MA4H4 Geometric Group Theory

Exercise sheet 3 - Solutions

If there are any corrections, comments or questions please email `alex@wendland.org.uk`.

Question 1 Let d be the distance on $\mathbb{R}$ defined by $d(x,y) = |x-y|^p$ for $p > 0$. Show that this is a metric for $p \leq 1$. Show that this is a length space if and only if $p = 1$.

Let $p \leq 1$, then we get: positivity $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}$ by definition, the identity property

$$\begin{aligned} d(x,y) = 0 &\Leftrightarrow |x-y|^p = 0 \\ &\Leftrightarrow |x-y| = 0 \\ &\Leftrightarrow x = y, \end{aligned}$$

reflexivity

$$\begin{aligned} d(x,y) =& |x-y|^p \\ =& |-1|^p |y-x|^p \\ =& d(y,x), \end{aligned}$$

and lastly triangle inequality from

$$\begin{aligned} (d(x,y) + d(y,z))^{1/p} \geq& d(x,y)^{1/p} + d(y,z)^{1/p} && \text{as } 1/p \geq 1 \\ =& |x-y| + |y-z| \\ \geq& d(x,z)^{1/p} && \text{by triangle inqueality on } \mathbb{R} \end{aligned}$$

giving $d(x,y) + d(y,z) \geq d(x,z)$. This makes this a metric for $p \leq 1$.

If $p = 1$ then given x, y the path $\gamma : [0, |x-y|] \to \mathbb{R}$, $\gamma(t) = x + \frac{y-x}{|y-x|} t$ is a geodesic from x to y.

Suppose $p < 1$ and consider $0, 1 \in \mathbb{R}$ with $d(0,1) = 1$. Suppose $\gamma : [0,1] \to \mathbb{R}$ is a geodesic from 0 to 1. Then

$$\begin{aligned} 1 =& \sum_{i=1}^{n} (\gamma\left(\frac{i-1}{n}\right) - \gamma\left(\frac{i}{n}\right)) \\ \leq& \sum_{i=1}^{n} |\gamma\left(\frac{i-1}{n}\right) - \gamma\left(\frac{i}{n}\right)| \\ =& \sum_{i=1}^{n} d(\gamma\left(\frac{i-1}{n}\right), \gamma\left(\frac{i}{n}\right))^{1/p} \\ =& \sum_{i=1}^{n} \left|\frac{i-1}{n} - \frac{i}{n}\right|^{1/p} && \text{as } \gamma \text{ is a geodesic} \\ =& n\left(\frac{1}{n}\right)^{1/p} \\ <& 1 \end{aligned}$$

giving us a contradiction, therefore it is a length space if and only if $p = 1$.

Question 2 Let $\gamma : [a, b] \to X$ be a geodesic. Show that length$(\gamma) = d(\gamma(a)), \gamma(b))$.

Given $a = t_0 < t_1 < \ldots < t_n = b$

$$\begin{aligned}\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) =& \sum_{i=1}^{n} |t_{i-1} - t_i| && \text{for all } t, u \in [a, b]\ d(\gamma(t), \gamma(u)) = |t - u| \\ =& t_n - t_0 \\ =& d(\gamma(a), \gamma(b)).\end{aligned}$$

Hence length$(\gamma) = \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) | a = t_0 < t_1 < \ldots < t_n = b\} = d(\gamma(a), \gamma(b))$.

Question 3 Suppose that $\gamma : [a, b] \to X$ is a path. Prove that length$(\gamma) = d(\gamma(a)), \gamma(b))$ if and only if $d(\gamma(t)), \gamma(v)) = d(\gamma(t)), \gamma(u)) + d(\gamma(u)), \gamma(v))$ for all $t, u, v \in [a, b]$ with $t \le u \le v$ (call this property $*$). If γ is also injective, show that it can be reparameterised as a (unit speed) geodesic.

Assume property $*$, then

$$\begin{aligned}\text{length}(\gamma) =& \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) | a = t_0 < t_1 < \ldots < t_n = b\} \\ =& \sup\{d(\gamma(a), \gamma(b)) | a = t_0 < t_1 < \ldots < t_n = b\} && \text{by property } * \\ =& d(\gamma(a), \gamma(b))\end{aligned}$$

giving us what is required.

Suppose that for some $t, u, v \in [a, b]$ with $t \le u \le v$, we had $d(\gamma(t), \gamma(v)) < d(\gamma(t), \gamma(u)) + d(\gamma(u), \gamma(v))$ then

$$\begin{aligned}d(\gamma(a), \gamma(b)) \le& d(\gamma(a), \gamma(t)) + d(\gamma(t), \gamma(v)) + d(\gamma(v), \gamma(b)) \\ <& d(\gamma(a), \gamma(t)) + d(\gamma(t), \gamma(u)) + d(\gamma(u), \gamma(v)) + d(\gamma(v), \gamma(b)) \\ \le& \text{length}(\gamma).\end{aligned}$$

Hence if $d(\gamma(a), \gamma(b)) = \text{length}(\gamma)$ we have property $*$.

Let $s : [a, b] \to [0, d(a, b)]$, where $s(t) = d(\gamma(a), \gamma(t))$, s is surjective. Suppose that $s(t) = s(u)$ for $t \le u$, however by $*$

$$\begin{aligned}s(u) =& d(\gamma(a), \gamma(u)) \\ =& d(\gamma(a), \gamma(t)) + d(\gamma(t), \gamma(u)) \\ =& s(t) + d(\gamma(t), \gamma(u))\end{aligned}$$

giving that $t = u$ as γ is injective. Therefore s is a bijection, so set $\overline{\gamma} = \gamma \circ s^{-1} : [0, d(a, b)] \to X$. Let $t, u \in [0, d(a, b)]$ then

$$\begin{aligned}d(\overline{\gamma}(t), \overline{\gamma}(u)) =& d(\gamma(s^{-1}(t)), \gamma(s^{-1}(u))) && \text{definition of } \overline{\gamma} \\ =& |d(\gamma(a), \gamma(s^{-1}(t))) - d(\gamma(a), \gamma(s^{-1}(u)))| && \text{by } * \\ =& |t - u| && \text{definition of } s\end{aligned}$$

giving that $\overline{\gamma}$ is a unit speed geodesic.

Question 4 Show that a length space X is proper (complete and locally compact) if and only if all closed balls are compact.

Suppose that all closed balls in X are compact. X is locally compact since every $x \in X$ has a compact neighbourhood - the closed ball $\overline{B}(x, \epsilon)$. Suppose $(x_n)_{n\in\mathbb{N}}$ is a cauchy sequence, then there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have $d(x_n, x_m) < 1$ therefore $(x_n)_{n>N}$ is fully contained in $\overline{B}(x_{N+1}, 1)$ which is compact therefore contains the limit point of $(x_n)_{n\in\mathbb{N}}$ by the shift rule.

The opposite direction is called the Hopf-Rinow theorem. Suppose X is proper for some $x \in X$ it suffices to show that $\overline{B}(x, r)$ is compact for any r. Let $I = \{r \mid \overline{B}(x, r) \text{ is compact}\}$, I is then an interval containing atleast a small neighbourhood of 0 due to local compactness. Our goal is to show that I is open and closed relative to $[0, \infty)$ and therefore $I = [0, \infty)$ and we have that all balls are compact.

Suppose $r \in I$ then use local compactness on $y \in \overline{B}(x, r)$ to cover $\overline{B}(x, r)$ by finitely many open neighbourhoods $B(x_i, r_i)$ such that $\overline{B}(x_i, r_i)$ are compact, however these cover a ball $\overline{B}(x, r + \delta)$ with $\delta > 0$ which is therefore compact, showing $r + \delta \in I$. So we get that I is open.

To prove that I is closed suppose we have $[0, R) \subset I$ with $R > 0$. Then let $(y_n)_{n\in bN}$ be any sequence in $B(x, R)$. Let $(\epsilon_i)_{i\in\mathbb{N}}$ be a decreasing sequence which converges to zero and $\epsilon_i < R$. Then as X is a length space there exists $x^i_j \in \overline{B}(x, R - \epsilon_i)$ such that $d(x^i_j, y_j) \leq \epsilon_i$ (geodesic between x and y_j). From compactness of $\overline{B}(x, R - \epsilon_1)$ the sequence $(x^1_j)_{j\in\mathbb{N}}$ has a convergent subsequence $(x^1_{j(1,k)})_{k\in\mathbb{N}}$. Next define $j(i + 1, k)$ inductively as follows suppose we have $j(i, k)$ then $(x^{i+1}_{j(i,k)})_{k\in\mathbb{N}}$ has a convergent subsequence $(x^{i+1}_{j(i+1,k)})_{k\in\mathbb{N}}$ due to compactness of $\overline{B}(x, R - \epsilon_{i+1})$. Then set $j(k) = j(k, k)$, the sequence $(x^i_{j(k)})_{k\in\mathbb{N}}$ converges for all $i \in \mathbb{N}$, and we claim that $(y_{j(k)})_{k\in\mathbb{N}}$ is cauchy. Let $\epsilon > 0$ and pick $\epsilon_i < \epsilon/3$. Then for sufficiently large k, l we have $d(x^i_{j(k)}, x^i_{j(l)}) < \epsilon/3$. It follows that

$$\begin{aligned} d(y_{j(k)}, y_{j(l)}) \leq& d(y_{j(k)}, x^i_{j(k)}) + d(x^i_{j(k)}, x^i_{j(l)}) + d(x^i_{j(l)}, y_{j(l)}) \\ \leq& \epsilon_i + \epsilon/3 + \epsilon_i \\ \leq& \epsilon. \end{aligned}$$

Since X is complete we get that $(y_{j(k)})_{k\in\mathbb{N}}$ converges, giving that $\overline{B}(x, R)$ is compact therefore $[0, R] \subset I$. So I is open and closed therefore $I = [0, \infty)$ and every closed ball is compact.

Question 5 Suppose Γ acts by isometries on a proper length space X. Show that the following are equivalent:

1. The action is cocompact.
2. Some orbit is cobounded.
3. Every orbit is cobounded.

First show (1) $\Rightarrow$ (3). Consider the open cover of X/Γ given by $B([x], 1)$ for all $x \in X$, since the action is cocompact there is a finite subcover $\cup_{i=1}^n B([x_i], 1/2)$. Now take any point two points $x, y \in X$, then $[y]$ and $[x]$ are in one of the balls in our finite cover of X/Γ. Since the balls cover X/Γ to get to $[x]$ from $[y]$ involves at worst crossing all of the balls once, so $d([y], [x]) \leq n$. That is there exists $g \in \Gamma$ such that $d(y, gx) \leq n$, so the orbit of x is cobounded.

Note that (3) $\Rightarrow$ (2) is immediate, so we show (2) $\Rightarrow$ (1). Suppose that there exists $x_0 \in X$ and $R > 0$ such that for all $y \in X$ there exists $g \in \Gamma$ where $d(y, gx_0) < R$. This is equivalent to saying for each $y \in X$ there exists $g \in \Gamma$ such that $gy \in \overline{B}(x, R)$. Hence the image of $\overline{B}(x, R)$ under the quotient map is the whole space X/Γ. Let $\cup_{\alpha\in A} U_\alpha$ be an open cover of X/Γ, the preimage of U_α is an open cover of X which via restriction gives an open cover $\cup_{\alpha\in A} \overline{U_\alpha}$ of $\overline{B}(x, R)$. However as the space is proper we know $\overline{B}(x, R)$ is compact so we can choose a finite subcover $\cup_{i=1}^n \overline{U_{\alpha_i}}$ which when taking quotients again gives us a finite subcover of X/Γ namely $\cup_{i=1}^n U_{\alpha_i}$.

Question 6 Show that quasi-isometry is an equivalence relation.

Note it is clearly reflexive as the identity is a quasi-isometry with constants 1 and 0. Suppose we have $f : X \to Y$ and $g : Y \to Z$ quasi-isometries. Then we have the following

- $k_1 d_X(x, x') - k_2 \leq d_Y(f(x), f(x')) \leq k_3 d_X(x, x') + k_4$ with $N(f(X), k_5) = Y$, and
- $l_1 d_Y(y, y') - l_2 \leq d_Z(g(y), g(y')) \leq l_3 d_Y(y, y') + l_4$ with $N(g(Y), l_5) = Z$.

where combining these we get

$$l_1 k_1 d_X(x, x') - (l_1 k_2 + l_2) \leq d_Z(gf(x), gf(x')) \leq l_3 k_3 d_X(x, x') + (l_3 k_4 + l_4), \text{ with } N(fg(X), k_5 l_3 + l_4 + l_5) = Z.$$

So quasi-isometries are transitive, lastly show symmetric. Let $f : X \to Y$ be a quasi-isometry with constants as above. For all $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leq k_5$, choose such an x for each y and set $g(x) = y$. Then

$$\begin{aligned} d_Y(y, y') \leq& d_Y(y, f(x)) + d_Y(f(x), f(x')) + d_Y(f(x'), y') \\ \leq& k_3 d_x(g(y), g(y')) + (k_4 + 2k_5) \\ d_Y(y, y') \geq& d_Y(f(x), f(x')) - d_Y(y, f(x)) - d_Y(f(x'), y') \\ \geq& k_1 d_X(g(y), g(y')) - (k_2 + 2k_5) \end{aligned}$$

then rearanging this gives

$$\frac{1}{k_3} d_Y(y, y') - \frac{k_4 + 2k_5}{k_3} \leq d_X(g(y), g(y')) \leq \frac{1}{k_1} d_Y(y, y') + \frac{k_2 + 2k_5}{k_1}.$$

For any $x \in X$ $gf(x) = x' \in X$ such that $d_Y(f(x), f(x')) \leq k_5$

$$\begin{aligned} d_X(x, g(f(x))) =& d_X(x, x') \\ \leq& k_1 d_Y(f(x), f(x')) + k_2 \\ \leq& k_1 k_5 + k_2. \end{aligned}$$

Giving that g is a quasi-isometry and that we have symmetry.

Question 7 Show that the Cayley graphs of $\mathbb{Z}$ with respect to the generating sets $\{a, a^2\}$ and $\{a^2, a^3\}$ are quasi-isometric to $\mathbb{R}$.

First let $e_{a^k, a^{k+1}} \cong [0, 1]$ be the edge from the generator a connecting a^k to a^{k+1} and let $e_{a^k, a^{k+1}} : [0, 1] \to \Delta(\mathbb{Z}, \{a, a^2\}) =: \Gamma$ be the unit speed geodesic connecting a^k to a^{k+1}. Define map $f : \mathbb{R} \to \Gamma$ by $f(k + i) = e_{a^k, a^{k+1}}(i)$ where $k \in \mathbb{Z}$ and $i \in [0, 1]$. Then as the $e_{a^k, a^{k+1}}$ are unit speed geodesics we get that $d_\Gamma(f(x), f(y)) \leq d_\mathbb{R}(x, y)$ however also $1/2 d_\mathbb{R}(x, y) \leq d_\Gamma(f(x), f(y))$ as they can at worst be connected by a path $1/2|x - y|$ a^2 edges of length 1. Then $\Gamma = N(f(\mathbb{R}), 1/2)$ as the only elements of Γ not mapped to are the a^2 edges which are at most $1/2$ distance from a vertex.

Using similar notation as before let $\Gamma := \Delta(\mathbb{Z}, \{a^2, a^3\})$ and $e_{a^k, a^{k+t}} : [0, 1] \to \Gamma$ represent unit speed geodesics. Then map $f : \Gamma \to \mathbb{R}$ by the following $f(e_{a^k, a^{k+t}}(i)) = k + ti$. Note that this is surjective so $f(\Gamma) = \mathbb{R}$. To get the isometric inequality,

$$d_\Gamma(x, y) - 3 \leq d_\mathbb{R}(f(x), f(y)) \leq 3 d_\Gamma(x, y),$$

note as above the fastest the distances can change in comparison to $\mathbb{R}$ is by a factor of 3 coming from a^3 edges. To get the lower bound notice that going along a a^3 edge then going one and half ways through a^{-2} edge gets you back to the same point in $\mathbb{R}$ but not in Γ however this worst that happens.

Question 8 Let X be a geodesic space. A subset $Y \subset X$ is called r-quasi-convex if for all $x, y \in Y$, any geodesic from x to y (in X) is in the r-neighbourhood of Y. We simply say Y is quasi-convex if it is r-quasi-convex for some $r \geq 0$.

Question 8a Show that a finite subgroup of a f.g. group of Γ is quasi-convex inside any Cayley graph of Γ.

Let $F \leq \Gamma$ be a finite subgroup and S the generating set of Γ. Then take $m := max_{f \in F}|f|_S$, then

$$d_{\delta(\Gamma,S)}(f_1, f_2) \leq d_{\delta(\Gamma,S)}(f_1, 1) + d_{\delta(\Gamma,S)}(1, f_2) \leq 2m$$

so contained in $N(F, m)$.

Question 8b Show that this also holds for finite index subgroups of Γ.

Let Γ be generated by S with $G \leq \Gamma$ be of finite index and let the cosets be represented by $c_1G, \dots c_nG$ set $m = max_{1 \leq i \leq n}|c_i|_S$. Any element of $\gamma \in \Gamma$ can be written as $\gamma = c_i g$ where $d_{\Delta(\Gamma,S)}(\gamma, g) \leq m$. So we have that G is $m/2$-dense in $\Delta(\Gamma, s)$, giving that any geodesic must lie in $N(G, m/2)$.

Question 9a Consider $\mathbb{Z}^2$ with generating set $S_1 = \{(0,1), (1,0)\}$. Show that the subgroup generated by $\{(0,1)\}$ is quasi-convex in $\Delta(\mathbb{Z}^2, S_1)$.

Let $\Gamma_1 := \Delta(\mathbb{Z}^2, S_1)$ (note here that d_{Γ_1} could be realised by the induced metric on the latice when viewing it as a subspace of $\mathbb{R}^2$ with the l_1 metric). Observe that $d_{\Gamma_1}((0,a),(0,b)) = |b-a|$ with the only geodesic being realised by the straight line on the y-axis connecting them. This makes the subgroup $\langle (0,1) \rangle$ 1/2-quasi-dense.

Question 9b Is the subgroup G generated by $S = \{(1,1)\}$ quasi-convex in $\Delta(\mathbb{Z}^2, S_1)$?

Note that $d_{\Gamma_1}((a,a),(b,b)) = 2|b-a|$ but a geodesic can be realised in many different ways. Choose the triangle geodesic i.e the path $(a,a) - (a,b) - (b,b)$ then the point (a,b) is distance at least $|a-b|$ from G. Assuming it is R-quasi-convexity then examine the traingle geodesic from $(0,0)$ to $(\lceil R \rceil, \lceil R \rceil)$ however the point $(0, \lceil R \rceil)$ is distance $\lceil R \rceil$ away from G contradicting quasi-convexity.

Question 9c Let $S_2 = \{(0,1), (1,0), (1,1)\}$. Is G quasi-convex in $\Delta(\mathbb{Z}^2, S_2)$?

Let $\Gamma_2 = \Delta(\mathbb{Z}^2, S_2)$ then just like in question 9a $d_{\Gamma_2}((a,a),(b,b)) = |b-a|$ which is realised only by a geodesic which is the straight line from the generator $(1,1)$, therefore making it 1/2-quasi-convex.

Question 9d Is the natural inclusion a quasi-isometric embedding from $\Delta(G, S)$ to $\Delta(\mathbb{Z}^2, S_1)$ or $\Delta(\mathbb{Z}^2, S_2)$?

The natural map from $\Gamma_3 := \Delta(G, S)$ to Γ_2 is an inclusion map therefore an isometry. Suppose we have the closest point contraction mapping of $f : \Gamma_3 \to \Gamma_2$ then this is a quasi-isometry as

$$d_{\Gamma_3}(x,y) - 1 \leq d_{\Gamma_2}(f(x), f(y)) \leq 2d_{\Gamma_3}(x,y).$$

The lower inequality holds as you contract a length of at most 1 to a point, and the inequality comes from the calculations above.

Question 9e Show that $\Delta(\mathbb{Z}^2, S_1)$ and $\Delta(\mathbb{Z}^2, S_2)$ are quasi-isometric. Is quasi-convexity a property preserved under quasi-isometries?

Include $i : \Gamma_1 \to \Gamma_2$, then as Γ_1 is a subgraph of Γ_2 we have that $d_{\Gamma_1}(x, y) \leq d_{\Gamma_2}(i(x), i(y))$ and the quickest short cut that can be made by $(1, 1)$ halves the distance, so we get

$$d_{\Gamma_1}(x, y) \leq d_{\Gamma_2}(i(x), i(y)) \leq d_{\Gamma_1}(x, y)/2.$$

The image is 1/2-dense as the only thing not mapped to are edges of length 1, giving $N(i(\Gamma_1), 1/2) = \Gamma_2$.

Quasi-convexity is not a property preserved by quasi-isometries from this example.