \mathbb{R}^n$ compact. Extend f to $\hat{f} : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ by setting $\hat{f}(\infty) = \infty$. Note that $\hat{f}$ is continuous as $f^{-1}(K) = K'$ is compact so $f^{-1}(\mathbb{R}^n \backslash K) = \mathbb{R}^n \backslash K'$. Suppose f is not surjective, then we have a continuous map $\hat{f}' : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\} \backslash \{x\}$. However let $g : S^n \to \mathbb{R}^n \cup \{\infty\}$ and $h : \mathbb{R}^n \cup \{\infty\} \backslash \{x\} \to \mathbb{R}^n$ be homeomorphism.

So $h \circ \hat{f} \circ g : S^n \to \mathbb{R}^n$ is a continuous map, so by the BorsukUlam theorem it identifies two antipodal points of S^n, p and p'. We can choose the homeomorphism $g : S^n \to \mathbb{R}^n \cup \{\infty\}$ so that any pair of anti-podal points of S^n are arbitarily far apart in $\mathbb{R}^n \cup \infty$ namely larger that k. However $diam(f^{-1}(x)) \leq k$ contradicting what was assumed.

Question 6 Show that any quasi-isometric map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a quasi-isometry.

Notice that we can assume $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, to do this put a simplicial structure on $\mathbb{R}^n$ where the verticies are the lattice points $\mathbb{Z}^n$. Then if $\phi' : \mathbb{R}^n \to \mathbb{R}^n$ is not continuous define ϕ setting $\phi(z) = \phi'(z)$ for all $z \in \mathbb{Z}$ however map the higher dimensional simplicies by the linear extensions of the integer points. Then a point in the image of ϕ lies at most $(k_3 + k_4)^n$ distance from the image of an integer point of ϕ'. So it suffices to show ϕ is surjective. However look at $\phi^{-1}(K)$ where K is compact, therefore closed and bounded, as ϕ is continuous $\phi^{-1}(K)$ is closed and as ϕ is a quasi-isometry $\phi^{-1}(K)$ is bounded $(d_{\mathbb{R}^n}(\phi^{-1}(y), \phi^{-1}(y')) \leq d_{\mathbb{R}^n}(y, y')/k_1 + k_1/k_2)$ and therefore compact, so ϕ is proper. Now consider $\phi^{-1}(x)$, well if $\phi(y) = x = \phi(y')$ then $d(y, y') \leq k_2/k_1$ and so $diam(f^{-1}(x))$ is bounded. Therefore by the previous exercise ϕ is surjective and a quasi-isometry.

Question 7 Show that the relation commensurability of groups is transitive.

Suppose $\Gamma_1 \approx \Gamma_2$ and $\Gamma_2 \approx \Gamma_3$ then there exists $G' \leq \Gamma_1$, $G, H \leq \Gamma_2$ and $H' \leq \Gamma_3$ all of finite index such that $G \cong G'$ and $H \cong H'$. Then we claim that $G \cap H \leq \Gamma_2$ is of finite index. Let $x, y \in aH \cap bG$ then $x^{-1}y \in H$ since $x, y \in aH$ and $x^{-1}y \in G$ since $x, y \in bG$ hence x, y are in the same coset of $H \cap G$. Therefore there is at most $[\Gamma_2 : H][\Gamma_2 : G]$ cosets. Then there is a corresponding $H \cap G \cong N \leq G' \leq \Gamma_1$ of finite index and $H \cap G \cong N' \leq H' \leq \Gamma_3$ giving that $\Gamma_1 \approx \Gamma_3$.

Question 8 Let Γ be a group acting on a geodesic space X. The action is said to be quasi-convex if the orbits are quasi-convex.

Question 8a Show that any isometric $\mathbb{Z}$ action on $\mathbb{R}^2$ is quasi-convex. What about $\mathbb{Z}^n$ on $\mathbb{R}^m$?

Let $\mathbb{Z} = \langle g \rangle$ act on $\mathbb{R}^2$, let $r := \min\{d(x, gx) | x \in \mathbb{R}^2\}$ and $K := \{x \in \mathbb{R} | d(x, gx) = r\}$. We claim that K is a translation of a subspace of $\mathbb{R}^2$. Suppose $x, y \in K$ then note as g acts by isometries that for any point z in a line between x and y gets mapped to a point on the line between gx and gy thus mapped by r and so $z \in K$. Now take any point $x \in \mathbb{R}^2$ it is of finite distance away from K therefore as g acts by isometries any image of x lies the same distance, d, from K. However notice that the closest point to x on K, x' gets translated by some vector v of length r therefore $g^n x'$ all lie on a geodesic, with $g^n x'$ being the closest point to $g^n x$. For any two points $g^n x$ and $g^m x$ there exists a quasi-geodesic going from $g^n x$ to $g^n x'$ along to $g^m x'$ crossing all intermittent powers of $g^k x'$ then to $g^m x$ which is distance at most $k + r$ away from $\mathbb{Z}x$, however as quasi-geodesics are bounded distance b from geodesics we get that $[g^n x, g^m x]$ is a bounded distance $d + r + b$ from $\mathbb{Z}x$.

Notice what was done above didn't depend on $\mathbb{R}^2$ so could have been said for $\mathbb{R}^m$, also if $\mathbb{Z}^n = \langle g_1, \ldots, g_n \rangle$ we can define analogous $r_i := \min\{d(x, g_i x) | x \in \mathbb{R}^2\}$ and $K_i := \{x \in \mathbb{R} | d(x, g_i x) = r_i\}$ for each g_i and the K_i are still translated subspaces. One can also observe that $g_j K_i = K_i$ consider some point $g_j x \in g_j K_i$ then

$$\begin{aligned} d(g_j x, g_i g_j x) =& d(g_j x, g_j g_i x) && \text{as } g_i \text{ and } g_j \text{ commute} \\ =& d(x, g_i x) && \text{as } g_j \text{ acts by isometry} \\ =& r_i && \text{as } x \in K_i \end{aligned}$$

giving that $g_j K_i \subset K_i$ however $g_j K_i$ is a subspace of the same dimension so we get equality (Note: you can further show that without loss of generality $K_i \subset K_j$ or K_i and K_j are perpendicular). However as this holds for all i, j we know that points under the action any element $z \in \mathbb{Z}$ must stay the same distance away from all spaces K_i, so this is more of a restrictive condition on the orbit of any point, therefore let $d = \max_i d(x, K_i)$ and $r = \max_i r_i$ then every orbit is $(d+r)+b$-quasi-convex by the same reasons as above.

Question 8b Give an example of a quasi-convex action of $\mathbb{Z}$ on $\mathbb{H}^2$. Is every action on $\mathbb{H}^2$ quasi-convex?

Let $\mathbb{Z}$ act by any hyperbolic reflection otherwise a trivial action. No, consider a parabolic element (translation on a horoball).

Question 8c Assume X is proper and suppose the action of Γ by isometries on X is proper and quasi-convex. Prove that Γ is finitely generated and for any $x_0 \in X$, the map sending $y \in \Gamma$ to $yx_0 \in X$ is a quasi-isometric embedding. (Hint: If Γx_0 is r-quasi-convex, consider the set of nonidentity elements of Γ which move x_0 by at most $2r+1$.)

Let $S := \{g \in \Gamma | d_X(x_0, gx_0) \leq 2r+1\}$ which is finite as the group action is proper and so is the space (consider a ball around the point x_0 of radius $2r+1$). We want to show that S generates Γ, so consider some element of $\gamma \in \Gamma$, and look at a geodesic $\alpha := [x_0, \gamma x_0]$. We know that the action of Γ is r-quasi convex so $\alpha \subset N(\Gamma x_0, r)$. Choose $a_n \in \alpha$ such that $d(a_{n-1}, a_n) \leq 1$ with $a_0 = x_0$ and $a_N = \gamma x_0$ we know there exists a $\gamma_n \in \Gamma$ such that $d_X(\gamma_n x_0, a_n) \leq r$ with $\gamma_0 = 1$. Therefore

$$\begin{aligned} d_X(\gamma_{n-1}x_0, \gamma_n x_0) \leq & d_X(\gamma_{n-1}, a_{n-1}) + d_X(a_{n-1}, a_n) + d_X(a_n, \gamma_n x_0) \\ \leq & 2r+1 \end{aligned}$$

giving that $d_X(x_0, \gamma_{n-1}^{-1}\gamma_n x_0) \leq 2r+1$ and so $\gamma_{n-1}^{-1}\gamma_n \in S$ therefore we can write $\gamma = (\gamma_0^{-1}\gamma_1)(\gamma_1^{-1}\gamma_2)\dots(\gamma_{N_1}^{-1}\gamma_N)$ in terms of S. Notice that $G := \Delta(\Gamma, S)$ embeds in X with the map $f : \gamma \mapsto \gamma x_0$ where we map edges in linearly across geodesics connecting the vertices. Then set $m := min_{s \in S} d(x_0, sx_0)$ and we get

$$md_G(x, y) \leq d_X(f(x), f(y)) \leq (2r+1)d_G(x, y).$$