MA4H4 Geometric Group Theory

Exercise sheet 2 - Solutions

If there are any corrections, comments or questions please email `alex@wendland.org.uk`.

Question 1 Let F_n be generated by the letters $\{x_1, x_2, \ldots, x_n\}$. Show that $[F_n, F_n]$ consists exactly of those elements whose reduced word representative contains an equal number of x_i's as x_i^{-1}'s, for each i.

Let $F_n' = \{w \in F_n | \; w$ contains an equal number of x_i's as x_i^{-1}'s, for each $i\}$. Note that any generator of $[F_n, F_n]$ has the property that it contains an equal number of x_i's as x_i^{-1}'s, for each i therefore $[F_n, F_n] \subset F_n'$. Observe that $w_1 w_2 = w_2 w_1 [w_1^{-1}, w_2^{-1}]$. Suppose we have some word $w \in F_n'$ then

$$\begin{aligned}
w &= w_{1,1} x_{k_1}^{\pm 1} w_{1,2} x_{k_1}^{\mp 1} \\
&= w_{1,1} w_{1,2} [w_{1,2}^{-1}, x_{k_1}^{\pm 1}] \\
&= w_{2,1} x_{k_2}^{\pm 1} w_{2,2} x_{k_2}^{\mp 1} [w_{1,2}^{-1}, x_{k_1}^{\pm 1}] \\
&= w_{2,1} w_{2,2} [w_{2,2}^{-1}, x_{k_2}^{\pm 1}][w_{1,2}^{-1}, x_{k_1}^{\pm 1}] \\
&\vdots \\
&= [w_{r,2}^{-1}, x_{k_r}^{\pm}] \ldots [w_{1,2}^{-1}, x_i^{\pm 1}].
\end{aligned}$$

Note we eventually get no letters on the right as there are an equal number of x_i's as x_i^{-1}'s allowing us to pair them in the above fashion. This gives $F_n' \subset [F_n, F_n]$, therefore $F_n' = [F_n, F_n]$ as required.

Question 2 Show that $F_n/[F_n, F_n] \cong \mathbb{Z}^n$.

Pick the standard basis of $\mathbb{Z}^n$ using notation e_i. Define homomorphism $\phi : F_n \to \mathbb{Z}^n$ by $\phi(x_i) = e_i$, this is well defined from the universal property of free groups and surjective by definition. Suppose $w \in F_n$ belongs to $ker(\phi)$, then for $\phi(w)$ to have a zero e_i coefficient w has to have as many x_i's as x_i^{-1}'s therefore $w \in [F_n, F_n]$. However also if $w \in [F_n, F_n]$ then $\phi(w) = \overline{0}$, therefore $ker(\phi) = [F_n, F_n]$ so by the first isomorphism theorem $F_n/[F_n, F_n] = \mathbb{Z}^n$.

Question 3 Let $S = \{x_i\}_{i \in \mathbb{N}}$ be a countably infinite set indexed by $\mathbb{N}$. Let R be the set of relations of the form $x_{i+1} = x_j x_i x_j^{-1}$, for $i, j \in \mathbb{N}$ with $j < i$. Let $T = \langle S | R \rangle$ be the "Thompson group". (This is actually Thompson group F - there are also Thompson's group T and V). Show that:

$$T = \langle x_0, x_1 | [x_0^{-1} x_1, x_0 x_1 x_0^{-1}] = [x_0^{-1} x_1, x_0^2 x_1 x_0^{-2}] = 1 \rangle$$

(Hint: First show that all the x_n, for $n \geq 2$, can be expressed in terms of x_0 and x_1. Write some expressions for x_3 and x_4 and conjugate the appropriate ones by x_0 or x_0^2 to show that the required commutativity relations hold. Finally, show that these imply the original relations). This shows that T is indeed finitely presented.

Lets call relation $x_{i+1} x_j = x_j x_i$ relation $R_{i,j}$. We show by induction that $x_n = x_0^{n-1} x_1 x_0^{1-n}$, note that for $n = 2$ this is a relation in R. Suppose it is true for $n = k$ then

$$\begin{aligned}
x_{k+1} &= x_0 x_k x_0^{-1} && \text{from relation } R_{k,0} \\
&= x_0 (x_0^{k-1} x_1 x_0^{1-k}) x_0^{-1} && \text{from the induction hypothesis} \\
&= x_0^k x_1 x_0^{-k}
\end{aligned}$$

giving it for $n = k + 1$.

We will show that the two relations given are equivalent to $R_{1,2}$ and $R_{1,3}$. Given

$$\begin{aligned}1 =& x_0^{-1}(x_1 x_k x_1^{-1} x_{k+1}^{-1})x_0 \\ =& x_0^{-1} x_1 x_0^{k-1} x_1 x_0^{k-1} x_1^{-1} x_0^{k} x_1^{-1} x_0^{-(k-1)} \\ =& (x_0^{-1}x_1)(x_0^{k-1}x_1x_0^{k-1})(x_0^{-1}x_1)^{-1}(x_0^{k-1}x_1x_0^{-(k-1)})^{-1} \\ =& [x_0^{-1}x_1, x_0^{k-1}x_1x_0^{k-1}]\end{aligned}$$

so when $k = 2, 3$ we get what is required. So now we require to show $R_{i,j}$ from just $R_{1,2}$, $R_{1,3}$ and that $x_{i+1} = x_0^i x_1 x_0^{-i}$. Note that $R_{i,0}$ are true from definition of x_i, $x_{i+1} = x_0 x_i x_0^{-1}$. Also note that if we have $R_{i,j}$ for $i > j > 0$ we get by conjugating by x_0^k, $R_{i+k,j+k}$, so it suffices to show $R_{k,1}$ for higher $k > 3$. Show this by induction, we have it for $k = 3$, so lets assume it for all $k < n$. Then

$$\begin{aligned} x_2 x_n x_1 &= x_{n-1} x_2 x_1 && \text{using } R_{n-1,2} \\ &= x_{n-1} x_1 x_3 && \text{using } R_{2,1} \\ &= x_1 x_n x_3 && \text{using } R_{n-1,1} \\ &= x_1 x_3 x_{n+1} && \text{using } R_{n,3} \\ &= x_2 x_1 x_{n+1} && \text{using } R_{2,1} \end{aligned}$$

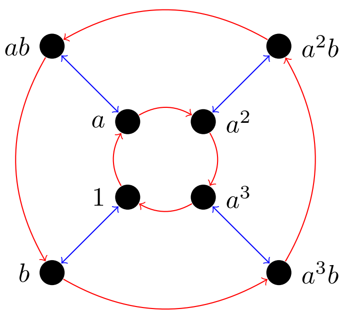
which by right multiplication of x_2^{-1} gives us $R_{n,1}$.

Question 4 Draw the Cayley graph of the dihedral group D_n with presentation

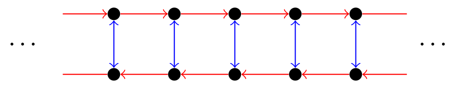
$$a, b | a^n = b^2 = (ab)^2 = 1\rangle$$

and the infinite dihedral group D_∞ with presentation

$$\langle a, b | b^2 = (ab)^2 = 1\rangle.$$

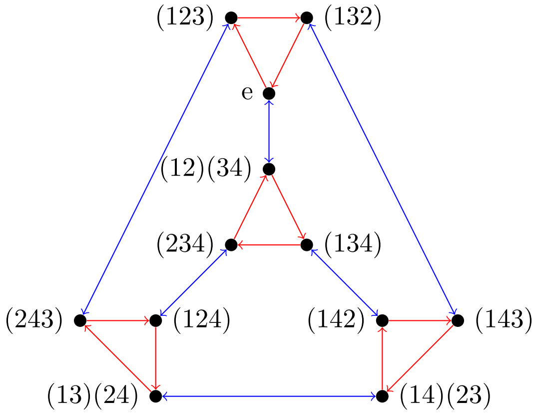


Cay$\langle a, b | a^4, b^2, (ab)^2\rangle$



Cay$\langle a, b | b^2, (ab)^2\rangle$

Question 5 Draw the Cayley graph of the alternating group A_4 with generators (123) and (12)(34).

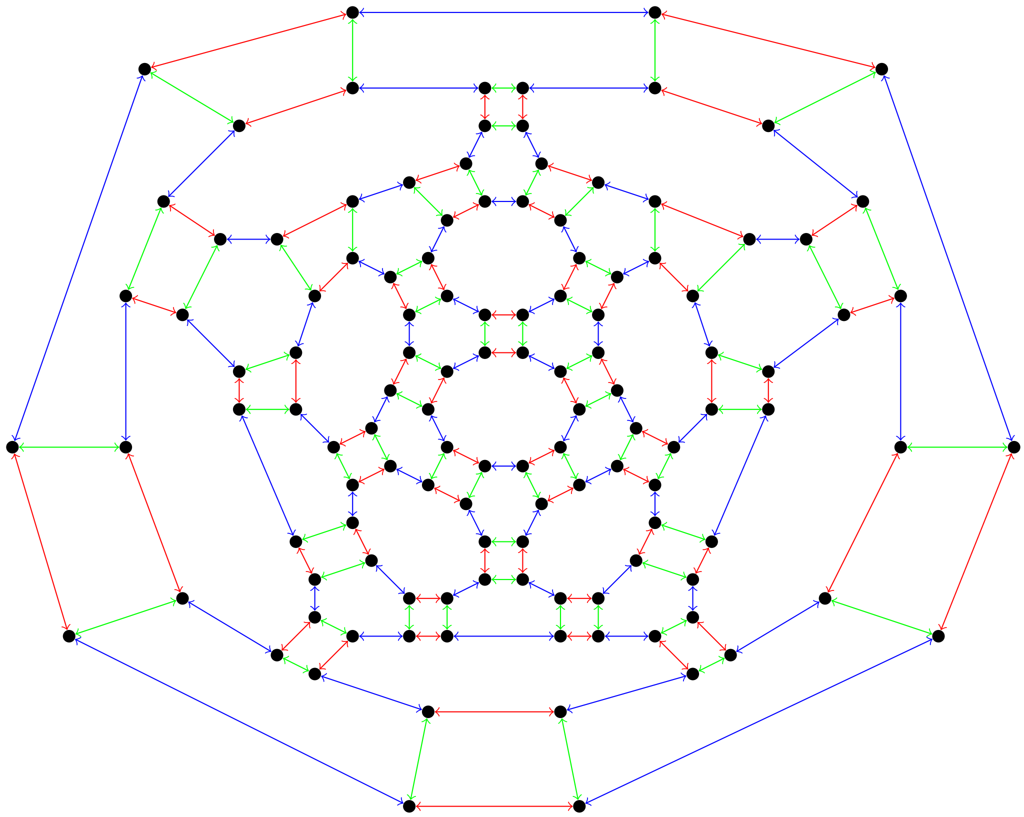


$\langle a = (123), b = (12)(34) | a^3, b^2, (ab)^3 \rangle$

Question 6 The triangle group $\Delta(p, q, r)$ has presentation

$$\langle a, b, c | a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle.$$

Question 6a Draw the Cayley graph of the icosahedral group $\Delta(2, 3, 5)$.



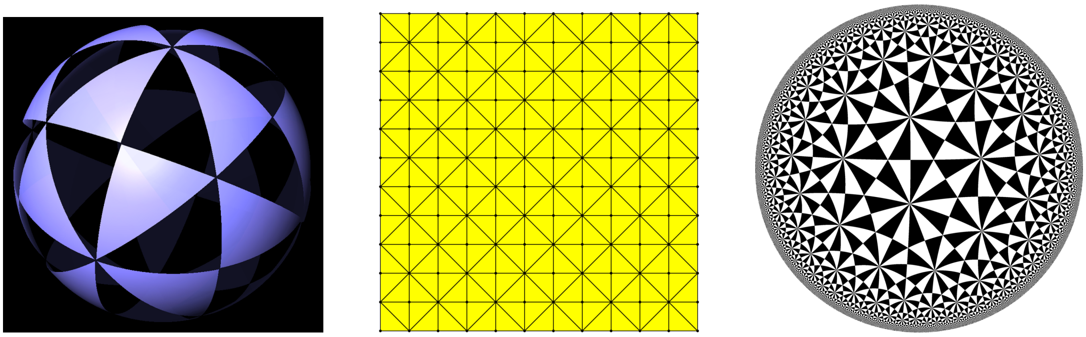
$\langle a, b, c | a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ca)^5 = 1 \rangle$

Question 6b Classify the triples (p, q, r) for which $1/p + 1/q + 1/r = 1$. What do their Cayley graphs look like?

Assuming $p \leq q \leq r$ one can show that the only triples are $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$. Their Cayley graphs are tilings of $\mathbb{E}^2$.

Question 6c Try to find a relationship between the Cayley graph of $\Delta(p,q,r)$ and a tessellation of a suitable space with congruent triangles. Why should the traingle groups with $1/p+1/q+1/r>1$ be finite?

Take triangles with angles π/p, π/q and π/r at vertices x_p, x_q and x_r. Let a act on our triangle by reflection over the x_px_r edge, b over the x_px_q edge and c over the x_qx_r edge. Use this action to tile a surface note that the relations hold as (ab) acts by rotation around vertex x_p by $2\pi/p$ and similar for other products. We get that if $1/p+1/q+1/r>1$ we get positive curvature so tile a sphere, $1/p+1/q+1/r=1$ we get zero curvature so tile a euclidean plane and $1/p+1/q+1/r<1$ negative curvature so tile the hyperbolic plane.



Tilings associated to $\Delta(2,3,4)$, $\Delta(2,4,4)$ and $\Delta(2,4,7)$,
find more at: `https://en.wikipedia.org/wiki/Triangle_group`.

Question 7 Let S be a finite generating set for G and suppose $S' \subset S$. We can form a subgraph $\Delta(G;S') \subset \Delta(G;S)$. Describe the connected components of $\Delta(G;S')$.

Let $G' = \langle S' \rangle \leq G$ then each connected component of $\Delta(G;S')$ is isomorphic to $\Delta(G';S')$ and the connected component are in bijection with the cosets $[G:G']$. This is because if $x,y \in xG'$ then $x = yg'$ where g' can be written in terms of S' therefore are connected by a path. Equally if x and y are connected then from following such a path we get that $x = yg'$ with $g' \in G'$.

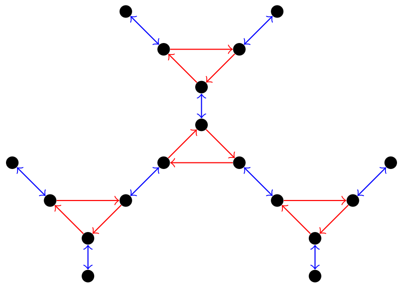
Question 8 Let $G_1 = \langle S_1 | R_1 \rangle$ and $G_2 = \langle S_2 | R_2 \rangle$ (with S_1 and S_2 disjoint). The free product $G_1 * G_2$ has presentation $\langle S_1 \cup S_2 | R_1 \cup R_2 \rangle$. (Check the the definition doesn't actually depend on the presentation of G_1 and G_2).

Given a different presentation $G_1 = \langle S_1' | R_1 \rangle$ there exists a maps $\phi : F_{S_1} \to F_{S_1'}$ and $\phi' : F_{S_1'} \to F_{S_1}$ such that they are inverses and notably $\phi(r) \in \langle\langle R_1' \rangle\rangle$ and $\phi'(r') \in \langle\langle R_1 \rangle\rangle$ for all $r' \in R'$ and $r \in R$. Then map $\phi^* : \langle \S_1 \cup S_2 | R_1 \cup R_2 \rangle \to \langle S_1' \cup S_2 | R_1' \cup R_2 \rangle$ by $\phi_*(s_1) = \phi(s_1)$ and $\phi^*(s_2) = s_2$ for $s_1 \in S_1$ and $s_2 \in S_2$ and ϕ_*' similarly. These are well defined homomorphisms by what is above and inverses to one another, giving us an isomorphism.

Question 8a What is $\mathbb{Z} * \mathbb{Z}$?

Given the standard presentation of $\mathbb{Z}$ we have $\mathbb{Z} * \mathbb{Z} = \langle a, b | \emptyset \rangle = F_2$.

Question 8b Draw the Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$ (using their standard presentations).



$\langle a, b | a^3, b^2 \rangle$

Question 8c Describe (informally) the Cayley graph of a free product. (You may take the factor subgroups to be finite for concreteness).

For $G_1 * G_2$ we have $\mathrm{Cay}(G_1 * G_2, S_1)$ and $\mathrm{Cay}(G_1 * G_2, S_2$ both look like disjoint unions of $\mathrm{Cay}(G_i, S_i$ when considering $\mathrm{Cay}(G_1 * G_2, S_1 \cup S_2)$ these disjoint copies form a tree like structure where on copy of $\mathrm{Cay}(G_1, S_1)$ has a single copy of $\mathrm{Cay}(G_2, S_2)$ attached to each vertex however this branches in a tree like manner to never connect up.

Question 8d Discuss whether the following statement should be true: every element of $G_1 * G_2$ can be written uniquely as an alternating product of non-trivial elements of G_1 and G_2.

This is true because of this tree like structure if an element could be written in two forms alternating forms, this would give us a relationship between the two groups which would contradict the original definition.