MA4H4 Geometric Group Theory

Exercise sheet 4

May 19, 2011

1. Show that there is exactly one n-regular tree up to isomorphism for any $n \in \mathbb{N}$.

2. Show that $T_n \sim T_m$ if $m, n \geq 3$. In fact, if T is any tree all of whose vertices have degree between 3 and n for some $n \geq 3$, then $T \sim T_3$. What could happen if we don't impose an upper bound on the degree of the vertices?

3. Show that $\mathbb{R}$ is not q.i. to T_3.

4. Show that $\mathbb{R}^2$ is not q.i. to T_3. (Hint: suppose that $\phi : R^2 \to T_3$ is a q.i.. Consider the image of a large equilateral triangle.)

5. Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a proper continuous map. ("Proper" means that $f^{-1}K$ is compact for all compact K.) Suppose there is some $k \geq 0$ such that for all $x \in \mathbb{R}^n$, $\mathrm{diam}(f^{-1}(x)) \leq k$. Then f is surjective.

   (The idea of the proof is to extend f to a continuous map between the one-point compactifications $f : \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$, and using appropriate identifications of $\mathbb{R}^n \cup \{\infty\}$ with the sphere, S^n, we can apply the Borsuk-Ulam theorem to get a contradiction.)

6. Show that any quasi-isometric map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a quasi-isometry.

7. Show that the relation of commensurablity of groups is transitive.

8. Let γ be a group acting on a geodesic space X. The action is said to be *quasi-convex* if the orbits are quasi-convex.

   (a) Show that any isometric action of $\mathbb{Z}$ on $\mathbb{R}^2$ is quasi-convex. What about $\mathbb{Z}^n$ on $\mathbb{R}^m$?

   (b) Give an example of a quasi-convex action of $\mathbb{Z}$ on $\mathbb{H}^2$. Is every action of $\mathbb{Z}$ on $\mathbb{H}^2$ quasi-convex?

   (c) Assume X is proper and suppose the action of Γ on X is proper and quasi-convex. Prove that Γ is finitely generated and for any $x_0 \in X$, the map sending $\gamma \in \Gamma$ to $\gamma x_0 \in X$ is a quasi-isometric embedding. (Hint: If Γx_0 is r-quasi-convex, consider the set of non-identity elements of Γ which move x_0 by at most $2r + 1$.)