MA4H4 Geometric Group Theory

Exercise sheet 3

January 27, 2017

1. Let d be the distance on $\mathbb{R}$ defined by $d(x, y) = |x - y|^p$ for $p > 0$. Show that this is a metric for $p \leq 1$. Show that this is length space if and only if $p = 1$.

2. Let $\gamma : [a, b] \longrightarrow X$ be a geodesic. Show that length$(\gamma) = d(\gamma(a), \gamma(b))$.

3. Suppose that $\gamma : [a, b] \longrightarrow X$ is a path. Prove that length$(\gamma) = d(\gamma(a), \gamma(b))$ if and only if $d(\gamma(t), \gamma(v)) = d(\gamma(t), \gamma(u)) + d(\gamma(u), \gamma(v))$ whenever $t, u, v \in [a, b]$ with $t \leq u \leq v$. If γ is also injective, show that it can be reparameterised as a (unit speed) geodesic.

4. Show that a length space is proper (complete and locally compact) if and only if all closed balls are compact.

5. Suppose Γ acts by isometries on a proper length space X. Show that the following are equivalent:

   - The action is cocompact.

   - Some orbit is cobounded.

   - Every orbit is cobounded.

6. Show that quasi-isometry is an equivalence relation (i.e. prove it is reflexive, symmetric and transitive).

7. Show that the Cayley graphs of $\mathbb{Z}$ with respect to the generating sets $\{a, a^2\}$ and $\{a^2, a^3\}$ are quasi-isometric to $\mathbb{R}$.

8. Let X be a geodesic space. A subset $Y \subset X$ is called r-*quasi-convex* if for all $x, y \in Y$, any geodesic from x to y (in X) is in the r-neighbourhood of Y. We simply say Y is quasi-convex if it is r-quasi-convex for some $r \geq 0$.

    (a) Show that a finite subgroup of a f.g. group Γ is quasi-convex inside any Cayley graph of Γ.

    (b) Show that this also holds for finite index subgroups of Γ.

9. (a) Consider $\mathbb{Z}^2$ with generating set $S_1 = \{(0,1), (1,0)\}$. Show that the subgroup generated by $\{(0,1)\}$ is quasi-convex in $\Delta(\mathbb{Z}^2, S_1)$.

    (b) Is the subgroup G generated by $S = \{(1,1)\}$ quasi-convex in $\Delta(\mathbb{Z}^2, S_1)$?

    (c) Let $S_2 = \{(0,1), (1,0), (1,1)\}$. Is G quasi-convex in $\Delta(\mathbb{Z}^2, S_2)$?

    (d) Is the natural inclusion a quasi-isometric embedding from $\Delta(G, S)$ to $\Delta(\mathbb{Z}^2, S_1)$ or $\Delta(\mathbb{Z}^2, S_2)$?

    (e) Show that $\Delta(\mathbb{Z}^2, S_1)$ and $\Delta(\mathbb{Z}^2, S_2)$ are quasi-isometric. Is quasi-convexity a property preserved under quasi-isometries?