MA4H4 Geometric Group Theory

Exercise sheet 6

May 19, 2011

1. Let (X, d) be a length space. We shall write xy for $d(x, y)$. Recall the Gromov product: for $x, y, z \in X$, write
$$\langle x, y \rangle_z = \frac{1}{2}(xz + yz - xy).$$
   (a) Check that the Gromov product is always non-negative. When does it vanish? (i.e. how are x, y and z positioned relative to one another).

   (b) What does the Gromov product represent if X is a tree?

2. Let Δ be geodesic triangle on vertices x, y, z in a length space X. Define a "tripod" $T(\Delta)$: this is a metric tree with one vertex of degree 3 and three vertices of degree 1, and whose edge lengths are $\langle x, y \rangle_z$, $\langle y, z \rangle_x$ and $\langle z, x \rangle_y$. We will allow for degenerate cases where some of the edge lengths are zero. Let O_Δ be the central vertex of $T(\Delta)$. (See lecture notes for a diagram).

   (a) Show that there exists a map $\chi_\Delta : \Delta \to T(\Delta)$ which is an isometry when restricted to each side of Δ. The map is unique modulo isometries of $T(\Delta)$ to itself.

   (b) Show that X is k-hyperbolic for some k if and only if there exists k' such that for any geodesic triangle Δ in X,
$$\mathrm{diam}(\chi_\Delta^{-1}(O_\Delta)) \leq k'.$$

(c) We call a triangle Δ k''-*thin* if $\mathrm{diam}(\chi_\Delta^{-1}(p)) \leq k''$ for all $p \in T(\Delta)$. Show that the condition in the previous part is equivalent to the following: there exists a k'' such that all geodesic triangles in X are k''-thin.

3. A geodesic triangle in a length space X is called k-*slim* if each side is contained in the k-neighbourhood of the union of the other two sides.

   (a) Show that any k-thin triangle is also k-slim.

   (b) Show than any k-slim triangle is k'-thin, where k' depends only on k.