MA4H4 Geometric Group Theory

Exercise sheet 1 - Solutions

If there are any corrections, comments or questions please email `alex@wendland.org.uk`.

Question 1 Show that $\mathbb{Z}^n \cong \mathbb{Z}^m$ implies $n = m$.

We know these are f.g. groups where the standard basis form a minimal generating set (for a contradiction take a smaller one and expand to $\mathbb{R}^n$, use linear algebra), therefore by mapping under the given isomorphism form a generating set in the other group, giving $n \leq m$ and $m \leq n$ therefore $n = m$.

Question 2 Show that a finitely generated torsion-free abelian group is isomorphic to $\mathbb{Z}^n$, for some $n \geq 0$.

By the fundamental theorem of finitely generated abelian groups we have it is of the form $\mathbb{Z}^n \oplus_{i=1}^m \mathbb{Z}/k_i\mathbb{Z}$ however as it is torsion free there are no $\mathbb{Z}/k_i\mathbb{Z}$ factors. Therefore we get it is of the form $\mathbb{Z}^n$.

Question 3 Prove that any finitely generated subgroup of $(\mathbb{Q}, +)$ is either trivial or infinite cyclic.

Suppose $S \leq \mathbb{Q}$ is non-trivial finitely generated subgroup. Prove the statement by induction on the number of generators.

Suppose S is generated by n_1/d_1 and n_2/d_2 where n_i and d_i are coprime positive integers (we can assume the generators are positive as multiplication by the unit -1 doesn't change the subgroup they form). Let $Q = gcd(d_1, d_2)$, $L = lcm(d_1, d_2) = d_1d_2/Q$ and $D = gcd(n_1d_2/Q, n_2d_1/Q)$ then by the Euclidean algorithm we can write $D = an_1d_2/Q + bn_2d_1/Q$ for some $a, b \in \mathbb{Z}$. The we claim $S = \langle D/L \rangle$, first note that

$$\begin{aligned} D/L =& (an_1d_2/Q + bn_2d_1/Q)/(d_1d_2/Q) \\ =& an_1/d_1 + bn_2/d_2 \in S. \end{aligned}$$

Lastly that for a generic $\alpha, \beta \in \mathbb{Z}$ we have that

$$\begin{aligned} \alpha n_1/d_1 + \beta n_2/d_2 =& (\alpha n_1d_2/Q + \beta n_2d_1/Q)/(d_1d_2/Q) \\ =& \zeta D/L \in \langle D/L \rangle. \end{aligned}$$

Proving the statement for 2 generated subgroups. Assuming the $k-1$ case and showing it for k generators just apply the inductive assumption to the first $k-1$ generators then the 2 generator case.

Question 4a If $N \triangleleft \Gamma$ and Γ is f.g., then Γ/N is f.g.

Suppose $g_1, \dots g_n \in \Gamma$ generate, then there images in $\overline{g_1}, \dots \overline{g_n} \in \Gamma/N$ still generate, therefore Γ/N is f.g.

Question 4b If $N \triangleleft \Gamma$, N is f.g. and Γ/N is f.g., then Γ is f.g.

Let $n_1, \dots, n_k$ generate N and $\overline{g_1}, \dots, \overline{g_n}$ generate Γ/N. Pick any representative $g_i \in G$ of $\overline{g_i}$ then every element in Γ can be written as $gn \in gN$ where g is a word in g_i and $n \in N$ so a word in n_i therefore $n_1, \dots, n_k, g_1, \dots, g_n$ generate.

Question 4c If $H \leq \Gamma$ and $[\Gamma : H] < \infty$ ("finite index") then Γ is f.g. if and only if H is f.g.

If H is f.g. by $h_1, \ldots, h_n$ and Γ/H has cosets $c_1H, \ldots, c_mH$ then $h_1, \ldots, h_n, c_1, \ldots, c_m$ generate by a similar reason to 4b.

Suppose Γ is f.g. by $g_1, \ldots, g_n$ where this is a symmetric generating set and let Γ/H have cosets $c_1H, \ldots, c_mH$ where $c_1 = 1$. For each $1 \leq i \leq m$ and $1 \leq j \leq n$, define $h_{i,j} \in H$ to be such that $c_i g_j = h_{i,j} c_{k_{i,j}}$ for some $k_{i,j}$. Then for any $h \in H$ this can be written as $h = g_{i_1} \ldots g_{i_t}$ in the generators of Γ. Then

$$\begin{aligned}
g_{i_1} \cdots g_{i_t} =& c_1 g_{i_1} \cdots g_{i_t} \\
=& h_{1,i_1} c_{k_{1,i_1}} g_{i_2} \cdots g_{i_t} \\
& \vdots \\
=& (\text{product of } h_{i,j}s) \cdot c_*.
\end{aligned}$$

However $c_* = 1$ as $h \in H$, therefore H is generated by $h_{1,1} \ldots h_{m,n}$.

Question 5 Show that if $F(B)$ is f.g. then B is also finite.

Suppose $F(B)$ is f.g. by $x_1, \ldots x_n$ then write each x_i as words in B which only takes finite letters $b_1, \ldots b_m$. Suppose there was a $b \in B \backslash \{b_1, \ldots b_n\}$ then $b \in F(B)$ so can be written as a word in $x_1, \ldots x_m$ and therefore a reduced word in $b_1 \ldots b_n$ lets call w. However, $b^{-1}w$ is a reduced word in $F(B)$ but is equal to the identity, contradicting the definition of the free group and the existent of b therefore B is finite.

Question 6 Prove that every element in $F(S)$ can be written uniquely as a reduced word in $S \cup S^{-1}$.

This will be solved geometrically later in the course, see introduction to topology or presentations of groups for a rigorous algebraic proof. Intuitively, a word in $F(S)$ corresponds to a path on a Rose graph (this has one vertex and a loop for every generator in S), then the statement is equivalent to saying this has a unique homotopic representative of shortest length.

Question 7 Let $F_2 = \langle a, b \rangle$. Let $S = \{b^n a b^{-n} | n \in \mathbb{N}\}$. Show (combinatorially) that $\langle S \rangle$ is freely generated by S. Deduce that $\langle S \rangle$ is not f.g. (Note that $\langle S \rangle$is isomorphic to $F(S)$, since they are both freely generated by S.)

Define $x_n = b^n a b^{-n}$. Suppose w is a word in S such that w represents the identity. Note that $x_n^k = b^n a^k n^{-n}$, for $k \in \mathbb{Z}$. Write

$$\begin{aligned}
w =& x_{i_1}^{k_1} x_{i_2}^{k_2} \ldots x_{i_r}^{k_r}, & i_j \neq i_{j+1} \text{ and } k_j \neq 0 \\
=& b^{i_1} a^{k_1} b^{i_2 - i_1} a^{k_2} b^{i_3 - i_2} \ldots b^{i_r - i_{r-1}} a^{k_r} b^{-i_r}.
\end{aligned}$$

Since $i_j \neq i_{j+1}$, no $i_{j+1} - i_j = 0$ so this is a reduced word in F_2 and empty if and only if w is. Hence $\langle S \rangle$ is freely generated by S so not f.g. by Question 5.

Remark: A subgroup of a finitely generated group need not be finitely generated.

Remark: Let $T \subset \mathbb{N}$ this proof also verifies that if $S = \{b^n a b^{-n} | n \in T\}$ then $\langle S \rangle$ is freely generated by S showing that $F_n \leq F_2$ for all $n \in \mathbb{N}$.

Question 8 Show that there are only countably many f.p. groups up to isomorphism.

Note that if there are countably many finite presentations using n generators then there are countably many finite presentations, as a countable number of countable sets is countable. So consider the set of words in $x_1, x_2, \ldots, x_n$ and the set of subsets of finite size, this is countable as the set of finite subsets of a countable set is countable. Therefore there are countably many f.p. of groups and so countably many up to isomorphism.

Question 9 Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ be matrices in SL(2,$\mathbb{R}$). Let $X = \{(x, y) \in \mathbb{R}^2 | \ |x| > |y|\}$ and $Y = \{(x, y) \in \mathbb{R}^2 | \ |x| < |y|\}$.

Question 9a Describe the sets $A^n(Y)$ and $B^n(X)$ for $n \neq 0$. How do these sets relate to X and Y?

Note that,

$$A^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2ny \\ y \end{pmatrix}, \text{ and } B^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + 2nx \end{pmatrix}.$$

So if $(x, y) \in Y$ then $|y| > |x|$ giving $|x + 2ny| > |y|$ for all $n \neq 0$, hence $A^n(Y) \subset X$. Similarly $B^n(X) \subset Y$.

Question 9b Show that

$$W = A^{n_1} B^{n_2} A^{n_3} \ldots A^{n_{k-2}} B^{n_{k-1}} A^{n_k}$$

is not the identity, where $k \geq 1$ and $n_i \neq 0$ for all i. (Hint: Look at $W(Y)$).

From above

$$\begin{aligned} A^{n_k}(Y) &\subset X \\ B^{n_{k-1}} A^{n_k}(Y) &\subset Y \\ &\vdots \\ A^{n_1} B^{n_2} A^{n_3} \ldots A^{n_{k-2}} B^{n_{k-1}} A^{n_k}(Y) &\subset X \\ w(Y) &\subset X \end{aligned}$$

Since X and Y are disjoint, this implies w is not the identity.

Question 9c Deduce that all non-trivial reduced words in A and B do not represent the identity. (There are 4 cases: the words must begin with either an A or B and end with either an A or B).

Case 1, it starts and ends in A. This is done it part (b).

Case 2, it starts and ends in B. This is done similarly to part (b).

Case 3, it starts in A and ends in B. Let $w = A^{n_1} B^{n_2} \ldots A^{n_{k-1}} B^{n_k}$ and suppose $w = id$ then $A^{n_1} w A^{-n_1} = A^{2n_1} B^{n_2} \ldots A^{n_{k-1}} B^{n_k} A^{-n_1} = id$ which reduces to case 1.

Case 4, it starts in B and ends in A. It is conjugate to case 1 by similar reasons to above.

Question 9d Hence, show that A and B generate a free subgroup of rank 2 in SL(2,$\mathbb{R}$).

From 9c no non-empty reduced word in $\{A, B\}$ can represent the identity element, so A and B freely generate a free subgroup of rank 2 in $SL(2, \mathbb{R})$.

Further Question: This is an example of the table tennis lemma or (ping-pong lemma) - a method often used in geometric group theory to prove that a certain set of elements of a group freely generate a free subgroup. Formulate a statement and prove it. Try to generalise to the case of n generators.

Ping Pong Lemma Let G act on X, and $g_1, \dots, g_n \in G$. Let $X_1, \dots, X_n$ be disjoint, non-empty subsets of X such that for all i and $j \neq i$, $g_i^k(X_j) \subset X_i$ for all $k \neq 0$. Then $\langle g_1, \dots, g_n \rangle \leq G$ is free of rank n.

Let w be a reduced word starting and ending with g_1. For all $x \in X_2$, $w(x) \in X_1$ since $X_1 \cap X_2 = \emptyset$ we have that w is a non-trivial element of G. Now any other non-empty reduced word in the g_i is conjugate to a word starting and ending in g_1 therefore is also non-trivial, giving what is required.