MA4H4 Geometric Group Theory

Exercise sheet 5

May 19, 2011

1. Show that a f.g. virtually Abelian group is virtually $\mathbb{Z}^n$.

2. Fill in the details in the proof of the following theorem given in lecture: show that if a group is q.i. to $\mathbb{Z}$ then it is virtually $\mathbb{Z}$.

3. (a) Give some examples of subgroups of F_2 isomorphic to F_n.

   (b) Show that if K is a finite graph then $\pi_1(K) \cong F_n$ where $n = E - V + 1$.

   (c) Show that a finite index subgroup of F_n is isomorphic to F_m for some $m \geq n$.

4. Let $F_2 = \langle a, b \rangle$. Let $S = \{b^n a b^{-n} \mid n \in \mathbb{Z}\}$. Show that $\langle S \rangle$ is the normal closure of $\{a\}$ in F_2.

5. Recall the triangle group $\Delta(p, q, r)$ with presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle.$$

   (a) Draw that Cayley graph of $\Delta(7, 3, 2)$.

   (b) Decide for which triples (p, q, r) is $\Delta(p, q, r)$ q.i. to a point, a euclidean space, or hyperbolic space. (Note: This question may require some knowledge of basic hyperbolic geometry).

   (c) Describe a proper discontinuous action of $\Delta(p, q, r)$ by isometries on either the sphere, euclidean plane or hyperbolic plane. There should be examples where the fundamental domain is a triangle.